COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 41, nº 2 (1980), p. 287-295

<http://www.numdam.org/item?id=CM_1980__41_2_287_0>

COMPOSITIO MATHEMATICA, Vol. 41, Fasc. 2, 1980, pag. 287–295

Printed in the Netherlands

A NORMALIZED WEAKLY NULL SEQUENCE WITH NO SHRINKING SUBSEQUENCE IN A BANACH SPACE NOT CONTAINING ℓ_1

E. Odell*

1. Introduction

An important question in Banach space theory is "does every infinite dimensional Banach space contain ℓ_1 or an infinite dimensional subspace with separable dual" (equivalently, a shrinking basic sequence [3])? A natural approach to this question is to ask for something even stronger: does every weakly null normalized basic sequence in a Banach space have a shrinking subsequence? This conjecture is false. One example is the unit vector basis of a Lorentz space $d_{w,1}$ (see also Maurey and Rosenthal [5]). $d_{w,1}$ is hereditarily ℓ_1. In this note we give an example of a normalized weakly null sequence (f^n) so that

(i) if $(f^{n'})$ is any subsequence of (f^n), then its closed linear span has nonseparable dual (hence no subsequence is shrinking), and

(ii) the closed linear span of (f^n) does not contain an isomorph of ℓ_1.

Our example is thus another example of a normalized weakly null sequence with no unconditional subsequence. Our construction does not use the Maurey–Rosenthal technique. Instead we are indebted to J. Bourgain [1] for our immediate inspiration. The example also draws upon two other beautiful works: Rosenthal's basic result on ℓ_1 [7] and James' tree space, *JT* [2]. *JT* was one of the first examples of a separable space not containing ℓ_1 with nonseparable dual (another example was discovered simultaneously by Lindenstrauss and Stegall [4]).

* Research partially supported by NSF Grant MCS 78-01344.

0010–437X/80/05/0287–09$00.20/0

The effect of this example is that in order to answer the question posed in the first paragraph of this paper it will be necessary to work with more general block bases of a given basis than just its subsequences. On the one hand this is unfortunate since much progress in general Banach space theory has dealt with subsequences of a given basis. The combinatorial result known as Ramsey's theorem has been especially useful in this regard (see [6] for a brief discussion). The combinatorics needed to handle general block bases is extremely complicated. On the positive side, we may take consolation in the fact that these questions are truly deep and their solution should give us great understanding of general Banach spaces.

2. The example

We first shall define a Banach space Y and then choose a particular weakly null sequence $(f^n)_{n=1}^\infty \subseteq Y$. The space Y will be a sort of James tree space with the nodes infinite dimensional rather than one dimensional. Some notation is necessary.

Let $K_{ij} = [(j-1)/2^i, j/2^i]$ be the closed dyadic intervals in $[0,1]$ for $0 \le i < \infty$ and $1 \le j \le 2^i$. Let $C_0(K_{ij}) = \{f \in C(K_{ij}) : f((j-1)/2^i) = f(j/2^i) = 0\}$. If $g \in C_0(K_{ij})$, let $\tilde{g} \in C[0,1]$ be the linear extension of g which is identically 0 outside of K_{ij}.

Let T be the dyadic tree whose nodes are indexed by $\{(i,j) : 0 \le i < \infty \text{ and } 1 \le j \le 2^i\}$ with the natural order. Thus $(i,j) \le (i',j')$ if $K_{i'j'} \subseteq K_{ij}$. By a *segment* β in T we mean a linearly ordered set of the form $\{(i,j_1), (i+1,j_2), \ldots, (i+n,j_n)\}$.

If $(g_{ij})_{(i,j)\in T}$ is a set of functions with $g_{ij} \in C_0(K_{ij})$ for all $(i,j) \in T$ and all but a finite number of g_{ij} identically 0, we define

$$\|(g_{ij})\| = \sup\left(\sum_{k=1}^{\ell} \Big\| \sum_{(i,j)\in\beta_k} \tilde{g}_{ij}\Big\|_\infty^2\right)^{1/2}$$

where the "sup" is taken over all collections of disjoint segments, $(\beta_k)_{k=1}^{\ell}$.

Y is the completion of the set of all such (g_{ij}).

Before we define (f^n) we shall require a simple lemma.

Let $\delta_i \downarrow 0$ so that for all i_0,

$$\delta_{i_0}\left(\sum_{i=0}^{\infty} 5^{-i}\right) 5^{i_0} 2^{i_0/2} \le \tfrac{1}{4}. \tag{1}$$

Recall that a Lorentz sequence space $d_{w,p}$ where $w = (w_i)$ satisfies

(2)
$$w_i \downarrow 0,\ w_1 = 1 \text{ and } \sum_1^\infty w_i = \infty$$

has norm defined by

$$\|(a_i)\|_{w,p} = \left(\sum_{i=1}^\infty |\bar{a}_i|^p w_i\right)^{1/p}$$

where $(\bar{a}_i)$ is the decreasing rearrangement of $|a_i|$. Note that if (g_i) is any block basis of the unit vector basis for $d_{w,p}$ then

(3)
$$\left\|\sum_1^k g_i\right\| \leq \left(\sum \left\|g_i\right\|^p\right)^{1/p}.$$

LEMMA: *There exists a collection of Lorentz sequence spaces $(d_{w^{ij},2})_{(i,j)\in T}$ satisfying the following condition ($(f_{ij}^n)_{n=1}^\infty$ denotes the unit vector basis of $d_{w^{ij},2}$): for each $(i_0, j_0) \in T$ there exist scalars $(a_n)_{n=1}^\ell$ such that*

(4)
$$\left\|\sum_{n=1}^\ell a_n f_{i_0,j_0}^n\right\| = 1 \text{ and}$$

(5)
$$\left\|\sum_{n=1}^\ell a_n f_{i,j}^n\right\| < \delta_{i_0} \text{ if } (i,j) \neq (i_0, j_0).$$

PROOF: Actually, given $(i_0, j_0) \in T$, $(a_n)_{n=1}^\ell$ will be a constant sequence. Thus we wish to choose w^{ij} so that given (i_0, j_0) there is some $\ell = \ell(i_0, j_0)$ so that

(6)
$$\left\|\sum_{n=1}^\ell f_{i,j}^n\right\|_{d_{w^{ij},2}} \Big/ \left\|\sum_{n=1}^\ell\right\|_{d_{w^{i_0j_0},2}} < \delta_{i_0},$$

if $(i,j) \neq (i_0, j_0)$.

We use induction. Choose $w^{0,1}$ to satisfy (2) and choose $\ell(0,1)$ so that $\|\Sigma_{n=1}^{\ell(0,1)} f_{0,1}^n\| > 2\delta_0^{-1}$. Thus to satisfy (6) for $(i_0, j_0) = (0,1)$ we need only require that if $(i,j) \neq (0,1)$, then $\|\Sigma_{n=1}^{\ell(0,1)} f_{ij}^n\| \leq 2$. This is easily accomplished by requiring $w_n^{ij} \leq 3/\ell(0,1)$ for $2 \leq n \leq \ell(0,1)$. Choose $w_1^{1,1} = 1$ and $3/\ell(0,1) \geq w_2^{1,1} \geq \ldots \geq w_{\ell(0,1)}^{1,1} = w > 0$. Choose $\ell(1,1)$ so large that if $w_n^{1,1} = w$ for $\ell(0,1) \leq n \leq \ell(1,1)$ then (6) is satisfied for $(i_0, j_0) = (1,1)$ and $(i,j) = (0,1)$. This is possible because $\lim_{n\to\infty} w_n^{0,1} = 0$. Let $w_n^{1,1}$ for $n > \ell(1,1)$ be chosen to satisfy (2).

The inductive procedure is now clear. If $(w^{ij})_{(i,j)\in F}$ have been chosen for a finite set F, we need only require later $w_n^{i,j}$'s to be sufficiently small for a finite number of n's ($n > 2$) in order to satisfy (6) for $(i_0, j_0) \in F$. Then we keep $w_n^{i,j}$ constant for a long interval to get the other half of (6).

From now on we shall regard each $d_{w^{ij},2}$ to be isometrically embedded into $C_0(K_{ij})$. Define

$$f^n = (5^{-i}2^{-i/2}f_{ij}^n)_{(i,j)\in T}.$$

Note that for a fixed i,

$$\|(2^{-i/2}f_{i,j}^n)_{j=1}^{2^i}\|_Y = 1,$$

and thus $f^n \in Y$. Also (f^n) is semi-normalized $(1 \le \|f^n\| \le \Sigma_{i=0}^{\infty}5^{-i})$ which is sufficient for our purposes.

It remains to show that (f^n) has the desired properties. We begin with the easiest one.

(f^n) *is weakly null*: It suffices to show that for all subsequences (f^{n_i}) and for all k,

$$\left\|\sum_{s=1}^{k} f^{n_s}\right\| \le 2k^{1/2}. \tag{7}$$

If i is fixed, then by (3)

$$\left\|\sum_{s=1}^{k} (2^{-i/2}f_{ij}^{n_s})_{j=1}^{2^i}\right\|_y \le k^{1/2}.$$

Thus

$$\left\|\sum_{s=1}^{k} f^{n_s}\right\| = \left\|\sum_{s=1}^{k} (5^{-i}2^{-i/2}f_{ij}^{n_s})_{(i,j)\in T}\right\| \le k^{1/2}\left(\sum_{i=0}^{\infty} 5^{-i}\right) < 2k^{1/2},$$

and (7) is proved.

Now let (f^{n_s}) be any subsequence of (f^n), and let $[(f^{n_s})]$ be the closed linear subspace it generates.

$[(f^{n}{}_s)]^*$ *is nonseparable*: We first define a large collection of functionals. For $t \in [0, 1]$ and $g = (g_{ij}) \in Y$, let

$$\delta_t(g) = \sum_{(i,j)\in\beta_t} g_{ij}(t) \tag{8}$$

where β_t is the infinite branch determined by t. Thus if t is not a dyadic point, then $\beta_t = \{(i,j) : t \in K_{ij}\}$ is uniquely determined. If t is dyadic then there are 2 branches naturally determined by t, however, both branches yield the same value in (8). Clearly δ_t is in the unit ball of Y^*.

Claim. There exists $\delta > 0$ and $C < \infty$ so that for all $(i_0, j_0) \in T$ there exists $u \in [(f^{n_s})]$ with $\|u\| \leq C$, $\delta_t(u) > \delta$ for some $t \in K_{i_0,j_0}$ and $\delta_s(u) \leq \delta/2$ if $s \notin K_{i_0,j_0}$.

Let us believe the claim for the moment and see why this implies $[(f^{n_s})]^*$ is not separable. Note that if $t_n \to t$ in $[0, 1]$, then $\delta_{t_n} \to \delta_t$ (weak*). Thus if $u \in [(f^{n_s})]$, then the function $\bar{u}$ given by $\bar{u}(t) = \delta_t(u)$ is in $C[0, 1]$. By the claim we may inductively choose a subtree $T' \subseteq T$ and $(u_{ij})_{(i,j)\in T'} \subseteq [(f^{n_i})]$ with $\|u_{ij}\| \leq C$, $\bar{u}_{ij}(t) > \delta$ if $t \in K_{ij}$ with $(ij) \in T'$ and $\bar{u}_{ij}(s) \leq \delta/2$ if $s \in K_{i'j'}$ with $(i', j') \in T'$ and $K_{ij} \cap K_{i'j'} = \emptyset$. Now for each branch β of T', let $t_\beta = \bigcap_{(i,j)\in\beta} K_{ij}$. It is clear that $\|\delta_{t_\beta} - \delta_{t_\gamma}\| \geq \delta/2C$ if $\beta \neq \gamma$, where the norm of $\delta_{t_\beta} - \delta_{t_\gamma}$ is calculated in $[(f^{n_s})]^*$. Thus $[(f^{n_s})]^*$ is not separable.

Proof of the Claim: Fix $(i_0, j_0) \in T$ and let $(a_n)_{n=1}^{\ell}$ satisfy (4) and (5) above. Define

$$u = 5^{i_0} 2^{i_0/2} \left(\sum_{s=1}^{\ell} a_s f^{n_s} \right) = \left(5^{i_0 - i} 2^{(i_0-i)/2} \sum_{s=1}^{\ell} a_s f_{ij}^{n_s} \right)_{(i,j)\in T}.$$

Fix $i \neq i_0$. Then by (5) we have

$$\left\| \left(5^{i_0-i} 2^{(i_0-i)/2} \sum_{s=1}^{\ell} a_s f_{ij}^{n_s} \right)_{j=1}^{2^i} \right\| \leq 5^{i_0-i} 2^{i_0/2} \delta_{i_0}.$$

If $i = i_0$ then by (4) and (5)

$$\left\| \left(\sum_{s=1}^{\ell} a_s f_{i_0,j}^{n_s} \right)_{j=1}^{2^{i_0}} \right\| \leq \delta_{i_0} 2^{i_0/2} + 1.$$

It follows that

$$\|u\| \leq \delta_{i_0} \left(\sum_{i=0}^{\infty} 5^{i_0-i} \right) 2^{i_0/2} + 1.$$

And so by (1), $\|u\| \leq 2$.

We shall show that $\delta_t(u) \geq 1/2$ for some $t \in K_{i_0,j_0}$ and $\delta_s(u) \leq 1/4$ if $s \notin K_{i_0,j_0}$. Choose t so that

$$\left| \sum_{s=1}^{\ell} a_s f_{i_0,j_0}^{n_s}(t) \right| = 1 \quad (\text{use}(4)).$$

Thus

$$|\delta_t(u)| \geq 1 - 5^{i_0}2^{i_0/2}\left[\sum_{i=0}^{\infty} 5^{-i}2^{-i/2}\delta_{i_0}\right] \geq 1/2 \quad \text{(use (5) and (1))}.$$

Furthermore if $s \notin K_{i_0,j_0}$ then

$$|\delta_s(u)| \leq 5^{i_0}2^{i_0/2}\left[\sum_{i=1}^{\infty} 5^{-i}2^{-i/2}\delta_{i_0}\right] \leq 1/4.$$

This proves the claim.

$[(f^n)]$ *contains no isomorph of* ℓ_1: This is the most complex part of the example. Suppose $(g^m) \subseteq [(f^n)]$ is equivalent to the unit vector basis of ℓ_1. We may assume that (g^m) is a normalized block basis of (f^n) of the form $g^m = \Sigma_{n=p_m}^{p_{m+1}-1} b_n f^n$ where (b_n) are scalars and $p_1 < p_2 < \cdots$. Fix $(i, j) \in T$. Then since $\|g^m\| = 1$, for $(i, j) \in T$ we have

$$\left\|\sum_{n=p_m}^{p_{m+1}-1} b_n 5^{-i}2^{-i/2} f_{ij}^n\right\|_{d_w ij,2} \leq 1$$

and thus for all k by (3),

$$\left\|\sum_{m=1}^{k}\left(\sum_{n=p_m}^{p_{m+1}-1} b_n f_{ij}^n 5^{-i}2^{-i/2}\right)\right\|_{d_w ij,2} \leq k^{1/2}. \tag{9}$$

However (g^m) is equivalent to the unit vector basis of ℓ_1 and thus there is a constant $\gamma > 0$ so that $\|\Sigma_1^k g^m\| \geq \gamma k$ for all k. Now the expression in (9) is the (i, j)-coordinate of $\Sigma_1^k g^m$ in Y. Thus by taking long averages of the form $\Sigma_1^k g^m / \|\Sigma_1^k g^m\|$ and using the standard perturbation arguments we obtain a normalized sequence $(h^m) \subseteq Y$ which is equivalent to the unit vector basis of ℓ_1 and satisfies

(10) for all i_0 there exists m_0 so that if $m \geq m_0$ and $h^m = (h_{ij}^m)$ then $h_{ij}^m = 0$ if $i \leq i_0$.

We shall show this is impossible.

Claim: There exists a subsequence $(h^{m'})$ of (h^m) so that $(\delta_t(h^{m'}))_{m=1}^{\infty}$ converges for all $t \in [0, 1]$.

Indeed as above let $\bar{h}^m \in C[0, 1]$ be given by $\bar{h}^m(t) = \delta_t(h^m)$. Then $\|\bar{h}^m\|_{C[0,1]} \leq 1$ and we shall show $(\bar{h}^m)$ has a pointwise convergent subsequence. Suppose not. Then by Rosenthal's theorem [7] there exists a subsequence $(\bar{h}^{m'})$, $r \in R$ and $\delta > 0$ so that if $A_m = \{t : \bar{h}^{m'}(t) > r + \delta\}$ and $B_m = \{t : \bar{h}^{m'}(t) < r\}$ then (A_m, B_m) is Boolean independent. This means that if $\epsilon A_m = A_m$ for $\epsilon = +1$ and $\epsilon A_m = B_m$ for $\epsilon = -1$, then $\bigcap_{i=1}^{k} \epsilon_i A_i \neq \emptyset$ for all k and all $\epsilon_i = \pm 1$, $1 \leq i \leq k$. We may assume $r + \delta > 0$.

Since $(\bar{h}^{m'})$ are continuous, A_m and B_m are open sets. Fix k and consider the 2^k disjoint nonempty open sets $0_\epsilon = \bigcap_{m=1}^{k} \epsilon_m A_m$ $(\epsilon = (\epsilon_m)_1^k)$. Choose i_0 sufficiently large so that the oscillation of $\bar{h}^{m'}$ on $K_{i_0,j}$ is smaller than $\delta/2$ for each $1 \leq m \leq k$ and $1 \leq j \leq 2^{i_0}$. Thus for each j, $K_{i_0,j} \cap 0_\epsilon \neq \emptyset$ for at most one $\epsilon = (\epsilon_i)_1^k$. Choose m so large that if $h^{m'} = (h_{ij}^{m'})$ then $h_{ij}^{m'} = 0$ if $i \leq i_0$. Since $0_\epsilon \cap A_m \neq \emptyset$ for all ϵ, $\bar{h}^{m'}$ is greater than $r + \delta$ at some point in at least 2^k distinct $K_{i_0,j}$'s. It follows that $1 = \|h^{m'}\| \geq (r + \delta) 2^{k/2}$. But k was arbitrary and thus the claim is proved.

Form a new sequence (x^m) by taking differences of $(h^{m'})$ and normalizing. (x^m) is thus equivalent to the unit vector basis of ℓ_1, satisfies (10) and the functions $(\bar{x}^m)$ are weakly null in $C[0,1]$. Some sequence of convex combinations of $(\bar{x}^m)$ converges to 0 in norm in $C[0,1]$. The corresponding functions (U^m) in Y are still equivalent to the unit vector basis of ℓ_1, satisfy (10) and $\|\bar{U}^m\|_{C[0,1]} \to 0$.

Choose a subsequence (U^{m_s}) of (U^m), integers $i_s \to \infty$ and $\epsilon_s \to 0$ so that

$$\sum_{k=1}^{\infty} 2^{i_k} \left(\sum_{s=k+1}^{\infty} \epsilon_s \right) \leq 1, \tag{11}$$

$$\left\| \bar{U}^{m_s} \right\|_{C[0,1]} \leq \epsilon_s, \quad \text{and} \tag{12}$$

$$\text{if } U^{m_s} = (U_{ij}^{m_s}), \quad \text{then} \quad U_{ij}^{m_s} = 0 \text{ if } i \notin (i_{s-1}, i_s). \tag{13}$$

This may be done inductively. Let $m_1 = 1$, $i_0 = 0$ and $\epsilon_1 = \|\bar{U}^{m_1}\|$. Choose i_1 so that (13) holds for $s = 1$ (we may assume that each U^m is supported only on a finite number of nodes in T.) Let ϵ_2 be very small (according to (11)) and choose m_2 so that (12) holds. Choose i_2 so that (13) holds and continue in this manner.

We shall show that $\|\Sigma_1^k U^{m_s}\|$ is of the order $k^{1/2}$ and thus obtain a contradiction to the fact that (U^{m_s}) is equivalent to the unit vector basis of ℓ_1.

Fix k and suppose

$$\left\| \sum_{s=1}^{k} U^{m_s} \right\| = \left(\sum_{r=1}^{\ell} \left\| \sum_{(i,j) \in \beta_r} \left(\sum_{s=1}^{k} \tilde{U}_{ij}^{m_s} \right) \right\|_\infty^2 \right)^{1/2}$$

for a certain collection of disjoint segments $(\beta_r)_1^\ell$. Replace each segment β_r by three segments as follows. Suppose $\beta_r = \{(i_0, j_1), (i_0+1, j_2), \ldots, (i_0+n, j_n)\}$, with $i_s \le i_0 < i_{s+1}$ and $i_{s'} \le i_0 + n < i_{s'+1}$. Let

$$\beta_{r,1} = \{(i,j) \in \beta_r : i < i_{s+1}\},$$

$$\beta_{r,2} = \{(i,j) \in \beta_r : i_{s+1} \le i < i_{s'}\} \text{ and}$$

$$\beta_{r,3} = \{(i,j) \in \beta_r : i_{s'} \le i\}.$$

Thus we have split β_r into initial, middle and terminal segments, some of which may be empty. Then

$$\left\|\sum_{s=1}^k U^{m_s}\right\| \le 3 \sum_{p=1}^3 \left(\sum_{r=1}^\ell \left\| \sum_{(i,j)\in\beta_{r,p}} \left(\sum_{s=1}^k \tilde{U}_{ij}^{m_s}\right)\right\|_\infty^2\right)^{1/2}.$$

If $p = 1$ or 3 we have

$$\left(\sum_{r=1}^\ell \left\| \sum_{(i,j)\in\beta_{r,p}} \left(\sum_{s=1}^k \tilde{U}_{ij}^{m_s}\right)\right\|_\infty^2\right)^{1/2} \le k^{1/2},$$

since at most one U^{m_s} has non-zero coordinates at $(i,j) \in \beta_{r,p}$. Also

$$\left(\sum_{r=1}^\ell \left\| \sum_{(i,j)\in\beta_{r,2}} \left(\sum_{s=1}^k \tilde{U}_{ij}^{m_s}\right)\right\|_\infty^2\right)^{1/2} < \sum_{s=1}^\infty 2^{i_s - 1}\left(\sum_{j=s}^\infty \epsilon_j\right) \le 3.$$

This is because by (13) each $\beta_{r,2}$ is a segment which either passes entirely through the support of U^{m_s} or is disjoint from it, and (12) and (11).

Consequently for all k, $\|\sum_{s=1}^k U^{m_s}\| \le 3(2k^{1/2}+3)$, which was to be shown.

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(Oblatum 20-IV-79)

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