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ON LARGE SUBSPACES OF THE SCHATTEN p-CLASSES*

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Abstract

Let C_p denote the Schatten p-class of operators on Hilbert space. We prove that if X is a subspace of C_p $(1 < p < \infty)$ which is isomorphic to C_p, then X contains a further subspace Y which is also isomorphic to C_p, and it is complemented in C_p. As a consequence, we get that every complemented subspace of C_p which contains an isomorphic copy of C_p, is actually isomorphic to C_p.

A related result is that for $1 < p < \infty$, C_p is primary.

1. Introduction

The Schatten p-classes C_p $(1 \leq p \leq \infty)$ of operators on the separable Hilbert space ℓ_2 are defined as follows.

For $1 \leq p < \infty$ *let* C_p be the Banach space of all compact operators x on ℓ_2, so that

$$\|x\|_p = (\text{trace}(x^*x)^{p/2})^{1/p} < \infty.$$

C_∞ denotes the Banach space of all compact operators x on ℓ_2 with the operator-norm induced from $B(\ell_2)$, the space of all bounded operators on ℓ_2,

$$\|x\|_\infty = \sup\{\|x\xi\|_{\ell_2};\ \xi \in \ell_2, \|\xi\|_{\ell_2} \leq 1\}.$$

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Our main interest in this paper are the spaces C_p for $1<p<\infty$, but some partial results are stated and proved also for $p = 1, \infty$. The main result of this work is the following theorem.

THEOREM 1.1: *Let X be a subspace of C_p, $1<p<\infty$, which is isomorphic to C_p. Then there is a subspace Y of X so that Y is isomorphic to C_p and Y is complemented in C_p.*

Let us turn first to notation and background material. We use [7] as a general reference to Banach space theory. By "subspace" we shall always mean a closed subspace. If $\{x_n\}_{n=1}^\infty$ is a basic sequence in the Banach space X, then we denote by $[x_n]_{n=1}^\infty$ the subspace spanned by $\{x_n\}_{n=1}^\infty$ in X. The basic sequence $\{x_n\}_{n=1}^\infty$ is said to be λ-equivalent to the basic sequence $\{y_n\}_{n=1}^\infty$ if there exist $0<\lambda_1, \lambda_2<\infty$ so that $\lambda_1 \cdot \lambda_2 \leq \lambda$, and for all scalars $\{t_n\}_{n=1}^\infty$ we have

$$\lambda_2^{-1} \left\| \sum_n t_n y_n \right\| \leq \left\| \sum_n t_n x_n \right\| \leq \lambda_1 \left\| \sum_n t_n y_n \right\|.$$

A subspace Y of X is λ-complemented in X if there exists a projection P from X onto Y with $\|P\| \leq \lambda$. A complemented subspace is a subspace which is λ-complemented for some $\lambda < \infty$. If X, Y are isomorphic Banach spaces, then we denote

$$\mathrm{d}(X, Y) = \inf\{\|T\| \cdot \|T^{-1}\|;\ T \text{ is an isomorphism from } X \text{ onto } Y\}.$$

$X \approx Y$ means that X is isomorphic to Y. We use in several places what we call "standard perturbation arguments". By this we mean the appropriate analogue of [1, Proposition 1.a.9].

We refer to [5] and [8] for the elementary properties of C_p, and to [1], [2] and [10] for the study of C_p from the point of view of the geometry of Banach spaces.

Let us establish the notation which we will use along this work. Given orthonormal bases $\{e_i\}_{i=1}^\infty$ and $\{f_i\}_{i=1}^\infty$ for ℓ_2, we represent every $x \in B(\ell_2)$ as a matrix $x = (x(i,j))_{i,j=1}^\infty$, where $x(i,j) = (xf_j, e_i)$.

The *standard unit matrices* associated with the pair $(\{e_i\}_{i=1}^\infty, \{f_i\}_{i=1}^\infty)$ are

$$(1.1) \qquad e_{i,j} = (\cdot, f_j)e_i, \quad 1 \leq i, j < \infty.$$

Note that $e_{i,j}(k,\ell) = \delta_{i,k} \cdot \delta_{j,\ell}$. In the ordering

$$(1.2)\qquad \begin{gathered} e_{1,1}, e_{2,1}, e_{2,2}, e_{1,2}, e_{3,1}, e_{3,2}, e_{3,3}, e_{2,3}, e_{1,3}, \ldots, \\ e_{n,1}, e_{n,2}, \ldots, e_{n,n}, e_{n-1,n}, e_{n-2,n}, \ldots, e_{1,n}, \ldots \end{gathered}$$

the $\{e_{i,j}\}_{i,j=1}^{\infty}$ form a Schauder basis of C_p for every p.

For every n let P_n and E_n be the following projections in C_p.

$$(1.3)\qquad (P_n x)(i,j) = \begin{cases} x(i,j); & 1 \le i, j \le n \\ 0; & \text{otherwise} \end{cases}$$

$$(1.4)\qquad (E_n x)(i,j) = \begin{cases} x(i,j); & 1 \le \min\{i,j\} \le n \\ 0; & \text{otherwise.} \end{cases}$$

We use also the notation

$$(1.5)\qquad E_{n,m} = E_m - E_n, \quad P_{n,m} = P_m - P_n; \quad n < m$$

and

$$(1.6)\qquad E^n = 1 - E_n, \quad P^n = 1 - P^n.$$

Clearly $\|P_n\| = 1$ and $\|E_n\| \le 2$ for every n.

Another important projection is the triangular projection P_T, defined by

$$(1.7)\qquad (P_T x)(i,j) = \begin{cases} x(i,j); & 1 \le j \le i < \infty \\ 0; & \text{otherwise.} \end{cases}$$

P_T is bounded in C_P if and only if $1 < p < \infty$ (see [6, p. 121]). Denote the space of all lower triangular matrices in C_p by

$$(1.8)\qquad T_p = \{x \in C_p ; P_T x = x\}.$$

By [1, Proposition 1], $C_p \approx T_p$ if and only if $1 < p < \infty$. So in proving Theorem 1.1 for $1 < p < \infty$ we can use T_p instead of C_p.

The spaces C_p and T_p admit finite dimensional Schauder decompositions

$$(1.9)\qquad C_p = \sum_{n=1}^{\infty} \oplus P_{n-1,n} C_p, \quad T_p = \sum_{n=1}^{\infty} \oplus P_{n-1,n} T_p.$$

These decompositions enjoy the property of being reproducible (see Proposition 2 in [1] and the Definition that preceeds it). The version of the reproducibility which will be used below for T_p is the following. Let V be an isomorphism of T_p into itself, then there exist increasing sequences $\{n_k\}_{k=1}^{\infty}$ and $\{m_k\}_{k=1}^{\infty}$ of positive integers, so that $V([e_{n_k,j}]_{j=1}^{k})$ is almost contained in $P_{m_k,m_{k+1}}T_p$ (so, if $Ue_{k,j}=e_{n_k,j}$, then VU is an isomorphism of T_p into itself with $VU(P_{k-1,k}T_p)$ almost contained in $P_{m_k,m_{k+1}}T_p$).

Let $\{A_k\}_{k=1}^{\infty}$ and $\{B_k\}_{k=1}^{\infty}$ be two sequences of subsets of the natural numbers, so that for $k\neq \ell$,

$$A_k\cap A_\ell=\emptyset=B_k\cap B_\ell.$$

Let $P(\{A_k\},\{B_k\})$ be defined by

$$[P(\{A_k\},\{B_k\})x](i,j)=\begin{cases}x(i,j); & \text{if } (i,j)\in A_k\times B_k \text{ for some } k\\ 0; & \text{otherwise,}\end{cases} \tag{1.10}$$

then for every $x\in C_p$,

$$\|P(\{A_k\},\{B_k\})x\|_p=\Big(\sum_k\|x_k\|_p^p\Big)^{1/p}\le\|x\|_p \tag{1.11}$$

where $x_k(i,j)=x(i,j)$ if $(i,j)\in A_k\times B_k$, and $x_k(i,j)=0$ otherwise. If each of the $\{A_k\}_{k=1}^{\infty}$ and $\{B_k\}_{k=1}^{\infty}$ is infinite, we get that the range of the contractive projection $P(\{A_k\},\{B_k\})$ is isometric to $(C_p\oplus C_p\oplus\cdots\oplus C_p\oplus\cdots)_{\ell_p}$. By the decomposition method (see [5], page 54) we get that $C_p\approx(C_p\oplus C_p\oplus\cdots\oplus C_p\oplus\cdots)_{\ell_p}$. The same proof shows also that $T_p\approx(T_p\oplus T_p\oplus\cdots\oplus T_p\oplus\cdots)_{\ell_p}$ for every $1\le p\le\infty$. Here, if $p=\infty$, the infinite direct sum is taken in the sense of c_0, and "$(\Sigma_k|t_k|^p)^{1/p}$" means $\sup_k|t_k|$.

Two elements $x,y\in B(\ell_2)$ are said to have disjoint supports if there exists a matrix representation in which

$$x=\begin{pmatrix}\tilde{x}&0\\0&0\end{pmatrix},\quad y=\begin{pmatrix}0&0\\0&\tilde{y}\end{pmatrix},$$

where $\tilde{x}$ and $\tilde{y}$ are the appropriate restrictions of x and y respectively. If $\{x_k\}_{k=1}^{n}$ are pairwise disjointly supported elements of C_p, then $\|\Sigma_{k=1}^{n}x_k\|_p=(\Sigma_{k=1}^{n}\|x_k\|_p^p)^{1/p}$.

Let us denote by $r(x)$, for every $x\in B(\ell_2)$, the orthogonal projection from ℓ_2 onto $(\ker x)^{\perp}$. Then for $x,y\in B(\ell_2)$, $r(x)\cdot r(y)=0$ if and only if there exists a matrix representation in which

$$x = \begin{pmatrix} x_{1,1} & 0 \\ x_{2,1} & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & y_{1,2} \\ 0 & y_{2,2} \end{pmatrix},$$

with $x_{i,j}$ and $y_{i,j}$ appropriate restrictions of x and y respectively.

For $1 < p < \infty$, the dual of C_P is C_q, $q^{-1} + p^{-1} = 1$. The duality is given by $\langle x, y\rangle = \operatorname{trace}(xy^\star)$, $x \in C_p$ and $y \in C_q$. C_2, the space of all Hilbert–Schmidt operators, is a Hilbert space under the inner product $\langle \cdot, \cdot \rangle$, and thus its structure is well known. Therefore, the index $p = 2$ will be omitted in the sequel (Theorem 1.1 is trivial in this case). Also, C_1 is isometric to $C_\infty^\star$ and $B(\ell_2)$ is isometric to $C_1^\star$, where the duality is given again by $\langle x, y \rangle = \operatorname{trace}(xy^\star)$.

If $0 \neq x \in C_p$, $1 \leq p < \infty$, and if $x = v(x) \cdot |x|$ is the standard polar decomposition of x (i.e. $|x| = (x^\star x)^{1/2}$ and $v(x)$ is a partial isometry with $\ker x = \ker v(x)$), then we define $n_p(x) = v(x)|x|^{p-1}$. Clearly, $n_p(x) \in C_q$ if $1 < p < \infty$, and $n_1(x) = v(x) \in B(\ell_2)$ if $x \in C_1$. Also

$$\langle x, n_p(x)\rangle = \|x\|_p^p = \|x\|_p \cdot \|n_p(x)\|_q.$$

Another piece of information concerns the behaviour of Rademacher averages of elements of C_p (cf. [8] and [10]): for $1 \leq p < \infty$ there is a constant K_p such that for every choice of $\{x_i\}_{i=1}^n$ in C_p,

$$(1.12) \qquad K_p \left(\sum_{i=1}^n \|x_i\|_p^2\right)^{1/2} \leq \left(\int_0^1 \left\|\sum_{i=1}^n r_i(t) x_i\right\|_p^p \mathrm{d}t\right)^{1/p} \leq \left(\sum_{i=1}^n \|x_i\|_p^p\right)^{1/p} \quad \text{if } 1 \leq p \leq 2,$$

$$(1.13) \qquad \left(\sum_{i=1}^n \|x_i\|_p^p\right)^{1/p} \leq \left(\int_0^1 \left\|\sum_{i=1}^n r_i(t) x_i\right\|_p^p \mathrm{d}t\right)^{1/p} \leq K_p \left(\sum_{i=1}^n \|x_i\|_p^2\right)^{1/2} \quad \text{if } 2 \leq p < \infty,$$

(here, the $r_i(t)$ are the Rademacher functions). So C_p, $1 \leq p < \infty$, is of type s and of cotype r, where $s = \min\{p, 2\}$ and $r = \max\{p, 2\}$.

Let us mention some results on tensor product of operators. Here we use the notations $C_p(H)$ for the class C_p of operators on the Hilbert space H. Let $\ell_2 \otimes \ell_2$ be the Hilbert-space tensor product of ℓ_2 with itself. If $x, y \in B(\ell_2)$, then there is a unique element $x \otimes y \in B(\ell_2 \otimes \ell_2)$ satisfying $(x \otimes y)(\xi \otimes \eta) = x\xi \otimes y\eta$ for every $\xi, \eta \in \ell_2$. If $x, y \in C_p(\ell_2)$, $1 \leq p \leq \infty$, then $x \otimes y \in C_p(\ell_2 \otimes \ell_2)$ and

$$(1.14) \qquad \|x \otimes y\|_p = \|x\|_p \cdot \|y\|_p.$$

Moreover, $C_p(\ell_2 \otimes \ell_2)$ is spanned by the elements $x \otimes y$ with $x, y \in C_p(\ell_2)$. We therefore denote $C_p(\ell_2 \otimes \ell_2)$ by $C_p(\ell_2) \otimes C_p(\ell_2)$, or simply by $C_p \otimes C_p$.

Let $\{e_i\}_{i=1}^{\infty}$ and $\{f_i\}_{i=1}^{\infty}$ be two orthonormal bases of ℓ_2, let $N = \bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} B_k$ be two partitions of the set N of positive integers into pairwise disjoint infinite subsets, and let $\varphi_k : N \to A_k$, $\psi_k : N \to B_k$ be one-to-one and onto mappings, $1 \le k < \infty$. Since $\{f_\ell \otimes f_j\}_{\ell,j=1}^{\infty}$ and $\{e_k \otimes e_i\}_{k,i=1}^{\infty}$ are orthonormal bases for $\ell_2 \otimes \ell_2$, there exist isometries u, w of ℓ_2 onto $\ell_2 \otimes \ell_2$ so that for every $i, j, k, \ell \in N$:

$$wf_{\psi_\ell(j)} = f_\ell \otimes f_j; \quad ue_{\varphi_k(i)} = e_k \otimes e_i.$$

Define for $x \in B(\ell_2)$

$$Vx = uxw^{-1}.$$

Then V is an isometry of C_p onto $C_p \otimes C_p$, $1 \le p \le \infty$. In the sequel we shall therefore identify C_p with $C_p \otimes C_p$; the identification will always be made in the way described above, and usually it will be clear from the context how the identification is made. We call this identification a "tensor product representation" of C_p as $C_p \otimes C_p$. Obviously, we can identify in an analogous manner C_p with $C_p \otimes C_p \otimes C_p$, with $C_p \otimes C_p \otimes C_p \otimes C_p$, etc.

Let us give an example to illustrate the use of the tensor product notation. Let $\{A_k\}_{k=1}^{\infty}$ and $\{B_k\}_{k=1}^{\infty}$ be subsets of the natural numbers, so that $A_k \cap A_\ell = \emptyset = B_k \cap B_\ell$ if $k \ne \ell$. Let $x_{k,\ell} \in B(\ell_2)$ be such that $x_{k,l}(i,j) \ne 0$ only for $(i,j) \in A_k \times B_\ell$. Then there exists a tensor product representation of C_p as $C_p \otimes C_p$, in which the $x_{k,\ell}$ have the form $x_{k,\ell} = e_{k,\ell} \otimes y_{k,\ell}$ for some $y_{k,\ell} \in B(\ell_2)$. If, moreover, $A_k = \{n_i^{(k)}\}_{i=1}^{\infty}$ and $B_k = \{m_i^{(k)}\}_{i=1}^{\infty}$ and $x_{k,\ell}(n_i^{(k)}, m_j^{(\ell)}) = x_{1,1}(n_i^{(1)}, m_j^{(1)})$ for every i, j, k and ℓ, then the tensor product representation can be chosen so that for some $y \in B(\ell_2)$ we have $x_{k,\ell} = e_{k,\ell} \otimes y$ for every k and ℓ.

2. Preliminaries

Proposition 2.1: *Let $x_{i,j} \in C_p$ so that $x_{i,j} \ne 0$ only for finitely many pairs (i,j). Then*

$$(2.1) \qquad \left(\sum_{i,j} \|x_{i,j}\|_p^2\right)^{1/2} \le \left\|\sum_{i,j} e_{i,j} \otimes x_{i,j}\right\|_p \le \left(\sum_{i,j} \|x_{i,j}\|_p^p\right)^{1/p}$$

if $1 \le P \le 2$:

$$(2.2) \qquad \Big(\sum_{i,j} \|x_{i,j}\|_p^p\Big)^{1/p} \le = \Big\|\sum_{i,j} e_{i,j} \otimes x_{i,j}\Big\|_p \le \Big(\sum_{i,j} \|x_{i,j}\|_p^2\Big)^{1/2}$$
$$\textit{if } 2 \le p \le \infty.$$

PROOF: If $x, y \in B(\ell_2)$ satisfy $x^\star y = 0$ (i.e., they have orthogonal ranges), then

$$(2.3) \qquad \|x+y\|_\infty = \sup_{\substack{\xi \in \ell_2 \\ \|\xi\| \le 1}} \|(x+y)\xi\| = \sup_{\substack{\xi \in \ell_2 \\ \|\xi\| \le 2}} (\|x\xi\|^2 + \|y\xi\|^2)^{1/2}$$
$$\le (\|x\|_\infty^2 + \|y\|_\infty^2)^{1/2}.$$

This implies, by induction, that if $x_j \in B(\ell_2)$ and $x_i^\star x_j = 0$ for $i \neq j$, then $\|\Sigma_j x_j\|_\infty \le (\Sigma_j \|x_j\|_\infty^2)^{1/2}$. Similarly, if $x_i x_j^\star = 0$ for $i \neq j$, then $\|\Sigma_j x_j\|_\infty \le (\Sigma_j \|x_j\|_\infty^2)^{1/2}$. Using these facts we get for any $x_{i,j} \in B(\ell_2)$, and in particular for $x_{i,j} \in C_\infty$, that

$$(2.4) \qquad \Big\|\sum_{i,j} e_{i,j} \otimes x_{i,j}\Big\|_\infty = \Big\|\sum_i \sum_j e_{i,j} \otimes x_{i,j}\Big\|_\infty \le \Big(\sum_i \Big\|\sum_j e_{i,j} \otimes x_{i,j}\Big\|_\infty^2\Big)^{1/2}$$
$$\le \Big(\sum_i \sum_j \|e_{ij} \otimes x_{i,j}\|_\infty^2\Big)^{1/2} = \Big(\sum_{i,j} \|x_{i,j}\|_\infty^2\Big)^{1/2}.$$

This proves the right inequality for $p = \infty$, while the left inequality is trivial.

If $p = 1$ and $x_{i,j} \in C_1$, choose $y_{i,j} \in B(\ell_2)$ so that $\|y_{i,j}\|_\infty = \|x_{i,j}\|_1$ and $\langle x_{i,j}, y_{i,j}\rangle = \|x_{i,j}\|_\infty^2$. So,

$$(2.5) \qquad \Big\|\sum_{i,j} e_{i,j} \otimes x_{i,j}\Big\|_1 \ge \Big|\Big\langle \sum_{i,j} e_{i,j} \otimes x_{i,j}, \sum_{k,\ell} e_{k,\ell} \otimes y_{k,\ell} \Big/ \Big\|\sum_{k,\ell} e_{k,\ell} \otimes y_{k,\ell}\Big\|_\infty\Big\rangle\Big|$$
$$\ge \sum_{i,j} \langle x_{i,j}, y_{i,j}\rangle \Big/ \Big\|\sum_{k,\ell} e_{k,\ell} \otimes y_{k,\ell}\Big\|_\infty$$
$$\ge \sum_{i,j} \|x_{i,j}\|_1^2 \Big/ \Big(\sum_{k,\ell} \|y_{k,\ell}\|_\infty^2\Big)^{1/2} = \Big(\sum_{i,j} \|x_{i,j}\|_1^2\Big)^{1/2}.$$

This establishes the left inequality for $p = 1$, while the right inequality in this case is just the triangle inequality.

Using the cases $p = 1, \infty$ and the generalized Riesz–Thorin theorem for the spaces C_p (see [3]), we get the desired inequalities for every $1 \le p \le \infty$. Note that for $p = 2$ we actually have an equality (the spaces $e_{i,j} \otimes C_2$ are pairwise orthogonal). □

REMARK: One can prove (2.1) and (2.2) for $1 < p < \infty$, $p \neq 2$, without the interpolation techniques of [3], by using (1.12), (1.13) and [2, Proposition 3.1].

Next, we want to study the subspaces of C_p which are isomorphic to ℓ_2. By [1, Proposition 4], [4, Theorem 2], if $x \subset C_p$, $p \neq 2$, then $x \approx \ell_2$ if and only if there exists an n such that $E_{n|X}$ is an isomorphism. We first establish the following quantitative strenghtening of this result.

PROPOSITION 2.2: *Let $X \subset C_p$, $1 \le p \le \infty$, $p \neq 2$, and assume that* $\mathrm{d}(X, \ell_2) = M < \infty$.

(i) *If $1 \le p < 2$ and $\epsilon > 0$, there exists an n such that $\|E^n{}_{|X}\| \le \epsilon$.*

(ii) *If $2 < p \le \infty$ and $0 < \delta < (3M)^{-1}$, there exists an n such that $\|E_n x\|_p \ge \delta \|x\|_p$ for every $x \in X$.*

PROOF:

(i) If there is no such n for a given $0 < \epsilon$, we can find an increasing sequence $\{n_k\}_{k=1}^{\infty}$ of positive integers, and a sequence $\{x_k\}_{k=1}^{\infty}$ of normalized elements of X so that for every k:

$$\|E^{n_k} x_k\|_p > \epsilon, \quad \|P^{n_{k+1}} x_k\|_p \le 2^{-k}. \tag{2.6}$$

Since X is reflexive, we can assume (by passing to a subsequence and using standard perturbation arguments) that $x_k = x + y_k$, $x \in C_p$, and

$$P_{n_1} x = x, \quad y_k = P_{n_k, n_{k+1}} y_k, \quad \|E^{n_k} y_k\|_p > \epsilon. \tag{2.7}$$

For every m, $\Sigma_{\ell=1}^{m} E^{n_\ell} P_{n_\ell, n_{\ell+1}}$ is a projection of the form (1.10). Using (1.11) and (2.7) we get

$$\begin{aligned} (2.8) \quad Mm^{1/2} &= \left(\int_0^1 \left\| \sum_{k=1}^{m} r_k(t) x_k \right\|_p^2 \mathrm{d}t \right)^{1/2} \\ &\ge \left(\int_0^1 \left\| \sum_{k=1}^{m} r_k(t) y_k \right\|_p^2 \mathrm{d}t \right)^{1/2} - m^{1/2} \|x\|_p \\ &\ge \left(\int_0^1 \left\| \left(\sum_{\ell=1}^{m} E^{n_\ell} P_{n_\ell, n_{\ell+1}} \right) \left(\sum_{k=1}^{m} r_k(t) y_k \right) \right\|_p^2 \mathrm{d}t \right)^{1/2} - m^{1/2} \|x\|_p \\ &= \left(\int_0^1 \left\| \sum_{k=1}^{m} r_k(t) E^{n_k} P_{n_k, n_{k+1}} y_k \right\|_p^2 \mathrm{d}t \right)^{1/2} - m^{1/2} \|x\|_p \\ &= \left(\sum_{k=1}^{m} \|E^{n_k} P_{n_k, n_{k+1}} y_k\|_p^p \mathrm{d}t \right)^{1/p} - m^{1/2} \|x\|_p \\ &\ge m^{1/p} \epsilon - m^{1/2} \|x\|_p. \end{aligned}$$

Since $p < 2$, this leads to a contradiction if m is large enough.

(ii) If there is no such n for a given $0 > \delta > (3M)^{-1}$, we can find an increasing sequence $\{n_k\}_{k=1}^{\infty}$ of positive integers, and a sequence $\{x_k\}_{k=1}^{\infty}$ of normalized elements in X, so that for every k:

$$\|E_{n_k}x_k\|_p < \delta, \quad \|P^{n_{k+1}}x_k\|_p \leq 2^{-k}. \tag{2.9}$$

Using the reflexivity of X and a standard perturbation argument, we can assume that $x_k = x + y_k$, $x \in C_p$, $P_{n_1}x = x$, and for every k,

$$y_k = P_{n_k,n_{k+1}}y_k, \quad \|E_{n_k}(x + y_k)\|_p < \delta. \tag{2.10}$$

Using Proposition 2.1, we get for every m

$$\begin{aligned} m^{1/2}M^{-1} &\leq \left(\int_0^1 \left\|\sum_{k=1}^m r_k(t)x_k\right\|_p^2 \mathrm{d}t\right)^{1/2} \\ &= \left(\int_0^1 \left\|\sum_{k=1}^m r_k(t)(x + E_{n_k}y_k + E^{n_k}y_k)\right\|_p^2 \mathrm{d}t\right)^{1/2} \\ &\leq m^{1/2}\|x\|_p + \left(\int_0^1 \left\|\sum_{k=1}^m r_k(t)P_T E_{n_k}y_k\right\|_p^2 \mathrm{d}t\right)^{1/2} \\ &\quad + \left(\int_0^1 \left\|\sum_{k=1}^m r_k(t)(1 - P_T)E_{n_k}y_k\right\|_p^2 \mathrm{d}t\right)^{1/2} \\ &\quad + \left(\int_0^1 \sum_{k=1}^m r_k(t)E^{n_k}y_k\Big\|_p^2 \mathrm{d}t\right)^{1/2} \\ &\leq m^{1/2}\delta + m^{1/2}\delta + m^{1/2}\delta + m^{1/p} = 3\delta m^{1/2} + m^{1/p}. \end{aligned} \tag{2.11}$$

(We use the fact that $\|P_T E_{n_k}y_k\|_p \leq \|E_{n_k}y_k\|_p \leq \delta$ and $\|(1 - P_T)E_{n_k}y_k\|_p \leq E_{n_k}y_k\|_p \leq \delta$). Since $\delta < (3M)^{-1}$, this leads to a contradiction if m is chosen large enough. □

If $X \subset C_p$ and $X \approx \ell_2$, then by [1, Prop. 4], [4, Prop. 3], X is complemented in C_p. If $p = 2$, this is trivial. If $p \neq 2$ and $V = E_{n|X}$ is an isomorphism, let P be the orthogonal projection from E_nC_2 onto E_nX. Then $Q = V^{-1}PE_n$ is a projection from C_p onto X. Since $\mathrm{d}(E_nC_p, E_nC_2) = \mathrm{d}(E_nC_p, \ell_2) \approx n^{|1/p-1/2|}$, the norm of the projection Q might be very bad. However, by passing to a subspace of X, we can get better results. Precisely, we shall show below that, given $0 < \epsilon$, there is a subspace Y of X which is $1 + \epsilon$-isomorphic to ℓ_2 and $1 + \epsilon$-complemented in C_p.

Let us first establish the following proposition:

PROPOSITION 2.3: *Let $\{m_k\}_{k=1}^{\infty}$ be an increasing sequence of natural* *numbers, and let* $\{y_k\}_{k=1}^{\infty}$ *be normalized elements of* C_p, $1 \le p \le \infty$, *so that for some natural number n we have, for every k,*

$$(2.12) \qquad y_k = E_n P_{m_k, m_{k+1}} y_k.$$

Then for every $0 < \epsilon$ there exists a subsequence $\{y_{k_j}\}_{j=1}^{\infty}$ which is $1+\epsilon$-equivalent to the unit vector basis of ℓ_2, so that $[y_{k_j}]_{j=1}^{\infty}$ is $1+\epsilon$-complemented in C_p. Moreover, the $\{y_{k_j}\}_{j=1}^{\infty}$ can be taken to be arbitrarily close to normalized elements of C_p of the form $z_j = e_{j+1,1} \otimes a + e_{1,j+1} \otimes b$.

Proof: Without loss of generality, we can assume that $n < m_1$ and that $\epsilon < 1$. In an appropriate tensor product represemation, we can write assumption (2.12) as

$$(2.13) \qquad y_k = e_{k+1,1} \otimes a_k + e_{1,k+1} \otimes b_k,$$

with $a_k, b_k \in C_p^n = C_p(\ell_2^n)$, and $(\|a_k\|_p^p + \|b_k\|_p^p)^{1/p} = 1$ (we use the fact that $\text{rank}(P_T y_k) \le n$ and $\text{rank}((1-P_T)y_k) \le n$). By compactness of the unit ball of C_p^n there exist elements $a, b \in C_p^n$ with $(\|a\|_p^p + \|b\|_p^p)^{1/p} = 1$, and an increasing sequence $\{k_j\}_{j=1}^{\infty}$, so that

$$(2.14) \qquad \|a - a_{k_j}\|_p < \epsilon \cdot 8^{-j}, \quad \|b - b_{k_j}\|_p < \epsilon \cdot 8^{-j}.$$

If we put $z_j = e_{k_j+1,1} \otimes a + e_{1,k_j+1} \otimes b$, then $\Sigma_{j=1}^{\infty} \|z_j - y_{k_j}\|_p \le \Sigma_{j=1}^{\infty} 2\epsilon 8^{-j} \le 2\epsilon/7$. Since $\{z_j\}_{j=1}^{\infty}$ is isometrically equivalent to the unit vector basis of ℓ_2 and $[z_j]_{j=1}^{\infty}$ is 1-complemented in C_p (see [2, Theorem 2.2]), and since $(1+2\epsilon/7)(1-2\epsilon/7)^{-1} \le 1+\epsilon$, we get by standard perturbation arguments that $\{y_{k_j}\}_{j=1}^{\infty}$ is $1+\epsilon$-equivalent to the unit vector basis of ℓ_2, and that $[y_{k_j}]_{j=1}^{\infty}$ is $1+\epsilon$-complemented in C_p. □

Lemma 2.4: *Let* $1 \le p < 2$ *and let* $\{x_k\}_{k=1}^{\infty}$ *be a normalized sequence in* C_p *which is equivalent to the unit vector basis of* ℓ_2. *Then for every* $0 < \epsilon < 1$ *there exists a subsequence* $\{x_{k_j}\}_{j=1}^{\infty}$ *which is* $1+\epsilon$-*equivalent to the unit vector basis of* ℓ_2 *and so that* $[x_{k_j}]_{j=1}^{\infty}$ *is* $1+\epsilon$-*complemented in* C_p. *Moreover, given any sequence* $\{\alpha_i\}_{i=1}^{\infty}$ *with* $0 < \alpha_i < 1$, *there exist normalized elements* $\{v_i\}_{i=1}^{\infty}$ *of* C_p *and sequences* $\{a_j\}_{j=1}^{\infty}$ *and* $\{b_j\}_{j=1}^{\infty}$ *in* C_p *with*

$$(2.15) \qquad v_i = \sum_{j=1}^{i-1} (e_{i,j} \otimes a_j + e_{j,i} \otimes b_j) + e_{i,i} \otimes c_i,$$

so that for $j \le 2$

$$(2.16) \qquad (\|a_j\|_p^p + \|b_j\|_p^p)^{1/p} \le \alpha_{j-1}, \quad \|c_j\|_p \le \alpha_{j-1}$$

$$(2.17) \qquad 1 - \alpha_1 \le (\|a_1\|_p^p + \|b_1\|_p^p)^{1/p} \le 1$$

and

$$(2.18) \qquad \|v_i - x_i\|_p \le \alpha_i.$$

PROOF: Since $x_k \to 0$ weakly as $k \to \infty$, we can assume (by passing to a subsequence if necessary, and by using perturbation arguments) that for some increasing sequence $\{m_k\}_{k=0}^{\infty}$ of positive integers with $m_0 = 0$, we have

$$(2.19) \qquad x_k = P_{m_{k-1},m_k} x_k, \quad k = 1, 2, \ldots.$$

We may assume that the given $\{\alpha_i\}_{i=1}^{\infty}$ satisfies

$$(2.20) \qquad \sum_{k=1}^{\infty} \alpha_k \le \epsilon/30.$$

By Proposition 2.2(i) we have

$$(2.21) \qquad \lim_{m\to\infty} (\sup_k \|E^m x_k\|_p) = 0.$$

We can therefore assume that besides (2.19) we have also

$$(2.22) \qquad \sup_k \|E^{m_j} x_k\|_p \le \alpha_j, \quad j = 1, 2, \ldots.$$

For $1 \le j < k$, put

$$(2.23) \qquad y_{k,j} = P_T E_{m_{j-1},m_j} x_k, \quad z_{k,j} = (1 - P_T) E_{m_{j-1},m_j} x_k,$$

and for every k let

$$(2.24) \qquad u_k = E_{m_{k-1},m_k} x_k.$$

We now change the matrix representation so that for some increasing sequence $\{n_k\}_{k=0}^{\infty}$ of positive integers with $n_0 = 0$ and $n_{k-1} + m_k \le n_k$, we have, in terms of the new P_n's and E_n's,

$$(2.25) \qquad y_{k,j} = P_T E_{n_{j-1},n_j} P_{n_{k-1},n_{k-1}+n_j} y_{k,j}, \quad 1 \le j < k,$$

$$(2.26) \qquad z_{k,j} = (1 - P_T) E_{n_{j-1},n_j} P_{n_{k-1},n_{k-1}+n_j} z_{k,j}, \quad 1 \le j < k,$$

(2.27) $$u_k = E_{n_{k-1},n_k} P_{n_{k-1},n_k} u_k$$

and

(2.28) $$\|y_{k,j} + z_{k,j}\|_p = (\|y_{k,j}\|_p^p + \|z_{k,j}\|_p^p)^{1/p} \le \alpha_{j-1}, \quad 2 \le j < k.$$

Indeed, let $n_0 = 0$ and $n_1 = m_1$, and for every $x \in B(\ell_2)$ denote by $R(x)$ the range of x. Choose orthonormal sequences $\{e_i\}_{i=1}^{n_1}$ and $\{f_i\}_{i=1}^{n_1}$ so that

(2.29) $$[e_i]_{i=1}^{n_1} \supset R(u_1) \cup \bigcup_{k=2}^{\infty} R(z_{k,1})$$

(2.30) $$[f_i]_{i=1}^{n_1} \supset R(u_1^*) \cup \bigcup_{k=2}^{\infty} R(y_{k,1}^*).$$

Since $\operatorname{rank}(y_{2,1}) \le n_1$ and $\operatorname{rank}(z_{2,1}) \le n_1$, there exists an $n_2 \ge n_1 + m_2$ and orthonormal sequences $\{e_i\}_{i=n_1+1}^{n_2}$ and $\{f_i\}_{i=n_1+1}^{n_2}$, so that $\{e_i\}_{i=1}^{n_2}$ and $\{f_i\}_{i=1}^{n_2}$ are orthonormal, and so that

(2.31) $$[e_i]_{i=n_1+1}^{2n_1} \supset R(y_{2,1}), \quad [f_i]_{i=n_1+1}^{2n_1} \supset R(z_{2,1})$$

(2.32) $$[e_i]_{i=n_1+1}^{n_2} \supset R(u_2) \cup \bigcup_{k=3}^{\infty} R(z_{k,2})$$

(2.33) $$[f_i]_{i=n_1+2}^{n_2} \supset R(u_2^*) \cup \bigcup_{k=3}^{\infty} R(y_{k,2}^*).$$

Similarly, since $\operatorname{rank}(y_{3,j}) \le n_j - n_{j-1}$ and $\operatorname{rank}(z_{3,j}) \le n_j - n_{j-1}$ for $j = 1, 2$, there is some $n_3 \ge n_2 + m_3$, and there exist orthonormal sequences $\{e_i\}_{i=n_2+1}^{n_3}$ and $\{f_i\}_{i=n_2+1}^{n_3}$ so that $\{e_i\}_{i=1}^{n_3}$ and $\{f_i\}_{i=1}^{n_3}$ are orthonormal, so that for $j = 1, 2$,

(2.34) $$[e_i]_{i=n_2+1}^{n_2+n_j} \supset R(y_{3,j}), \quad [f_i]_{i=n_2+1}^{n_2+n_j} \supset R(z_{3,j})$$

and

(2.35) $$[e_i]_{i=n_2+1}^{n_3} \supset R(u_3) \cup \bigcup_{k=4}^{\infty} R(z_{k,3})$$

(2.36) $$[f_i]_{i=n_2+1}^{n_3} \supset R(u_3^*) \cup \bigcup_{k=4}^{\infty} R(y_{k,3}^*).$$

We continue inductively in the obvious way. If the new P_n's and E_n's are defined by means of formulas (1.3) and (1.4), using the new matrix representation associated with the pair of the above constructed orthonormal bases $(\{e_i\}_{i=1}^{\infty}, \{f_i\}_{i=1}^{\infty})$, then we clearly have (2.25)–(2.28). Note that we still have

$$x_k = \sum_{j=1}^{k-1} (y_{k,j} + z_{k,j}) + u_k, \quad k = 1, 2, \ldots. \tag{2.37}$$

Let $C_p^{n,m}$ denote the space of all $n \times m$-complex matrices with the norm induced from C_p. Passing to tensor product notations, we obtain from (2.25)–(2.28),

$$y_{k,j} = e_{k,j} \otimes a_{k,j}, \quad a_{k,j} \in C_p^{n_j, n_j - n_{j-1}}, \quad 1 \le j < k \tag{2.38}$$

$$z_{k,j} = e_{j,k} \otimes b_{k,j}, \quad b_{k,j} \in C_p^{n_j - n_{j-1}, n_j}, \quad 1 \le j < k \tag{2.39}$$

$$u_k = e_{k,k} \otimes \tilde{c}_k, \quad \tilde{c}_k \in C_p^{n_k - n_{k-1}, n_k - n_{k-1}} \tag{2.40}$$

$$(\|a_{k,j}\|_p^p + \|b_{k,j}\|_p^p)^{1/p} \le \alpha_{j-1}, \quad 2 \le j < k. \tag{2.41}$$

Clearly, for $k \ge 2$ we have $\|u_k\| \le \alpha_{k-1}$ and

$$1 - \alpha_1 \le \|E_{n_1} x_k\|_p = (\|a_{k,1}\|_p^p + \|b_{k,1}\|_p^p)^{1/p} \le 1. \tag{2.42}$$

As in the proof of Proposition 2.3, there exist elements $\tilde{a}_j \in C_p^{n_j, n_j - n_{j-1}}$ and $\tilde{b}_j \in C_p^{n_j - n_{j-1}, n_j}$ with $1 - \alpha_1 \le (\|\tilde{a}_1\|_p^p + \|\tilde{b}_1\|_p^p)^{1/p} \le 1$ and $(\|\tilde{a}_j\|_p^p + \|\tilde{b}_j\|_p^p)^{1/p} \le \alpha_{j-1}$ for $2 \le j$, and there exists a subsequence $\{x_{k_i}\}_{i=1}^{\infty}$ with $k_1 > 1$ so that, if we define

$$v_i = \sum_{j=1}^{k_i - 1} (e_{k_i,j} \otimes \tilde{a}_j + e_{j,k_i} \otimes \tilde{b}_j) + e_{k_i,k_i} \otimes \tilde{c}_{k_i}, \tag{2.43}$$

then $\|x_{k_i} - v_i\|_p \le \alpha_i$. Now, if

$$w_i = e_{k_i,1} \otimes \tilde{a}_1 + e_{1,k_i} \otimes \tilde{b}_1, \tag{2.44}$$

then the $\{w_i / \|w_i\|_p\}_{i=1}^{\infty}$ are isometrically equivalent to the unit vector basis of ℓ_2 and $[w_i]_{i=1}^{\infty}$ is 1-complemented in C_p. Let $\{t_i\}_{i=1}^{\infty}$ be scalars so that $\Sigma_{i=1}^{\infty} |t_i|^2 = 1$. Then by (2.20) and (2.43),

$$\begin{aligned} \left\| \sum_{i=1}^{\infty} t_i (x_{k_i} - w_i) \right\|_p &\le \sum_{i=1}^{\infty} |t_i| \alpha_i + \left\| \sum_{i=1}^{\infty} t_i (v_i - w_i) \right\|_p \\ &\le \sum_{i=1}^{\infty} \alpha_i + \sum_{i=1}^{\infty} |t_i| \, \|u_{k_i}\|_p + \left\| \sum_{i=1}^{\infty} \sum_{j=2}^{k_i - 1} t_i (e_{k_i,j} \otimes \tilde{a}_j + e_{j,k_i} \otimes \tilde{b}_j) \right\|_p \end{aligned} \tag{2.45}$$

$$\leq 2\sum_{i=1}^{\infty}\alpha_i + \sum_{j=2}^{\infty}(\|\tilde{a}_j\|_p^p + \|\tilde{b}_j\|_p^p)^{1/p}\left(\sum_{j<k_i}|t_i|^2\right)^{1/2}$$

$$\leq 2\sum_{i=1}^{\infty}\alpha_i + \sum_{j=2}^{\infty}\alpha_{j-1} \leq 3\sum_{i=1}^{\infty}\alpha_i \leq \epsilon/10.$$

Since $1 \geq \|w_i\|_p \geq 1-\alpha_1 > 1-\epsilon/30$, we get that $\{x_{k_i}\}_{i=1}^{\infty}$ is λ-equivalent to the unit vector basis of ℓ_2, and that $[x_{k_i}]_{i=1}^{\infty}$ is λ-complemented in C_p, where

(2.46) $$\lambda \leq (1+\epsilon/10)(1-\epsilon/30)^{-1}(1-\epsilon/10)^{-1} \leq 1+\epsilon.$$

Finally, let us define $c_i = \tilde{c}_{k_i}$ and

(2.47) $$a_j = \sum_{\ell=k_{j-1}}^{k_j-1} e_{1,\ell}\otimes\tilde{a}_\ell, \quad j=1, \quad j=1,2,\ldots,$$

(2.48) $$b_j = \sum_{\ell=k_{j-1}}^{k_j-1} e_{\ell,1}\otimes\tilde{b}_\ell, \quad j=1,2,\ldots.$$

By (2.22) we clearly have (2.16) and (2.17). Also, it is clear how to choose a new tensor product representation so that (2.15) holds (use (2.43) and the definition of the a_j, b_j and c_i in terms of the $\tilde{a}_j$, $\tilde{b}_j$ and $\tilde{c}_{k_i}$). Clearly, (2.18) still holds. □

Remark: Let $\lambda > 0$ and consider the sequence $x_k = (\lambda e_{k,1} + e_{k,k})(\lambda^2+1)^{-1/2}$ in C_p, $2<p<\infty$. The equivalence constant of every subsequence $\{x_{k_j}\}_{j=1}^{\infty}$ to the unit vector basis of ℓ_2 behaves like λ^{-1} (which might be very large). Thus, the analogue of Lemma 2.4 is false for $2<p<\infty$. It can also be easily verified that if $X = [x_{k_j}]_{j=1}^{\infty}$ for some increasing sequence $\{k_j\}$, then for every $n: \|E^n|_X\| \geq (1+\lambda^2)^{-1/2}$. Therefore the analogue of Proposition 2.2(i) is also false for $2<p<\infty$. There are, however, averages of these $\{x_k\}$ which behave in a better way. Precisely, let $0<\epsilon$, and choose an increasing sequence $\{k_j\}_{j=1}^{\infty}$ of positive integers, so that if $\Delta k_j = k_{j+1}-k_j$, then

(2.49) $$\Delta k_j \geq (8^j\lambda^{-1}\epsilon^{-1})^{2p/(p-2)}.$$

Define for $j=1,2,\ldots,$

(2.50) $$\tilde{y}_j = \sum_{k=k_j+1}^{k_{j+1}} x_k, \quad y_j = \tilde{y}_j/\|\tilde{y}_j\|_p.$$

Then

$$(2.51)\qquad \|E^1 y_j\|_p \le \lambda^{-1}(1+\lambda^2)^{1/2}(\Delta k_j)^{-1/2}\left\|E^1 \sum_{k=k_j+1}^{k_{j+1}} x_k\right\|_p \le \lambda^{-1}(\Delta_{kj})^{(2-p)/2p} \le \epsilon \cdot 8^{-j}.$$

Using Proposition 2.1 we get, for every scalar $\{t_j\}_{j=1}^n$,

$$(2.52)\qquad \left(\sum_j |t_j|^2\right)^{1/2} \ge \left\|\sum_j t_j y_j\right\|_p \ge \left\|\sum_j t_j E_1 y_j\right\|_p - \sum_j |t_j|\,\|E^1 y_j\|_p$$
$$\ge \left(\sum_j |t_j|^2 (1-\|E^1 y_j\|_p)^2\right)^{1/2} - \left(\sum_j |t_j|^2\right)^{1/2}$$
$$\times \left(\sum_j (\epsilon \cdot 8^{-j})^2\right)^{1/2} \ge \left(\sum_j |t_j|^2\right)^{1/2} (1+\epsilon)^{-1}.$$

So $\{y_j\}_{j=1}^\infty$ are $1+\epsilon$-equivalent to the unit vector basis of ℓ_2. Since $Px = \sum_{j=1}^\infty \langle x, n_p(E_1 y_j)\rangle E_1 y_j / \|E_1 y_j\|_p^p$ is a contractive projection from C_p onto $[E_1 y_j]_{j=1}^\infty$, we get by (2.51) that $[y_j]_{j=1}^\infty$ is $1+\epsilon$-complemented in C_p.

The idea of using averages of the for (2.50) in order to "kill the ℓ_2-part" of a sequence in C_p, $2<p<\infty$, which is equivalent to the unit vector basis of ℓ_2 is due to Odell [9]. This is the heart of the proof of the following lemma, which is essentially [9, Lemma 5].

LEMMA 2.5: *Let $\{x_k\}_{k=1}^\infty$ be a normalized sequence in C_p, $2<p<\infty$, which is equivalent to the unit vector basis of ℓ_2. Let $\epsilon>0$, then there exists a subsequence $\{x_{k_j}\}_{j=1}^\infty$ and an increasing sequence $\{j_\ell\}_{\ell=1}^\infty$ of positive integers, so that if we define*

$$(2.53)\qquad \tilde{y}_\ell = \sum_{j=j_\ell+1}^{j_{\ell+1}} x_{k_j}, \quad y_\ell = \tilde{y}_\ell / \|\tilde{y}_\ell\|_p,$$

then the $\{y_\ell\}_{\ell=2}^\infty$ are $1+\epsilon$-equivalent to the unit vector basis of ℓ_2, and $[y_\ell]_{\ell=2}^\infty$ is $1+\epsilon$-complemented in C_p.

Moreover, given any sequence $\{\alpha_i\}_{i=1}^\infty$ with $0<\alpha_i<1$, the $\{y_\ell\}_{\ell=2}^\infty$ can be chosen so that there exists normalized elements $\{v_\ell\}_{\ell=2}^\infty$ of C_p of the form

$$(2.54)\qquad v_\ell = \sum_{i=1}^{\ell-1} (e_{\ell,i} \otimes a_i + e_{i,\ell} \otimes b_i) + e_{\ell,\ell} \otimes c_\ell$$

with $\max\{\|a_i\|_p, \|b_i\|_p\} \le \alpha_{i-1}$ and $\|c_i\| \le \alpha_i$ for $2 \le i$ and $(\|a_1\|_p^p + \|b_1\|_p^p)^{1/p} \ge 1-\alpha_1$, so that $\|y_\ell - v_\ell\|_p \le \alpha_\ell$ for $\ell = 2, 3, \ldots$.

PROOF: Let $X = [x_k]_{k=1}^{\infty}$ and $M = d(X, \ell_2) < \infty$. Fix $0 < \delta < (3M)^{-1}$ and choose, by Proposition 2.2(i), a natural number N so that $\|E_N x\|_p \geq \delta \|x\|_p$ for every $x \in X$. Since $x_k \to 0$ weakly as $k \to 0$, there is no loss of generality if we assume that for some subsequence $\{x_{k_j}\}_{j=1}^{\infty}$ and some increasing sequence $\{m_j\}_{j=0}^{\infty}$ of integers with $m_0 = 0$ and $m_1 = N$, we have

$$x_{k_j} = P_{m_{j-1}, m_j} x_{k_j}, \quad j = 1, 2, \ldots. \tag{2.55}$$

Put $y_j = P_T E_{m_{j-1}} x_{k_j}$ and $z_j = (1 - P_T) E_{m_{j-1}} x_{k_j}$ and note that $\|E_{m_{j-1}} x_{k_j}\|_p = (\|y_j\|_p^p + \|z_j\|_p^p)^{1/p}$, and that $\|y_j\|_p, \|z_j\|_p \leq \|x_{k_j}\|_p = 1$. Using a standard diagonal process, we can pass to a further subsequence which we continue to denote by $\{x_{k_j}\}_{j=1}^{\infty}$ for convenience, so that for every $n < \ell$, $\lim_{j \to \infty} \|E_{n,\ell} x_{k_j}\|_p$ exists.

CLAIM: If $\alpha > 0$, then there is some $n = n(\alpha)$ so that if $n < \ell$, then the set

$$A_{n,\ell,\alpha} = \{j : \|E_{n,\ell} x_j\|_p > \alpha\}$$

is finite.

PROOF OF THE CLAIM: Indeed, if the claim is false for some $\alpha > 0$, there exist integers

$$1 \leq n_1 < \ell_1 < n_2 < \ell_2 < \cdots < n_i < \ell_i < \cdots$$

so that the complement of each $A_{n_i, \ell_i, \alpha/2}$ is finite. Let m be such that $m > (4/\alpha)^p$, and choose j so that $m_{j-1} > \ell_m$ and that $j \in \bigcap_{i=1}^{m} A_{n_i, \ell_i, \alpha/2}$. Using Proposition 2.1 we get the desired contradiction:

$$\begin{aligned} 1 = \|x_j\|_p &\geq \Big\| \sum_{i=1}^{m} E_{n_i, \ell_i} x_j \Big\|_p \\ &\geq \Big(\sum_{i=1}^{m} \|E_{n_i, \ell_i} y_j\|_p^p + \|E_{n_i, \ell_i} z_j\|_p^p \Big)^{1/p} \\ &= \Big(\sum_{i=1}^{m} \|E_{n_i, \ell_i} x_j\|_p^p \Big)^{1/p} \geq (\alpha/2) m^{1/p} > 2, \end{aligned} \tag{2.56}$$

thus proving the claim.

Let $\{\alpha_i\}_{i=1}^{\infty}$ be any sequence with $0 < \alpha_i < 1$. We may assume that

$$\sum_{i=1}^{\infty} \alpha_i \leq \epsilon/100, \quad \text{and} \quad \sum_{i=m+1}^{\infty} \alpha_i \leq \alpha_m \text{ for every } m. \tag{2.57}$$

Using the claim, we can pass to a subsequence of $\{x_{k_j}\}_{j=1}^{\infty}$, which we continue to denote by $\{x_{k_j}\}_{j=1}^{\infty}$ for convenience, so that for some increasing sequence $\{n_j\}_{j=0}^{\infty}$ with $n_0 = 0$ and $n_1 > N$, we have

$$x_{k_j} = P_{n_{j-1},n_j} x_{k_j} \tag{2.58}$$

$$\|E_{n_{i-1},n_i} x_{k_j}\|_p \leq \alpha_{i-1}/16M, \quad 2 \leq i \leq j-1. \tag{2.59}$$

Passing to an appropriate tensor product representation, we have

$$x_{k_j} = \sum_{i=1}^{j-1} (e_{j,i} \otimes y_{j,i} + e_{i,j} \otimes z_{j,i}) + e_{j,j} \otimes u_j \tag{2.60}$$

with $y_{j,i} \in C^{n_i,n_i-n_{i-1}}$, $z_{j,i} \in C^{n_i-n_{i-1},n_i}$, and

$$(\|y_{j,1}\|_p^p + \|z_{j,1}\|_p^p) \geq \delta, \quad j = 2, 3, \ldots, \tag{2.61}$$

and for $2 \leq j < j$,

$$(\|y_{j,i}\|_p^p + \|z_{j,i}\|_p^p)^{1/p} = \|E_{n_{i-1},n_i} x_{k_j}\|_p \leq \alpha_{i-1}/16M. \tag{2.62}$$

As in the proof of Proposition 2.3, we can obtain, by standard diagonal process, a subsequence $\{x_{k_{j_\nu}}\}_{\nu=1}^{\infty}$ and elements $\tilde{a}_i \in C_p^{n_i,n_i-n_{i-1}}$, $\tilde{b}_i \in C_p^{n_i-n_{i-1},n_i}$ so that $\|y_{j_\nu,i} - a_i\|_p$ and $\|z_{j_\nu,i} - b_i\|_p$ tend to zero as $\nu \to \infty$ arbitrarily fast. Without loss of generality we assume, therefore, that the $\{x_{k_j}\}_{j=1}^{\infty}$ themselves are given by

$$x_{k_j} = \sum_{i=1}^{j-1} (e_{j,i} \otimes \tilde{a}_i + e_{i,j} \otimes \tilde{b}_i) + e_{j,j} \otimes u_j \tag{2.63}$$

with

$$(\|\tilde{a}_i\|_p^p + \|\tilde{b}_i\|_p^p)^{1/p} \leq \alpha_{i-1}/8M, \quad 2 \leq i < \infty \tag{2.64}$$

and

$$(\|\tilde{a}_1\|_p^p + \|\tilde{b}_1\|_p^p)^{1/p} \geq \delta/2. \tag{2.65}$$

Here the norms $\|x_{k_j} - (e_{j,1} \otimes \tilde{a}_{1,j} + e_{1,j} \otimes \tilde{b}_1)\|_p$ need not be small, since for the norms of the u_j we have only the trivial estimate $\|u_j\|_p \leq (1 - \delta/2)^p)^{1/p}$. Now, the $\{e_{j,j} \otimes u_j\}_{j=1}^{\infty}$ are pairwise disjointly supported, and thus

$$\left\|\sum_{j=k}^{m} e_{j,j}\otimes u_j\right\|_p = \left(\sum_{j=k}^{m} \|u_j\|_p^p\right)^{1/p}$$

for every $k < m$. Since $2 < p < \infty$, we can "kill" these "ℓ_p-parts" by taking long averages as in the example which precedes the statement of Lemma 2.5. Precisely, let $\{j_\ell\}_{\ell=1}^{\infty}$ be an increasing sequence of integers with $j_1 = 1$, so that the differences $\Delta_\ell = j_{\ell+1} - j_\ell$ satisfy

$$\Delta_\ell^{(2-p)/2p} \le \alpha_\ell / 2M. \tag{2.66}$$

Set

$$\tilde{y}_\ell = \sum_{j=j_\ell+1}^{j_{\ell+1}} x_{k_j}, \quad y_\ell = \tilde{y}_\ell / \|\tilde{y}_\ell\|_p. \tag{2.67}$$

Then

$$y_\ell = \sum_{i=1}^{j_\ell} (y_\ell^{(i)} + z_\ell^{(i)}) + w_\ell, \tag{2.68}$$

where

$$y_\ell^{(i)} = \sum_{j=j_\ell+1}^{j_{\ell+1}} e_{j,i}\otimes \tilde{a}_i / \|\tilde{y}_\ell\|_p, \tag{2.69}$$

$$z_\ell^{(i)} = \sum_{j=j_\ell+1}^{j_{\ell+1}} e_{i,j}\otimes \tilde{b}_i / \|\tilde{y}_\ell\|_p, \tag{2.70}$$

$$w_\ell = \left\{\sum_{j=j_\ell+1}^{j_{\ell+1}} e_{j,j}\otimes u_j + \sum_{i=j_\ell}^{j_{\ell+1}} \sum_{j=i+1}^{j_{\ell+1}} (e_{j,i}\otimes \tilde{a}_i + e_{i,j}\otimes \tilde{b}_i)\right\} \Big/ \|\tilde{y}_\ell\|_p. \tag{2.71}$$

Now, for every $2 \le m \le j_\ell$,

$$\begin{aligned} \left\|\sum_{i=m}^{j_\ell} y_\ell^{(i)} + w_\ell\right\|_p &\le M\Delta_\ell^{-1/2}\left\{\sum_{i=m}^{j_\ell}\sum_{j=j_\ell+1}^{j_{\ell+1}} \|\tilde{a}_i\|_p^2 + \|w_\ell\|_p^2\right\}^{1/2} \\ &\le M\sum_{i=m}^{j_\ell} \alpha_{i-1}/8M + M\Delta_\ell^{-1/2}\cdot\Delta_\ell^{1/p} \\ &\quad + M\Delta_\ell^{-1/2}\left(\sum_{i=j}^{j_{\ell+1}-1}\sum_{j=i+1}^{j_{\ell+1}} (\|\tilde{a}_i\|_p^2 + \|\tilde{b}_i\|_p^2)\right)^{1/2} \\ &\le \sum_{i=m}^{\infty} \alpha_i/4 + M\Delta_\ell^{(2-p)/2p} \le \alpha_{m-1}/4 + \alpha_\ell/2. \end{aligned} \tag{2.72}$$

Similarly, for every $2 \le m \le j_\ell$,

$$\left\| \sum_{i=m}^{j_\ell} z_\ell^{(i)} + w_\ell \right\|_p \leq \alpha_{m-1}/4 + \alpha_\ell/2. \tag{2.73}$$

Using Proposition 2.3 again (and passing to a subsequence of $\{y_\ell\}_{\ell=1}^\infty$ if necessary) we can assume that in some other tensor product representation we have normalized elements

$$v_\ell = \sum_{i=1}^{\ell-1} (e_{\ell,i} \otimes a_i + e_{i,\ell} \otimes b_i) + e_{\ell,\ell} \otimes c_\ell, \tag{2.74}$$

with $\max\{\|a_i\|_p, \|b_i\|_p\} \leq \alpha_{i-1}$ and $\|c_i\|_p \leq \alpha_i$ for $2 \leq i$, $(\|a_1\|_p^p + \|b_1\|_p^p)^{1/p} \geq 1 - \alpha_1$, and so that $\|y_\ell - v_\ell\|_p \leq \alpha_\ell$ for $\ell = 2, 3, 4, \ldots$.

Set $\tilde{v}_\ell = (e_{\ell,1} \otimes a_1 + e_{1,\ell} \otimes b_1)/(\|a_1\|_p^p + \|b_1\|_p^p)^{1/p}$, $\ell = 2, 3, \ldots$, and note that $\{\tilde{v}_\ell\}_{\ell=2}^\infty$ are isometrically equivalent to the unit vector basis of ℓ_2, and that $\{\tilde{v}_\ell\}_{\ell=2}^\infty$ is 1-complemented in C_p. If $\{t_\ell\}_{\ell=2}^\infty$ are scalars with $\Sigma_{\ell=2}^\infty |t_\ell|^2 = 1$, then

$$\begin{aligned} \left\| \sum_{\ell=2}^{\infty} t_\ell (y_\ell - \tilde{v}_\ell) \right\|_p &\leq \sum_{\ell=2}^{\infty} |t_\ell| \alpha_\ell + \left\| \sum_{\ell=2}^{\infty} t_\ell (v_\ell - \tilde{v}_\ell) \right\|_p \\ &\leq \epsilon/100 + \sum_{\ell=2}^{\infty} |t_\ell| \, \|v_\ell - \tilde{v}_\ell\|_p \\ &\leq \epsilon/100 + 4 \sum_{i=1}^{\infty} \alpha_i \leq \epsilon/20. \end{aligned} \tag{2.75}$$

This implies that $\{y_\ell\}_{\ell=2}^\infty$ is $1+\epsilon$-equivalent to the unit vector basis of ℓ_2 and $[y_\ell]_{\ell=2}^\infty$ is $1+\epsilon$-complemented in C_p.

The next proposition follows from [2, Theorem 2.2].

PROPOSITION 2.6: *Let $x \in C_p$, $\|x\|_p = 1$, $1 \leq p \leq \infty$, and let $x_{i,j} = e_{i,j} \otimes x$, $1 \leq i, j < \infty$. Then*

(i) *the $\{x_{i,j}\}_{i,j=1}^\infty$ are isometrically equivalent to the standard unit matrices $\{e_{i,j}\}_{i,j=1}^\infty$ in C_p, and there is a contractive projection from C_p onto $[x_{i,j}]_{i,j=1}^\infty$.*

(ii) *the $\{x_{i,j}\}_{1 \leq j < i < \infty}$ are isometrically equivalent to the standard unit matrices $\{e_{i,j}\}_{1 \leq j < j < \infty}$ of T_p, and there is a projection of norm ≤ 2 from T_p onto $[x_{i,j}]_{1 \leq j < j < \infty}$.*

PROOF: Assertion (i) is actually a part of [2, Theorem 2.2], and the first statement in (ii) follows from (i). If P is the contractive projection from C_p ($\equiv C_p \otimes C_p$) onto $[x_{i,j}]_{i,j=1}^\infty$ constructed in [2], and if D is the canonical contractive projection from C_p onto $\Sigma_i \oplus (e_{i,i} \otimes C_p)$ (it is a projection of the form (1.10)), then $Q = (1 - D)P_{|T_p}$ is a projection from T_p onto its subspace $[x_{i,j}]_{1\le j<i<\infty}$ and $\|Q\|\le 2$. Clearly, the $\{x_{i,j}\}_{1\le j<i<\infty}$ are isometrically equivalent to the $\{e_{i,j}\}_{1\le j<j>\infty}$. □

A triangular sequence is a double sequence of the form $\{x_{i,j}\}_{1\le j\le i<\infty}$. In short, we denote it also by $\{x_{i,j}\}_{j\le i}$ and call it simply a ***triangle***. A ***subtriangle*** of $\{x_{i,j}\}_{j\le 1}$ is a triangle of the form $\{x_{i_k,j_\ell}\}_{\ell\le k}$, where $\{i_k\}_{k=1}^\infty$ and $\{j_\ell\}_{\ell=1}^\infty$ are increasing sequences of positive integers with $i_k\ge j_k$ for every k. When we consider a triangle $\{x_{i,j}\}_{j\le i}$ of elements of a Banach space as a basic sequence, we shall always mean that it is a basic sequence in the following (lexicographic) ordering:

$$x_{1,1}, x_{2,1}, x_{2,2}, x_{3,1}, x_{3,2}, x_{3,3}, x_{4,1}, x_{4,2}, x_{4,3}, x_{4,4}, \ldots$$

A triangle $\{x_{i,j}\}_{j\le i}$ in X is M-equivalent to a triangle $\{y_{i,j}\}_{j\le 1}$ in Y if $\{x_{i,j}\}_{j\le i}$ and $\{y_{i,j}\}_{j\le i}$ are basic sequences which are M-equivalent in the usual sense.

In what follows we shall use several times a procedure of passing to a subtriangle $\{x_{i_k,j_\ell}\}_{\ell\le i}$ (which has nice properties) starting with a triangle $\{x_{i,j}\}_{j\le i}$. The general scheme of a such procedure is the following. Assume that A is an infinite set of naturals numbers, and that for every $j\in A$, every subsequence of $\{x_{i,j}\}_{i=j}^\infty$ has a further, "nice" subsequence. Let j_1 be the first element of A, and let $\{x_{i_k^{(1)},j_1}\}_{k=1}^\infty$ be a "nice" subsequence of $\{x_{i,j_1}\}_{i=j_1}^\infty$ and $i_1^{(1)}\ge j_1$. Assume that $j_1<j_2<\cdots<j_m$ have been chosen from A, and that we have already defined increasing sequences $\{i_k^{(\ell)}\}_{k=\ell}^\infty$, $1\le\ell\le m$, so that $\{x_{i_k^{(\ell)},j_\ell}\}_{k=\ell}^\infty$ is a "nice" subsequence of $\{x_{i_k^{(\ell-1)},j_\ell}\}_{j_\ell\le i_k^{(\ell-1)}}$ and $i_\ell^{(\ell)}\ge j_\ell$.

Let j_{m+1} be the first element of $A\sim(j_1,\ldots,j_m\}$ which is greater than j_m, and let $\{x_{i_k^{(m+1)},j_{m+1}}\}_{k=m+1}^\infty$ be a "nice" subsequence of $\{x_{i_k^{(m)},j_{m+1}}\}_{j_{m+1}\le i_k^{(m)}}$ with $i_{m+1}^{(m+1)}\ge j_{m+1}$. If we write $i_k=i_k^{(k)}$, then, clearly, $\{x_{i_k,j_\ell}\}_{\ell\le k}$ is a subtriangle of $\{x_{i,j}\}_{j\le i}$; and each column $\{x_{i_k,j_\ell}\}_{k=\ell}^\infty$ is "nice".

3. Proof of theorem 1.1

In proving Theorem 1.1 we shall treat separately the cases $1<p<2$ and $2<p<\infty$ (since C_2 is a Hilbert space, Theorem 1.1 is trivial for $p=2$). Let us establish first the following lemma, whose proof is the same for every $1\le p\le\infty$. Recall that $C_p^{n,m}$ denotes the space of all $n\times m$ complex matrices with the norm induced from C_p.

LEMMA 3.1: *Let $1\le p\le\infty$, let N be a natural number, let $\{m_n\}_{n=1}^\infty$*

be an increasing sequence of integers with $m_1 = 1$ *and* $m_{n+1} - m_n > N$ *for every* n, *and let* $\{x_{i,j}\}_{j \le i}$ *be normalized elements in* T_p *which satisfy*

$$x_{i,j} = E_{m_j, m_{j+1}} P_{m_i, m_i + N} x_{i,j}. \tag{3.1}$$

Then, for every $1 > \epsilon > 0$ *there exists a subtriangle* $\{x_{i_k, j_\ell}\}_{\ell \le k}$ of $\{x_{i,j}\}_{j \le i}$, *which is* $1+\epsilon$*-equivalent to the triangle* $\{e_{k,\ell}\}_{\ell \le k}$ *of the standard unit matrices of* T_p, *and so that* $[x_{i_k, j_\ell}]_{\ell \le k}$ *is* $2+\epsilon$*-complemented in* T_p.

PROOF: It is clear that in an appropriate tensor product representation of C_p as $C_p \otimes C_p$, assumption (3.1) can be written as

$$x_{i,j} = e_{i,j} \otimes y_{i,j}, \quad j \le i, \tag{3.2}$$

where $y_{i,j} \in C_p^{N, m_{j+1} - m_j}$ and $\|y_{i,j}\|_p = 1$.

Now, for a fixed j the sequence $\{y_{i,j}\}_{i=j}^{\infty}$ is contained in the unit ball of a finite dimensional space, so it has a norm-convergent subsequence. We obtain, therefore, normalized elements $y_j \in C_p^{N, m_{j+1} - m_j}$ and increasing sequences of positive integers $\{i_k^{(j)}\}_{k=j}^{\infty}$, $j = 1, 2, 3, \ldots$, so that $\{i_k^{(j+1)}\}_{k=j+1}^{\infty}$ is a subsequence of $\{i_k^{(j)}\}_{k=j}^{\infty}$, and so that

$$y_{i_k^{(j)}, j} \xrightarrow[k \to \infty]{} y_j$$

in norm. Let $i_k = i_k^{(k)}$ be the diagonal sequence. Since $y_{i_k, j} \xrightarrow[k \to \infty]{} y_j$ for every j, there is no loss of generality in assuming that for every j and every $j \le k$, we have

$$\|y_{i_k, j} - y_j\|_p \le \epsilon \cdot 8^{-k-j}.$$

Recall that for every bounded operator x in the Hilbert space H, we denote by $r(x)$ the orthogonal projection from H onto $(\ker x)^\perp$. Now, if $j_1 \ne j_2$ and k_1, k_2 are arbitrary, then

$$r(e_{i_{k_1}, j_i} \otimes y_{j_1}) \cdot r(e_{i_{k_2}, j_2} \otimes y_{j_2}) = 0.$$

Moreover, each y_j is an operator of rank $\le N$, as an element of $C_p^{N, m_{j+1} - m^j}$. It follows that we can change the matrix representation (by choosing a new orthonormal basis for the domain of the operators, while keeping the orthonormal basis for their range unchanged) so that $y_j \in C_p^{N,N}$ for every j, and so that (3.1) is still valid with the new E_n's and P_n's. Again, by the compactness of the unit ball of $C_p^{N,N}$, there is an element $y \in C_p^{N,N}$ with $\|y\|_p = 1$ and there is a subsequence $\{y_{j_\ell}\}_{\ell=1}^\infty$ of $\{y_j\}_{j=1}^\infty$ so that $\|y_{j_\ell} - y\|_p \le \epsilon \cdot 8^{-\ell}$ for every ℓ. By passing to a subsequence of $\{i_k\}_{k=1}^\infty$ if necessary, we can assume that $\ell \le k$ always implies $j_\ell \le j_k$.

Let $\{t_{k,\ell}\}_{\ell\le k}$ be scalars, and let $s = \min\{p, 2\}$. Then, using Proposition 2.1, we obtain

$$
\begin{aligned}
(3.3)\qquad & \Big\|\sum_{\ell\le k} t_{k,\ell}(x_{i_k,j_\ell} - e_{i_k,j_\ell}\otimes y)\Big\|_p \\
&\le \Big\|\sum_{\ell\le k} t_{k,\ell}e_{i_k,j_\ell}\otimes(y_{i_k,j_\ell} - y_{j_\ell})\Big\|_p \\
&\quad + \Big\|\sum_{\ell\le k} t_{k,\ell}e_{i_k,j_\ell}\otimes(y_{j_\ell} - y)\Big\|_p \\
&\le \sum_{\ell\le k} |t_{k,\ell}|\epsilon\cdot 8^{-k-\ell} + \Big(\sum_{\ell=1}^\infty \Big\|\sum_{k=\ell}^\infty t_{k,\ell}e_{i_k,j_\ell}\otimes(y_{j_\ell} - y)\Big\|_p^s\Big)^{1/s} \\
&\le \frac{\epsilon}{50}\cdot\sup_{\ell\le k}|t_{k,\ell}| + \Big(\sum_{\ell=1}^\infty (\epsilon\cdot 8^{-\ell})^s\Big(\sum_{k=\ell}|t_{k,\ell}|^2\Big)^{s/2}\Big)^{1/s} \\
&\le \Big[\frac{\epsilon}{50} + \Big(\sum_{\ell=1}^\infty (\epsilon\cdot 8^{-\ell})^s\Big)^{1/s}\Big]\cdot\Big\|\sum_{\ell\le k} t_{k,\ell}e_{k,\ell}\Big\|_p \\
&\le \frac{\epsilon}{6}\cdot\Big\|\sum_{\ell\le k} t_{k,\ell}e_{k,\ell}\Big\|_p = \frac{\epsilon}{6}\cdot\Big\|\sum_{\ell\le k} t_{k,\ell}e_{i_k,j_\ell}\otimes y\Big\|_p.
\end{aligned}
$$

From this it follows that the triangle $\{x_{i_k,j_\ell}\}_{\ell\le k}$ is $1+\epsilon$-equivalent to the triangle $\{e_{k,\ell}\}_{\ell\le k}$. Also, inequality (3.3) and the existence of a projection from T_p onto $[e_{i_k,j_\ell}\otimes y]_{\ell\le k}$ with norm ≤ 2 (see Proposition 2.6) imply the existence of a projection from T_p onto $[x_{i_k,j_\ell}]_{\ell\le k}$ of norm $\le 2+\epsilon$. □

In proving Theorem 1.1 we prefer, for convenience, to work in T_p instead of in C_p (since $T_p \approx C_p$ for $1 < p < \infty$, this is permissible). Our proof works also for $p = 1$, and it gives an almost isometric result. Therefore Theorem 1.1 is the consequence of the following theorem.

THEOREM 3.2: *Let X be a subspace of T_p, $1 \le p < \infty$, so that X is isomorphic to T_p, and let $0 < \theta < 1$. Then there exists a subspace Y of X so that* $d(Y, T_p) \le 1+\theta$, *and so that Y is $2+\theta$-complemented in T_p.*

Let us sketch first the two main steps in the proof of Theorem 3.2. We start with a triangle $\{x_{i,j}\}_{j\le i}$ which is equivalent to the triangle $\{e_{i,j}\}_{j\le i}$ of the standard unit matrices of T_p, and so that $[x_{i,j}]_{j\le 1} = X$. In the first, lengthy, step of the proof we construct from the $x_{i,j}$ a triangle of normalized elements of X which is an arbitrarily small perturbation of a triangle of the form $\{e_{k,1} \otimes z_\ell\}_{\ell \le k}$ and is still equivalent to $\{e_{k,\ell}\}_{\ell \le k}$.

In the second step we use the fact that the sequence $\{z_\ell\}_{\ell=1}^\infty$ is equivalent to the unit vector basis of ℓ_2, and thus, using Lemmas 2.4 and 2.5, we can replace the $\{z_\ell\}_{\ell=1}^\infty$ by elements $\{v_\ell\}_{\ell=1}^\infty$ which are essentially of the form $v_\ell = e_{1,\ell} \otimes b$, $\|b\|_p = 1$. Thus we construct a triangle $\{y_{k,\ell}\}_{\ell \le k}$ of elements of X which is a very small perturbation of a triangle of the form $\{e_{k,1} \otimes e_{1,\ell} \otimes b\}_{\ell \le k}$.

If we put $Y = [y_{k,\ell}]_{\ell \le k}$, then Y is a subspace of X, and by Proposition 2.6 Y is $2+\theta$-complemented in T_p, and $\mathrm{d}(Y, T_p) \le 1+\theta$, provided the perturbations are small enough.

Proof of Theorem 3.2 for $1 \le p < 2$: Let $\{x_{i,j}\}_{j \le i}$ be a triangle of elements of T_p which is M-equivalent to the triangle $\{e_{i,j}\}_{j \le i}$ of the standard unit matrices of T_p, and so that $X = [x_{i,j}]_{j \le i}$. For convenience we want the $x_{i,j}$ to be normalized. This we can obtain by passing to a subtriangle (by using perturbation arguments, and by slightly enlarging M) as follows.

Let $\epsilon > 0$. Using a procedure very similar to that which was used in the proof of Lemma 3.1, we can find numbers $\{\alpha_\ell\}_{\ell=1}^\infty$ and α in the interval $[M^{-1}, M]$, and a subtriangle $\{x_{i_k,j_\ell}\}_{\ell \le k}$ of $\{x_{i,j}\}_{j \le i}$, so that

$$\big|\,\|x_{i_k,j_\ell}\|_p - \alpha_\ell\big| \le \epsilon \cdot 8^{-k-\ell}, \quad |\alpha_\ell - \alpha| \le \epsilon \cdot 8^{-\ell}.$$

Write $x'_{k,\ell} = \alpha x_{i_k,j_\ell}/\|x_{i_k,j_\ell}\|_p$. Then a computation very similar to (3.3) shows that the triangle $\{x'_{k,\ell}\}_{\ell \le k}$ is equivalent to the triangle $\{x_{i_k,j_\ell}\}_{\ell \le k}$, and thus to $\{e_{k,\ell}\}_{\ell \le k}$. Therefore, $\{x'_{k,\ell}/\alpha\}_{\ell \le k}$ is a normalized triangle in X which is equivalent to $\{e_{k,\ell}\}_{\ell \le k}$. Note that by choosing ϵ small enough we can make the new equivalence constant arbitrarily close to M. We therefore assume simply that the original $x_{i,j}$ are normalized, and continue to denote the equivalence constant of $\{x_{i,j}\}_{j \le i}$ to $\{e_{i,j}\}_{j \le i}$ by M.

For every $j = 1, 2, 3, \ldots$ let us denote $X_j = [x_{i,j}]_{i=j}^\infty$. Note that $X = \Sigma_{j=1}^\infty \oplus X_j$, and that $\{x_{i,j}\}_{i=j}^\infty$ are M-equivalent to the unit vector basis of ℓ_2.

We now fix a number $0 < \delta < (M+1)^{-1}$, and we claim:

$$(*) \quad \left\{ \begin{array}{l} \text{For every } n \text{ there is a } k = k(n, \delta) \text{ so that for every} \\[1ex] x \in \displaystyle\sum_{j=k}^{\infty} \oplus X_j \text{ we have } \|E^n x\|_p \ge \delta \|x\|_p. \end{array} \right.$$

Indeed, if (*) is false, then for some n we can construct an increasing sequence of positive integers $\{k_j\}_{j=1}^\infty$ and normalized elements $x_j \in$

$\Sigma_{i=k_j+1}^{k_{j+1}} X_i$, so that $\|E^n x_j\|_p < \delta$. By passing to a subsequence of $\{x_j\}_{j=1}^\infty$ we clearly may assume that the sequence $\{x_j\}_{j=1}^\infty$ is $M+1$-equivalent to the unit vector basis of ℓ_p. Since $\mathrm{d}(E_n T_p, \ell_2) \le n^{1/2}$, we get by (1.12) that for every natural number m:

$$(3.4) \qquad (M+1)^{-1} m^{1/p} \le \left(\int_0^1 \left\| \sum_{j=1}^m r_j(t) x_j \right\|_p^p \mathrm{d}t \right)^{1/p}$$
$$\le \left(\int_0^1 \left\| \sum_{j=1}^m r_j(t) E_n x_j \right\|_p^p \mathrm{d}t \right)^{1/p}$$
$$+ \left(\int_0^1 \left\| \sum_{j=1}^m r_j(t) E^n x_j \right\|_p^p \mathrm{d}t \right)^{1/p}$$
$$\le n^{1/2} \cdot m^{1/2} + \left(\sum_{j=1}^m \|E^n x_j\|_p^p \right)^{1/p}$$
$$\le n^{1/2} \cdot m^{1/2} + \delta \cdot m^{1/p}.$$

Since $p < 2$, the desired contradiction follows from $\delta < (M+1)^{-1}$ by choosing an m large enough, So (*) is proved.

If $\{\epsilon_\ell\}_{\ell=1}^\infty$ are positive numbers so that $\Sigma_{\ell=1}^\infty \epsilon_\ell$ is very small, then using Proposition 2.2(i) and (*) we can construct increasing sequences of positive integers $\{n_\ell\}_{\ell=1}^\infty$ and $\{j_\ell\}_{\ell=1}^\infty$ with $n_1 = 0$, $j_1 = 1$, and so that

$$(3.5) \qquad \|E^{n_{\ell+1}}|_{X_{j_\ell}}\| \le \epsilon_\ell$$

and for every $x \in X_{j_\ell}$ we have

$$(3.5) \qquad \|E^{n_\ell} x\|_p \ge \delta \|x\|_p.$$

Since $\tilde{X} = \Sigma_{\ell=1}^\infty \oplus X_{j_\ell}$ is a Schauder decomposition (into infinite dimensional subspaces), and since $\Sigma_{\ell=1}^\infty \epsilon_\ell$ is arbitrarily small, we can apply standard perturbation arguments and assume, for convenience, that instead of (3.5) we have for every ℓ,

$$(3.7) \qquad E^{n_{\ell+1}}|_{X_{j_\ell}} = 0.$$

For each ℓ the sequence $\{x_{i,j_\ell}\}_{i=j_\ell}^\infty$ is M-equivalent to the unit vector basis of ℓ_2. Given $\epsilon > 0$, we obtain by Lemma 2.4 that $\{x_{i,j_\ell}\}_{\ell=1}^\infty$ has a subsequence which is $1+\epsilon$-equivalent to the unit vector basis of ℓ_2 and spans a $1+\epsilon$-complemented subspace of T_p. However, we want to choose these subsequences for $\ell = 1, 2, \ldots$, so that together they form a whole subtriangle. Let us make this precise.

First, we may assume that for some increasing sequences $\{m_k\}_{k=1}^\infty$

and $\{i_k\}_{k=1}^{\infty}$ of positive integers with $m_k > n_k$ and $i_k > j_k$, we have for every $\ell \leq k$:

$$x_{i_k,j_\ell} = P_{m_k,m_{k+1}} x_{i_k,j_\ell}. \tag{3.8}$$

Let us denote, for convenience, $u_{k,\ell} = x_{i_k,j_\ell}$, and write $\mu_k = \Sigma_{\ell=1}^{k} n_\ell$. We choose now a new matrix representation (by choosing a new orthonormal basis for the range space of the operators), so that the new m_k's and P_n's satisfy $m_{k+1} - m_k = \mu_{k+1}$, and instead of (3.8) we have the better expression

$$u_{k,\ell} = P_{m_k,m_k+\mu_{\ell+1}} u_{k,\ell}. \tag{3.9}$$

Indeed, for every $\ell \leq k$, $E_{n_{\ell+1}} u_{k,\ell} = u_{k,\ell}$. Therefore, $\operatorname{rank}(u_{k,\ell}) \leq n_{\ell+1}$. If $x \in B(\ell_2)$, let $R(x)$ denote the range of x. Let $\{f_i^{(k)}\}_{i=1}^{n_2}$ be orthonormal sequences with $k = 1, 2, 3, \ldots$, so that $R(u_{k,1}) \subseteq [f_i^{(k)}]_{i=1}^{n_2}$, and so that $\bigcup_{k=2}^{\infty} \{f_i^{(k)}\}_{i=1}^{n_2}$ is an orthonormal sequence. Since $\operatorname{rank}(u_{k,2}) \leq n_3$, there exist orthonormal sequences $\{f_i^{(k)}\}_{i=n_2+1}^{\mu_3}$ so that $R(u_{k,2}) \subseteq [f_i^{(k)}]_{i=1}^{\mu_3}$, and so that $\{f_i^{(1)}\}_{i=1}^{n_2} \cup (\bigcup_{k=2}^{\infty} \{f_i^{(k)}\}_{i=1}^{\mu_3})$ is an orthonormal sequence. Continuing in the obvious way, we can clearly redefine the m_k's and the P_n's so that $m_{k+1} - m_k = \mu_{k+1}$, and so that (3.9) holds for every $\ell \leq k$. Note that in the new matrix representation we still have (3.6) and (3.7) (assuming that $j_\ell = \ell$).

Let us denote for every $\ell \leq k$ and $1 \leq \nu \leq \ell$,

$$u_{k,\ell}^{(\nu)} = E_{n_\nu,n_{\nu+1}} u_{k,\ell}. \tag{3.10}$$

Clearly, $u_{k,\ell}^{(\nu)} = P_{m_k,m_k+\mu_{\ell+1}} u_{k,\ell}^{(\nu)} = P_{m_k,m_{k+1}} u_{k,\ell}^{(\nu)}$. We can therefore choose an appropriate tensor product representation in which

$$u_{k,\ell}^{(\nu)} = e_{k,\nu} \otimes a_{k,\ell}^{(\nu)}, \quad a_{k,\ell}^{(\nu)} \in C_p^{\mu_{\ell+1},n_{\nu+1}-n_\nu}. \tag{3.11}$$

Thus for $\ell \leq k$,

$$u_{k,\ell} = \sum_{\nu=1}^{\ell} u_{k,\ell}^{(\nu)} = \sum_{\nu=1}^{\ell} e_{k,\nu} \otimes a_{k,\ell}^{(\nu)}. \tag{3.12}$$

As in the proof of Lemma 3.1, we can assume (by passing to a subtriangle and using perturbation arguments) that for fixed $1 \leq \nu \leq \ell$, the $a_{k,\ell}^{(\nu)}$ are independent of k, i.e. that for some elements $a_\ell^{(\nu)} \in C_p$ we have, for every $1 \leq \nu \leq \ell \leq k$:

$$(3.13)\qquad a^{(\nu)}_{k,\ell} = a^{(\nu)}_{\ell}.$$

This allows us to pass to some other tensor product representation of C_p as $C_p^{(1)} \otimes C_p^{(2)}$, where $C_p^{(i)}$ are copies of C_p $(i = 1,2)$, so that the elements $u_{k,\ell}$ have the form

$$(3.14)\qquad u_{k,\ell} = e_{k,1} \otimes z_\ell, \quad \ell \le k,$$

where $z_\ell \in C_p^{(2)}$. Clearly, $\{z_\ell\}_{\ell=1}^\infty$ is M-equivalent to the unit vector basis of ℓ_2 (since for every finite sequence of scalars $\{t_\ell\}_{\ell=1}^k$ we have $\|\sum_{\ell=1}^k t_\ell z_\ell\|_p = \|\sum_{\ell=1}^k t_\ell u_{k,\ell}\|_p$).

Let $0 < \theta < 1$, and let $\{\alpha_i\}_{i=1}^\infty$ be a sequence of positive numbers such that

$$(3.15)\qquad \sum_{i=1}^\infty \alpha_i \le \min\left\{\theta/20M, \left(\frac{\delta}{2}\right)^{p+1} \Big/ 8M^2\right\}.$$

Using Lemma 2.4, we find elements $\{v_\ell\}_{\ell=1}^\infty$ of $C_p^{(2)}(C_p^{(2)}$ is identified here with $C_p \otimes C_p)$ of the form

$$(3.16)\qquad v_\ell = \sum_{i=1}^{\ell-1} (e_{\ell,i} \otimes a_i + e_{i,\ell} \otimes b_i) + e_{\ell,\ell} \otimes c_\ell$$

with

$$(3.17)\qquad \max\{\|a_i\|_p, \|b_i\|_p\} \le \alpha_{i-1} \quad \text{and} \quad \|c_i\|_p \le \alpha_{i-1}, \quad 2 \le i < \infty$$

$$(3.18)\qquad (\|a_1\|_p^p + \|b_1\|_p^p)^{1/p} \ge 1 - \alpha_1, \quad \|b_1\|_p \ge \delta/2,$$

so that for some subsequence of $\{z_\ell\}_{\ell=1}^\infty$, which we assume without loss of generality to be $\{z_\ell\}_{\ell=1}^\infty$ itself, we have

$$(3.19)\qquad \|z_\ell - v_\ell\|_p \le \alpha_\ell, \quad \ell = 1, 2, \ldots.$$

Write

$$(3.20)\qquad w_{k,\ell} = e_{k,1} \otimes v_\ell.$$

If $\{t_{k,\ell}\}_{\ell \le k}$ are scalars with $\|\sum_{\ell \le k} t_{k,\ell} u_{k,\ell}\|_p = 1$, then

$$(3.21)\qquad \left\|\sum_{\ell \le k} t_{k,\ell}(u_{k,\ell} - w_{k,\ell})\right\|_p = \left\|\sum_{\ell \le k} t_{k,\ell} e_{k,1} \otimes (z_\ell - v_\ell)\right\|_p$$

$$\le \sum_{\ell=1} \|z_\ell - v_\ell\|_p \cdot \left(\sum_{k=\ell}^{\infty} |t_{k,\ell}|^2\right)^{1/2}$$
$$\le \sum_{\ell=1} \alpha_\ell \cdot M \le \min\left\{\theta/20, \left(\frac{\delta}{2}\right)^{p+1} \Big/ 8M\right\}.$$

Note that $W = [w_{k,\ell}]_{2\le\ell\le k<\infty}$ is complemented in T_p. Indeed, it can be easily shown that the projection

$$Px = \sum_{2\le\ell\le k<\infty} \langle x, n_p(e_{k,1} \otimes e_{1,\ell} \otimes b_1)\rangle w_{k,\ell}/\|b_1\|_p^p \tag{3.22}$$

from T_p onto W has norm $\le 4M(2/\delta)^{p+1}$. So $U = [u_{k,\ell}]_{2\le\ell\le k<\infty}$ is $12M(2/\delta)^{p+1}$-complemented in T_p. The proof that U has a subspace Y which is $2+\theta$-complemented in T_p and satisfies $\mathrm{d}(Y, T_p) \le 1+\theta$ requires some additional work. By (3.21) it is enough to show that W has a subspace Z which is $2+\theta/2$-complemented in T_p, so that $\mathrm{d}(Z, T_p) \le 1 + \theta/2$. The behaviour of the $\{w_{k,\ell}\}_{\ell\le k}$ need not be improved by passing to a subtriangle (consider for example the triangle $\{e_{2k+1,1} + e_{2k,2\ell}\}_{\ell\le k}$). In order to "kill the ℓ_2-part" of the $\{w_{k,\ell}\}_{r\le k}$ (namely, the elements $e_{k,1} \otimes e_{\ell,1} \otimes a_1$), we pass to some averages in the ℓ_p-sense of the $\{w_{k,\ell}\}_{\ell\le k}$. Precisely, let m be such that

$$\left(\frac{2}{\delta}\right) m^{(p-2)/2p} \cdot \left\{1 + 3\sum_{i=1}^{\infty} \alpha_i\right\} \le \theta/10, \tag{3.23}$$

and define for $1 \le \mu \le \nu < \infty$,

$$z_{\nu,\mu} = \sum_{j=1}^{m} w_{\nu m+j,\mu m+j}/\|b_1\|_p m^{1/p} \tag{3.24}$$

$$h_{\nu,\mu} = \sum_{j=1}^{m} e_{\nu m+j,1} \otimes e_{1,\mu m+j} \otimes b_1/\|b_1\|_p m^{1/p}. \tag{3.25}$$

We claim that the subspace $Z = [z_{\nu,\mu}]_{\mu\le\nu}$ of W has the desired properties. Note that by Proposition 2.6, $\{h_{\nu,\mu}\}_{\mu\le\nu}$ is isometrically equivalent to $\{e_{\nu,\mu}\}_{\mu\le\nu}$ and $[h_{\nu,\mu}]_{\mu\le\nu}$ is 2-complemented in T_p. Let $\{t_{\nu,\mu}\}_{\mu\le\nu}$ by scalars, so that $\|\Sigma_{\mu\le\nu} t_{\nu,\mu} h_{\nu,\mu}\|_p = 1$. Then,

$$\left\|\sum_{\mu\le\nu} t_{\nu,\mu}(z_{\nu,\mu} - h_{\nu,\mu})\right\|_p \tag{3.26}$$

$$\le \|b_1\|_p^{-1} m^{-1/p} \left\{\left\|\sum_{\mu\le\nu} t_{\nu,\mu} \sum_{j=1}^{m} e_{\nu m+j,1} \otimes e_{\mu m+j,1} \otimes c_{\mu m+j}\right\|_p\right.$$

$$+\sum_{i=1}^{\infty}\Big\|\sum_{\mu\le\nu} t_{\mu,\nu}\sum_{j=1}^{m} e_{\nu m+j,1}\otimes e_{\mu m+j,1}\otimes a_i\Big\|_p$$

$$+\sum_{i=2}^{\infty}\Big\|\sum_{\mu\le\nu} t_{\nu,\mu}\sum_{j=1}^{m} e_{\nu m+j,1}\otimes e_{1,\mu m+j}\otimes b_i\Big\|_p\Big\}$$

$$\le \|b_1\|_p^{-1} m^{-1/p}\Big\{\sum_{j=1}^{m}\sum_{\mu=1}^{\infty}\|c_{\mu m+j}\|_p\Big(\sum_{\nu=\mu}^{\infty}|t_{\nu,\mu}|^2\Big)^{1/2}$$

$$+\sum_{i=1}^{\infty}\|a_i\|_p\Big\|\sum_{\mu\le\nu} t_{\nu,\mu}\sum_{j=1}^{m} e_{\nu m+j,1}\otimes e_{\mu m+j,1}\Big\|_p$$

$$+\sum_{i=2}^{\infty}\|b_i\|_p\Big\|\sum_{\mu\le\nu} t_{\nu,\mu}\sum_{j=1}^{m} e_{\nu m+j,1}\otimes e_{1,\mu m+j}\Big\|_p\Big\}$$

$$\le \|b_1\|_p^{-1} m^{-1/p}\Big\{\sum_{i=m+1}^{\infty}\|c_i\|_p+\sum_{i=1}^{\infty}\|a_i\|_p m^{1/2}+\sum_{i=2}^{\infty}\|b_i\|_p m^{1/p}\Big\}$$

$$\le \|b_1\|_p^{-1} m^{-1/p}\Big\{\sum_{i=m}^{\infty}\alpha_i+\Big(1+2\sum_{i=1}^{\infty}\alpha_i\Big)m^{1/2}\Big\}$$

$$\le\Big(\frac{2}{\delta}\Big) m^{(p-2)/2p}\Big\{1+3\sum_{i=1}^{\infty}\alpha_i\Big\}\le \theta/10.$$

By standard perturbation arguments, this implies that the triangle $\{z_{\nu,\mu}\}_{\mu\le\nu}$ is $1+\theta/2$-equivalent to the triangle $\{h_{\nu,\mu}\}_{\mu\le\nu}$ (and thus $d(Z,T_p)\le 1+\theta/2$) and that Z is $2+\theta/2$-complemented in T_p. This completes the proof of Theorem 3.2 for $1\le p<2$. □

PROOF OF THEOREM 3.2 FOR $2<p<\infty$: Let $\{x_{i,j}\}_{j\le i}$ be a triangle of elements of T_p which is M-equivalent to the triangle $\{e_{i,j}\}_{j\le i}$ of the standard unit matrices in T_p, and so that $[x_{i,j}]_{j\le i}=X$. As in the case $1\le p<2$, we can assume (by passing to a subtriangle if necessary) that $\|x_{i,j}\|_p=1$ for every $j\le i$. Write again $X_j=[x_{i,j}]_{i=j}^{\infty}$, and note that $X=\Sigma_{j=1}^{\infty}\oplus X_j$ and that for each j, $\{x_{i,j}\}_{i,j}^{\infty}$ is M-equivalent to the unit vector basis of ℓ_2.

CLAIM: For every n and $0<\epsilon$ there exists a $k=k(\epsilon,n)$ such that $\|E_n{}_{|\Sigma_{j=k}^{\infty}\otimes X_j}\|\le\epsilon$.

PROOF: If there is no such k for some n and ϵ, we can find an increasing sequence of natural numbers $\{k_j\}_{j=1}^{\infty}$ and a sequence $\{x_j\}_{j=1}^{\infty}$ of normalized elements of X so that

$$\|E_n x_j\|_p\ge\epsilon/2,\quad x_j\in\sum_{k=k_j+1}^{k_{j+1}} x_k. \tag{3.27}$$

It is clear that some subsequence $\{x_{j_\ell}\}_{\ell=1}^{\infty}$ of $\{x_j\}_{j=1}^{\infty}$ is equivalent to the unit vector basis of ℓ_p. Since for $p > 2$ every bounded operator from ℓ_p to ℓ_2 is compact (see [5, Proposition 2.c.3]), we get that

$$E_{n|[x_{j_\ell}]_{\ell=1}^{\infty}} : [x_{j_\ell}]_{\ell=1}^{\infty} \to E_n T_p \tag{3.28}$$

is compact. But $\|x_{j_\ell}\|_p = 1$, $x_{j_\ell} \to 0$ weakly as $\ell \to \infty$, and $\|E_n x_{j_\ell}\|_p \ge \epsilon/2$. This leads to a contradiction, and so the claim is proved.

Now set $0 < \delta < (3M)^{-1}$. Using the above claim, Proposition 2.2(ii) and the fact that for every fixed j, $x_{i,j} \to 0$ weakly as $i \to \infty$, we can assume (by passing to a subtriangle if necessary, and by using perturbation arguments as in the case $1 \le p < 2$) that for some increasing sequence of integers $\{\nu_i\}_{i=0}^{\infty}$ with $\nu_0 = 0$, we have for every $j \ge 1$:

$$E_{\nu_{j-1}} X_j = 0 \tag{3.29}$$

$$\|E_{\nu_j} x\|_p \ge \delta \|x\|_p, \quad x \in X_j \tag{3.30}$$

$$x_{i,j} = P_{\nu_{i-1},\nu_i} x_{i,j}, \quad j \le i < \infty. \tag{3.31}$$

We can also assume that for every $n < \ell$ and every $1 \le j$, $\lim_{i\to\infty} \|E_{n,\ell} x_{i,j}\|_p$ exists (since by a standard diagonal method we can pass to a subtriangle of $\{x_{i,j}\}_{j\le i}$ which satisfies this condition). For integers $0 \le n < \ell$ and $i \le j$, and for every number $0 < \alpha$, let

$$A(n, \ell, j, \alpha) = \{i;\, i \ge j, \|E_{n,\ell} x_{i,j}\|_p > \alpha\}. \tag{3.32}$$

As in the proof of Lemma 2.5, we have:

(+) {For every j and every $0 < \alpha$, there exists an $n = n(j, \alpha)$ so that if $n < \ell$ then $A(n, \ell, j, \alpha)$ is a finite set.

Let $0 < \theta < 1$, and let $\{\alpha_i\}_{i=1}^{\infty}$ be positive numbers, so that

$$2M \sum_{i=1}^{\infty} \alpha_i \le \theta/20, \text{ and } \sum_{i=\ell+1}^{\infty} \alpha_i \le \alpha_\ell \text{ for every } \ell. \tag{3.33}$$

Now we construct increasing sequences of natural numbers $\{i_k\}_{k=1}^{\infty}$, $\{m_k\}_{k=0}^{\infty}$ and $\{n_k\}_{k=0}^{\infty}$ with $i_1 = 1$, $m_0 = 0 = n_0$ and $m_k \le n_k$ for every k, so that for $\ell \le k$:

$$x_{i_k,i_\ell} = \sum_{j=\ell}^{k} x^{(j)}_{i_k,i_\ell}, \tag{3.34}$$

where

$$x^{(j)}_{i_k,i_\ell} = P_{m_{k-1},m_k} E_{n_{j-1},n_j} x_{i_k,i_\ell}, \tag{3.35}$$

and so that for every ℓ:

$$\|E_{n_{\ell-1},n_\ell} x_{i,i_\ell}\|_p \geq \delta, \quad i_\ell \leq i, \tag{3.36}$$

and for $\ell < j$ and $i \geq i_{\ell+2}$:

$$\|E_{n_{j-1},n_j} x_{i,i_\ell}\|_p \leq \alpha_j \alpha_\ell. \tag{3.37}$$

Indeed, let $m_0 = n_0 = 1$, $i_1 = 1$, $m_1 = \nu_1$, and write $n_1 = \max\{m_1, n(i_1, \alpha_1\alpha_2)\}$, where $n(\cdot, \cdot)$ is the function that appeared in (+). Then (3.34) and (3.35) are satisfied for $k = \ell = 1$, and (3.36) for $\ell = 1$ follows from (3.30) and from the fact that $n_1 \geq \nu_{i_1}$. Let $i_2 > i_1$ be such that $\nu_{i_2-1} \geq n_1$, write $m_2 = \nu_{i_2}$, and define

$$n_2 = \max\{m_2, n(i_1, \alpha_1\alpha_3), n(i_2, \alpha_2\alpha_3)\}. \tag{3.38}$$

Then by (3.29), (3.30) and (3.31) we obtain (3.34) and (3.35) for $1 \leq \ell \leq k \leq 2$, and (3.36) for $\ell = 2$. Indeed, if $i \geq i_2$, then

$$E_{n_1} x_{i,i_2} = E_{n_1} E_{\nu_{i_2}-1} x_{i,i_2} = 0 \tag{3.39}$$

and

$$\|E_{n_1,n_2} x_{i,i_2}\|_p \geq \|E_{\nu_{i_2-1},\nu_{i_2}} x_{i,i_2}\|_p \geq \delta. \tag{3.40}$$

By (+) the set $A(n_1, n_2, i_1, \alpha_1\alpha_2)$ is finite. Let $i_3 > i_2$ be such that $\nu_{i_3-1} \geq n_2$, and (3.31) holds for $\ell = 1$, $j = 2$ and every $i \geq i_3$. Write $m_3 = \nu_{i_3}$ and

$$n_3 = \max\{m_3, \alpha(i_1, \alpha_1\alpha_4), n(i_2, \alpha_2\alpha_4), n(i_3, \alpha_3\alpha_4)\}. \tag{3.41}$$

Again (3.34), (3.35) for $1 \leq \ell \leq k \leq 3$ and (3.36) for $\ell = 3$ follow from (3.29), (3.30) and (3.31).

We continue inductively in the same way. Assume that i_k, m_k and n_k were defined for $k \leq k_0$, so that (3.34) and (3.35) hold for $1 \leq \ell \leq k \leq k_0$, (3.36) holds for $1 \leq \ell \leq k_0$, and (3.37) holds for $1 \leq \ell \leq k_0 - 2$ and for every $1 \leq k \leq k_0$:

$$n_k = \max\{m_k, n(i_\ell, \alpha_\ell \cdot \alpha_{k+1})\}_{\ell=1}^{k}. \tag{3.42}$$

By (+) and (3.42) the sets $A(n_{k_0-1}, n_{k_0}, i_\ell, \alpha_\ell\alpha_{k_0})$, $1 \le \ell \le k_0 - 1$ are finite. Choose $i_{k_0+1} > i_{k_0}$ such that $\nu_{i_{k_0+1}-1} \ge n_{k_0}$, and such that (3.37) holds for $1 \le \ell \le k_0 - 1$, $j = k_0$ and $i \ge i_{k_0+1}$. Thus, trivially, (3.37) holds for every $1 \le \ell \le j \le k_0$ and $i \ge i_{\ell+2}$. Let $m_{k_0+1} = \nu_{i_{k_0+1}}$ and define n_{k_0+1} by (3.42), with $k_0 + 1$ instead of k. As before, (3.34) and (3.35) for $1 \le \ell \le k \le k_0 + 1$, and (3.36) for $\ell = k_0 + 1$, are easy consequences of (3.29), (3.30) and (3.31). This completes the inductive construction of the sequences $\{i_k\}_{k=1}^{\infty}$, $\{m_k\}_{k=0}^{\infty}$ and $\{n_k\}_{k=0}^{\infty}$, and so (3.34)–(3.37) are valid for all indices involved.

Let $u_{k,\ell} = x_{i_k, i_\ell}$, $1 \le \ell \le k < \infty$, then by passing to an appropriate tensor product representation we have

$$u_{k,\ell} = \sum_{j=\ell}^{k} e_{k,j} \otimes u_{k,\ell}^{(j)}, \tag{3.43}$$

with

$$\|u_{k,\ell}^{(\ell)}\|_p \ge \delta \tag{3.44}$$

and

$$\|u_{k,\ell}^{(j)}\|_p \le \alpha_j \alpha_\ell \quad \text{for } \ell + 1 \le j \le k - 1. \tag{3.45}$$

As we have done several times before, we can assume first that instead of just (3.36), we actually have

$$u_{k,\ell}^{(j)} \in C_p^{\mu_j, n_j - n_{j-1}} \tag{3.46}$$

for every $\ell \le j \le k$, where $\{\mu_j\}_{j=1}^{\infty}$ is some sequence of natural numbers. This implies that for every fixed $\ell \ge j$, every subsequence of $\{u_{k,\ell}^{(j)}\}_{k=j}^{\infty}$ has a further subsequence which converges in the norm. Therefore, by passing to a subtriangle if necessary, we can assume that

$$u_{k,\ell} = \sum_{j=\ell}^{k-1} e_{k,j} \otimes u_\ell^{(j)} + e_{k,k} \otimes \tilde{u}_{k,\ell}, \tag{3.47}$$

where $u_\ell^{(j)} \in C_p^{\mu_j, n_j - n_{j-1}}$ for $\ell \le j$, and for every ℓ we have

$$\|u_\ell^{(\ell)}\|_p \ge \delta/2, \tag{3.48}$$

(3.49) $$\|u_\ell^{(j)}\|_p \leq 2\alpha_j\alpha_\ell, \quad \ell < j.$$

Let $\{k_i\}_{i=1}^\infty$ be an increasing sequence of integers with $k_1 = 1$, so that if $\Delta_i = k_{i+1} - k_i$, then for every $i : \Delta_i^{(2-p)/2p} \leq \alpha_{i+1}$. By (3.33) we get for every ℓ:

(3.50) $$\left(\sum_{i=\ell}^\infty \Delta_i^{(2-p)/p}\right)^{1/2} \leq \sum_{i=\ell}^\infty \Delta_i^{(2-p)/2p} \leq \alpha_\ell.$$

Define for $1 \leq \ell \leq i < \infty$,

(3.51) $$v_{i,\ell} = \sum_{k=k_i+1}^{k_{i+1}} u_{k,\ell}/\Delta_i^{1/2}$$

and

(3.52) $$w_{i,\ell} = \sum_{k=k_i+1}^{k_{i+1}} e_{k,\ell} \otimes u_\ell^{(\ell)}/\Delta_1^{1/2}.$$

If $f_{i,\ell} = \Sigma_{k=k_i+1}^{k_{i+1}} e_{k,\ell}/\Delta_i^{1/2}$ for $\ell \leq i$, then $\{f_{i,\ell}\}_{\ell\leq i}$ is isometrically equivalent to $\{e_{i,\ell}\}_{\ell\leq i}$. Thus $\{v_{i,\ell}\}_{\ell\leq i}$ is M-equivalent to $\{e_{i,\ell}\}_{\ell\leq i}$. If $\{t_{i,\ell}\}_{\ell\leq i}$ are scalars such that $\|\Sigma_{\ell\leq i} t_{i,\ell}v_{i,\ell}\|_p = 1$, then by (3.33) and (3.50),

(3.53)
$$\begin{aligned}
&\left\|\sum_{\ell\leq i} t_{i,\ell}(v_{i,\ell} - w_{i,\ell})\right\|_p \leq \sum_{\ell=1}^\infty \left\|\sum_{i=\ell}^\infty t_{i,\ell}(v_{i,\ell} - w_{i,\ell})\right\|_p \\
&\leq \sum_{\ell=1}^\infty \left\|\sum_{i=\ell}^\infty t_{i,\ell}\Delta_i^{-1/2} \sum_{k=k_i+1}^{k_{i+1}} \left(\sum_{j=\ell+1}^{k-1} e_{k,j} \otimes u_\ell^{(j)} + e_{k,k} \otimes \tilde{u}_{k,k}\right)\right\|_p \\
&\leq \sum_{\ell=1}^\infty \left\{\sum_{i=\ell}^\infty |t_{i,\ell}|\Delta_i^{-1/2} \left\|\sum_{k=k_i+1}^{k_{i+1}} e_{k,k} \otimes \tilde{u}_{k,k}\right\|_p \right. \\
&\quad \left. + \left(\sum_{j=\ell+1}^\infty \left\|\sum_{i=\ell}^\infty t_{i,\ell}\Delta_i^{-1/2} \sum_{k=\max\{j+1,k_i+1\}} e_{k,j} \otimes u_\ell^{(j)}\right\|_p^2\right)^{1/2}\right\} \\
&\leq \sum_{\ell=1}^\infty \left\{\sum_{i=\ell}^\infty |t_{i,\ell}|\Delta_i^{(2-p)/2p} + \left(\sum_{j=\ell+1}^\infty \|u_\ell^{(j)}\|_p^2 \sum_{i=\ell}^\infty |t_{i,\ell}|^2\right)^{1/2}\right\} \\
&\leq M \sum_{\ell=1}^\infty \left\{\left(\sum_{i=\ell}^\infty \Delta_i^{(2-p)/p}\right)^{1/2} + \sum_{j=\ell+1}^\infty \alpha_j\alpha_\ell\right\} \\
&\leq M \sum_{\ell=1}^\infty (\alpha_\ell + \alpha_\ell^2) \leq M \sum_{\ell=1}^\infty 2\alpha_\ell \leq \theta/20.
\end{aligned}$$

This implies that $\{w_{i,\ell}\}_{\ell\leq i}$ is M'-equivalent to $\{e_{i,\ell}\}_{\ell\leq i}$, where $M' \leq \frac{21}{19}M$. Note that $v_{i,\ell} \in [u_{\nu,\mu}]_{\mu\leq\nu} \subset X$ for every $\ell \leq i$. Thus, in order to find a subspace Y of X which is $2+\theta$-complemented in T_p and satisfies

$d(Y, T_p) \le 1+\theta$, it is clearly enough to find a subspace Z of $W = [w_{i,\ell}]_{\ell\le i}$ which is $2+\theta/2$-complemented in T_p, and satisfies $d(Z, T_p) \le 1+\theta/2$.

We now pass to another tensor product representation of C_p as $C_p^{(1)} \otimes C_p^{(2)}$, where $C_p^{(i)}$ are copies of C_p $(i=1,2)$, so that (3.52) is written as

$$w_{i,\ell} = e_{i,1} \otimes z_\ell, \quad \ell \le i, \tag{3.54}$$

where the elements $z_\ell \in C_p^{(2)}$ satisfy $r(z_k)\cdot r(z_\ell) = 0$ for $k \ne \ell$. Since

$$\delta/2 \le \|u_\ell^{(\ell)}\|_p = \|z_\ell\|_p = \|w_{i,\ell}\|_p \le \|u_{\ell,\ell}\|_p \le 1, \tag{3.55}$$

we can assume (by passing to a subtriangle whose elements have almost constant norms, and by perturbation arguments) that $\|z_\ell\|_p = \|w_{i,\ell}\|_p = 1$ for every $\ell \le i$.

Clearly, $\{z_\ell\}_{\ell=1}^\infty$ is M'-equivalent to the unit vector basis of ℓ_2. Using Lemma 2.5 and the fact that $r(z_\ell)\cdot r(z_k) = 0$ for $\ell \ne k$, we get a subsequence $\{z_{\ell_\nu}\}_{\nu=1}^\infty$, so that for some averages of the form

$$\tilde z_j = \sum_{\nu=\sigma_j+1}^{\sigma_{j+1}} z_{\ell_\nu} \Big/ \Big\| \sum_{\nu=\sigma_j+1}^{\sigma_{j+1}} z_{\ell_\nu} \Big\|_p \tag{3.56}$$

and for some normalized elements of $C_p^{(2)}$ (represented as $C_p \otimes C_p$) of the form

$$v_j = \sum_{\nu=1}^{j-1} e_{\nu,j} \otimes b_\nu + e_{j,j} \otimes c_j \tag{3.57}$$

with $\|b_1\|_p \ge 1-\alpha_1$ and $\max\{\|b_\nu\|_p, \|c_\nu\|_p\} \le \alpha_{\nu-1}$ for $2 \le \nu$, we have

$$\|\tilde z_j - v_j\|_p \le \alpha_j. \tag{3.58}$$

Let $\tau_i = \ell_{\sigma_{i+1}}$, and define for $j \le i$,

$$z_{i,j} = \sum_{\nu=\sigma_j+1}^{\sigma_{j+1}} w_{\tau_i,\ell_\nu} \Big/ \Big\| \sum_{\nu=\sigma_j+1}^{\sigma_{j+1}} w_{\tau_i,\ell_\nu} \Big\|_p . \tag{3.59}$$

We claim that the subspace $Z = [z_{i,j}]_{j\le i}$ of W has the desired properties. Note first that

$$z_{i,j} = e_{\tau_i,1} \otimes \tilde z_j, \quad j \le i. \tag{3.60}$$

Write

$$h_{i,j} = e_{\tau_i,1} \otimes e_{1,j} \otimes b_1/\|b_1\|_p, \quad j \le i. \tag{3.61}$$

By proposition 2.6, $\{h_{i,j}\}_{j\le i}$ is isometrically equivalent to $\{e_{i,j}\}_{j\le i}$, and $[h_{i,j}]_{j\le i}$ is 2-complemented in T_p. Let $\{t_{i,j}\}_{j\le i}$ be scalars such that $\|\Sigma_{j\le i} t_{i,j}h_{i,j}\|_p = 1$. Then,

$$\begin{aligned}
\Big\|\sum_{j\le i} t_{i,j}(h_{i,j} - z_{i,j})\Big\|_p &\le \Big\|\sum_{j\le i} t_{i,j}e_{\tau_i,1} \otimes \Big(e_{i,j} \otimes \frac{b_1}{\|b_1\|_p} - v_j\Big)\Big\|_p \\
&\quad + \Big\|\sum_{j\le i} t_{i,j}e_{\tau_i,1} \otimes (v_j - \tilde{z}_j)\Big\|_p \\
&\le \alpha_1 + \Big\|\sum_{j\le i} t_{i,j}e_{\tau_i,1} \otimes \Big[\sum_{\nu=2}^{j-1} e_{\nu,j} \otimes b_\nu + e_{j,j} \otimes c_j\Big]\Big\|_p \\
&\quad + \sum_{j=1}^{\infty} \|v_j - \tilde{z}_j\|_p \cdot \Big\|\sum_{i=j}^{\infty} t_{i,j}e_{\tau_i,1}\Big\|_p \\
&\le \alpha_1 + \sum_{j=2}^{\infty} \Big\|\sum_{i=j}^{\infty} t_{i,j}e_{\tau_i,1}\Big\|_p \|c_j\|_p \\
&\quad + \sum_{\nu=2}^{\infty} \Big\|\sum_{\nu+1\le j\le i} t_{i,j}e_{\tau_i,1} \otimes e_{\nu,1}\Big\|_p \|b_\nu\|_p + \sum_{j=1}^{\infty} \alpha_j \\
&\le \alpha_1 + \sum_{j=2}^{\infty} \alpha_{j-1} + \sum_{\nu=2}^{\infty} \alpha_{\nu-1} + \sum_{j=1}^{\infty} \alpha_j \\
&\le 4\sum_{j=1}^{\infty} \alpha_j \le 2\Big(2M\sum_{j=1}^{\infty} \alpha_j\Big) \le \theta/10.
\end{aligned} \tag{3.62}$$

This implies that $\{z_{i,j}\}_{j\le i}$ is $1+\theta/2$-equivalent to $\{h_{i,j}\}_{j\le i}$ (and therefore, $d(Z, T_p) \le 1+\theta/2$), and that Z is $2+\theta/2$-complemented in T_p. This completes the proof of Theorem 3.2 for $2 < p < \infty$. □

4. Applications and concluding remarks

Our first corollary might be of importance in the classification of the complemented subspaces of C_p.

Corollary 4.1: *Let Z be a complemented subspace of T_p, $1 \le p < \infty$, which contains a subspace isomorphic to T_p. Then Z is isomorphic to T_p.*

Proof: Let $X \subset Z$ be such that $X \approx T_p$. By Theorem 3.2, there exists a subspace Y of X such that $Y \approx T_p$ and such that Y is complemented in T_p. In particular, Y is complemented in Z. So, for some Banach spaces U and W,

$$T_p \approx Z \oplus U, \quad Z \approx Y \oplus W \approx T_p \oplus W.$$

since $T_p \approx (T_p \oplus T_p \oplus \cdots \oplus T_p \oplus \cdots)_{\ell_p}$, by using the decomposition method (see [5, page 54]) we get that $Z \approx T_p$. □

Since for $1 < p < \infty$, T_p is isomorphic to C_p, we obtain

COROLLARY 4.2: *Let Z be a complemented subspace of C_p, $1 < p < \infty$, which contains a subspace isomorphic to C_p. Then Z is isomorphic to C_p.*

Theorem 1.1 and Corollary 4.2 imply by transposition and standard duality arguments the following two corollaries on quotient spaces of C_p.

COROLLARY 4.3: *Let X be a Banach space isomorphic to C_p, $1 < p < \infty$, and let Q_1 be any quotient map from C_p onto X. Then there exists a quotient map Q_2 from X onto some Banach space Y isomorphic to C_p, and there is an isomorphism V from Y into C_p so that $Q_1 Q_2 V$ is the identity operator on Y.*

COROLLARY 4.4: *Let X be a complemented subspace of C_p, $1 < p < \infty$, and assume that X has a quotient which is isomorphic to C_p. Then X is isomorphic to C_p.*

Recall that a Banach space X is called *primary* if for any bounded projection P defined on X, either PX or $(1-P)X$ is isomorphic to X.

THEOREM 4.5: *For $1 < p < \infty$, C_p is primary.*

Since a reflexive Banach space X is primary if and only if $X^\star$ is primary, clearly it is enough to prove Theorem 4.5 for $1 < p \leq 2$. Since the case $p = 2$ is trivial, and since $C_p \approx T_p$ for $1 < p < 2$, Theorem 4.5 will be the consequence of the following, somewhat stronger result.

THEOREM 4.6: *For $1 \leq p < 2$, T_p is primary.*

PROOF: Let P be a bounded projection in T_p, $1 \leq p < 2$. For $1 \leq j \leq i < \infty$, let $a_{i,j} = Pe_{i,j}$ and $b_{i,j} = (1-P)e_{i,j}$. Since $a_{i,j} + b_{i,j} = e_{i,j}$, either

$|a_{i,j}(i,j)| \geq \frac{1}{2}$, or $\|b_{i,j}(i,j)| \geq \frac{1}{2}$ (or both). By Ramsey's Theorem in combinatorics, there exist increasing sequences $\{i_k\}_{k=1}^{\infty}$ and $\{j_k\}_{k=1}^{\infty}$ of positive integers with $i_k > j_k$ for every k, so that either

$$|a_{i_k,j_\ell}(i_k, j_\ell)| \geq \tfrac{1}{2}, \quad \text{for every } \ell \leq k, \tag{4.1}$$

or

$$|b_{i_k,j_\ell}(i_k, j_\ell)| \geq \tfrac{1}{2}, \quad \text{for every } \ell \leq k. \tag{4.2}$$

Without loss of generality we assume that (4.1) holds (otherwise, we consider $I - P$ instead of P). Write for $\ell \leq k$,

$$x_{k,\ell} = a_{i_k,j_\ell} = Pe_{i_k,j_\ell}, \tag{4.3}$$

and let $X = \overline{\text{span}}\{x_{k,\ell}\}_{\ell \leq k}$. Since X is a subspace of the complemented subspace $Z = PT_p$ of T_p, in order to prove that $Z \approx T_p$ clearly it is enough, by Corollary 4.1, to find a subspace Y of X with $Y \approx T_p$. We shall construct below a subtriangle $\{x_{k_\nu,\ell_\mu}\}_{\mu \leq \nu}$ of $\{x_{k,\ell}\}_{\ell \leq k}$ so that, essentially, $x_{k_\nu,\ell_\mu}(i_{k_{\nu'}}, j_{\ell_{\mu'}}) = \delta_{\nu,\nu'} \cdot \delta_{\mu,\mu'} \cdot \lambda$ for some number λ with $\frac{1}{2} \leq |\lambda| \leq \|P\|$. Using this subtriangle we complete the proof as follows.

Write for $\mu \leq \nu$, $y_{\nu,\mu} = x_{k_\nu,\ell_\mu}$ and let $Y = \overline{\text{span}}\{y_{\nu,\mu}\}_{\mu \leq \nu}$. Let Q be the following contractive projection in C_p (it is a simple case of the projections described by (1.10)):

$$(Qx)(i,j) = \begin{cases} x(i,j); & \text{if } i = i_{k_\nu} \text{ and } j = j_{\ell_\mu} \text{ for some } \nu \text{ and } \mu \\ 0\;; & \text{otherwise.} \end{cases} \tag{4.4}$$

Then for every scalars $\{t_{\nu,\mu}\}_{\mu \leq \nu}$ with $t_{\nu,\mu} \neq 0$ only for finitely many pairs (ν, μ), we have

$$\begin{aligned} \Big\|\sum_{\mu \leq \nu} t_{\nu,\mu} e_{\nu,\mu}\Big\|_p &= \Big\|\sum_{\mu \leq \nu} t_{\nu,\mu} e_{i_{k_\nu},j_{\ell_\mu}}\Big\|_p \\ &\geq \|P\|^{-1} \Big\|\sum_{\mu \leq \nu} t_{\nu,\mu} y_{\nu,\mu}\Big\|_p \\ &\geq \|P\|^{-1} \Big\|\sum_{\mu \leq \nu} t_{\nu,\mu} Q y_{\nu,\mu}\Big\|_p \\ &= \|P\|^{-1} \Big\|\sum_{\mu \leq \nu} t_{\nu,\mu} y_{\nu,\mu}(i_{k_\nu}, j_{\ell_\mu}) e_{i_{k_\nu},j_{\ell_\mu}}\Big\|_p \\ &= |\lambda| \|P\|^{-1} \Big\|\sum_{\mu \leq \nu} t_{\nu,\mu} e_{i_{k_\nu},j_{\ell_\mu}}\Big\|_p \\ &\geq 2\|P\|^{-1} \Big\|\sum_{\mu \leq \nu} t_{\nu,\mu} e_{\nu,\mu}\Big\|_p . \end{aligned} \tag{4.5}$$

Therefore, $\{y_{\nu,\mu}\}_{\mu\le\nu}$ is $2\|P\|$-equivalent to $\{e_{\nu,\mu}\}_{\mu\le\nu}$, and thus the subspace Y of X is isomorphic to T_p.

So let us turn to the construction of the desired subtriangle $\{x_{k_\nu,\ell_\mu}\}_{\mu\le\nu}$. Note first that for every fixed ℓ, $x_{k,\ell}\to 0$ weakly as $k\to\infty$. By passing to a subtriangle of $\{x_{k,\ell}\}_{\ell\le k}$ and by standard perturbation arguments we can assume that for some increasing sequence of positive integers $\{m_k\}_{k=1}^{\infty}$ with $m_k < i_k \le m_{k+1}$, we have

$$x_{k,\ell} = P_{m_k,m_{k+1}} x_{k,\ell}, \quad \ell \le k. \tag{4.6}$$

Now, for fixed ℓ and every scalars $\{t_k\}_{k=\ell}^{N}$,

$$\begin{aligned}\|P\|\left(\sum_{k=\ell}^{N}|t_k|^2\right)^{1/2} &= \|P\|\left\|\sum_{k=\ell}^{N} t_k e_{i_k,j_\ell}\right\|_p \\ &\ge \left\|\sum_{k=\ell}^{N} t_k x_{k,\ell}\right\|_p \\ &\ge \left\|\sum_{k=\ell}^{N} t_k x_{k,\ell}(i_k,j_\ell) e_{i_k,j_\ell}\right\|_p \\ &= \left(\sum_{k=\ell}^{N}|t_k|^2 |x_{k,\ell}(i_k,j_\ell)|^2\right)^{1/2} \\ &\ge \left(\sum_{k=\ell}^{N}|t_k|^2\right)^{1/2}\Big/ 2.\end{aligned} \tag{4.7}$$

Therefore, $\{x_{k,\ell}\}_{k=\ell}^{\infty}$ is $2\|P\|$-equivalent to the unit vector basis of ℓ_2. Let $X_\ell = [x_{k,\ell}]_{k=\ell}^{\infty}$. Using Proposition 2.2(i), we can assume (by passing to a subsequence of $\{X_\ell\}_{\ell=1}^{\infty}$ and to a subsequence of $\{j_\ell\}_{\ell=1}$, and by perturbation arguments as in the proof of Theorem 3.2) that for some increasing sequence of positive integers $\{n_\ell\}_{\ell=1}^{\infty}$ with $n_\ell < j_\ell \le n_{\ell+1}$, we have

$$E^{n_{\ell+1}} X_\ell = 0. \tag{4.8}$$

By a standard diagonal process, there exists an increasing sequence $\{k_\nu\}_{\nu=1}^{\infty}$ such that the following limits exist for every $\ell' \le \ell$.

$$\lim_{\nu\to\infty} x_{k_\nu,\ell}(i_{k_\nu}, j_{\ell'}) = \lambda_{\ell,\ell'}. \tag{4.9}$$

Since $|x_{k,\ell}(i_k, j_\ell)| \ge \frac{1}{2}$ for every $\ell \le k$, we clearly have $|\lambda_{\ell,\ell}| \ge \frac{1}{2}$. Also, again by a diagonal process, there exists an increasing sequence of positive integers $\{\ell_\mu\}_{\mu=1}^{\infty}$ so that also the following limits exist:

(4.10) $$\lambda = \lim_{\mu\to\infty} \lambda_{\ell_\mu,\ell_\mu}, \quad |\lambda| \geq \tfrac{1}{2},$$

(4.11) $$\lambda_\sigma = \lim_{\mu\to\infty} \lambda_{\ell_\mu,\ell_\sigma}, \quad \sigma = 1, 2, \ldots.$$

By passing to a further subsequence of $\{k_\nu\}_{\nu=1}^\infty$, we may clearly assume that $k_\nu \geq \ell_\nu$ for every ν.

Now, by passing to a subtriangle of $\{x_{k_\nu,\ell_\mu}\}_{\mu\leq\nu}$ we can assume that the sequences in (4.9), (4.10) and (4.11) converge arbitrarily fast. Thus, by perturbation arguments, there is no loss of generality in assuming simply that

(4.12) $$x_{k_\nu,\ell_\mu}(i_{k_\nu}, j_{\ell_\sigma}) = \lambda_\sigma, \quad \sigma < \mu \leq \nu,$$

and

(4.13) $$x_{k_\nu,\ell_\mu}(i_{k_\nu}, j_{\ell_\mu}) = \lambda, \quad \mu \leq \nu.$$

Let $y_{\nu,\mu} = x_{k_\nu,\ell_\mu}$, $\mu \leq \nu$. As we have stated above, in order to complete the proof it is enough to show that

(4.14) $$y_{\nu,\mu}(i_{k_{\nu'}}, j_{\ell_{\mu'}}) = \delta_{\nu,\nu'} \cdot \delta_{\mu,\mu'} \cdot \lambda.$$

In view of (4.6), (4.8), (4.12) and (4.13), in order to prove (4.14) we only have to show that $\lambda_\sigma = 0$ for every σ.

Fix σ, let N be arbitrary, and let $\nu \geq N + \sigma$. Then,

(4.15) $$\|P\| N^{1/2} \geq \left\| \sum_{\mu=\sigma+1}^{\sigma+N} y_{\nu,\mu} \right\|_p \geq \left| \sum_{\mu=\sigma+1}^{\sigma+N} y_{\nu,\mu}(i_{k_\nu}, j_{\ell_\sigma}) \right| = |\lambda_\sigma| N.$$

Since N is arbitrary, this clearly implies that $\lambda_\sigma = 0$. Thus (4.14) holds, and this completes the proof of Theorem 4.6. □

In the proof of Theorem 3.2 we did not use the full force of the assumption that the triangle $\{x_{i,j}\}_{j\leq i}$ is equivalent to the triangle $\{e_{i,j}\}_{j\leq i}$ of the standard unit matrices of T_p. A careful check of the proof of Theorem 3.2 shows that what was relevant is the existence of a positive constant K, so that:

(a) For every fixed j, $\{x_{i,j}\}_{i=j}^\infty$ is K-equivalent to the unit vector basis of ℓ_2;

(b) For every fixed i, $\{x_{i,j}\}_{j=1}^i$ is K-equivalent to the unit bector basis of ℓ_2^i;

Let $X_j = [x_{i,j}]_{i=j}^\infty$, then, (c′) $(1 \le p < 2)$. For every n there is some $j = j(n)$, so that for every $x \in X_j$, $\|E^n x\|_p \ge (2K)^{-1}\|x\|_p$;

(c″) $(2 < p < \infty)$. For every n, $\liminf_{j\to\infty}\|E_{n|X_j}\| = 0$.

Conditions (c′) and (c″) are the consequence of the following condition:

(c) $(1 \le p < \infty,\ p \ne 2)$. If $\{x_j\}_{j=1}^{\infty}$ is a normalized sequence with $x_j \in X_j$ for every j, then some subsequence $\{x_{j_\ell}\}_{\ell=1}^{\infty}$ is K-equivalent to the unit vector basis of ℓ_p.

Thus, actually we have the following refinement of Theorem 3.2.

THEOREM 4.7: *Let $1 \le p < \infty$, $p \ne 2$, and let $\{x_{i,j}\}_{j\le i}$ be a triangle of elements of T_p. Assume that for some positive constant K, conditions (a), (b), and one of the conditions (c′) (for $1 \le p < 2$), (c″) (for $2 < p < \infty$), or (c) are satisfied.*

Then for every $0 < \theta < 1$ there exist a tensor product representation of C_p as $C_p \otimes C_p$, a normalized element $z \in C_p$, and an isomorphism V from a subspace Y of $X = \overline{\text{span}}\{x_{i,j}\}_{j\le i}$ onto $T_p \otimes z$, so that for every $y \in Y$ we have $\|Vy - y\|_p \le (\theta/5)\|y\|_p$. Thus $\mathrm{d}(Y, T_p) \le 1 + \theta$ and Y is $2 + \theta$-complemented in T_p.

Moreover, the construction can be made so that for some subtriangle $\{u_{k,\ell}\}_{\ell\le k}$ of $\{x_{i,j}\}_{j\le i}$, the elements $y_{\nu,\mu} = V(e_{\nu,\mu} \otimes z)$, $\mu \le \nu$, have the following form: for $1 \le p < 2$, there is a positive integer m, so that

$$y_{\nu,\mu} = \sum_{j=1}^{m} u_{\nu m+j,\mu m+j}/m^{1/p}. \tag{4.16}$$

For $2 < p < \infty$, there exists an increasing sequence of positive integers $\{\ell_\mu\}_{\mu=1}^{\infty}$, so that

$$y_{\nu,\mu} = \sum_{\ell=\ell_\mu+1}^{\ell_{\mu+1}} u_{\ell_{\nu+1},\ell}/(\ell_{\mu+1} - \ell_\mu)^{1/2}. \tag{4.17}$$

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