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ZIV RAN

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CYCLES ON FERMAT HYPERSURFACES

Ziv Ran*

The purpose of this paper is to verify the *Hodge conjecture* for certain "Fermat hypersurfaces", i.e., hypersurfaces V_m^n in $\mathbb{P}^{n+1}$ defined by $X_0^m + \cdots + X_{n+1}^m = 0$. Recall that the Hodge Conjecture [2], also known as the "Generalized Hodge Conjecture", states that, for a projective algebraic manifold V^n and integers r, k, a class $c \in H_r(V, \mathbb{Q})$ whose Poincaré dual has Hodge filtration k should lie in the image of $H_r(W, \mathbb{Q}) \to H_r(V, \mathbb{Q})$ for some subvariety W of codimension k in V.[1] In particular, for $2n - r = 2k$ this says that if c is a Hodge class, i.e., its Poincaré dual is a (k, k) class, then c is algebraic, i.e., can be represented by (a linear combination of) algebraic subvarieties of V. The latter statement is also sometimes called the Hodge conjecture.

Our results include the following:

(1) The Hodge conjecture is true on V_m^n for $r = n = 2k$ provided m satisfies a certain arithmetical condition; this is satisfied, for instance, if m is prime or $m = 9$. In fact, for m prime, the Hodge classes are generated over $\mathbb{Q}$ by the fundamental classes of linear spaces.

The way this result comes about is this: We introduce a natural geometric operation, $*: H_{n'}(V_m^{n'}) \otimes H_{n''}(V_m^{n''}) \to H_{n'+n''+2}(V_m^{n'+n''+2})$ which associates to $c' \otimes c''$ a sort of "ruled cycle" spanned by c' and c''; by definition, $*$ preserves algebraic cycles. One of our main results asserts the *injectivity* of $*$ (on primitive homology), which in turn allows us to determine its image exactly. This, combined with explicit information on the Hodge classes [6], [7], [3], gives result (1). It also gives:

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[1] This conjecture is, in general, false in this form and has to be weakened somewhat [1], but in the cases we shall consider, we shall actually verify it in the original form.

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(2) For fixed m, there are *finitely many* Hodge classes $c_i \in H_{n_i}(V_m^{n_i})$, such that, for all n, repeated $*$-products of the c_i generate the Hodge classes in $H_n(V_m^n)$, modulo the action of the symmetric group induced by permutations of the coordinates on V_m^n.

(3) If a certain (verifiable) arithmetical statement holds, the Hodge conjecture holds on V_m^n for $r = n = 2k+1$, m prime.

By a similar but slightly different method we also prove:

(4) With a similar arithmetical provision, the generalized Hodge conjecture holds on V_m^3 for $k = 1$.

We proceed with the detailed contents of the sections. §1 establishes some notation and preliminary facts. We discuss the action of roots of unity on Fermat hypersurfaces and their homology; this leads to a somewhat explicit determination of the Hodge filtration which, by work of W. Parry can, for certain degrees m (in particular m prime), in fact be made *very* explicit for the case of Hodge classes. Finally we discuss lines on the Fermat surface, and the space spanned by their fundamental classes.

In §2 we give a topological construction related to degenerations of certain kinds of hypersurfaces in $\mathbb{P}^n$. The main point here is to show that certain open sets in $\mathbb{P}^n$ are *contractible.*

In §3 we apply the result of §2 to study the homology of a smooth hypersurface X defined by the equation $g(X_0, \ldots, X_{n-1}) + X_n^m + X_{n+1}^m = 0$ in $\mathbb{P}^{n+1}$. We introduce certain operations Φ_k, s_ℓ which put together give us a precise and effective description of the homology of X in terms of that of $X \cap \{X_{n+1} = 0\}$ and that of $X \cap \{X_n = X_{n+1} = 0\}$; and, what's most important, Φ_k in fact preserves the rank of cycles; specializing to the Fermat case, we get result 4 above.

In §4 we introduce the operation $\star$ (in a more general context) by means of a projective-geometric construction. The main fact to prove about it is its *injectivity* (on primitive homology). We prove this in the Fermat case by an application of the machinery of §3, using the compatibility of $\star$ with Φ_k and s_ℓ; the more general case follows immediately from this by an easy deformation argument. Given the information about the Hodge filtration, this gives results 1, 2, 3.

Note: After seeing a preliminary draft of this paper, T. Shioda has given another proof of the injectivity of $*$, and has deduced a version of result 1 above with a similar but less restrictive arithmetical assumption. Cf., *Math. Ann.*, *245* (1979), 175–184.

Finally it is a pleasure to record here my thanks to Arthur Ogus for his generous advice and many keen suggestions related to this paper.

I would also like to thank Blaine Lawson and Robin Hartshorne for helpful discussions and encouragement.

§1. Notations and preliminary results

We begin by making a few remarks of a trivial nature concerning representations of finite abelian groups, for the convenience of the reader.

Let Γ be a finite abelian group, and let H be a finite dimensional complex Γ-vector space.

(1.1) We have $H = \bigoplus\{H_\chi : \chi \in \Gamma^\star\}$, where $\Gamma^\star = \mathrm{Hom}(\Gamma, \mathbf{C}^x)$ is the dual group and $H_\chi = \{c \in H : \gamma(c) = \chi(\gamma)c,\ \forall \gamma \in \Gamma\}$. In particular, if $\Gamma \simeq Z/mZ$ then $\Gamma^\star \simeq Z/mZ$ too, so that we have $H = \bigoplus_{r \in Z/mZ} H_r$.

(1.2) Any Γ-equivariant surjection $H \xrightarrow{\alpha} H'$ has a Γ-equivariant cross-section, so that $H \simeq H' \oplus \ker \alpha$, Γ-equivariantly.

(1.3) Consequently, H contains no nonzero vector fixed by $\Gamma \Leftrightarrow$ [in the above decomposition (1.1), $H_0 = 0$] $\Rightarrow$ [*in* (1.2) H' contains no nonzero vector fixed by Γ].

All our varieties will be projective and over $\mathbf{C}$, and will be considered as topological spaces in their complex topology. Our homology and cohomology will be with $\mathbf{C}$-coefficients, unless otherwise stated, and for nonsingular varieties we shall usually identify them via Poincaré duality, which will allow us to talk about pull-back and intersection product (= Poincaré dual of cup product) on homology.

For a variety V of dimension n in $\mathbf{P}^N$, possibly singular and reducible, we denote by $\omega_i \in H_i(V)$ the homology class of the intersection of V with a general $\mathbf{P}^{N-n+(i/2)}$ if i is even, 0 if i is odd. We denote by $P_i(V)$ "the primitive homology" of V which we define to be the kernel of $H_i(V) \to H_i(\mathbf{P}^N)$. These definitions of course depend on the projective embedding, but in all the varieties we shall be considering such an embedding will be implicit. In case V is nonsingular connected, $P_i(V)$ coincides with the Poincaré dual of usual primitive cohomology, i.e. the kernel of cup-product with the hyperplane section class. Thus $H_i(V) = P_i(V) \oplus \mathbf{C}\omega_i$. It is clear from our definitions that if we have a cartesian diagram

$$\begin{array}{ccc} V & \xrightarrow{f} & V' \\ \cap & & \cap \\ \mathbf{P}^N & \longrightarrow & \mathbf{P}^M \end{array}$$

then $f_\star P_i(V) \subseteq P_i(V')$ and $f_\star(\omega_i)$ = multiple of ω_i. Now fix an integer $m \geq 1$ and let $\zeta = \zeta_m = e^{2\pi\sqrt{-1}/m}$. Denote by σ_n the projective transformation of $\mathbf{P}^N$, $N \geq n$ defined by $[X_0, \ldots, X_n, \ldots, X_N] \rightarrow [X_0, \ldots, \zeta X_n, \ldots, X_N]$. Let Γ^n be the subgroup of $\mathrm{PGL}(n+1)$ generated by $\sigma_0, \ldots, \sigma_{n+1}$, so that $\Gamma^n \cong \bigoplus_0^{n+1} \mathbf{Z}/m\mathbf{Z}/(\text{diagonal})$ and $\Gamma^{n\star} = \{\chi = (\chi_0, \ldots, \chi_{n+1}) \in \bigoplus_0^{n+1} \mathbf{Z}/m\mathbf{Z} : \Sigma\, \chi_i = 0\}$; we have an obvious map $\Gamma^{n'} \rightarrow \Gamma^n$ for $n' \geq n$, namely $\sigma_i \mapsto \sigma_i$, $i \leq n$, $\sigma_i \mapsto 1$ otherwise. Also put $\Delta^n = \langle \sigma_0, \ldots, \sigma_{n-1}, \sigma_n \circ \sigma_{n+1} \rangle$,[1] $\Lambda^n = \langle \sigma_n, \sigma_{n+1} \rangle$. Now let X be a nonsingular hypersurface in $\mathbf{P}^{n+1}$ with homogeneous equation $f(X_0, \ldots, X_n) + X_{n+1}^m = 0$. Then σ_{n+1} acts on X and we claim:

LEMMA 1.4: *$\sigma_{n+1\star}$ fixes no nonzero vector in $P_i(X)$.*

PROOF: Note that via the projection $X \rightarrow \mathbf{P}^n : \pi[X_0, \ldots, X_{n+1}] = [X_0, \ldots, X_n]$, $\mathbf{P}^n$ becomes topologically the qutoient space of X under $\langle \sigma_{n+1} \rangle$. Therefore, identifying homology and cohomology, image $\pi^* = H_i(X)^{\langle \sigma_{n+1} \rangle}$. Since image $\pi^* = \mathbf{C}\omega_i$, Q.E.D. □

By (1.3) we can state the lemma also as:

(1.5) In the eigenspace decomposition $P_i(X) = \bigoplus_{r \in \mathbf{Z} \in \mathbf{Z}/m\mathbf{Z}} C_r$, $C_0 = 0$.

Note that this implies

(1.6) For $c \in P_i(X)$, $\Sigma_{j=0}^{m-1} \sigma_{n+1\star}^j(c) = 0$.

We now turn our attention to Fermat hypersurfaces V_m^n. Note that Γ^n acts on V_m^n; the next proposition, communicated by Ogus, describes the corresponding representation on $P_n(V)$ and its relation to the Hodge filtration. The result follows immediately from results in [6]; we have however made the (obvious) translation from cohomology to homology.

[1] $\langle\ \rangle$ means subgroup generated by.

PROPOSITION 1.7: (i) *With respect to the action of* Γ^n, *the character decomposition (eigenspace decomposition) of* $P_n(V_m)$ *is as follows*:

$$P_n(V_m^n) = \bigoplus\{H_\chi : \chi : (\chi_0, \ldots, \chi_{n+1}) \in \Gamma^{n\star}, \chi_j \neq 0, \forall j\}$$

with each H_χ *1-dimensional. (The* χ *appearing in the latter decomposition will be called "relevant characters.")*

(ii) *For the Hodge filtration* $F^{\cdot}$ *on* $P^n(V_m^n)$, *the Poincaré dual of* $F^kP^n(V_m^n)$ *is given by* $\bigoplus\{H_\chi : \chi$ *relevant and* $s(\chi) \leq (n-k+1)m\}$, *where* $s(\chi) = s(x_0, \ldots, \chi_{n+1}) = \Sigma \langle \chi_j \rangle$ *(recall that* $\langle \cdot \rangle$ *denote the unique integer representative between* 0 *and* $m-1$).

(iii) *Let* $(\mathbb{Z}/m\mathbb{Z})^x$ *act on* $\Gamma^{n\star}$ *by coordinatewise multiplication. Then the Poincaré dual of* $[F^kP^n(V_m^n) \cap H^n(V, \mathbb{Q})] \otimes \mathbb{C}$ is $\bigoplus\{H_\chi : \chi$ *is relevant and* $s(\epsilon\chi) \leq (n-k+1)m$, $\forall \epsilon \in (\mathbb{Z}/m\mathbb{Z})^x\}$. *In particular for* n *even, the "Hodge subspace," i.e. then Poincaré dual of* $[H^{n/2,n/2}(V_m^n) \cap H^n(V, \mathbb{Q})] \otimes \mathbb{C}$ *is* $\bigoplus\{H_\chi : \chi$ *relevant and* $s(\epsilon\chi) = \left(\frac{n}{2}+1\right)m$, $\forall \epsilon \in (\mathbb{Z}/m\mathbb{Z})^x\}$ *(these latter* χ *will be called "Hodge characters").* □

We shall need more precise information on the Hodge subspace and the Hodge filtration than Proposition 1.7 provides. Such information comes from recent arithmetical work of W. Parry [7], and we will now state his results in the form we shall use:

For $\chi' \in \Gamma^{n'\star}$, $\chi'' \in \Gamma^{n''\star}$ denote by $\chi' \star \chi'' \in \Gamma^{n'+n''+2\star}$ the character obtained from χ' and χ'' by "juxtaposition," i.e. if $\chi' = (\chi_0', \ldots, \chi_{n+1}')$, $\chi'' = (\chi_0'', \ldots, \chi_{n''+1}'')$ then $\chi' \star \chi'' = (\chi_0', \ldots, \chi_{n'+1}', \chi_0'', \ldots, \chi_{n''+1}'')$.

PROPOSITION 1.8: (i) *For* m *prime, the Hodge characters in all dimensions are generated under* $\star$, *up to permutations, by those in dimension* 0, *namely by* $(1, m-1), \ldots, (m-1, 1)$.

(ii) *For* $m = p^2$, p *prime, there are generators-up-to-permutations in dimensions* 0 *and* $p-1$. □

(Of course, the "dimension" of $\chi \in \Gamma^{n\star}$ is n.)

PROPOSITION 1.9: *For arbitrary* m *there is always a finite set of generators-up-to-permutations of the Hodge characters.* □

For the proof of these statements, it is convenient to represent a character χ by the function $f : [\mathbb{Z}/m\mathbb{Z}\backslash\{0\}] \to \mathbb{Z}$ which assigns to a residue class the number of times it occurs in χ. Then 1.8(i) follows immediately from the proposition in [3]: namely, they prove that an f

corresponding to χ Hodge can be written $f = \Sigma\, r_i f_i$ with f_i corresponding to $(1, m-1) \ldots \left(\left[\frac{m}{2}\right], m - \left[\frac{m}{2}\right]\right)$, $r_i \in \mathbb{Q}$. Since f_i have disjoint supports, it follows that r_i must be positive integers, which gives what we want. 1.8(ii) is more complicated, but we won't really use it. 1.9 follows from a general lemma of Gordan saying that in $\mathbb{Z}^k$ the semigroup of positive integral solutions to a finite number of linear equations with integer coefficients is finitely generated. Cf. *Vorlesungen über Invariantentheorie*, vol. 1, p. 199 (Leipzig: Teubner, 1885).

I also want to record here for future reference two statements, each of which, when true, will give us, as we shall later see, a case of the Hodge conjecture. (Of course, in view of Proposition 1.7, it is possible to check them by hand (or computer) in any given instance.)

STATEMENT 1.10_m: Suppose H_χ is contained in the Poincaré dual of $[F^1P^3(V_m^3) \cap H^3(V_m^3, \mathbb{Q})] \otimes \mathbb{C}$, $m \geq 3$. Then by a suitable permutation of coordinates we can arrange that $\chi_3 + \chi_4 \neq 0$ and that $(\chi_0, \chi_1, \chi_2, \chi_3 + \chi_4)$ be a Hodge character on V_m^2. □

In the second statement we take m prime.

STATEMENT $1.11_{m,k}$: The Poincaré dual of $[F^kP^{2k+1}(V_m^{2k+1}) \cap H^{2k+1}(V_m^{2k+1}, \mathbb{Q})] \otimes \mathbb{C}$ is $\bigoplus\{H_\chi : \chi = (\chi_0, \ldots, \chi_{2k+2})$ contains $k-1$ pairs of opposites$\}$. □

I have verified these statements for $m \leq 8$, $k \leq 2$.

REMARK 1.12: Take for example $k = 1$, $n = 3$. It is clear that if χ contains a pair of opposites, say $\chi_1 = -\chi_0$, then the conclusion of statement $(1.10)_m$ holds for for χ, since then if, say, $\chi_3 + \chi_4 \neq 0$ then $(\chi_0, \chi_1, \chi_2, \chi_3 + \chi_4)$ consists of two pairs of opposites, which makes it a Hodge character.

EXAMPLE 1.13: Take $n = 3$, $m = 8$. One verifies that the characters χ with $H_\chi \subset$ Poincaré dual of $[F^1P^3(V_m^3) \cap H^3(V_m^3, \mathbb{Q})] \otimes \mathbb{C}$ either contain a pair of opposites or are permutations of (14551) or (23722), and for the latter the conclusion of statement $(1.10)_m$ holds. In particular, statement $(1.10)_m$ holds for $m = 8$.

Finally I want to give a result for the Fermat *surface* which will later be generalized to higher dimensions.

On $V^2 = V^2_m$, let $L_{i,j}$ denote the homology class of the line with equations $X_0 = \xi\zeta^i X_1$, $X_2 = \xi\zeta^j X_3$, $\xi = m$-th root of -1.

PROPOSITION 1.14: *$\{M_{ij} := L_{i,j} - L_{i,j+1} - L_{i+1} + L_{i+1,j+1}\,; i, j = 0, \ldots, m-2\}$ forms a basis for $\bigoplus\{H_\chi : \chi_0 = -\chi_1, \chi_2 = -\chi_3\}$.*

PROOF: Since the latter space has dimension $(m-1)^2$ and clearly contains $M_{i,j}$, it suffices to prove that the latter are linearly independent, which will follow once we prove their intersection matrix to be nonsingular. Now from the facts that $L^2_{i,j} = 2 - m$ (adjunction formula) and that $L_{i,j}, L_{i',j'} = \delta_{i,i'} + \delta_{j,j'}$ for $(i, j) \neq (i', j')$, it follows that the intersection matrix of the $M_{i,j}$ has the form

$$A = m \cdot \begin{pmatrix} 2B & -B & 0 & & \bigcirc \\ -B & 2B & -B & & \\ & & \ddots & & \\ & \bigcirc & & & -B \\ & & & -B & 2B \end{pmatrix} \quad \text{where } B = \begin{pmatrix} -2 & 1 & 0 & & & \\ 1 & -2 & 1 & 0 & \bigcirc & \\ 0 & 1 & -2 & & & \\ & & & \ddots & & 1 \\ & \bigcirc & & & 1 & -2 \end{pmatrix}$$

$(m-1)B$ is the intersection matrix of $\{L_{0,j} - L_{0,j+1} : j = 0, \ldots, m-2\}$. Note that $A = mB \otimes B$, so to prove A nonsingular it suffices to prove B is. One way to see the latter is to note that $L_{0,j} \cdot L_{1,j'} = \delta_{j,j'}$, so $\{L_{0,j} : j = 0, \ldots, m-1\}$ are linearly independent. Therefore so are $\{L_{0,j} - L_{0,j+1} : j = 0, \ldots, m-2\}$. Hence by the Hodge index theorem their intersection matrix $(m-1)B$ is nonsingular. □

§2. The topological construction

By way of introduction to this section, consider the following problem: given a hypersurface D in $\mathbb{P}^n$, find a pencil of hypersurfaces $\{D_t\}_{t \in \mathbb{P}^1}$ with, say $D_1 = D$, such that for some path β in $\mathbb{P}^1$, if we let $L \overset{\text{def}}{=} \bigcup_{t \in \beta} D_t$, then $\mathbb{P}^n - L$ is *contractible*. (Motivation for considering this problem is provided in §3). Consideration of the case $n = 1$ clearly suggests what the answer might be: namely take $\{D_t\}$ to be a pencil with $D_0 =$ multiple hyperplane, $\beta =$ path from 1 to 0. This in fact turns out to be true, at least in some cases, as we shall see presently.

We shall consider in detail only the case which will be of use later

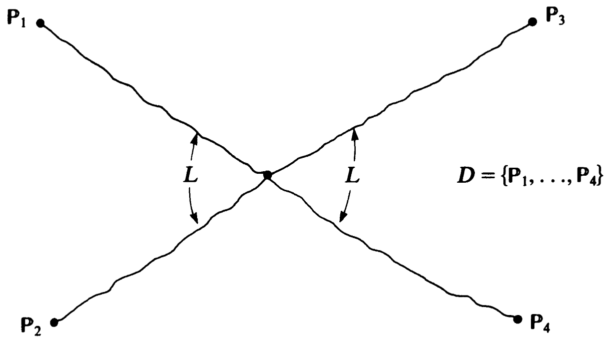

Fig. 1. $P^n - L$ is contractible; an illustration for $n = 1$.

on; so assume now that D has homogeneous equation of the form $g(X_0, \ldots, X_{n-1}) + X_n^m = 0$, where $X_0, \ldots, X_n$ are homogeneous coordinates in P^n. For $t \in C$ define $D_t \subset P^n$ by the equation $tg(X_0, \ldots, X_{n-1}) + X_n^m = 0$, and let $L \overset{\text{def}}{=} \bigcup_{t \in [0,1]} D_t$.

LEMMA 2.1: *For any neighborhoods N of D_0, N' of L there exists a diffeomorphism ψ of P^{r+1} such that $\psi|_{P^{r+1}-N'} =$ identity, and $\psi(L) \subseteq N$.*

PROOF: Let $\{\psi_s'\}$ be the flow on P^{r+1} given by $[X_0, \ldots, X_{r+1}] \to [X_0, \ldots, X_r, e^{-s}X_{r+1}]$. Then for $s \gg 0$, surely $\psi_s'(L) \subseteq N$. (Note that the fixed points of $\{\psi_s'\}$ consists of $[0, \ldots, 0, 1]$ ("repelling") plus the hyperplane $\{X_{r+1} = 0\}$ (which attracts everything).) Let $\{\psi_s'\}$ be given by a vector field v'. Let α be a C^∞ function on P^{r+1} which is $=1$ near L and $=0$ in an open neighborhood of $\overline{P^{r+1} - N'}$. Let $\{\psi_s\}$ be the flow determined by $\alpha v'$. Then $\psi_s|_{P^{r+1}-N'} =$ identity for all s; also since $\psi_s'(L) \subseteq L$ for $s \geq 0$, it follows that $\psi_s'|_L = \psi_s|_L$ for $s \geq 0$. So we can take $\psi = \psi_s$ for $s \gg 0$. □

A formal consequence of this is the following:

COROLLARY 2.2: $\pi_i(P^{r+1} - L) = H_i(P^{r+1} - L) = 0$ *for* $i \geqq 1$ *and* $P^{r+1} - L$ *is connected.*

PROOF: We give the proof for π_i; the rest is proved similarly. Let c be a representative of a class in $\pi_i(P^{r+1} - L)$. By compactness there is a neighborhood N' of L in P^{r+1}, so that $c \subseteq P^{r+1} - N'$. Now D_0 is a hyperplane in P^{r+1}, so $P^{r+1} - D_0 = C^{r+1}$ is contractible. Hence there is a homotopy F of c to a constant. Again by compactness there is a neighborhood N of D_0 such that image$(F) \subseteq P^{r+1} - N$. Now let ψ be given by Lemma 2.1. Then $\psi^{-1}(\mathbb{P}^{r+1}) \subseteq \mathbb{P}^{r+1} - L$ and $\psi^{-1}(c) = c$. Therefore $\psi^{-1} \circ F$ is a homotopy of c to a constant in $\mathbb{P}^{r+1} - L$. Q.E.D. □

The proof of the above corollary shows that it continues to hold whenever the following assumptions hold:

(1) D_0 is a (multiple) hyperplane.

(2) The pencil $\{D_t\}$ forms a "fibre space" in the following sense: there are finitely many points $p_1, \ldots, p_\ell \in \mathbb{P}^1$, with $1 \notin \{p_1, \ldots, p_\ell\}$ such that if $\{\tau_s\}_{s \in [0,1]}$ is an isotropy of $id_{\mathbb{P}^1}$, fixing a neighborhood of $\{p_1, \ldots, p_\ell\}$, then there is an isotropy $\{\tilde{\tau}_s\}$ of $\mathrm{id}_{\mathbb{P}^n}$ which "lifts $\{\tau_s\}$" in the sense that $\tilde{\tau}_s(D_t) = D_{\tau_s(t)}$.

In fact, standard deformation theory [5], [8] shows that assumption 2 is satisfied, for instance, whenever D_1 is smooth. Thus the conclusion of the corollary still holds whenever D_1 is smooth, D_0 is a multiple hyperplane and β is a simple path between 0 and 1, omitting all "bad points" except 0.

§3. A homology description

In this section V will denote a nonsingular hypersurface in $\mathbb{P}^{n+1}$ with equation $g(X_0, \ldots, X_{n-1}) + X_n^m + X_{n+1}^m$. We project V to $\mathbb{P}^n$:

$$\begin{array}{c} V \\ {\scriptstyle\pi}\big\downarrow \\ \mathbb{P}^n \end{array}$$

by $\pi([X_0, \ldots, X_{n+1}]) = [X_0, \ldots, X_n]$, and note that the branch locus of π in $\mathbb{P}^n$ is the hypersurface D with equation $g(X_0, \ldots, X_{n-1}) + X_n^m$. Thus we may apply the construction of §2 to D, as indeed we do, letting D_t and L be as there. Also set $\tilde{D}_t = \pi^{-1}D_t$, $\tilde{L} = \pi^{-1}L$. The point of that construction can now be stated:

PROPOSITION 3.1: *The map $H_j(\tilde{L}) \to H_j(V)$ induced by inclusion is an isomorphism for $j \leq 2n-2$ and is surjective for $j = 2n-1$.*

PROOF: Consider the sequence $H_{j+1}(V, \tilde{L}) \to H_j(\tilde{L}) \to H_j(V) \to H_j(V, \tilde{L})$. By Lefschetz duality, $H_i(V, \tilde{L}) = H^{2n-i}(V - \tilde{L})$. Now since L contains the branch locus of π, $\pi : V - \tilde{L} \to \mathbb{P}^n - L$ is a covering; but by Corollary 2.2 $\mathbb{P}^n - L$ is contractible. Thus $V - \tilde{L}$ is a union of m contractible pieces, so $H^{2n-i}(V - \tilde{L}) = 0$ for $i < 2n$, and the proposition follows. □

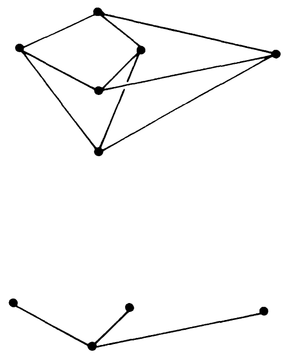

Fig. 2. Picture of L and $\tilde{L}$ for $n = 1$, $m = 3$.

We now turn to examining $\tilde{L}$. The picture we want to portray is of $\tilde{L}$ as essentially a fibre space over $[0, 1]$ except for some pinching down occurring (in different ways!) at both ends. This leads to a description of the homology of $\tilde{L}$ by cycles on the general fiber which "vanish" at both ends, à la Lefschetz. We now make this precise.

Note that on V, $\tilde{D}_t$ is defined by the equation $(t-1)X_n^m = tX_{n+1}^m$, and hence for $t \neq 0, 1$, $\tilde{D}_t$ is reducible with components $\tilde{D}_{t,i} = \{t^{1/m}X_{n+1} = \zeta^i(1-t)^{1/m}X_n\}$, for $i = 0, \ldots, m-1$, where $t^{1/m}$ denotes the positive m-th root and $\zeta = e^{2\pi\sqrt{-1}/m}$. Set $E = \tilde{D}_{1/2}$, $E_i = \tilde{D}_{1/2,i}$. Also put $B = V \cap \{X_n = X_{n+1} = 0\}$. Note that B is the base locus of the pencil $\tilde{D}_T$, i.e. $\tilde{D}_t \cap \tilde{D}_{t'} = B$ whenever $t \neq t'$. Also $B = \tilde{D}_{t,i} \cap \tilde{D}_{t,i'}$ whenever $i \neq i'$. B itself is a (nonsingular, by the Jacobian criterion) hypersurface in $\mathbf{P}^{n-1}$.

LEMMA 3.2: *Let* $\tilde{L}_0 = \bigcup_{0 \leq t \leq 3/4} \tilde{D}_t$, $\tilde{L}_{1/2} = \bigcup_{1/4 \leq t \leq 3/4} \tilde{D}_t$, $\tilde{L}_1 = \bigcup_{1/4 \leq t \leq 1} \tilde{D}_t$. *Then* $\tilde{D}_u$ *is a deformation retract of* $\tilde{L}_u$ *for* $u = 0, \frac{1}{2}, 1$.

PROOF: Define a map $Q: E \times [0, 1] \to \tilde{L}$ by

$$Q([X_0, \ldots, X_{n+1}], t) = [X_0, \ldots, X_{n-1}, 2^{1/m}t^{1/m}X_n, 2^{1/m}(1-t)^{1/m}X_{n+1}].$$

Note that Q has the properties:

(a) $Q(E, t) = \tilde{D}_t$.

(b) Q is topologically a quotient map (being a surjective map of compact Hausdorff spaces).

(c) If $Q(x, t) = Q(x', t')$ and $t, t' \neq 0, 1$ then either $(x, t) = (x', t')$ or $x = x' \in B$.

Now define homotopies F_s, G_s, H_s of the identity on $E \times [0, 1]$ by

$$
\begin{aligned}
F_s(x, t) &= (x, (1-s)t) \\
G_s(x, t) &= (x, \tfrac{1}{2} + (1-s)(t - \tfrac{1}{2})), \\
H_s(x, t) &= (x, s + (1-s)t).
\end{aligned}
$$

It is then a straightforward matter to check, using properties (a), (b) and (c) above that F_s, G_s, H_s descend via Q to define the requisite deformations, in the requisite order. □

REMARK 3.3: Note that, in the above notation, $P_0 \overset{\mathrm{def}}{=} F_{1|E} : E \to \tilde{D}_0$ and $P_1 \overset{\mathrm{def}}{=} H_{1|E} : E \to \tilde{D}_1$ are explicitly given by

$$
\begin{aligned}
P_0[X_0, \ldots, X_{n+1}] &= [X_0, \ldots, X_{n-1}, 0, 2^{1/m} X_{n+1}], \\
P_1[X_0, \ldots, X_{n+1}] &= [X_0, \ldots, 2^{1/m} X_n, 0].
\end{aligned}
$$

Note further that all the maps $\tilde{D}_t \to \tilde{D}_{t'}$ induced by the above deformations all have the form $[X_0, \ldots, X_{n+1}] \to [X_0, \ldots, X_{n-1}, aX_n, bX_{n+1}]$ for some a, b. □

Now consider the following diagram:

$$
\begin{tikzcd}
(\star) & H_n(\tilde{L}_{1/2}) \arrow[r, "\iota_\star \oplus -1_\star"] & H_n(\tilde{L}_0) \oplus & H_n(\tilde{L}_1) \arrow[r, "\iota_\star + \iota_\star"] & H_n(\tilde{L}) \arrow[r] & H_{n-1}(\tilde{L}_{1/2}) \arrow[r, "\iota_\star \oplus -\iota_\star"] & H_{n-1}(\tilde{L}_0) \oplus & H_{n-1}(\tilde{L}_1) \\
(\star\star) & H_n(E) \arrow[u, "\wr"'] \arrow[r, "p_{0\star} \oplus -p_{1\star}"] & H_n(\tilde{D}_0) \oplus \arrow[u, "\wr"'] & H_n(\tilde{D}_1) \arrow[u, "\wr"'] \arrow[r, "\iota_\star + \iota_\star"] & H_n(\tilde{L}) \arrow[u, "\wr"'] \arrow[r] \arrow[d, "i_\star"] & H_{n-1}(E) \arrow[u, "\wr"'] \arrow[r, "p_{0\star} \oplus -p_{1\star}"] & H_{n-1}(\tilde{D}_0) \oplus \arrow[u, "\wr"'] & H_{n-1}(\tilde{D}_1) \arrow[u, "\wr"'] \\
 & & & & H_n(V) & & &
\end{tikzcd}
$$

in which: $(\star)$ is the Mayer–Vietoris sequence for $\tilde{L} = \tilde{L}_0 \cup \tilde{L}_1$, p_0, p_1 are as above, i stands for the various inclusions, and the vertical isomorphisms are also induced by inclusions.

Now note that Λ^n acts compatibly on the various spaces above; $(\star)$, and therefore $(\star\star)$, are thus Λ^n-equivariant. Furthermore, $\tilde{D}_0$ and $\tilde{D}_1$ are both smooth hypersurfaces of dimension $n-1$, so that $H_n(\tilde{D}_i) = \mathbf{C}\omega_n$ and image$(H_n(\tilde{D}_0) \oplus H_n(\tilde{D}_1) \to H_n(V))$ is spanned by ω_n; also $i_\star : H_n(\tilde{L}) \to H_n(V)$ is an isomorphism for $n \geq 2$, by Proposition 3.1, so finally the map $H_n(\tilde{L}) \to H_{n-1}(E)$ is injective on $i_\star^{-1}(P_n(V))$ for all n. Taking its inverse where defined we obtain:

LEMMA 3.4: *There is a Λ^n-equivariant map*

$$\Psi: W_n \stackrel{\mathrm{def}}{=} \{\ker(p_{0\star}) \cap \ker(p_{1\star}) \subset H_{n-1}(E)\} \to P_n(V);$$

for $n \leq 2$, Ψ is an isomorphism while for $n = 1$, Ψ is surjective and its kernel equals the image of the composite $H_2(V, \tilde{L}) \to H_1(\tilde{L}) \to H_0(E)$. □

(The last statement of course comes from the fact that for $n = 1$, $\mathrm{Ker}(H_1(\tilde{L}) \to H_1(V)) = \mathrm{image}(H_2(V, \tilde{L}) \to H_1(\tilde{L}))$ by the proof of Proposition 3.1.) Geometrically ψ should be interpreted as taking an $(n-1)$-cycle c on $\tilde{D}_{1/2}$ to the n-cycle it describes as it is being dragged off to vanish at $\tilde{D}_0$ and $\tilde{D}_1$; thus it is a version of the "cone on a vanishing cycle" construction of Lefschetz.

REMARK 3.5: Note that Ψ will in fact also be equivariant with respect to any bigger group acting on the situation ($\star$), ($\star\star$) above. For instance in case V is a Fermat hypersurface the whole group Γ_n acts on that situation, so in this case Ψ is actually Γ_n-equivariant.

Next we turn to the construction of some (in fact, as we shall later see, of all) elements of W_n.

Let $q_i : D \to E_i$ be defined by

$$q_i([X_0, \ldots, X_n]) = [X_0, \ldots, X_{n-1}, 2^{1/m}X_n, \zeta^i 2^{-1/m} X_n], \quad i = 0, \ldots, m-1.$$

q_i is an isomorphism, so we may use it to identify E_i with D. Note the following identities:

$$\begin{aligned} &p_0 \circ q_i = \sigma_{n+1}^i \circ p_0 \circ q_0, && p_1 \circ q_i = p_1 \circ q_0, \\ &\sigma_n \circ q_i = q_{i-1} \circ \sigma_n, && \sigma_{n+1} \circ q_i = q_{i+1}. \end{aligned} \tag{3.6}$$

Denote by i_j the inclusion $E_j \to E$. Now σ_n acts on $P_{n-1}(D)$. For $r \in \mathbb{Z}/m\mathbb{Z}$ let C_r be the ζ^r-eigenspace; i.e. for $c \in C_r$, $\sigma_{n_\star}(c) = \zeta^r c$. Then by (1.5) we know $C_r \neq (0)$ only if $r \neq 0 \in \mathbb{Z}/m\mathbb{Z}$. Let $K_k = \bigoplus\{C_r : r \neq -k\} \subseteq P_{n-1}(D)$ for $k \neq 0$, and define

$$\begin{aligned} &\hat{\Phi}_k : K_k \to P_{n-1}(E) \text{ by} \\ &\hat{\Phi}_k(c) = \sum_{j=0}^{m-1} \zeta^{jk} i_{j_\star} q_{j_\star}(c). \end{aligned} \tag{3.7}$$

By (3.6), we obtain:

$$\sigma_n \circ \hat{\Phi}_k = \zeta^k \hat{\Phi}_k \circ \sigma_n, \quad \sigma_{n+1} \circ \hat{\Phi}_k = \zeta^{-k} \hat{\Phi}_k. \tag{3.8}$$

LEMMA 3.9: *$\hat{\Phi}_k$ goes into W_n.*

PROOF: We must show $p_{0\star}\hat{\Phi}_k$ and $p_{1\star}\hat{\Phi}_k$ vanish, so we just compute, for $c \in C_r$, $r \neq k$:

$$\begin{aligned}
p_{0\star}\hat{\Phi}_k(c) &= \sum_{i=0}^{m-1} \zeta^{ki}(p_0 \circ q_i)_\star(c) \\
&= \sum_{i=0}^{m-1} \zeta^{ki}\sigma^i_{n+1\star}(p_0 \circ q_0)_\star(c) \quad \text{(by (3.6))} \\
&= \sum_{i=0}^{m-1} \zeta^{ki}(p_0 \circ q_0)_\star(\sigma^i_{n\star}(c)) \\
&= \sum_{i=0}^{m-1} \zeta^{(k+r)i}(p_0 \circ q_0)_\star(c) = 0 \\
p_{1\star}\hat{\Phi}_k(c) &= \sum_{i=0}^{m-1} \zeta^{ki} p_{1\star} q_{i\star}(c) \\
&= \sum_{i=0}^{m-1} \zeta^{ki}(p_1 \circ q_0)_\star(c) = 0 \quad \text{(by (3.6))}. \qquad \square
\end{aligned}$$

To get the rest of W_n for $n \geq 2$ we have to make another construction. Consider for $\ell = 1, \ldots, m-1$ the Meyer–Vietoris sequence

$$\begin{aligned}
\ldots H_{n-1}(E_0) \oplus H_{n-1}(E_\ell) &\xrightarrow{\alpha} H_{n-1}(E_0 \cup E_\ell) \xrightarrow{\partial_\ell} H_{n-2}(B) \\
&\longrightarrow H_{n-2}(E_0) \oplus H_{n-2}(E_\ell) \ldots
\end{aligned}$$

The image of ∂_ℓ is clearly $P_{n-2}(B)$. Also $\omega_n \in H_{n-1}(E_0 \cup E_\ell)$ is clearly in the image of α. Hence $\partial_\ell : P_{n-1}(E_0 \cup E_\ell) \to P_{n-2}(B)$ is already onto. Now note that $\sigma_{n+1} \circ \sigma_n$ acts on the situation (the action of B, of course, being trivial). As $\sigma_{n+1} \circ \sigma_n$ generates a finite group $\mathbb{Z}/m\mathbb{Z}$, ∂_ℓ admits an equivariant cross-section $\hat{s}_\ell : P_{n-2}(B) \to P_{n-1}(E_0 \cup E_\ell)$. By abuse of notation, we get a map $\hat{s}_\ell : P_{n-2}(B) \to P_{n-1}(E)$. Note that $\hat{s}_\ell$ is only defined if $n \geq 2$.

REMARK 3.10: Note as before that $\hat{s}_\ell$ can in fact be made equivariant with respect to any bigger finite group acting on the above situation. For example, in the Fermat hypersurface case, Δ^n so acts, so in that case we may assume $\hat{s}_\ell$ is in fact Δ^n-equivariant.

LEMMA 3.11: *$\hat{s}_\ell$ goes into W_n.*

PROOF: By construction, $(\sigma_{n+1}\circ\sigma_n)_\star\circ\hat{s}_\ell = s_\ell$. Since $p_1\circ\sigma_{n+1}\circ\sigma_n = \sigma_n\circ p_1$, we get the image of $p_{1\star}\circ\hat{s}_\ell$ is fixed by $\sigma_{n\star}$. But that image is contained in $P_{n-1}(D)$, hence by (1.4) it is zero, i.e. $p_{1\star}\circ\hat{s}_\ell = 0$. The proof that $p_{0\star}\hat{s}_\ell = 0$ is similar. □

LEMMA 3.12:
(i) *The map $P_{n-1}(E_0)\oplus\cdots\oplus P_{n-1}(E_{m-1})\to P_{n-1}(E)$ is injective.*
(i) $\left(\sum_\ell \text{image}(\hat{s}_\ell)\right)\cap\left(\sum_r \text{image } i_{r\star}\right) = 0.$
(iii) *$\hat{\Phi}_k$, $\hat{s}_\ell$ are all injective and their images are linearly independent.*

PROOF: (i) We prove by induction on j that $P_{n-1}(E_0)\oplus\cdots\oplus P_{n-1}(E_j)\to P_{n-1}(\bigcup_{0\le r\le j} E_r)$ is injective. For $j=0$ this is easy. Assuming this is true for j, consider the Mayer–Vietoris sequence:

$$H_{n-1}(B)\to H_{n-1}\left(\bigcup_{0\le r\le j} E_r\right)\oplus H_{n-1}(E_{j+1})\to H_{n-1}\left(\bigcup_{0\le r\le j+1} E_r\right)\ldots$$

As $H_{n-1}(B)$ is spanned by ω_{n-1}, $P_{n-1}(\bigcup_{0\le r\le j} E_r)\oplus P_{n-1}(E_{j+1}) \hookrightarrow P_{n-1}(\bigcup_{0\le r\le j+1} E_r)$, hence the assertion holds for $j+1$. This proves (i).

(ii) Take $c\in\sum_\ell$ image $\hat{s}_\ell$ and suppose $c = \sum_{r=0}^{m-1} i_{r\star}(c_r)$; as c is primitive we may assume, replacing the c_r by their primitive parts, that $c_r\in P_{n-1}(E_r)$. Write $c_r = q_{r\star}(d_r)$, $d_r\in P_{n-1}(D)$. Now as $(\sigma_{n+1}\circ\sigma_n)_\star\circ\hat{s}_\ell = \hat{s}_\ell$, $(\sigma_{n+1}\circ\sigma_n)_\star(c) = c$ and since $\sigma_{n+1}\circ\sigma_n\circ i_r\circ q_r = i_r\circ q_r\circ\sigma_n$, we see by (i) that $\sigma_{n\star}(d_r) = d_r$. Hence by (1.4), $d_r = 0$, so $c = 0$.

(iii) Injectivity of $\hat{\Phi}_k$ follows from (i), and that of $\hat{s}_\ell$ from the definition. The independence of the images of $\hat{\Phi}_k$ for the different $k\neq 0$ follows because, by (3.8), $\sigma_{n+1\star}(c) = \zeta^{-k}(c)$ for $c\in$ image $\hat{\Phi}_k$ (and hence

$$\begin{aligned}\sum_{j=0}^{m-1}(\zeta^k\sigma_{n+1\star})^j(c) &= mc \quad \text{if } c\in\text{image }\hat{\Phi}_k\\ &= 0 \qquad \text{if } c\in\sum_{k'\neq k}\text{image }\Phi_{k'}.\end{aligned}$$

Of course by (ii) $(\Sigma \text{ image } \hat{\Phi}_k)\cap(\Sigma \text{ image } \hat{s}_\ell) = (0)$, so to complete the proof we have only to show (image $\hat{s}_\ell) \cap \left(\sum_{0<r<\ell} \text{image } \hat{s}_r\right) = (0)$ for all ℓ. So assume $\hat{s}_\ell(c) = \sum_{0<r<\ell} \hat{s}_r(c_r)$. By the injectivity of $P_{n-1}(\bigcup_{0\leq r\leq \ell} E_r) \to P_{n-1}(E)$ it follows that this relation already holds in $P_{n-1}(\bigcup_{0\leq r\leq \ell} E_r)$. Now from the Mayer–Vietoris sequence

$$H_{n-1}(E_0) \to H_{n-1}\left(\bigcup_{0\leq r\leq \ell-1} E_r\right) \oplus H_{n-1}(E_0 \cup E_\ell) \to H_{n-1}\left(\bigcup_{0\leq r\leq \ell} E_r\right)$$

it follows that $\hat{s}_\ell(c)$ is in the image of $i_{0\star}: H_{n-1}(E_0) \to H_{n-1}(E)$, so by (ii), $\hat{s}_\ell(c) = 0$. □

We now invoke the map Ψ from Lemma 3.4 and set $\Phi_k = \Psi \circ \hat{\Phi}_k$, $s_\ell = \Psi \circ \hat{s}_\ell$.

Now we can finally state the result we have been aiming for:

PROPOSITION 3.13: $\Phi_1, \ldots, \Phi_{m-1}, s_1, \ldots, s_{m-1}$ *put together give an isomorphism*

$$\left[\bigoplus_{k=1}^{m-1} K_k\right] \oplus \left[\bigoplus_{1}^{m-1} P_{n-2}(B)\right] \xrightarrow{\alpha} P_n(V).$$

PROOF: α is injective: if $n \geq 2$, Ψ is injective, so the injectivity of α follows immediately from 3.12(iii). For $n = 1$, there are no s_ℓ, and we must show that $(\Sigma \text{ image } \Phi_k) \cap \ker \Psi = 0$. By Lemma 3.4 $\ker \Psi = \text{image}(H_2(V, \tilde{L}) \to H_1(E))$. Now by Lefschetz duality $H_2(V, \tilde{L}) \cong H_0(V - \tilde{L})$ is generated by the components of $V - \tilde{L}$. $\mathbb{P}^1 - L$ is contained in the affine piece $X_1 \neq 0$ with coordinate $z = \dfrac{X_0}{X_1}$. As $\mathbb{P}^1 - L$ is simply connected, the nowhere-zero function $1 + z^m$ has on it a single-valued m-th root $\sqrt[m]{1+z^m}$. Thus the components of $V - \tilde{L}$ are just given by $y = \xi^i(\sqrt[m]{1+z^m})$, where $y \stackrel{\text{def}}{=} \dfrac{X_2}{X_1}$.

The upshot of this is that σ_0, which sends z to ζz, acts trivially on the set of components of $V - \tilde{L}$, hence on $\ker \Psi$. But on Σ image $\hat{\Phi}_k$, σ_0 acts with all eigenvalues $\neq 1$, by (1.5). Hence $\ker \Psi \cap (\Sigma \text{ image } \hat{\Phi}_k) = (0)$.

To show α is surjective we consider first the case where V is a Fermat hypersurface V_m^n. In this case $D = V_m^{n-1}$ and $B = V_m^{n-2}$ and by (3.11) and (3.8), we see, in the notation of (1.7), that

$\Phi_k(H_{(\chi_0,\ldots,\chi_n)}) = H_{(\chi_0,\ldots,\chi_n+k,-k)}$ for $\chi_n \neq -k$. Also by (3.11) and dimension counting $\bigoplus_\ell\{\text{image}(s_\ell)\} = \bigoplus\{H_\chi : \chi_n + \chi_{n+1} = 0\}$ (since $\subseteq$ is clear). Hence by counting again we get that α is surjective.

To show α is surjective in general we can proceed as follows: Form "the universal smooth hypersurface of degree m in $\mathbb{P}^{n-1}$", $\begin{matrix}\mathcal{B}\\ \downarrow^{\pi}\\ U\end{matrix}$, i.e. U is the set of homogeneous forms g of degree m defining smooth hypersurfaces, and

$$\begin{tikzcd} \mathcal{B} = \{(X,g) : g(X) = 0\} \subseteq \mathbb{P}^{n-1} \times U \arrow[dr, "\pi"'] & \\ & U \end{tikzcd}$$

$$\mathbb{P}^{n-1} \times U \xrightarrow{pr_2} U.$$

We similarly define $\begin{matrix}\mathcal{D}\\ \downarrow^{\pi'}\\ U\end{matrix}$, $\begin{matrix}\mathcal{V}\\ \downarrow^{\pi''}\\ U\end{matrix}$, globalizing D and V. Note that π is smooth, so that $\mathcal{H}_{n-2}(\mathcal{B}) \overset{\text{def}}{=} R^{n-2}\pi_\star(\mathbb{C})$ forms a local system over U. Note that other groups such as $P_{n-2}(B)$, K_k, also globalize to local systems denoted by $\mathcal{P}_{n-2}(\mathcal{B})$, $\mathcal{K}_k$, etc. Our maps Φ_k and s_ℓ also globalize to maps of the corresponding local systems. All in all, we get a map of local systems $\alpha : [\bigoplus \mathcal{K}_k] \oplus [\bigoplus \mathcal{P}_{n-1}(\mathcal{B})] \to \mathcal{P}_n(\mathcal{V})$. We know α is surjective at the point corresponding to the Fermat hypersurface. But U is connected, and a map of local systems over a connected space which is surjective at one point is surjective at every point. Q.E.D. $\square$

In the next lemma we take V to be a Fermat hypersurface. By Remark 3.10 we may then assume $\hat{s}_\ell$ is Δ^n-equivariant, and we shall do so. The lemma tells us that his requirement in fact determines $\hat{s}_\ell$ uniquely:

LEMMA 3.14: *Let A be a vector space on which Γ^{n-2} acts and for some j, σ_j fixes no nonzero element, and let $\lambda : A \to P_{n-2}(B)$ be a Γ^{n-2}-equivariant map. Note that Δ^n then also acts on A and $P_{n-2}(B)$, making λ Δ^n-equivariant. Then $\hat{s}_\ell \circ \lambda$ is the unique Δ^n-equivariant map $\mu : A \to H_{n-1}(E_0 \cup E_\ell)$ with $\partial_\ell \circ \mu = \lambda$.*

PROOF: In fact for such μ, $C \overset{\text{def}}{=} \text{image}(\hat{s}_\ell \circ \lambda - \mu) \subseteq \text{image}(H_{n-1}(E_0)$

$\oplus\ H_{n-1}(E_\ell) \to H_{n-1}(E_0 \cup E_\ell))$. Since for $c \in C$, $\sum_{i=0}^{m-1} \sigma_j^i(c) = 0$, we see that $C \subseteq P_{n-1}(E_0 \cup E_\ell)$, hence that $C \subseteq \text{image}(P_{n-1}(E_0) \oplus P_{n-1}(E_\ell) \to P_{n-1}(E_0 \cup E_\ell))$. Since $P_{n-1}(E_0) \oplus P_{n-1}(E_\ell) \hookrightarrow P_{n-1}(E)$ by 3.12(i), we can consider C as a subspace of $P_{n-1}(E)$. As C is elementwise fixed by $\sigma_{n+1} \circ \sigma_n$, the proof of (3.11) now shows that $C \subseteq W_n$. Now we have $n \geq 2$, so $\Psi: W_n \to P_n(V)$ is an isomorphism. Therefore by Proposition 3.13 and Lemma 3.12(ii), $\left(\sum_r \text{image } i_{r*}\right) \cap W_n = \bigoplus_k \hat{\Phi}_k$. We know that $\sigma_{n+1} \circ \sigma_n$ fixes no nonzero element in the right hand side, hence none in the left hand side. But $C \subseteq$ L.H.S. Therefore $C = 0$, i.e. $\mu = \hat{s}_\ell \circ \lambda$. Q.E.D. □

This completes setting up our machinery, and we now turn to some applications. First we want to note a remarkable property of the Φ_k, namely that they preserve the rank of cycles. This is a special feature of our set-up and cannot be expected to hold in general pencil situations, even though the constructions may generalize.

PROPOSITION 3.15: *Assume c has* rank r. *Then so does $\Phi_k(c)$.*

PROOF: c lies in the image $H_{n-1}(Y) \to P_{n-1}(D)$, hence in the image $P_{n-1}(Y) \to P_{n-1}(D)$, where Y is some subvariety of codimension r in D. Replacing Y if necessary by $\gamma^{-1}\gamma Y$, where $\gamma: D \to \mathbb{P}^{n-1}$ is the projection sending $[X_0, \ldots, X_n]$ to $[X_0, \ldots, X_{n-1}]$, we can assume that $Y = D \cap Z$, where Z is a subvariety of codimension r in $\mathbb{P}^N$ whose equations do not involve X_n. Let $\tilde{Z} = \pi^{-1}Z$, $\tilde{Y} = \tilde{Z} \cap V$ where $\pi: \mathbb{P}^{n+1} \to \mathbb{P}^n$ is projection $[X_0, \ldots, X_{n+1}] \to [X_0, \ldots, X_n]$, defined away from $[0, \ldots, 0, 1]$. Say $c \in K_k$. Then it is clear that c belongs to the subspace $K_k(Y)$ of $P_{n-1}(Y)$, analogous to K_k. Going through the construction one sees that we can still define in an analogous manner $\Phi_k: K_k(Y) \to P_n(\tilde{Y})$ (note especially that, by Remark 3.3, the deformations used to define Ψ leave $\tilde{Y}$ invariant), so that we have a commutative diagram

$$\begin{array}{ccc} K_k(Y) & \xrightarrow{\Phi_k} & P_n(\tilde{Y}) \\ \downarrow & & \downarrow \\ K_k & \xrightarrow{\Phi_k} & P_n(V) \end{array}$$

Since $\tilde{Y}$ has codimension r in V, Q.E.D. □

We apply this to the Hodge conjecture for Fermat 3-folds:

THEOREM 3.16: *Assuming statement* $(1.10)_m$ *holds, every class of Hodge filtration* 1 *in* $H_3(V_m^3, A)$ *has rank* 1.

PROOF: Indeed our assumption says exactly that every such class can be written after an appropriate permutation as $\Phi_k(c)$ with $c \in H_2(V_m^2, \mathbb{Q}) \cap H^{1,1}(V_m^2)$. By the Lefschetz (1, 1) theorem, c is algebraic, i.e. has rank 1. Hence by Proposition 3.15 so does $\Phi_k(c)$. $\square$

EXAMPLE 3.17: By Example 1.13, Statement $(1.10)_m$ holds for $m = 8$, hence the Hodge conjecture holds for V_8^3.

§4. On ruled cycles

Let $\mathbb{P}' = \mathbb{P}^{n'+1}$, $\mathbb{P}'' = \mathbb{P}^{n''+1}$, $\mathbb{P} = \mathbb{P}^{n'+n''+3}$ and embed $\mathbb{P}'$ and $\mathbb{P}''$ into $\mathbb{P}$ via $[X_0, \ldots, X_{n'+1}] \to [X_0, \ldots, X_{n'+1}, 0, \ldots, 0]$ and $[X_0, \ldots, X_{n''+1}] \to [0, \ldots, 0, X_0, \ldots, X_{n''+1}]$, respectively. Thus $\mathbb{P}' \cap \mathbb{P}'' = \emptyset$. Now let $V' \subseteq \mathbb{P}'$ and $V'' \subseteq \mathbb{P}''$ be nonsingular hypersurfaces of degree m with equations $f' = 0$, $f'' = 0$, respectively, and let $V \subseteq \mathbb{P}$ be the hypersurface with equation $f' + f'' = 0$.

Note that if $p' \in V'$ and $p'' \in V''$ then the line $\overline{p'p''}$ spanned by p' and p'' is entirely contained in V. Thus for subsets c' of V' and c'' of V'' we can form the subset $c' \star c''$ of V, the union of the lines spanned by a point in c' and a point in c''. Note that if c' is a point, then $c' \star c''$ is the projective cone on c'' with vertex c'. It is easy to see that this construction descends to homology, and thus we get a map $\star : H_i(V') \otimes H_j(V'') \to H_{i+j+2}(V)$. To describe this formally we proceed as follows: consider the incidence correspondence

(4.1)
$$\begin{tikzcd} & I \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ G & & \mathbb{P} \end{tikzcd}$$

where $G =$ Grassmannian of lines in $\mathbb{P}$, $I = \{(\ell, p) \in G \times \mathbb{P} : p \in \ell\}$. Note that I forms a $\mathbb{P}^1$-bundle over G. Now via the map $V' \times V'' \to G$ defined by $(p', p'') \to \overline{p'p''}$, we can consider $V' \times V''$ as embedded in G. Let $I_0 = p_1^{-1}(V' \times V'')$, so that $p_2(I_0) \subset V$ so we have

(4.2)
$$\begin{tikzcd}[column sep=small] & I_0 \arrow[dl, "p_2"'] \arrow[dr, "p_2"] & \\ V' \times V'' & & V \end{tikzcd}$$

and we define $\star$ as the composite

$$H_i(V') \otimes H_j(V'') \xrightarrow{K} H_{i+j}(V' \times V'') \xrightarrow{p_1^\star} H_{i+j+2}(I_0) \xrightarrow{p_{2\star}} H_{i+j+2}(V),$$

where K is the Künneth map. Note that $\star$ preserves primitive homology, so from now on we shall only consider $\star$ as a map

$$\star : P_{n'}(V') \otimes P_{n''}(V'') \to P_{n'+n''+2}(V).$$

It is obvious by construction that if c' and c'' are algebraic then so is $c' \star c''$.

The main fact to prove about $\star$ is that it is *injective* (on primitive homology); here we shall first deduce this in case V' and V'' are Fermat hypersurfaces, using the machinery of §3, and the compatibility of $\star$ with this machinery. In fact, as we shall see, it is essentially only this compatibility that we shall use (plus Proposition 1.14), rather than the precise nature of $\star$. Then the general case follows easily from this using a similar argument as in the last part of the proof of Proposition 3.13. So assume until further notice that V' and V'' and hence also V are Fermat hypersurfaces.

Note that since p_1 is a $\mathbb{P}^1$-bundle projection, the map $p_1^\star$ is defined whether or not the base is a manifold. This allows us to define $\star$ in a compatible way also for (possibly singular) subvarieties of V' and V''. In particular we can define

$$\hat{\star} : P_{n'}(V') \otimes H_{n''-1}(E_0^{(n'')} \cup E_\ell^{(n'')}) \to H_{n-1}(E_0^{(n)} \cup E_\ell^{(n)})$$
$$\star : P_{n'}(V') \otimes P_{n''-2}(B^{(n'')}) \to P_{n-2}(B^{(n)})$$

where $E_r^{(s)}$, $B^{(s)}$ means the E_r or B we constructed on V_m^s.

Note also that $\star$ is equivariant in the appropriate sense: namely

$$(4.3) \qquad \begin{aligned} \sigma_{j\star}(c' \star c'') &= \sigma_{j\star}(c') \star c'', & 0 \le j \le n'+1,\\ &= c' \star \sigma_{j-n'-2\star}(c''), & n'+2 \le j \le n. \end{aligned}$$

Now we can state the above-mentioned compatibility:

LEMMA 4.4: *For V', V'' Fermat hypersurfaces, we have*
(i) $c' \star \Phi_k(c'') = \Phi_k(c' \star c'')$.
(ii) $c' \star s_\ell(c'') = s_\ell(c' \star c'')$.

(This is a slight abuse of notation as the Φ_k and s_ℓ on the L.H.S. are on V'', while those on the R.H.S. are on V; in (i) we are also assuming $c'' \in K_k$ on V'', in which case clearly $c' \star c'' \in K_k$ on V.)

PROOF: (i) is immediate by construction (note that the whole construction on V restricts to that on V''). To prove (ii), we also have to make use of Lemma 3.14: let $\lambda : P_{n''-2}(B^{(n'')}) \to P_{n-2}(B^{(n)})$ be $c'' \mapsto c' \star c''$ and $\mu : P_{n''-2}(B^{(n'')}) \to P_{n-1}(E_0^{(n)} \cup E_\ell^{(n)})$ be $c'' \mapsto c' \hat{\star} \hat{s}_\ell(c'')$, and consider $P_{n''-2}(B^{(n'')})$ as a Γ^{n-2}-space in the obvious way. Then, by construction, the hypotheses of Lemma 3.14 are satisfied, and therefore $\mu = \hat{s}_\ell \circ \lambda$. Hence $c' \star s_\ell(c'') = s_\ell(c' \star c'')$.

COROLLARY 4.5: $c' \star \alpha(c'') = \alpha(c' \star c'')$. □

PROPOSITION 4.6: *For V' and V'' Fermat hypersurfaces, $\star$ is injective.*

PROOF: For $n' = n'' = 0$, our assertion is just (1.14): namely note that if $p_i = [1, \xi\zeta^i] \in V_m^0$, then a basis for $P_0(V_M^0)$ is given by (the classes of) $p_i - p_0$, $i = 1, \ldots, m-1$. Now $(p_i - p_0) \star (p_j - p_0) = M_{ij}$, which are linearly independent. For general n', n'' we induce on $n' + n'' > 0$. By permuting coordinates we may assume $n'' > 0$. Take $c' \in P_{n'}(V')$, $c'' \in P_{n''}(V'')$ with c', $c'' \neq 0$. By Proposition 3.13 we can write $c'' = \alpha(d)$ for an appropriate d, so that by Corollary 4.5, $c' \star c'' = \alpha(c' \star d)$. By induction, $c' \star d \neq 0$. Hence by (3.13) again $c' \star c'' \neq 0$. □

COROLLARY 4.7: *$H_{\chi'} \star H_{\chi''} = H_{\chi' \star \chi''}$, where $\chi' \star \chi''$ is obtained from χ', χ'' by juxtaposition (in fact $\subseteq$ is clear by* (4.3)*).* □

Now we pass to the general case:

THEOREM 4.8: *For arbitrary smooth hypersurfaces V', V'', $\star$ is injective.*

PROOF: We just copy the last part of the proof of (3.13): form the universal smooth hypersurfaces $\begin{smallmatrix}\mathcal{V}'\\ \downarrow\\ U'\end{smallmatrix}$, $\begin{smallmatrix}\mathcal{V}''\\ \downarrow\\ U''\end{smallmatrix}$ (here we do *not* projectivize U',

U'') and also $\begin{array}{c}\mathcal{V}\\ \downarrow\\ U'\times U''\end{array}$, giving rise to local systems $\mathcal{P}_{n'}(\mathcal{V}')$, $\mathcal{P}_{n''}(\mathcal{V}'')$ (hence $\mathcal{P}_{n'}(\mathcal{V}')\otimes\mathcal{P}_{n''}(\mathcal{V}'')$ over $U'\times U''$) and $\mathcal{P}_n(\mathcal{V})$ and to a map $\mathcal{P}_{n'}(\mathcal{V}')\otimes\mathcal{P}_{n''}(\mathcal{V}'')\xrightarrow{\star}\mathcal{P}_n(\mathcal{V})$. By (4.3) this map is injective at the "Fermat point," and hence it is injective at every point. □

Combining Corollary 4.7 with explicit information about characters, namely (1.8)–(1.11), gives us results concerning the Hodge conjecture for Fermat hypersurfaces:

THEOREM 4.9: *If the Hodge characters for m are generated by those in dimensions* 0 *and* 2, *e.g.*, m *is prime or* $m = 0$, *then the Hodge conjecture is true on* V_m^n *for n even. Moreover, if m is prime, the Hodge subspace is generated by the homology classes of linear spaces.*

THEOREM 4.10: *Fixing m arbitrary, there are finitely many Hodge classes* $c_i \in P_{n_i}(V_m^{n_i})$, *such that for all n, the Hodge classes in* $P_n(V_m^n)$ *are generated up to permutations by repeated* $\star$*-products of the* c_i. □

THEOREM 4.11: *Assuming* $(1.11)_{(m,k)}$ *holds, the Poincaré dual of* $F^k p^{2k+1}(V_m^{2k+1}) \cap H^{2k+1}(V_m^{2k+1}, \mathbb{Q})$ *consists of cycles of rank k.* □

REMARK: Note that for $m = 8$, $k = 1$, for example, $(1.11)_{(m,k)}$ is not satisfied, so that Theorem 4.11 does not apply; however we have seen before ((1.13) and (3.17)) that $(1.10)_m$ is atisfied, so that Theorem 3.16 does apply to give the desired conclusion.

REFERENCES

[1] A. GROTHENDIECK: Hodge's general conjecture is false for trivial reasons. *Topology*, *8* (1969) 299–303.

[2] W.V.D. HODGE: The topological invariants of algebraic varieties, *Proc. Int'l Cong. Math., Cambridge, 1950*, Vol. 1.

[3] N. KOBLITZ and A. OGUS: Algebraicity of some products of values of the Γ function. *Proc. Symp. Pure Math.*, *33* (1979), part 2, 343–345.

[4] S. LEFSCHETZ: *L'Analysis Situs et la Géométrie Algébrique*, Gauthier-Villars, Paris, 1924.

[5] A. LANDMAN: On the Picard–Lefschetz transformation for algebraic varieties acquiring general singularities, *TAMS*, *181* (1973) 89–126.

[6] A. OGUS: Griffiths transversality in crystalline cohomology, §3. *Ann. of Math.*, *108*, Sept., 1978.

[7] W. PARRY: Personal communication; (to appear).
[8] A. WALLACE: *Homology Theory on Algebraic Varieties.* Pergamon Press, Oxford–London–New York, 1958.

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School of Mathematics
The Institute for Advanced Study
Princeton, N.J. 08540
U.S.A.