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JOEL ROBERTS

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SOME PROPERTIES OF DOUBLE POINT SCHEMES

Joel Roberts*

0. Introduction

Let k be an algebraically closed field (of arbitrary characteristic), and let $f: X \to Y$ be a morphism of nonsingular algebraic varieties over k. Let $\pi: (X \times X)' \to X \times X$ be obtained by blowing up the diagonal $\Delta \subset X \times X$, and let $E = \pi^{-1}(\Delta)$ be the exceptional locus. We define the *double point scheme* $Z = Z(f) \subset (X \times X)'$ exactly as in [12, Section 4] or [11, Chapter V, Section C]. (See Section 1 below.) In particular, if $z \in (X \times X)' - E$, then $z \in Z$ if and only if $\pi(z) = (x, y)$, where $f(x) = f(y)$. (Recall that π induces an isomorphism $(X \times X)' - E \tilde{\to} X \times X - \Delta$.) On the other hand, the points of $\pi^{-1}(x, x)$ are in 1:1 correspondence with the 1-dimensional subspaces of the Zariski tangent space $T(X)_x$. If $\pi(z) = (x, x)$, then $z \in Z$ if and only if z corresponds to a 1-dimensional subspace of the kernel of the linear map $T(X)_x \to T(Y)_{f(x)}$.

Consider a projective embedding $X \subset \mathbb{P}^n$ and a morphism $f: X \to \mathbb{P}^m$ ($m \geqq \dim(X)$) induced by projection from a linear subspace $L \subset \mathbb{P}^n$, with $\operatorname{codim}(L) = m + 1$ and $L \cap X = \emptyset$. Kleiman proved the following theorem in [11], using the techniques of [10].

THEOREM (0.1): *If* $\operatorname{char}(k) = 0$ *and* L *is in general position, then* $Z(f)$ *is smooth over* k. *Moreover, every irreducible component of* $Z(f)$ *has dimension* $= 2 \cdot \dim(X) - m$, *or else* $Z(f) = \emptyset$.

To obtain similar results over a field of arbitrary characteristic we have had to impose some conditions on the embedding $X \subset \mathbb{P}^n$. For a closed point $x \in X$, let $t_{X,x} \subset \mathbb{P}^n$ be the embedded tangent space at x. If

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b is a positive integer, we say that $\mathcal{O}_{X,x}/\mathfrak{m}_{X,x}^b$ is *spanned by linear coordinate functions* if the following condition holds.

(*) If H is a hyperplane in $\mathbb{P}^n$ such that $x \notin H$ and if $\mathbb{P}^n - H$ is identified with Spec $k[T_1, \ldots, T_n]$, then every element of $\mathcal{O}_{X,x}/\mathfrak{m}_{X,x}^b$ is a linear combination of the residues of 1, $T_1, \ldots, T_n$.

The definition is independent of the choice of the hyperplane $H \not\ni x$. (See [16, Section 6].)

THEOREM (0.2): *Let X be an (irreducible) nonsingular subvariety of $\mathbb{P}^n$. Assume that $t_{X,x} \cap t_{X,y} = \emptyset$ whenever $x \neq y$ and that $\mathcal{O}_{X,x}/\mathfrak{m}_{X,x}^3$ is spanned by linear coordinate functions for every $x \in X$. If $f: X \to \mathbb{P}^m$ is induced by projection from a linear subspace $L \subset \mathbb{P}^n$ in general position, then $Z(f)$ is smooth and of pure dimension $= 2 \cdot \dim(X) - m$, or else $Z(f) = \emptyset$.*

The results of [16, Section 6] imply that every nonsingular projective variety has an embedding which satisfies the hypotheses of Theorem (0.2). A relatively elementary proof of Theorem (0.2), using the methods of [16], will be given in Section 2. In [11, Chapter V, Section D], Kleiman sketched a proof of a result that is similar to Theorem (0.2). In Section 3, I have worked out the details of that proof. Thus, we obtain a second proof of Theorem (0.2).

If $f: X \to Y$ and $Z = Z(f)$ are as before, let $g: Z \to X$ be induced by $p_2 \circ \pi$, where $p_2: X \times X \to X$ is projection to the second factor. Suppose that x_1, x_2, and x_3 are distinct points of X such that $f(x_1) = f(x_2) = f(x_3)$. If $\pi(z_1) = (x_1, x_3)$ and $\pi(z_2) = (x_2, x_3)$, then $g(z_1) = g(z_2)$. Thus, triple points of f induce double points of g. This observation is the basis of the study of triple points in [11, Chapter V, Section D]. It is also interesting to ask what happens for "limiting positions" of triple points, where some or all of the points x_1, x_2, and x_3 coincide. This is the motivation for Section 4, 5, and 6 below.

If f is ramified at x and $f(x) = f(x')$ for some $x' \neq x$, then we have a *stationary point* of f. If $\pi(z) = (x, x')$, then g is ramified at z by Corollary (4.2) below. Our main local results about stationary points are Theorem (4.3) and Proposition (6.2). Theorem (6.3) gives a formula for the rational equivalence class of the stationary point cycle.

Finally, a point $x \in S_1^{(2)}(f) - S_1^{(3)}(f)$ can be regarded as a "limiting position of triple points" because the local ring at x on the fibre $f^{-1}(f(x))$ has length $= 3$. (The singularity subschemes $S_1^{(q)}(f)$ are defined as in [16, Section 3].) These points give rise to ramified points of g which lie on the exceptional locus E. In fact, g induces isomorphism

$$(S_1^{(q)}(g) - S_1^{(q+1)}(g)) \cap E \tilde{\to} S_1^{(q+1)}(f) - S_1^{(q+2)}(f)$$

for all $q \geqq 1$, by Theorem (4.5). Theorem (5.6) gives a formula for the rational equivalence class of the cycle associated to the subscheme $S_1^{(2)}(f) \subset X$. It is the same as the characteristic 0 formula. The proof, however, works in all characteristics because it uses a formalism that is based on Theorem (4.5). In order to be able to work in the rational equivalence ring, we have restricted to the case where Z is smooth. Thus, the result is valid for generic projections. It also seems likely that one could work with the Chow homology theory of [3] and prove a more general result.

Terminology and notation will be similar to what is used in [11] and [16].

1. The basic constructions

Let $f: X \to Y$ be a morphism of non-singular varieties, and let Δ_X and Δ_Y be the diagonals in $X \times X$ and $Y \times Y$ respectively. Then $\Delta_X \subset (f \times f)^{-1}(\Delta_Y)$ so that $E \subset ((f \times f) \circ \pi)^{-1}(\Delta_Y)$, where $\pi: (X \times X)' \to X \times X$ is as in the introduction and $E = \pi^{-1}(\Delta_X)$. Therefore $J \subset I$, where J is the sheaf of ideals in $\mathcal{O}_{(X\times X)'}$ defining the subscheme $((f \times f) \circ \pi)^{-1}(\Delta_Y) \subset (X \times X)'$ and $I = \mathcal{O}_{(X\times X)'}(-E)$ is the invertible sheaf of ideals defining the exceptional locus E. (Recall that E is a divisor in $(X \times X)'$.) The *double point scheme* $Z(f)$ is defined to be the subscheme of $(X \times X)'$ defined by the sheaf of ideals $I^{-1}J$.

LEMMA (1.1): *π induces an isomorphism $Z(f) - E \tilde{\to} X \times_Y X - \Delta_X$.*

PROOF: We recall that $X \times_Y X \cong (f \times f)^{-1}(\Delta_Y) \subset X \times X$. Thus, the result follows because π induces an isomorphism $(X \times X)' - E \tilde{\to} X \times X - \Delta_X$. Q.E.D.

It is well-known that there is a commutative diagram

$$\begin{array}{ccc} \mathbb{P}(\Omega_X^1) & \xrightarrow{j} & (X \times X)' \\ \downarrow & & \downarrow \pi \\ X & \xrightarrow{\delta} & X \times X \end{array},$$

where $\delta(x) = (x, x)$, Ω_X^1 is the cotangent bundle, and j maps $\mathbb{P}(\Omega_X^1)$ isomorphically onto E. Thus, for each closed point $x \in X$, j induces a

1:1 correspondence between $\pi^{-1}(x, x)$ and the set of 1-dimensional subspaces of the Zariski tangent space $T(X)_x$.

LEMMA (1.2): *Let $z \in E$. Then $z \in Z(f)$ if and only if z corresponds to a 1-dimensional subspace of the kernel of the linear map $(df)_x : T(X)_x \to T(Y)_{f(x)}$, where $(x, x) = \pi(z)$.*

PROOF: We have $\mathcal{O}_{(X\times X)',z} \supset \mathcal{O}_{X\times X,(x,x)} \supset \mathcal{O}_{X,x} \otimes_k \mathcal{O}_{X,x}$. By choosing a suitable generating set $\{t_1, \ldots, t_r\}$ for the maximal ideal $m_{X,x} \subset \mathcal{O}_{X,x}$, we may assume that the maximal ideal $m_{(X\times X)',z} \subset \mathcal{O}_{(X\times X)',z}$ is generated by $\{1 \otimes t_1, \ldots, 1 \otimes t_r, t_1 \otimes 1, \xi_2, \ldots, \xi_r\}$, where ξ_i satisfies $t_i \otimes 1 - 1 \otimes t_i = \xi_i(t_1 \otimes 1 - 1 \otimes t_1)$, $i = 2, \ldots, r$. Then I_z is generated by $t_1 \otimes 1 - 1 \otimes t_1$.

Consider the mapping $f^* : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$. Then $I^{-1}J_z$ is generated by the elements $(f^*(u) \otimes 1 - 1 \otimes f^*(u))/(t_1 \otimes 1 - 1 \otimes t_1)$, as u runs through $\mathcal{O}_{Y,f(x)}$. If $f^*(u) \equiv \alpha_0 + \alpha_1 t_1 + \cdots + \alpha_r t_r (\mathrm{mod}\ m_{X,x}^2)$, then it is easily verified that

$$(f^*(u) \otimes 1 - 1 \otimes f^*(u))/(t_1 \otimes 1 - 1 \otimes t_1) \equiv \alpha_1 (\mathrm{mod}\ m_{(X\times X)',z}).$$

(See Lemma (2.1) below, also.)

In other words, $I^{-1}J_z$ is the unit ideal if and only if t_1 occurs to the first power in the expansion of some $f^*(u)$. This is equivalent to the conclusion of the lemma. Q.E.D.

EXAMPLE (1.3): Let $f: \mathbb{A}^2 \to \mathbb{A}^3$ be the map which sends $(t_1, t_2) \to (t_1, t_1 t_2, t_2^2 + t_2^3)$. The Jacobian matrix is:

$$\begin{pmatrix} 1 & t_2 & 0 \\ 0 & t_1 & 2t_2 + 3t_2^2 \end{pmatrix}.$$

Thus, f is ramified only at $(0, 0)$, and if $\mathrm{char}(k) \neq 2, 3$ at $(0, -\frac{2}{3})$. At a ramified point, the tangent space map has rank = 1. Hence $Z(f) \cap E$ consists of either one or two points, depending on $\mathrm{char}(k)$. We write $\mathbb{A}^2 \times \mathbb{A}^2 = \mathrm{Spec}(k[s_1, s_2]) \times \mathrm{Spec}(k[t_1, t_2]) = \mathrm{Spec}\ k[s_1, s_2 t_1, t_2]$, so that the defining ideal of Δ is generated by $s_1 - t_1$ and $s_2 - t_2$. After blowing up Δ, we work in an affine open set with coordinates s_1, s_2, t_2 and ξ, where $s_1 - t_1 = \xi(s_2 - t_2)$. The local equation of E is $s_2 - t_2 = 0$, and the defining ideal of $Z(f)$ is generated by:

$$(s_1 - t_1)/(s_2 - t_2) = \xi$$
$$(s_1 s_2 - t_1 t_2)/(s_2 - t_2) = s_1 + \xi t_2, \quad \text{and}$$
$$(s_2^2 - t_2^2 + s_2^3 - t_2^3)/(s_2 - t_2) = s_2 + t_2 + s_2^2 + s_2 t_2 + t_2^2.$$

(Note that $s_1s_2 - t_1t_2 = s_1(s_2 - t_2) + (s_1 - t_1)t_2$.) Thus, $Z(f)$ is a smooth curve. The scheme-theoretic intersection $Z(f) \cap E$ has equations $\xi = s_1 = s_2 - t_2 = 2t_2 + 3t_2^2 = 0$. If $\mathrm{char}(k) \neq 2, 3$, then $Z(f) \cap E$ consists of $(0,0,0,0)$ and $(0,0,-\frac{2}{3},-\frac{2}{3})$, both with multiplicity 1. If $\mathrm{char}(k) = 3$, then $(0,0,0,0)$ has multiplicity 1 and the other point moves away to infinity. If $\mathrm{char}(k) = 2$, then only $(0,0,0,0)$ is present, with multiplicity 2. Correspondingly, the ramification scheme $\bar{S}_1$ is smooth if $\mathrm{char}(k) \neq 2$ but is not smooth if $\mathrm{char}(k) = 2$.

REMARK (1.4): When $\mathrm{char}(k) = 2$, the above map is the canonical form for the pinch points arising from a generic projection of a nonsingular surface to $\mathbb{P}^3$. This is the "missing case" in Kleiman's proof. It is a general fact that the equations defining $Z(f)$ "look the same whether or not $\mathrm{char}(k) = 2$". This simple fact is what makes it possible to give a proof of Theorem (0.2) that makes no reference to the characteristic.

We now fix a nonsingular variety $X \subset \mathbb{P}^n$ and an integer m such that $\dim(X) < m < n$. We recall from [16, Section 8] that there is a dense open subset $S \subset \mathbb{P}^N$ (where $N = (m+1)(n+1) - 1$) and a morphism $\Phi : X \times S \to \mathbb{P}^m \times S$ such that:

(i) A closed point $s \in S$ is an $(m+1)$-tuple $(\ell_0, \ldots, \ell_m)$ of independent linear forms in $n+1$ variables (up to common scalar multiple) such that $L_s \cap X = \emptyset$, where L_s is the linear subspace of $\mathbb{P}^n$ given by $\ell_0 = \cdots = \ell_m = 0$.

(ii) If $x = (t_0, \ldots, t_n) \in X$, then $\Phi(x, s) = (\Phi_s(x), s)$, where $\Phi_s(x) = (\ell_0(t_0, \ldots, t_n), \ldots, \ell_m(t_0, \ldots, t_n))$.

In particular, Φ_s is projection from L_s.

Let $\Phi \times_S \Phi : X \times X \times S \to \mathbb{P}^m \times \mathbb{P}^m \times S$ be the morphism which sends $(x_1, x_2, s) \to (\Phi_s(x_1), \Phi_s(x_2), s)$. Let $\mathcal{J} \subset \mathcal{O}_{(X \times X)' \times S}$ be the sheaf of ideals that defines the subscheme $((\Phi \times_S \Phi) \circ (\pi \times \mathrm{id}_S))^{-1}(\Delta_{\mathbb{P}^m} \times S)$, and let $\mathcal{I}$ be the (invertible) sheaf of ideals that defines the subscheme $E \times S \subset (X \times X)' \times S$. Thus $\mathcal{I} \supset \mathcal{J}$ and the following makes sense.

DEFINITION (1.5): The *relative double-point scheme* $Z_S(\Phi)$ is the subscheme of $(X \times X)' \times S$ defined by the sheaf of ideals $\mathcal{I}^{-1}\mathcal{J}$.

The reason for the terminology is the obvious one: $Z_S(\Phi)$ is the natural analogue, in the category of schemes over S, of the usual double point scheme.

LEMMA (1.6): *Every irreducible component of $Z_S(\Phi)$ has codimension $\leqq m$ in $(X \times X)' \times S$.*

PROOF: It is well known that $\Delta_{\mathbb{P}^m}$ is a local complete intersection in $\mathbb{P}^m \times \mathbb{P}^m$. This implies that the stalk $\mathcal{J}_{(z,s)}$ is generated by m elements at every point $(z, s) \in (X \times X)' \times S$. Thus, it follows that $\mathcal{I}^{-1}\mathcal{J}$ is also generated (locally) by m elements at each point of $Z_S(\Phi)$. Together with the Krull altitude theorem, this implies the conclusion of the lemma.

PROPOSITION (1.7): $Z_S(\Phi) \cap ((X \times X)' \times \{s\}) = Z(\Phi_s) \times \{s\}$ *for every closed point* $s \in S$.

The verification is straightforward and elementary; details are left to the reader. The result can also be obtained as a consequence of known facts about the behavior of blowing up with respect to base changes. (See [7, Section 3].)

PROPOSITION (1.8): $Z_S(\Phi)$ *is nonsingular and of dimension* $= \dim(S) + 2 \cdot \dim(X) - m$. *In fact, if* $p : Z_S(\Phi) \to (X \times X)'$ *is induced by the projection* $(X \times X)' \times S \to (X \times X)'$, *then* p *is surjective and* $p^{-1}(z)$ *is nonsingular and of dimension* $= \dim(S) - m$ *for every* $z \in (X \times X)'$.

No assumption on the embedding $X \subset \mathbb{P}^n$ is needed in the proof of Proposition (1.8). The smoothness of $p^{-1}(z)$ for $z \in E$ will be proved in the next section. To prove that $p^{-1}(z)$ is smooth when $z \notin E$, we begin by observing that $\pi(z) = (x, y) \in X \times X - \Delta$ and choosing homogeneous coordinates in $\mathbb{P}^n$ such that $x = (1, 0, \ldots, 0)$ and $y = (0, 1, 0, \ldots, 0)$. We may assume that $\Phi : X \times S \to \mathbb{P}^m \times S$ sends $(t_0, \ldots, t_n, s) \to (\ell_0(t), \ldots, \ell_m(t), s)$, where $\ell_i(t) = \Sigma_{j=0}^n a_{ij} t_j$, $i = 0, \ldots, m$, and a_{ij} are the homogeneous coordinates of $s \in \mathbb{P}^{(m+1)(n+1)-1}$. Then $p^{-1}(z) \cong U \subset V$, where $V \subset \mathbb{P}^{(m+1)(n+1)-1}$ consists of the points (a_{ij}) such that the 2×2 minors (i.e. subdeterminants) of the $(m+1) \times 2$ submatrix $(a_{ij})_{0 \leqq i \leqq m, j=0,1}$ all vanish, and U consists of all $s \in V$ such that $L_s \cap X = \emptyset$. Thus, $U \cap \mathrm{Sing}(V) = \emptyset$. (In fact, $x \notin L_s$ or $y \notin L_s$ is sufficient for this.) Since V has codimension m in $\mathbb{P}^{(m+1)(n+1)-1}$, this part of the proof is complete.

For a dominant morphism $q : T \to S$ of nonsingular varieties, the *nonsmooth locus* $NS(q)$ consists of all $t \in T$ such that $q^{-1}(q(t))$ fails to be smooth and of dimension $= \dim(T) - \dim(S)$ at t. As noted in [16, Sections 2 and 10], it can be described as the first order singularity $\bar{S}_\nu(q)$, where $\nu = \dim(T) - \dim(S) + 1$, of the morphism q. (In the definition of the first order singularities S_i and $\bar{S}_i$, we follow [11]. Thus, $\bar{S}_i(q)$ consists of all closed points $x \in T$ where the tangent space map $(dq)_x : T(T)_x \to T(S)_{q(x)}$ has rank $\leqq \dim(T) - i$, while

$S_i(q) = \bar{S}_i(q) - \bar{S}_{i+1}(q)$ consists of all points where the rank is exactly $\dim(T) - i$.) In particular, the nonsmooth locus is closed in T. The following result will be proved in the next section.

PROPOSITION (1.9): *Let $q: Z_S(\Phi) \to S$ be induced by the projection $(X \times X)' \times S \to S$, and assume that $\mathcal{O}_{X,x}/m_{X,x}^3$ is spanned by linear coordinate functions for every closed point $x \in X$. Then* $\dim(NS(q) \cap (E \times S)) \leqq \dim(S) - 1$.

REMARK (1.10): If $q: T \to S$ is an arbitrary dominant morphism of nonsingular varieties, then every irreducible component of $NS(q)$ actually has dimension $\geqq \dim(S) - 1$. (The proof uses the exact sequence $q * \Omega_S^1 \to \Omega_T^1 \to \Omega_{T/S}^1 \to 0$ and standard properties of Fitting ideals.)

2. The proof of Theorem (0.2)

We assume that the embedding $X \subset \mathbb{P}^n$ satisfies the hypotheses of Theorem (0.2). Since $t_{X,x} \cap t_{X,y} = \emptyset$ when $x \neq y$, Theorem 7.6 of [16] implies that $Z(f) - E$ is smooth if $f: X \to \mathbb{P}^m$ is a generic projection. In fact, Lemma (1.1) implies that $Z(f) - E \cong \Sigma_2(f)$, in the notation of [16].

To complete the proof of Theorem (0.2), we must study $Z(f) \cap E$. We observe that Propositions (1.7), (1.8), and (1.9) imply that all points of $Z(\Phi_s) \cap E$ are smooth points of $Z(\Phi_s)$, provided that s lies in a suitable dense open subset of S. (Specifically, this open subset is the complement of $q(NS(q))$, where $NS(q)$ is the nonsmooth locus of q.) Since there is a dominant morphism $S \to G(n, n-m-1)$, it is enough to prove Propositions (1.8) and (1.9).

Let us fix a closed point $x \in X$. We choose homogeneous coordinates $T_0, \ldots, T_n$ on $\mathbb{P}^n$ such that $x = (1, 0, \ldots, 0)$ and $t_{X,x}$ is the subspace $T_{r+1} = \cdots = T_n = 0$, where $r = \dim(X)$. Thus, $t_1, \ldots, t_r$ generate the maximal ideal $m_{X,x} \subset \mathcal{O}_{X,x}$, and $t_i \in m_{X,x}^2$, $i = r+1, \ldots, n$. (Here, $t_i = T_i/T_0$ along X.) We also fix a point $z \in E$ that lies above (x, x). Then z corresponds to a line in $t_{X,x}$ that contains x; after a change in coordinates we may assume that this line is $T_2 = \cdots = T_n = 0$.

We have an inclusion $\mathcal{O}_{X,x} \otimes \mathcal{O}_{X,x} \subset A$, where (A, m_A) is the local ring on $(X \times X)'$ at z. To further simplify the notation, we set $t_i = 1 \otimes t_i$ and $s_i = t_i \otimes 1$, $i = 1, \ldots, n$. Then m_A is generated by $\{s_1, t_1, \ldots, t_r,$

$\xi_2, \ldots, \xi_r\}$, where $\xi_i = (s_i - t_i)/(s_1 - t_1)$. (Note that $\xi_2, \ldots, \xi_n$ must vanish at z, because z corresponds to the line $T_2 = \cdots = T_N = 0$.)

LEMMA (2.1): *If* $v \in \mathfrak{m}_{X,x}^d$, *then* $(v \otimes 1 - 1 \otimes v)/(s_1 - t_1) \in \mathfrak{m}_A^{d-1}$. *Moreover*:

(i) $(s_i - t_i)/(s_1 - t_1) = \xi_i$, $i = 2, \ldots, r$,

(ii) $(s_1^2 - t_1^2)/(s_1 - t_1) = s_1 + t_1$,

(iii) $(s_1 s_j - t_1 t_j)/(s_1 - t_1) \equiv s_j \equiv t_j \pmod{\mathfrak{m}_A^2}$,

for $j = 2, \ldots, r$, *and*

(iv) $(s_i s_j - t_i t_j)/(s_1 - t_1) \in \mathfrak{m}_A^2$ *when* $2 \leqq i < j \leqq r$.

PROOF: The first statement is proved by induction on d. If $v = ab$, where $a \in \mathfrak{m}_{X,x}$ and $b \in \mathfrak{m}_{X,x}^{d-1}$, then

$$(ab \otimes 1 - 1 \otimes ab)/(s_1 - t_1) = (a \otimes 1)[(b \otimes 1 - 1 \otimes b)/(s_1 - t_1)] + (1 \otimes b)[(a \otimes 1 - 1 \otimes a)/(s_1 - t_1)].$$

The expression inside the first pair of rectangular brackets lies in $\mathfrak{m}_A^{d-2}$, by the inductive hypothesis. Moreover, $1 \otimes b \in \mathcal{O}_{X,x} \otimes \mathfrak{m}_{X,x}^{d-1} \subset \mathfrak{m}_A^{d-1}$ and similarly $a \otimes 1 \in \mathfrak{m}_A$.

The last two statements follow from the identities

$$(s_i s_j - t_i t_j)/(s_1 - t_1) = (s_i(s_j - t_j) + t_j(s_i - t_i))/(s_1 - t_1) = s_i \xi_j + t_j \xi_i. \qquad \text{Q.E.D.}$$

We also consider closed points $s \in S$ such that $\Phi_s : X \to \mathbb{P}^m$ sends $(1, t_1, \ldots, t_n) \to (\ell_0(t), \ldots, \ell_m(t))$, where $\ell_i(t) = \ell_i(1, t_1, \ldots, t_n) = a_{i0} + a_{i1} t_1 + \cdots + a_{in} t_n$, $i = 1, \ldots, m$, and $\ell_0(t) = 1 + a_{01} t_1 + \cdots + a_{0n} t_n$. Thus, we can regard the a_{ij}, $(i, j) \in \{0, \ldots, m\} \times \{0, \ldots, n\} - \{(0, 0)\}$, as representing coordinates on some affine open subset of $\mathbb{P}^N$. Given a particular point s_0 in this affine open set, such that the point $x = (1, 0, \ldots, 0)$ does not lie in the linear space L_{s_0}, we may change coordinates so that $a_{01}, \ldots, a_{0n}, a_{10}, \ldots, a_{m0}$ all vanish at s_0. In fact, we change coordinates in $\mathbb{P}^n$ so that the hyperplane at infinity contains L_{s_0}, change coordinates in $\mathbb{P}^m$ so that $\Phi_{s_0}(x) = (1, 0, \ldots, 0) \in \mathbb{P}^m$, and then change coordinates in $\mathbb{P}^{(m+1)(n+1)-1}$ as in Lemma 8.3 of [16]. In other words, we may assume that $s_0 \in W$, where W is the intersection of S with the linear subspace $a_{01} = \cdots = a_{0n} = a_{10} = \cdots = a_{m0} = 0$ in $\mathbb{P}^{(m+1)(n+1)-1}$. Hence, properties of $Z_S(\Phi)$ which hold at a point of $\{z\} \times W$ actually hold at all points of $E \times S$.

For $s \in W$, the projection $\Phi_s : X \to \mathbb{P}^m$ sends $(1, t_1, \ldots, t_n) \to$

$(1, \ell_1(t), \ldots, \ell_m(t))$, where $\ell_i(t) = a_{i1}t_1 + \cdots + a_{in}t_n$, $i = 1, \ldots, m$. Let (B, m_B) be the local ring on $(X \times X)' \times W$ at (z, s). The local defining ideal, in B, of $Z_S(\Phi)$ is generated by the elements $(\ell_i(1, s_1, \ldots, s_n) - \ell_i(1, t_1, \ldots, t_n))/(s_1 - t_1)$, $i = 1, \ldots, m$. Lemma (2.1) implies that

$$(\ell_i(1, s_1, \ldots, s_n) - \ell_i(1, t_1, \ldots, t_n))/(s_1 - t_1) \equiv a_{i1} (\mathrm{mod}\ m_A B).$$

(The ring homomorphism $A \to B$ comes from the projection $(X \times X)' \times W \to (X \times X)'$.) Thus, for any $s \in W$, the requirement that our particular point z should lie in $Z(\Phi_s) \cong Z_S(\Phi) \cap (X \times \{s\})$ forces $a_{11}, \ldots, a_{m1}$ to vanish at s. In other words, the above congruences imply that $Z_S(\Phi) \cap (\{z\} \times W)$ is nonsingular and has codimension m in $\{z\} \times W$. Together with Lemma (1.6) and its proof, this implies that $p^{-1}(z) = Z_S(\Phi) \cap (\{z\} \times S)$ is nonsingular and has codimension m in $\{z\} \times S$. This proves Proposition (1.8).

We will now prove Proposition (1.9). We will use all of the notation introduced above. Since $\mathcal{O}_{X,x}/m_{X,x}^3$ is spanned by linear coordinate functions, we may also assume:

(a) $t_{r+1} \equiv t_1^2 (\mathrm{mod}\ m_{X,x}^3)$,

(b) $t_{r+i} \equiv t_1 t_i (\mathrm{mod}\ m_{X,x}^3)$, $i = 2, \ldots, r$, and

(c) the monomials $t_1^2, t_1 t_2, \ldots, t_1 t_r$ do not occur in the expansions of the linear coordinate functions $t_{2r+1}, \ldots, t_n$ modulo $m_{X,x}^3$.

These assumptions and Lemma (2.1) lead to the following congruences modulo $m_A^2 B$:

$$\begin{aligned}(\ell_i(1, s_1, \ldots, s_r) - \ell_i(1, t_1, \ldots, t_r))/(s_1 - t_1) \equiv\ & a_{i1} + a_{i2}\xi_2 + \cdots + a_{ir}\xi_r \\ & + a_{i,r+1}(s_1 + t_1) \\ & + a_{i,r+2}t_2 + \cdots + a_{i,2r}t_r.\end{aligned}$$

for $i = 1, \ldots, m$. The elements on the left sides of these congruences generate the defining ideal, in B, of the relative double point scheme. Thus, as before, the requirement that our particular point z should lie in $Z(\Phi_s)$ forces $a_{11}, \ldots, a_{m1}$ to vanish at s. The additional requirement that z should be a nonsmooth point of $Z(\Phi_s)$ imposes the further condition that the maximal minors of the $m \times (2r - 1)$ matrix

$$(a_{ij})_{1 \leqq i \leqq m, 2 \leqq j \leqq 2r}$$

should all vanish at s. But the elements a_{ij} are all coordinate functions on an affine space. Thus, the locus of common zeros of these maximal minors has codimension $= (2r - 1) - m + 1 = 2r - m$, and we have

$$\begin{aligned}\dim(NS(q) \cap (\{z\} \times W)) &= \dim(W) - m - (2r - m) \\ &= \dim(W) - 2r.\end{aligned}$$

Therefore, $NS(q) \cap (E \times S)$ has codimension $\geqq 2r$ in $E \times S$. Since $\dim(E \times S) = \dim(S) + 2r - 1$, this completes the proof of Proposition (1.9), and also of Theorem (0.2).

REMARK (2.2): Our proof of Proposition (1.9) shows that $(\{z\} \times S) \cap NS(q)$ has dimension $= \dim(S) - 2r$ for every $z \in E$. Moreover, the proof of Theorem 7.6 of [16] implies that $\dim((\{z\} \times S) \cap NS(q)) \leq \dim(S) - 2r - 1$ if $z \in (X \times X)' - E$. By Remark (1.10), we conclude that $\dim(NS(q)) = \dim(S) - 1$. In particular, one cannot rule out the possibility that $E \times S$ contains an irreducible component of $NS(q)$.

3. A different proof

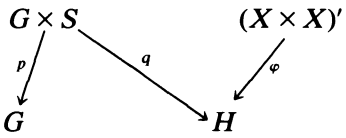
In this section, we give a proof of Theorems (0.1) and (0.2) that uses properties of group actions. The techniques are due to Kleiman.

Let $H = G(n, 1)$ be the Grassmann variety that parameterizes lines in $\mathbb{P}^n$. If $X \subset \mathbb{P}^n$, then the map $X \times X - \Delta \to H$ (which sends (x, y) to the line joining x and y) extends to a morphism $\varphi : (X \times X)' \to H$. If L is a linear subspace of codimension $m + 1$ in $\mathbb{P}^n$, then we have the (locally closed) Schubert subvariety $S \subset H$ consisting of all lines $\lambda \subset \mathbb{P}^n$ such that $\lambda \cap L \neq \emptyset$ and $\lambda \not\subset L$. Then S is smooth. In fact, the nonsmooth locus of the closure $\bar{S}$ is $\{\lambda \mid \lambda \subset L\}$. If $L \cap X = \emptyset$, then we consider the projection from L, specifically $f : X \to \mathbb{P}^m$. The following well-known lemma gives the relationship between S and the double point scheme $Z(f)$.

LEMMA (3.1): *With the notation as above, we have* $Z(f) = \varphi^{-1}(S)$.

We also consider the projective linear group $G = \mathrm{Aut}(\mathbb{P}^n)$. There is a transitive group action $G \times H \to H$, sending $(g, \lambda) \to g\lambda$ = translate of λ by g. The following result is basically a special case of a theorem proved by Kleiman.

PROPOSITION (3.2): *Consider the diagram*



where p is the projection to the first factor and $q(g, \lambda) = g\lambda$.

(i) *If γ is the generic point of G, then $p^{-1}(\gamma) \times_H (X \times X)'$ is either empty or regular and of pure dimension $= 2r - m$.* (*As above,* $r = \dim(X)$.)

(ii) *If $p^{-1}(\gamma) \times_H (X \times X)'$ is geometrically regular (i.e. smooth) over $\kappa(\gamma) = \mathcal{O}_{G,\gamma}$, then there is a dense open subset $U \subset G$ such that $\varphi^{-1}(gS)$ is smooth and of pure dimension $2r - m$ (or else empty) for every closed point $g \in U$.*

In fact, (i) is a special case of the theorem proved in section (2) of [10]. To prove (ii), one begins by showing that $q: G \times S \to H$ is a smooth morphism (in any characteristic). Hence, the projection $(G \times S) \times_H (X \times X)' \to (X \times X)'$ is smooth. It follows that $(G \times S) \times_H (X \times X)'$ is smooth. If $\rho: (G \times S) \times_H (X \times X)' \to G \times S$ is the projection to the first factor, then $(p \circ \rho)^{-1}(g) \cong p^{-1}(g) \times_H (X \times X)'$ for every point (closed or not) of G. Thus the hypothesis of (ii) says that the generic fibre of $p \circ \rho$ is smooth. Therefore, $(p \circ \rho)^{-1}(g)$ is smooth for every point g of some dense open subset $U \subset G$. This proves (ii).

The following theorem is the main result of this section. It clearly implies Theorem (0.1) and Theorem (0.2).

THEOREM (3.3): *Let G, H, S, and $\varphi: (X \times X)' \to H$ be as above. If* $\mathrm{char}(k) = 0$ *or if the embedding $X \subset \mathbb{P}^n$ satisfies the hypotheses of Theorem* (0.2), *then there is a dense open subset $U \subset G$ such that $\varphi^{-1}(gS)$ is smooth and either empty or of pure dimension $2r - m$ (where $r = \dim(X)$) for every closed point $g \in U$.*

PROOF: If $\mathrm{char}(k) = 0$, the result follows immediately from Proposition (3.2) because regularity and geometric regularity are equivalent in this case.

If the embedding of X in $\mathbb{P}^n$ satisfies the hypotheses of Theorem (0.2), we conclude as at the beginning of Section 2 that $Z(f) - E$ is smooth if $f: X \to \mathbb{P}^m$ is a generic projection. Now, for any fixed $\lambda \in H$, we have the (surjective) orbit map $G \times \{\lambda\} \to H$. Thus, we can use Lemma (3.1) to conclude that $\varphi^{-1}(gS) - E$ is smooth for g in a dense open subset of G.

We must also determine whether $\varphi^{-1}(gS) \cap E$ is smooth. The results of [15] and [16] imply that a generic projection $f: X \to \mathbb{P}^m$ has the following properties:

(1) $\bar{S}_i(f)$ has pure codimension $i(m - r + i)$ in X [or is empty] for every $i \geqq 1$;

(2) $S_1(f) = \bar{S}_1(f) - \bar{S}_2(f)$ has pure codimension $m - r + 1$ in X [or is empty] and is smooth if $\mathrm{char}(k) \neq 2$. If $\mathrm{char}(k) = 2$, there are only finitely many nonsmooth points.

(3) If $\mathrm{char}(k) = 2$, $r = 2$, and $m = 3$, then the local homomorphism at a point of $S_1(f)$ has a canonical form exactly like the morphism studied in Example (1.3) above. (For a more precise statement, see Theorem 3 of [15].)

Recall that S_i and $\bar{S}_i$ are defined as in [11]; see the discussion at the end of Section 1.

(The assumptions on the embedding that were used in [16] are stronger than the hypotheses of Theorem (0.2). However, the stronger hypotheses were used only for verifying the smoothness of the higher order singularity subschemes $S_1^{(q)}$. It is not hard to verify that the techniques of [16] do indeed yield a proof of (2), under the hypotheses of Theorem (0.2).)

LEMMA (3.4): *The structural map* $\lambda : \mathbb{P}(\Omega^1_{X/Y}) \to X$ *induces an isomorphism* $\lambda^{-1}(X - \bar{S}_2(f)) \tilde{\to} S_1(f)$.

PROOF: If $x \in X - \bar{S}_2(f)$, then there is an open neighborhood V of x and a surjection $\mathcal{O}_V \to \Omega^1_{X/Y \mid V}$. Thus, λ induces an isomorphism of $\lambda^{-1}(X - \bar{S}_2(f))$ and $\mathrm{Im}(\lambda) \cap (X - \bar{S}_2(f))$. But the existence of a surjection $\mathcal{O}_{X,x} \to \Omega^1_{X/Y,x}$ and standard properties of Fitting ideals imply that $\mathrm{Im}(\lambda) \cap (X - \bar{S}_2(f)) = S_1(f)$. Q.E.D.

Except in the case where $\mathrm{char}(k) = 2$, $r = 2$, and $m = 3$, Lemma (3.4), the isomorphism $Z(f) \cap E \cong \mathbb{P}(\Omega^1_{X/\mathbb{P}^m})$, and property (2) above imply that $Z(f) \cap E$ is smooth except possibly along a closed subset of dimension $\leqq 2r - m - 2$. Since the local defining ideal of the subscheme $Z(f) \subset (X \times X)'$ is generated by m elements at every point, we conclude that $Z(f)$ is smooth except possibly along a closed subset of dimension $\leqq 2r - m - 2$. In the remaining case ($r = 2$, $m = 3$, $\mathrm{char}(k) = 2$), one uses the canonical form of the local homomorphism $\hat{\mathcal{O}}_{\mathbb{P}^3,\pi(x)} \to \hat{\mathcal{O}}_{X,x}$ at a point $x \in S_1(f)$ to verify that $Z(f)$ is smooth in this case.

The results obtained so far imply that the generic translate $W = p^{-1}(\gamma) \times_H (X \times X)'$ is a regular scheme of pure dimension $2r - m$ and that its nonsmooth locus has dimension $\leqq 2r - m - 2$. If we can show that the generic translate is smooth, then the proof will be complete, by (ii) of Proposition (3.2). So assume that W is not smooth. Then there exists a finite algebraic extension K of $\kappa(\gamma)$ such that $W_K = \mathrm{Spec}(K) \times_{\mathrm{Spec}(\kappa(\gamma))} W$ is not a regular scheme. In particular, the base extension morphism $W_K \to W$ is not étale. However, the base extension morphism can fail to be étale only above the nonsmooth locus of W, which has codimension $\geqq 2$ in W. Therefore the Zariski–Nagata

theorem on purity of the branch locus [6, Exposé 8, Théorème 3.4] implies that W is smooth. (In fact, since $W_K \to W$ is finite, the simpler theorem of purity upstairs will suffice. See [2, Theorem 1 and Remark 2].) Q.E.D.

4. $Z(f)$ viewed as a scheme over X

We consider a morphism $f: X \to Y$ of nonsingular varieties over k (algebraically closed, as usual). Let $g: Z(f) \to X$ be the unique morphism which makes the following diagram commutative.

$$\begin{array}{ccc} (X \times X)' & \hookleftarrow & Z(f) \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle g} \\ X \times X & \xrightarrow[p_2]{} & X \end{array}$$

The *singularity subschemes* $S_1^{(q)}(f) \subset X$ are defined exactly as in [16, Section 3]. In particular, the closed points of $S_1^{(q)}(f)$ are all $x \in X - \bar{S}_2(f)$ such that the local ring of $f^{-1}(f(X))$ at x has length $\geqq q+1$. The $S_1^{(q)}(f)$ are a particular type of Thom–Boardman singularity. There are close relationships between the singularity subschemes of g and the singularity subschemes of f. These are described by Theorems (4.3) and (4.5) below. We will also state a couple of elementary propositions about the fibres of g that show why we expected the main results to be true. The first one is due to D. Laksov [12, Proposition 21].

PROPOSITION (4.1): *Let f be as above, and let $\pi_i: X \times_Y X \to X$ be the projection to the i-th factor, $i = 1, 2$. If x is a closed point of X, then π_1 induces isomorphisms $\pi_2^{-1}(x) - \Delta_X \tilde{\to} f^{-1}(f(x)) - \{x\}$ and $g^{-1}(x) - E \tilde{\to} f^{-1}(f(x)) - \{x\}$.*

The fact that the two isomorphisms are equivalent is an immediate consequence of Lemma (1.1).

COROLLARY (4.2): *Let $z \in Z(f)$. If $\pi(z) = (x, y) \in X \times X - \Delta$, then $z \in \bar{S}_i(g)$ if and only if $x \in \bar{S}_i(f)$, and $z \in S_1^{(q)}(g)$ if and only if $x \in S_1^{(q)}(f)$.*

We recall that $\Sigma_2(f; q, 0) = (S_1^{(q)}(f) \times X) \cap ((X \times_Y X) - \Delta) \subset X \times X$.

(See [16, Definition 1.1].) In particular, $\Sigma_2(f; q, 0)$ is an open subscheme of the fibre product $S_1^{(q)}(f) \times_Y X$. Corollary (4.2) says that $\pi : (X \times X)' \to X \times X$ induces a bijection {closed points of $S_1^{(q)}(g) - E$} $\to$ {closed points of $\Sigma_2(f; q, 0)$}.

THEOREM (4.3): *Let $f : X \to Y$ be a morphism of nonsingular varieties, and assume that $Z(f)$ is smooth and of pure dimension = $2 \cdot \dim(X) - \dim(Y)$. Then $\pi : (X \times X)' \to X \times X$ induces an isomorphism $S_1^{(q)}(g) - E \xrightarrow{\sim} \Sigma_2(f; q, 0)$.*

PROOF: As before, let $\pi_i : X \times_Y X \to X$ be the projection to the i-th factor. We must show that $\pi_1^{-1}(S_1^{(q)}(f)) - \Delta = S_1^{(q)}(\pi_2) - \Delta$. As noted above, these two subschemes of $X \times_Y X$ have the same closed points. We must show that their defining ideals coincide.

Let $z \in S_1^{(q)}(g) - E$, and let $(x', x) = \pi(z)$. Set $(A, \mathfrak{m}_A) = \mathcal{O}_{Y,f(x)}$, $(B, \mathfrak{m}_B) = \mathcal{O}_{X,x}$, and $(B', \mathfrak{m}_{B'}) = \mathcal{O}_{X,x'}$. Let C be the local ring on $X \times X$ at (x', x), and let R be the local ring on $X \times_Y X$ at (x', x). Thus $B' \otimes_k B \subset C$; in fact C is a localization of $B' \otimes_k B$. Let $f^* : A \to B$ and $f^{*\prime} : A \to B'$ be induced by f; let $\pi_1^* : B' \to R$ and $\pi_2^* : B \to R$ be induced by π_1 and π_2 respectively.

The ideal in B' which corresponds to the subscheme $S_1^{(q)}(f) \subset X$ is $\Delta_q(P^q_{B'/A})$, the highest nonunit Fitting ideal of the algebra of principal parts $P^q_{B'/A}$. On the other hand, the ideal in R which corresponds to the subscheme $S_1^{(q)}(\pi_2) \subset \Sigma_2$ is $\Delta_q(P^q_{R/B})$. The proof will thus be complete if we can show that $\pi_1^*(\Delta_q(P^q_{B'/A})) \cdot R = \Delta_q(P^q_{R/B})$.

Because R is a localization of $B' \otimes_A B$, Proposition 16.4.5 of [5] implies that $P^q_{R/B} \cong R \otimes_{B'} P^q_{B'/A}$. (In forming this tensor product, we use the left B'-module structure of $P^q_{B'/A}$.) Using this isomorphism and Lemma 2.7 of [16], we obtain the desired equality of ideals. Q.E.D.

We will also study the intersection of the singularity subschemes of g with the exceptional locus $E \subset (X \times X)'$. The following result is an easy consequence of Lemma (1.2).

PROPOSITION (4.4): *Let $z \in Z(f) \cap E$, and let $x = g(z)$ so that $\pi(z) = (x, x) \in \Delta$. If $x \in S_i(f)$ for some $i \geq 2$, then $\dim g^{-1}(x) \cap E = i - 1$. In particular $z \in \bar{S}_{i-1}(g)$. If $x \in S_2(f)$ and $z \notin \bar{S}_2(g)$, then $z \in \bigcap_{q=0}^{\infty} S_1^{(q)}(g)$.*

If $z \in Z(f) \cap E$ and $g(z) \in S_2(f)$, then we can have either $z \in S_1(g)$ or $z \in S_2(g)$. I do not know any useful criterion (in terms of the

singularities of $f: X \to Y$) which distinguishes between these two cases, even when Z is smooth. On the other hand, the following result gives a very detailed description of what happens when $g(z) \notin \bar{S}_2(f)$.

THEOREM (4.5): *Let $f: X \to Y$ be a morphism of nonsingular varieties. Then $g: Z \to X$ induces isomorphisms*

$$g^{-1}(X - \bar{S}_2(f)) \cap E \tilde{\to} S_1(f) \quad \textit{and}$$
$$(S_1^{(q)}(g) - S_1^{(q+1)}(g)) \cap E \tilde{\to} S_1^{(q+1)}(f) - S_1^{(q+2)}(f)$$

for all $q \geqq 1$.

In particular, these isomorphisms are valid when $q = 0$ and $q = 1$. Since $S_1^{(0)}(g) = Z - \bar{S}_2(g)$ and $S_1^{(1)}(g) = S_1(g)$, we see that $(Z - \bar{S}_1(g)) \cap E \tilde{\to} S_1(f) - S_1^{(2)}(f)$ and $(S_1(g) - S_1^{(2)}(g)) \cap E \tilde{\to} S_1^{(2)}(f) - S_1^{(3)}(f)$.

PROOF: The fact that g induces the first isomorphism follows from the isomorphism $Z(f) \cap E \cong \mathbb{P}(\Omega^1_{X/Y})$ (formula (V, 60) of [11]), Lemma (3.4), and Proposition (4.4).

In proving that g induces the second isomorphism, we fix $z \in Z \cap E$ with $z \notin \bigcap_{q=1}^{\infty} S_1^{(q)}(g) \cup \bar{S}_2(g)$. Let $x = g(z)$. Proposition (4.4) implies that $x \in S_1(f)$. We consider the local rings $(A, \mathfrak{m}_A) = \mathcal{O}_{Y,y}$, $(B, \mathfrak{m}_B) = \mathcal{O}_{X,x}$, $(C, \mathfrak{m}_C) = \mathcal{O}_{(X \times X)',z}$, and $(R, \mathfrak{m}_R) = \mathcal{O}_{Z,z}$. There are local homomorphisms $f^*: A \to B$ and $g^*: B \to R$ induced by $f: X \to Y$ and $g: Z \to X$ respectively. The ideal in B (respectively R) which corresponds to the subscheme $S_1^{(q+1)}(f) \subset X$ (resp. $S_1^{(q)}(g) \subset Z$) is the Fitting ideal $\Delta_{q+1}(P^{q+1}_{B/A})$ (resp. $\Delta_q(P^q_{R/B})$).

Let $\bar{I} \subset R$ be the (principal) ideal which corresponds to the subscheme $Z \cap E \subset Z$. We will show that $g^*: B \to R$ induces an isomorphism $B/\Delta_{q+1}(P^{q+1}_{B/A}) \tilde{\to} R/(\bar{I} + \Delta_q(P^q_{R/B}))$. [In fact, $\bar{I} + \Delta_q(P^q_{R/B}) = \bar{I} + g^*(\Delta_{q+1}(P^{q+1}_{B/A})) \cdot R$.] In view of the first isomorphism, this will imply that $(S_1^{(q)}(g) - g^{-1}(\bar{S}(f))) \cap E \tilde{\to} S_1^{(q+1)}(f)$. Since $(S_1^{(q)}(g) - S_1^{(q+1)}(g)) \cap E$ and $g^{-1}(\bar{S}_2(f))$ are disjoint, this implies that g induces the second isomorphism.

Since $x \in S_1(f)$, $\dim \mathfrak{m}_B/(f^*(\mathfrak{m}_A)B + \mathfrak{m}_B^2) = 1$. Thus, we can choose minimal generating sets $\{u_1, \ldots, u_m\} \subset \mathfrak{m}_A$ and $\{t_1, \ldots, t_r\} \subset \mathfrak{m}_B$ such that $f^*(u_i) = t_{i+1}$ for $i = 1, \ldots, r-1$ and $f^*(u_i) \in \mathfrak{m}_B^2$ for $i \geqq r$. We recall that $P^{q+1}_{B/k} = (B \otimes_k B)/\mathcal{L}^{q+2}$, where $\mathcal{L}$ is the kernel of the multiplication map $B \otimes_k B \to B$. We identify $b \in B$ with the element $\overline{b \otimes 1} \in P^{q+1}_{B/k}$, and define $d^{q+1}_{B/k}: B \to P^{q+1}_{B/k}$ by setting $d^{q+1}_{B/k}(b) = \overline{1 \otimes b}$. Now, $P^{q+1}_{B/k}$ is freely generated as B-module by the monomials of degree $\leqq q+1$ in

$\zeta_1, \ldots, \zeta_r$, where $\zeta_i = d^{q+1}_{B/k}(t_i) - t_i$. Let $\varphi_\ell : P^{q+1}_{B/k} \to B$ be the B-linear map such that $\varphi_\ell(\zeta_1^\ell) = 1$ and $\varphi_\ell(\zeta_1^{i_1} \ldots \zeta_r^{i_r}) = 0$ if $(i_1, \ldots, i_r) \neq (\ell, 0, \ldots, 0)$. Then $D^{(\ell)} = \varphi_\ell \circ d^{q+1}_{B/k} : B \to B$ is a differential operator (by definition). Lemma 3.7 and Proposition 3.10 of [16] imply:

$$D^{(\ell)}_{B/k}(t_1^{i_1} \ldots t_r^{i_r}) = \binom{i_1}{\ell} t_1^{i_1 - \ell} t_2^{i_2} \ldots t_r^{i_r}. \tag{4.5.1}$$

[If $\ell > i_1$, we set $\binom{i_1}{\ell} = 0$.]

(4.5.2) $\Delta_{q+1}(P^{q+1}_{B/A})$ is generated by
$\{D^{(\ell)}_{B/k}(f^* u_i) \mid 1 \leqq \ell \leqq q+1 \text{ and } r \leqq i \leqq m\}$.

(4.5.3) If $b_1 \equiv b_2 (\mathrm{mod}\ \mathfrak{m}_B^n)$, then $D^{(\ell)}_{B/k}(b_1) \equiv D^{(\ell)}_{B/k}(b_2) (\mathrm{mod}\ \mathfrak{m}_B^{n-\ell})$.

We have $B \otimes_k B \subset C$, and $(p_2 \circ \pi)^* : B \to C$ identifies B with the subring $k \otimes_k B \subset B \otimes_k B$. We denote $1 \otimes t_i$ by t_i and set $s_i = t_i \otimes 1$, $i = 1, \ldots, r$. Using standard facts about the blowing-up process, we check that $\mathfrak{m}_C = (t_1, \ldots, t_r, s_1, \xi_2, \ldots, \xi_r)$, where $s_i - t_i = \xi_i(s_1 - t_1)$, $i = 2, \ldots, r$. Furthermore, $s_1 - t_1$ generates the ideal $I \subset C$ which corresponds to the subscheme $E \subset (X \times X)'$. The subscheme $Z \subset (X \times X)'$ corresponds to $I^{-1}J$, where $J \subset C$ is generated by $\{f^*(u_i) \otimes 1 - 1 \otimes f^*(u_i) \mid i = 1, \ldots, m\}$. Since $f^*(u_i) = t_{i+1}$, $i = 1, \ldots, r-1$, we have $\xi_i \in I^{-1}J$, $i = 2, \ldots, r$. Hence, there is a commutative diagram of local homomorphisms:

$$\begin{array}{ccc} C & \xrightarrow{\nu} & R_0 = C/(\xi_2, \ldots, \xi_r)C \\ {\scriptstyle (p_2 \circ \pi)^*}\uparrow & \nearrow_{\gamma} & \downarrow{\scriptstyle \nu_0} \\ B & \xrightarrow[g^*]{} & R = C/J \end{array} \tag{4.5.4}$$

where ν and ν_0 are residue class maps, and $\gamma = \nu \circ (p_2 \circ \pi)^*$. Clearly, $\dim(R_0) = r + 1$, and $\mathfrak{m}_{R_0} = (\tau_1, \ldots, \tau_r, \sigma_1)R_0$, where $\tau_i = \nu(t_i)$ and $\sigma_i = \nu(s_i)$. To simplify the notation, we set $w = \sigma_1 - \tau_1$.

LEMMA (4.6): *R_0 is a smooth B-algebra. In particular, $P^q_{R_0/B}$ is free of rank $q+1$ as an R_0-module. The set $\{(d^q_{R_0/B}(w) - w)^i \mid 0 \leqq i \leqq q\}$ is a minimal generating set.*

The verification of this lemma is standard, and we will omit it. For $\ell = 0, \ldots, q$, let $\psi_\ell : P^q_{R_0/B} \to R_0$ be the R_0-linear map such that $\psi_\ell((d^q_{R_0/B}(w) - w)^i) = 1$ (resp. 0) if $i = \ell$) (resp. $i \neq \ell$). We will consider the differential operators $D^{(\ell)}_{R_0/B} = \psi_\ell \circ d^q_{R_0/B} : R_0 \to R_0$. (In particular, $D^{(0)}$ is the identity map.) We also have:

(4.6.1) $$D^{(\ell)}_{R_0/B}(\gamma(b)w^i) = \binom{i}{\ell}\gamma(b)w^{i-\ell} \text{ for every } b \in B, \text{ and}$$

(4.6.2) *if* $r_1 \equiv r_2 (\mathrm{mod}\ m^n_{R_0})$, *then* $D^{(\ell)}_{R_0/B}(r_1) \equiv D^{(\ell)}_{R_0/B}(r_2) (\mathrm{mod}\ m^{n-\ell}_{R_0})$.

In fact, to verify (4.6.1) we write:

$$d^q_{R_0/B}(w^i) = ((d^q_{R_0/B}(w) - w) + w)^i$$
$$= \sum_{j=0}^{i} \binom{i}{j} w^{i-j}(d^q_{R_0/B}(w) - w)^j.$$

and apply the definition of $D^{(\ell)}_{R_0/B}$. (Also, note that $d^q_{R_0/B}$ is B-linear.) As for (4.6.2), it is a standard property of differential operators of order $\leqq \ell$.

LEMMA (4.7): *The elements* $\nu_0(D^{(\ell)}_{R_0/B}(\nu[(f^*u_i \otimes 1 - 1 \otimes f^*u_i)/(s_1 - t_1)]))$, *where* $\ell = 1, \ldots, q$ *and* $i = r, \ldots, m$, *generate the ideal* $\Delta_q(P^q_{R/B}) \subset R$.

LEMMA (4.8): *If* $b \in B$, *then*

$$D^{(\ell)}_{R_0/B}(\nu[(b \otimes 1 - 1 \otimes b)/(s_1 - t_1)]) \equiv \gamma(D^{(\ell+1)}_{B/k}(b)) (\mathrm{mod}(\sigma_1 - \tau_1)R_0)$$

for $\ell = 0, \ldots, q$.

In both lemmas, the notation is as in (4.5.4). We will prove Lemmas (4.7) and (4.8) below, after using them to finish the proof of Theorem (4.5). First of all, recall that $D^{(0)}_{R_0/B}$ is the identity map. Since $(f^*u_i \otimes 1 - 1 \otimes f^*u_i)/(s_1 - t_1) \in I^{-1}J$, we conclude from Lemma (4.8) that $g^*(D^{(1)}_{B/k}(f^*u_i)) \in \nu_0(\sigma_1 - t_1)R$. But $\nu_0(\sigma_1 - t_1)R$ is the ideal $\bar{I} = \nu_0(\nu(I))$ which corresponds to the subscheme $E \cap Z \subset Z$, while $\{g^*(D^{(1)}_{B/k}(f^*u_i)) \mid r \leqq i \leqq m\}$ generates the ideal in R which corresponds to the subscheme $g^{-1}(S_1(f))$. [Note that $D^{(1)}_{B/k}(f^*u_i) = \partial(f^*u_i)/\partial t_1$.] Thus, we recover the identity $g^{-1}(X - \bar{S}_2(f)) \cap E = g^{-1}(S_1(f)) \cap E$.

The proof of Theorem (4.5) is similar. Thus, (4.5.2), Lemma (4.7) and Lemma (4.8) imply that $\bar{I} + \Delta_q(P^q_{R/B}) = \bar{I} + g^*(\Delta_{q+1}(P^{q+1}_{B/A})) \cdot R$. As noted above, this equality of ideals implies Theorem (4.5). Q.E.D.

PROOF OF LEMMA (4.7): As noted above, $P^q_{R_0/B}$ is freely generated as R_0-module by $\{1, \zeta, \ldots, \zeta^q\}$, where $\zeta = d^q_{R_0/B}(w) - w$. Therefore, $R \otimes_{R_0} P^q_{R_0/B}$ is freely generated as R-module by $\{1 \otimes 1, 1 \otimes \zeta, \ldots, 1 \otimes \zeta^q\}$. Furthermore, there is a surjection $\lambda : R \otimes_{R_0} P^q_{R_0/B} \to P^q_{R/B}$ of R-algebras such that $\lambda(r \otimes d^q_{R_0/B}(r_0)) = r \cdot d^q_{R/B}(\nu(r_0))$. (See [5, Proposition 16.4.20]; in forming this tensor product, we use the left structure of $P^q_{R_0/B}$.) Thus, starting from elements $v_i = \Sigma^q_{j=0} r_{ij}(1 \otimes \zeta^j) = \Sigma^q_{j=0} r_{ij} \otimes \zeta^j$, $i = 1, \ldots, N$, which generate $\mathrm{Ker}(\lambda)$ as an R-module, we use the definition of Fitting ideals to show that $\{r_{ij} \mid 0 \leqq i \leqq N$ and $0 \leqq j \leqq q\}$ generates $\Delta_q(P^q_{R/B})$.

The Proposition from [5] cited above also says that $\mathrm{Ker}(\lambda)$ is generated as an ideal in the R-algebra $R \otimes_{R_0} P^q_{R_0/B}$ by the elements $1 \otimes d^q_{R_0/B}(r_0)$, where r_0 ranges through the ideal $\mathrm{Ker}(\nu_0) \subset R_0$. As an R-module generating set we can therefore take

$$\{1 \otimes \zeta^\ell d^q_{R_0/B}(\nu[(f^*u_i \otimes 1 - 1 \otimes f^*u_i)/(s_1 - t_1)]) \mid 0 \leqq \ell \leqq q \text{ and } r \leqq i \leqq m\}.$$

[Note that J is generated by the elements $(f^*u_i \otimes 1 - 1 \otimes f^*u_i)/(s_1 - t_1)$, where $1 \leqq i \leqq m$.] For $i = 1, \ldots, r-1$, $\nu((f^*u_i \otimes 1 - 1 \otimes f^*u_i)/(s_1 - t_1)) = \nu(\xi_{i+1}) = 0$. Hence $\mathrm{Ker}(\nu_0)$ is generated by $\{\nu((f^*u_i \otimes 1 - 1 \otimes f^*u_i)/(s_1 - t_1)) \mid r \leqq i \leqq m\}$.] Furthermore,

$$1 \otimes \zeta^\ell \cdot d^q_{R_0/B}(r_0) = 1 \otimes \sum_{j=0}^{q-\ell} D^{(j)}(r_0) \cdot \zeta^{\ell+j} = \sum_{j=0}^{q-\ell} \nu_0(D^{(j)}(r_0)) \otimes \zeta^{\ell+j}$$

for any $r_0 \in R_0$. (In limiting the range of summation, note that $\zeta^{q+1} = 0$.) Applying this with $r_0 = \nu((f^*u_i \otimes 1 - 1 \otimes f^*u_i)/(s_1 - t_1))$, we show that $\Delta_q(P^q_{R/B})$ is generated by the elements listed in the conclusion of Lemma (4.7). Q.E.D.

PROOF OF LEMMA (4.8): We first consider the case where b is of the form $t_1^{i_1} \ldots t_r^{i_r}$. For any $i > 0$ we have the Taylor expansion

$$s_1^i - t_1^i = \sum_{j=1}^{i} \binom{i}{j} t_1^{i-j}(s_1 - t_1)^j = \sum_{j=1}^{i} D^{(j)}_{B/k}(t_1^i)(s_1 - t_1)^j.$$

in C. We can divide by $s_1 - t_1$ to get

$$(s_1^i - t_1^i)/(s_1 - t_1) = \sum_{j=0}^{i-1} D^{(j+1)}_{B/k}(t_1^i)(s_1 - t_1)^j.$$

Since $\sigma_i = \tau_i$ in R_0 for $i = 2, \ldots, r$ and $\sigma_1 - \tau_1 = w$ (by definition), we have

$$(\sigma_1^{i_1} \dots \sigma_r^{i_r} - \tau_1^{i_1} \dots \tau^{i_r})/(\sigma_1 - \tau_1) = [(\sigma_1^{i_1} - \tau_1^{i_1})/(\sigma_1 - \tau_1)] \cdot \tau_2^{i_2} \dots \tau_r^{i_r}$$
$$= [(\sigma_1^{i_1} - \tau_1^{i_1})/(\sigma_1 - \tau_1)]\gamma(t_2^{i_2} \dots t_r^{i_r}).$$

All of this leads to the identity

$$(\sigma_1^{i_1} \dots \sigma_r^{i_r} - \tau_1^{i_1} \dots \tau_r^{i_r})/(\sigma_1 - \tau_1) = \gamma(t_2^{i_2} \dots t_r^{i_r}) \cdot \sum_{j=0}^{i_1-1} \gamma(D_{B/k}^{(j+1)}(t_1^{i_1})) \cdot w^j$$
$$= \sum_{j=0}^{i_1-1} \gamma(D_{B/k}^{(j+1)}(t_1^{i_1} \dots t_r^{i_r})) \cdot w^j$$

in R_0, by (4.5.1). Referring to the commutative diagram (4.5.4), we see that this can be written as:

$$\nu[(b \otimes 1 - 1 \otimes b)/(s_1 - t_1)] = \sum_{j=0}^{i_1-1} \gamma(D_{B/k}^{(j+1)}(b)) \cdot w^j.$$

where $b = t_1^{i_1} \dots t_r^{i_r}$.

Hence, (4.6.1) implies:

(4.8.1)
$$D_{R_0/B}^{(\ell)}(\nu((b \otimes 1 - 1 \otimes b)/(s_1 - t_1)])$$
$$= \sum_{j=\ell}^{i_1-1} \binom{j}{\ell} \gamma(D_{B/k}^{(j+1)}(b)) \cdot w^{j-\ell},$$

where $b = t_1^{i_1} \dots t_1^{i_r}$. In particular, this gives:

(4.8.2) $$D_{R_0/B}^{(\ell)}(\nu[(b \otimes 1 - 1 \otimes b)/(s_1 - t_1)]) \equiv \gamma(D_{B/k}^{(\ell+1)}(b))(\mathrm{mod}(w)R_0)$$

whenever b is a monomial in $t_1, \dots, t_r$.

Given an integer $n \geqq \ell + 1$, an arbitrary element $b \in B$ is congruent modulo $\boldsymbol{m}_B^n$ to a k-linear combination of monomials in $t_1, \dots, t_r$. Therefore Lemma (2.1), (4.5.3), and (4.6.2) imply that we can "replace" (4.8.1) by the congruence:

(4.8.3)
$$D_{R_0/B}^{(\ell)}(\nu[(b \otimes 1 - 1 \otimes b)/(s_1 - t_1)])$$
$$\equiv \sum_{j=\ell}^{n-1} \binom{j}{\ell} \gamma(D_{B/k}^{(j+1)}(b)) \cdot w^{j-\ell} (\mathrm{mod}\ \boldsymbol{m}_{R_0}^{n-\ell-1})$$

for every $b \in B$. Since this holds for every $n > 0$, we find that the congruence (4.8.2) holds for *every* $b \in B$. Since $w = \sigma_1 - \tau_1$, this is equivalent to the conclusion of Lemma (4.8). Q.E.D.

REMARK (4.9): It might seem surprising that we do not assume that $Z(f)$ is smooth in Theorem (4.5). In algebraic terms, we did not assume that $R = \mathcal{O}_{Z,z}$, in the diagram (4.5.4), is a smooth k-algebra. The reason is that R_0 is a smooth k-algebra (and even a smooth B-algebra), and is possible to regard $P^q_{R/B}$ as a quotient of the free R-module $R \otimes_{R_0} P^q_{R_0/B}$ in calculating the Fitting ideal $\Delta_q(P^q_{R/B})$. Thus, there is no reason to assume that $P^q_{R/k}$ is a free R-module.

EXAMPLE (4.10). Let $f: \mathbb{A}^2 \to \mathbb{A}^2$ be a map of the form $(t_1, t_2) \to (t_1, \varphi(t_1, t_2))$. Thus, $\varphi = a_0 + a_1 t_2 + \cdots + a_n t_2^n$, where $a_i \in k[t_1]$, $i = 0, \ldots, n$. We write $\mathbb{A}^2 \times \mathbb{A}^2 = \operatorname{Spec} k[s_1, s_2, t_1, t_2]$. Let $U = \operatorname{Spec} k[s_2, t_1, t_2, \xi]$, where $\xi = (s_1 - t_1)/(s_2 - t_2)$. Using the methods of Example (1.3), we show that $Z(f)$ is the closed subscheme of U defined by the equations $\xi = 0$ and

$$(*) \qquad (\varphi(s_1, s_2) - \varphi(t_1, t_2))/(s_1 - t_1) = 0.$$

Let V be the subvariety of U where $\xi = 0$. Then $Z(f)$ is the subscheme of V where (*) holds. Since $s_1 = t_1$ along V, it is easy to check that $Z(f)$ is defined as a subscheme of V by $\psi(s_2, t_1, t_2) = 0$, where

$$(**) \qquad \begin{aligned} \psi(s_2, t_1, t_2) &= \sum_{i=1}^{n} a_i(t_1) \cdot (s_2^i - t_2^i)/(s_2 - t_2) \\ &= \sum_{i=0}^{n-1} D^{(i+1)}\varphi(t_1, t_2) \cdot (s_2 - t_2)^i. \end{aligned}$$

Here, the $D^{(j)}\varphi$ are divided partial derivatives with respect to t_2. Thus $D^{(j)}(a_i(t_1)t_2^i) = \binom{i}{j} a_i(t_1) t_2^{i-j}$. Moreover, $V = \operatorname{Spec} k[s_2, t_1, t_2] = \mathbb{A}^1 \times \mathbb{A}^2$, and $p: Z(f) \to \mathbb{A}^2$ is induced by the projection $\mathbb{A}^1 \times \mathbb{A}^2 \to \mathbb{A}^2$. We also note that $V \cap E$ is the subvariety of V where $s_2 = t_2$. Thus, the projection induces an isomorphism $V \cap E \xrightarrow{\sim} \mathbb{A}^2$.

It is easily checked (see [16, Lemma 3.8]) that $S^{(3)}_1(f)$ is the subscheme of $\mathbb{A}^2 = \operatorname{Spec} k[t_1, t_2]$ defined by $D^{(1)}\varphi = D^{(2)}\varphi = D^{(3)}\varphi = 0$. On the other hand, let $B = k[t_1, t_2]$, $R_0 = k[s_2, t_1, t_2]$, and $R = \Gamma(Z(f), \mathcal{O}_{Z(f)})$. Then $P^{(2)}_{R_0/B}$ is freely generated as an R_0-module by $\{(d^2_{B/A}(w) - w)^i \mid i = 0, 1, 2\}$, where $w = s_2 - t_2$. There is a surjection $\lambda: R \otimes_{R_0} P^2_{R_0/B} \to P^2_{R/B}$ such that $\operatorname{Ker}(\lambda)$ is the ideal in the R-algebra $R \otimes_{R_0} P^2_{R_0/B}$ generated by

$$1 \otimes d^2_{R_0/B}\psi = \nu(\mathcal{D}^{(1)}\psi) \otimes (d^2_{R_0/B}(w) - w) + \nu(\mathcal{D}^{(2)}\psi) \otimes (d^2_{R_0/B}(w) - w)^2.$$

(See [5, Proposition 16.4.20].) Here, $\nu: R_0 \to R = R_0/(\psi)R_0$ is the resi-

due class map, and $\mathscr{D}^{(j)}$ denotes the divided partial derivative with respect to $s_2 - t_2$. Thus, $\mathscr{D}^{(j)}(h(t_1, t_2) \cdot (s_2 - t_2)^i) = \binom{i}{j} h(t_1, t_2) \cdot (s_2 - t_2)^{i-j}$. The reader can use these facts to verify that the ideal $\Delta_2(P^2_{B/A})$, which defines $S^{(2)}_1(g)$ as a subscheme of $Z(f) = \mathrm{Spec}(R)$, is generated by $\nu(\mathscr{D}^{(1)}\psi)$ and $\nu(\mathscr{D}^{(2)}\psi)$. Thus, $S^{(2)}_1(g)$ is the subscheme of V where $\psi = \mathscr{D}^{(1)}\psi = \mathscr{D}^{(2)}\psi = 0$. Furthermore, (**) implies

$$(***) \qquad \begin{aligned} \mathscr{D}^{(1)}\psi &= \sum_{i=1}^{n-1} iD^{(i+1)}\varphi(t_1, t_2) \cdot (s_2 - t_2)^{i-1}, \quad \text{and} \\ \mathscr{D}^{(2)}\psi &= \sum_{i=2}^{n-1} \binom{i}{2} D^{(i+1)}\varphi(t_1, t_2) \cdot (s_2 - t_2)^{i-2}. \end{aligned}$$

In describing $S^{(2)}_1(g) \cap E$, we only need the initial terms. Thus, by putting (**) and (***) together, we see that $S^{(2)}_1(g) \cap E \tilde{\to} S^{(3)}_1(f)$. Similarly, $S^{(q)}_1(g) \cap E \tilde{\to} S^{(q+1)}_1(f)$ for all $q \geqq 0$. Since $\bar{S}_2(f) = \emptyset$, this is equivalent to the conclusion of Theorem (4.5) for this particular type of map.

5. Application to an enumerative question

As usual, $f: X \to Y$ is a morphism of nonsingular projective varieties; $Z = Z(f)$ and $g: Z \to X$ are defined as in sections 1 and 4 respectively. Laksov and Fulton gave a formula for the rational equivalence class of the *double point cycle* $[M_2(f)] = p_{2*}\pi_*[Z]$, where $[Z]$ is the cycle associated to Z. (See [12], [4], and [11, Chapter V].) In [11, Chapter V], Kleiman also obtained a formula for the rational equivalence class of the *triple point cycle* $[M_3] = g_*[M_2(g)]$, under the assumption that Z is smooth.

In this section, we obtain a formula for the rational equivalence class of $[S^{(2)}_1(f)]$. For simplicity, we assume that Z is smooth over k, and we work with the rational equivalence rings $\mathscr{A}(X)$, $\mathscr{A}(Z)$, etc. However, some generalization should be possible. (See the footnote on p. 385 of [11].) The formalism here and in Section 6 is based on the formalism of [11, Chapter V, Sections C and D], supplemented by Lemma (5.1), Lemma (5.3), and Proposition (5.5).

If E and F are locally free sheaves, the formal difference $E - F$ is called a *virtual bundle.* Its *total Chern class* is defined to be $c(E - F) = c(E)s(F)$, where $c(E) = 1 + c_1(E) + \cdots$ is the total Chern class of E and $s(F)$ is the *total Segre class* of F, i.e. the formal inverse of the total Chern class: $s(F) = c(F)^{-1}$. The component of $c(E - F)$ in $\mathscr{A}_i(X)$ is the *i-th Chern class* of $E - F$:

$$c_i(E-F) = c_i(E) + c_{i-1}(E)s_1(F) + \cdots + s_i(F).$$

If E, F, and G are locally free sheaves, then it is clear that $c(E-G) = c(E-F)c(F-G)$. The *rank* of $E-F$ is defined to be $\operatorname{rank}(E) - \operatorname{rank}(F)$. If L is an invertible sheaf on X, we set $(E-F)\otimes L = E\otimes L - F\otimes L$.

LEMMA (5.1): *Let the notation be as above. Then*

$$c_p((E-F)\otimes L) = \sum_{j=0}^{p} \binom{d-p+j}{j} c_{p-j}(E-F)c_1(L)^j$$

for every $p \geqq 0$, *where* $d = \operatorname{rank}(E-F)$.

In this lemma, the case $E=0$ is a formula for $s_p(F\otimes L)$, which one proves by using the splitting principle and induction on $\operatorname{rank}(F)$. After doing this, one proves the general case by using the splitting principle and induction on $\operatorname{rank}(E)$.

Given $f: X \to Y$ as above, the *virtual normal bundle* of f is defined to be $\nu_f = \nu_{X/Y} = f^*\tau_Y - \tau_X$, where τ_X and τ_Y are the tangent bundles on X and Y respectively. The following lemma is a special case of Porteous' formula. (See [9, Corollary 11].)

LEMMA (5.2): *Let* $r = \dim(X)$ *and* $m = \dim(Y)$. *If* $S_1(f)$ *is empty or of pure codimension* $m-r+1$ *in* X, *then* $[S_1(f)] = c_{m-r+1}(\nu_f)$. *If* $S_2(f)$ *is empty or of pure dimension* $2(m-r+2)$ *in* X, *then* $[S_2(f)] = c_{m-r+2}(\nu_f)^2 - c_{m-r+1}(\nu_f)c_{m-r+3}(\nu_f)$.

LEMMA (5.3): *Let* $\pi: (X\times X)' \to X\times X$ *be obtained by blowing up the diagonal, let* E *be the exceptional locus, and let* $p_i: X\times X \to X$ *be projection to the* i*-th factor. Then*

$$c(\nu_\pi) = (1+e)^{-1}c(\pi^*p_2^*\tau_X)s(I\otimes \pi^*p_2^*\tau_X)$$

in $\mathcal{A}((X\times X)')$, *where* $e = [E]$ *and* $I = \mathcal{O}_{(X\times X)'}(-E)$ *is the sheaf of ideals defining the divisor* E. *Furthermore*

$$c(\nu_{p_2\pi}) = (1+e)^{-1}s(\pi^*p_1^*\tau_X)c(\pi^*p_2^*\tau_X)s(I\otimes \pi^*p_2^*\tau_X).$$

PROOF: The second formula follows from the first one because

$$c(\nu_{(X\times X)'/X}) = c(\pi^*p_2^*\tau_X - \pi^*\tau_{X\times X})c(\pi^*\tau_{X\times X} - \tau_{(X\times X)'})$$

$$= s(\pi^* p_1^* \tau_X) c(\nu_\pi).$$

To prove the first formula, we recall that there is a commutative diagram

$$\begin{array}{ccc} \mathbb{P}(\Omega^1_X) \cong E & \xrightarrow{j} & (X \times X)' \\ \rho \downarrow & & \downarrow \pi \\ X & \xrightarrow{\delta} & X \times X \end{array}$$

and use the following two exact sequences (See [14] and [22].)

(a) an exact sequence of locally free sheaves on E:

$$o \to \mathcal{O}_{E/X}(-1) \to \rho^* \tau_X \to \mathcal{F} \to 0$$

where $\mathcal{O}_{E/X}(1)$ is the canonical quotient line bundle on $E \cong \mathbb{P}(\Omega^1_X)$.

(b) an exact sequence of sheaves on $(X \times X)'$:

$$0 \to \tau_{(X \times X)'} \to \pi^* \tau_{X \times X} \to j_* \mathcal{F} \to 0.$$

From (b) we obtain

(i) $$c(\nu_\pi) = c(j_* \mathcal{F}).$$

Since $j: E \to (X \times X)'$ is an embedding, j_* preserves exactness, so that there is an exact sequence:

(a′) $$0 \to j_* \mathcal{O}_{E/X}(-1) \to j_* \rho^* \tau_X \to j_* \mathcal{F} \to 0.$$

Furthermore, basic properties of blowing up imply that $j_* \mathcal{O}_{E/X}(-1) = \mathcal{O}_E(E)$ fits into an exact sequence:

(c) $$0 \to \mathcal{O}_{(X \times X)'} \to \mathcal{O}_{(X \times X)'}(E) \to \mathcal{O}_E(E) \to 0.$$

The last two exact sequences yield the formula

(ii) $$c(j_* \mathcal{F}) = c(j_* \rho^* \tau_X) s(\mathcal{O}_E(E)) = (1 + e)^{-1} c(j_* \rho^* \tau_X).$$

To evaluate the last factor, we write $\rho^* \tau_X = \rho^* \delta^* p_2^* \tau_X = j^* \pi^* p_2^* \tau_X$, so that $j_* \rho^* \tau_X = \mathcal{O}_E \otimes \pi^* p_2^* \tau_X$. This implies

(iii) $$c(j_* \rho^* \tau_X) = c(\pi^* p_2^* \tau_X) s(I \otimes \pi^* p_2^* \tau_X).$$

The conclusion of the lemma follows directly from (i), (ii), and (iii).

LEMMA (5.4), [11, formula (V, 64)]: *Let $f: X^r \to Y^m$ be a morphism of nonsingular projective varieties, let $Z = Z(f)$, and let $\theta: Z \to (X \times X)'$ be the inclusion map. If Z is smooth and of pure dimension $2r-m$, then $\nu_\theta = I \otimes g^* f^* \tau_Y = \theta^*(I \otimes \pi^* p_2^* f^* \tau_Y)$, where $g = p_2 \circ \pi \circ \theta: Z \to X$.*

As an immediate consequence of the last two lemmas, we obtain:

PROPOSITION (5.5): *Let the assumptions and notation be as in Lemma (5.4). Then*

$$c(\nu_g) = \theta^*((1+e)^{-1} c(\pi^* p_2^* \tau_X - \pi^* p_1^* \tau_X) c(I \otimes \pi^* p_2^* \nu_f))$$

in $\mathcal{A}(Z)$, where $e = [E] \in \mathcal{A}((X \times X)')$ and $I = \mathcal{O}_{(X\times X)'}(-E)$.

THEOREM (5.6): *Let $f: X^r \to Y^m$ be a morphism of nonsingular projective varieties, where $r \leqq m$. Assume that $Z(f)$ is empty or smooth and of pure dimension $2r-m$, that $S_1^{(2)}(f)$ is empty or of pure codimension $2(m-r+1)$ in X, that $\Sigma_2(f; 1, 0)$ is empty or of pure codimension $2m-r+1$ in $X \times X$ and that $S_i(f)$ is of dimension $\leq 3r-2m-i-2$ for all $i \geqq 2$. Then the rational equivalence class of $S_1^{(2)}(f)$ satisfies*

$$[S_1^{(2)}(f)] = c_{m-r+1}(\nu_f)^2 + \sum_{i=1}^{m-r+1} 2^{i-1} c_{m-r+1+i}(\nu_f) c_{m-r+1-i}(\nu_f).$$

REMARK: In the case $k = \mathbf{C}$, this formula is due to F. Ronga [20, p. 33]. Thus the same formula holds in all characteristics, at least for sufficiently generic maps. The proofs given by Ronga and by Lascoux [13] depend on an iterative definition of $S_1^{(2)}(f)$ which is not the correct definition in the case $\mathrm{char}(k) = 2$. (The point is that if $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) > q$, then one can use the methods of [16] to check that $S_1^{(q)}(f) \cong S_1(f')$, where $f': S_1^{(q-1)}(f) \to Y$ is the restriction of f.)

PROOF: Let $g = p_2 \circ \pi \circ \theta: Z \to X$ as before. We have assumed that $\dim S_1^{(2)}(f) = r - 2(m-r+1) = 3r-2m-2$ and that $\dim \Sigma_2(f; 1, 0) = 2r-(2m-r+1) = 3r-2m-1$. Thus, Theorem (4.3) and Theorem (4.5) imply that all irreducible components of $S_1(g) \cap E$ (resp. $S_1(g)$) have dimension $= 3r-2m-2$ (resp. $3r-2m-1$) unless some irreducible component of $S_1(g) \cap E$ is contained in $g^{-1}(\bar{S}_i(f)) \cap E$ for

some $i \geqq 2$. However every irreducible component of $S_1(g) \cap E$ has dimension $\geqq 3r - 2m - 2 = 2\dim(Z) - \dim(X) - 2$ by [16, Proposition 2.8], while $\dim g^{-1}(S_i(f) \cap E) \leqq 3r - 2m - 3$ by Proposition 4.4 and our assumption about $\dim S_i(f)$. Therefore this last possibility is ruled out. (Observe that our assumption about $\dim S_i(f)$ holds if the irreducible components of $S_i(f)$ have the minimum possible dimension, viz. $r - i(m - r + i)$.)

We conclude that the codimensions in Z of $S_1(g) \cap E$ and of $S_1(g)$ are $m - r + 2 = \dim(X) - \dim(Z) + 2$ and $m - r + 1 = \dim(X) - \dim(Z) + 1$ respectively. Thus, Lemma (5.2) implies that $[S_1(g)] = c_{m-r+1}(\nu_g)$ in $\mathscr{A}(Z)$. (Observe that $\dim(X) - \dim(Z) = m - r$.) Moreover, $S_1(g)$ and E intersect properly. Another application of Theorem (4.5) implies that

$$[S_1^{(2)}(f)] = g_*([S_1(g)]\theta^*[E]) = g_*(c_{m-r+1}(\nu_g)\theta^*[E]).$$

Hence, we can complete the proof by calculating the right hand side of this equation.

Now $[E]c(\pi^*p_2^*\tau_X - \pi^*p_1^*\tau_X) = [E]$ in $\mathscr{A}((X \times X)')$, so that Proposition (5.5) yields

$$c(\nu_g)\theta^*(e) = \theta^*(e(1+e)^{-1}c(I \otimes \pi^*p_2^*\nu_f)).$$

Consider the diagram:

(*)
$$\begin{array}{ccccc} \mathbf{P}(\Omega^1_X) \cong E & \xhookrightarrow{j} & (X \times X)' & \xhookleftarrow{\theta} & Z \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle g} \\ X & \xhookrightarrow{\delta} & X \times X & \xrightarrow{p_2} & X \end{array}$$

Applying the projection formula to θ and then to j, we obtain:

$$\begin{aligned} g_*(c(\nu_g)\theta^*(e)) &= p_{2*}\pi_*\theta_*\theta^*(e(1+e)^{-1}c(I \otimes \pi^*p_2^*\nu_f)) \\ &= p_{2*}\pi_*([Z]e(1+e)^{-1}c(I \otimes \pi^*p_2^*\nu_f)) \\ &= p_{2*}\pi_*j_*j^*([Z](1+e)^{-1}c(I \otimes \pi^*p_2^*\nu_f)) \\ &= \rho_*j^*([Z](1+e)^{-1}c(I \otimes \pi^*p_2^*\nu_f)). \end{aligned}$$

By [11, formula (V, 60)], $Z \cap E \cong \mathbf{P}(\Omega^1_{X/Y})$. This implies

$$j^*[Z] = \sum_{i=0}^{m} c_{m-i}(\rho^*f^*\tau_Y)\xi^i,$$

where $\xi = c_1(\mathcal{O}_{E/X}(1))$, or equivalently:

$$j^*[Z] = c_m((\rho^* f^* \tau_Y)(1)).$$

All of this gives:

$$g_*(c(\nu_g)\theta^*(e)) = \rho_*((1-\xi)^{-1} c_m((\rho^* f^* \tau_Y)(1)) c((\rho^* \nu_f)(1))),$$

where $F(1)$ means $F \otimes \mathcal{O}_{E/X}(1)$. (Note, in particular, that $j^*(I) = \mathcal{O}_{X/E}(1)$. Thus $j^*(e) = -\xi$.) A similar computation yields the following lemma.

LEMMA (5.6.1): *Assume that Z is smooth and of pure dimension $2r-m$. If $i_1, \ldots, i_d$ are positive integers, then*

$$g_*(c_{i_1}(\nu_g) \ldots c_{i_d}(\nu_g)\theta^*(e)) = \rho_*(c_m((\rho^* f^* \tau_Y)(1)) a_{i_1} \ldots a_{i_d}),$$

where $a_k = c_k((\rho^* \nu_f)(1) - \mathcal{O}_{E/X}(-1))$.

The following lemmas will also be useful. In all of these lemmas, the notation is as in the diagram (*).

LEMMA (5.6.2): *If $j \geqq 0$ and $\xi = c_1(\mathcal{O}_{E/X}(1))$, then*

$$\rho_*(c_m((\rho^* f^* \tau_Y)(1)) \xi^j) = c_{m-r+1+j}(\nu_f).$$

LEMMA (5.6.3): *If a_k is defined as above, then*

$$\rho_*(c_m((\rho^* f^* \tau_Y)(1)) a_k) = \sum_{i=0}^{k} \sum_{j=0}^{i} \binom{m-r-k+i}{j} c_{m-r+1+i}(\nu_f) c_{k-i}(\nu_f),$$

for every $k \geqq 0$.

We see immediately that Theorem (5.6) is a consequence of Lemma (5.6.1) and Lemma (5.6.3). In particular, we take $k = m-r+1$ in Lemma (5.6.3) and note that $\Sigma_{j=0}^{i} \binom{i-1}{j} = 1$ (resp. 2^{i-1}) if $i = 0$ (resp. $i \geqq 1$). Thus, it only remains to prove the last two lemmas.

PROOF OF LEMMA (5.6.2): We consider the map $\rho_*: \mathcal{A}(\mathbf{P}(\Omega_X^1)) \to \mathcal{A}(X)$. It is well known that $\rho_*(\xi^i) = 0$ for $i \leqq r-2$ while $\rho_*(\xi^i) = s_{i-r+1}(\tau_X)$ for $i \geqq r-1$. As a special case of Lemma (5.1) we have $c_m((\rho^* f^* \tau_Y)(1)) = \Sigma_{i=0}^{m} c_{m-i}(\rho^* f^* \tau_Y) \xi^i$. Thus we can obtain the desired

identity by multiplying by ξ^j and applying the projection formula. Q.E.D.

PROOF OF LEMMA (5.6.3): By definition, we have $a_k = \Sigma_{\lambda=0}^{k} c_{k-\lambda}((\rho^*\nu_f)(1))\xi^\lambda$. Furthermore,

$$c_{k-\lambda}((\rho^*\nu_f)(1)) = \sum_{j=0}^{k-\lambda} \binom{n-k+\lambda+j}{j} c_{k-\lambda-j}(\rho^*\nu_f)\xi^j$$

by Lemma (5.1), where $n = m - r$. These equations yield the identity

$$\begin{aligned} a_k &= \sum_{j+\lambda \leqq k} \binom{n-k+\lambda+j}{j} c_{k-\lambda-j}(\rho^*\nu_f)\xi^{\lambda+j} \\ &= \sum_{i=0}^{k} \sum_{j=0}^{i} \binom{n-k+i}{j} c_{k-i}(\rho^*\nu_f)\xi^i. \end{aligned}$$

To finish the proof, we multiply by $c_m((\rho^*f^*\tau_Y)(1))$ and apply Lemma (5.6.2) and the projection formula. Q.E.D.

If $\dim(Y) = \dim(X)$, then the formula in Theorem (5.6) becomes $[S_1^{(2)}(f)] = c_1(\nu_f)^2 + c_2(\nu_f)$. In particular, if $\dim(X) = r$ and $Y = \mathbb{P}^r$, let h be the class of a hyperplane in $\mathscr{A}(\mathbb{P}^r)$ and let $c_i(X)$ and $s_i(X)$ denote $c_i(\tau_X)$ and $s_i(\tau_X)$ respectively. In this case:

$$\begin{aligned} (5.7) \qquad [S_1^{(2)}(f)] &= \tfrac{1}{2}(r+1)(3r+2)f^*(h^2) + 3(r+1)s_1(X)f^*(h) \\ &\quad + s_1(X)^2 + s_2(X), \end{aligned}$$

where $s_1(X) = -c_1(X)$ *and* $s_2(X) = c_1(X)^2 - c_2(X)$.

If X is a suitably embedded nonsingular surface and $f: X \to \mathbb{P}^2$ is a generic projection, then $S_1^{(2)}(f)$ is a finite set of points. The number of points is the degree of the cycle $S_1^{(2)}(f)$. Each point is counted with multiplicity $= 1$ if $\mathrm{char}(k) \neq 3$. (To check this, use Theorem 4.1 of [16].) If $\mathrm{char}(k) = 3$, then each point is counted with a higher multiplicity, probably $= 3$. If $\mathrm{char}(k) \neq 2, 3$, then the points of $S_1^{(2)}(f)$ are mapped to ordinary cusps of the branch curve $f(S_1(f)) \subset \mathbb{P}^2$. Our formula for the number of points in $S_1^{(2)}(f)$ agrees with Iversen's formula for the number of cusps of $f(S_1(f))$. (See [8, p. 968].

If $\dim(Y) = \dim(X) + 1$, then the formula becomes $[S_1^{(2)}(f)] = c_2(\nu_f)^2 + c_1(\nu_f)c_3(\nu_f) + 2c_4(\nu_f)$. If $\dim(X) = r$ and $Y = \mathbb{P}^{r+1}$, then we use the same notation as above and obtain:

$$\begin{aligned} [S_1^{(2)}(f)] &= \tfrac{1}{2}(r+1)^3(r+2)f^*(h^4) + (r+1)(r+2)(2r+3)s_1(X)f^*(h^3) \\ (5.8) \qquad &\quad + (\tfrac{1}{2}(r+2)(3r+5)s_1(X)^2 + (r+2)(3r+4)s_2(X))f^*(h^2) \\ &\quad + 3(r+2)(s_1(X)s_2(X) + s_3(X))f^*(h) + s_2(X)^2 + s_1(X)s_3(X) + 2s_4(X), \end{aligned}$$

where $s_3(X) = -c_1(X)^3 + 2c_1(X)c_2(X) - c_3(X)$ *and*
$s_4(X) = c_1(X)^4 - 3c_1(X)^2c_2(X) + 2c_1(X)c_3(X) + c_2(X)^2 - c_4(X)$.

If X is a suitably embedded 4-dimensional variety and $f: X \to \mathbb{P}^5$ is a generic projection, then $S_1^{(2)}(f)$ is a finite set of points. If $\text{char}(k) \neq 3$, then each one occurs with multiplicity 1 in the cycle $S_1^{(2)}(f)$.

REMARKS (5.7), (5.8), (5.9): Let $Z = Z(f)$ and $g: Z \to X$ be as above. If g satisfies the hypotheses of Theorem (5.6), then we have a formula for $[S_1^{(2)}(g)]$ in $\mathscr{A}(Z)$. If $S_1^{(2)}(g)$ also intersects E properly, then one can conclude that $[S_1^{(3)}(f)] = g_*([S_1^{(2)}(g)]\theta^*(e))$ (cf. Theorem (4.5)) and get a formula for $[S_1^{(3)}(f)]$ that is valid over a ground field of any characteristic. Thus, if $\dim(Y) = m = r = \dim(X)$, then $\dim(Z) = \dim(X)$, and we have $[S_1^{(3)}(f)] = g_*((c_1(\nu_g)^2 + c_2(\nu_g))\theta^*(e))$ if the hypotheses are all satisfied. This leads to:

$$[S_1^{(3)}(f)] = c_1(\nu_f)^3 + 3c_1(\nu_f)c_2(\nu_f) + 2c_3(\nu_f),$$

which agrees with known formulas when $\text{char}(k) = 0$. (See [13] or [21]). However, when $\dim(Y) \geqq \dim(X) + 1$, this procedure does not always work! The difficulty seems to be related to the fact that $S_1^{(2)}(g)$ and $S_1^{(2)}(g) \cap E$ may not have the correct dimension, even if $S_1^{(3)}(f)$ has the correct dimension. Suppose, for example, that $S_2(f)$ is nonempty and has the correct (i.e. minimum possible) dimension $(= r - 2(m - r + 2))$ and that $g^{-1}(S_2(f)) \cap E \not\subset S_2(g)$. (If $m = r + 1$, the hypothesis on $S_2(f)$ implies that $r \geqq 6$.) Proposition (4.4) implies that $\dim(S_1^{(2)}(g) \cap E) = 3r - 2m - 3$. If $m \geqq r + 1$, this is larger than the correct dimension of this intersection $(= 4r - 3m - 3)$.

6. The stationary locus

As usual, we set $Z = Z(f)$ and let $g: Z \to X$ be defined as in section 4. We define the *stationary locus* of f, denoted N_2 or $N_2(f)$, to be the subscheme $g(S_1(g)) \subset X$. (If f maps X birationally onto its image $X' = f(X) \subset Y$, the points of $f(N_2)$ are sometimes called stationary points of X'. See [19, p. 1].) After establishing some basic local properties of N_2, we will give a formula for the rational equivalence class $[N_2]$ of the cycle associated to N_2.

First of all, if x and y are distinct points of X such that $f(x) = f(y)$ and $x \in S_1(f)$, then Corollary (4.2) implies that $y \in N_2(f)$.

PROPOSITION (6.1): *Let $f: X \to Y$ be a morphism of nonsingular varieties, let $q \geqq 0$, and let $f': S_1^{(q)}(f) \to Y$ and $g': S_1^{(q)}(g) \to X$ be the restrictions of f and of $g: Z(f) \to X$ respectively. If x is a closed point of X, then there is an isomorphism $g'^{-1}(x) - E \tilde{\to} f'^{-1}(f(x)) - \{x\}$.*

PROOF: Let $\Sigma = \Sigma_2(f; q, 0)$. By definition, Σ is the inverse image of Δ_Y under the morphism $S_1^{(q)}(f) \times X - \Delta \to Y \times Y$ induced by $f \times f$. More generally, let T be an arbitrary subscheme of X, let D be the inverse image $h^{-1}(\Delta_Y)$, where $h: T \times X - \Delta_X \to Y \times Y$ is induced by $f \times f$, and let $p: D \to X$ be induced by projection of $T \times X$ to the second factor. An easy modification of Laksov's proof of [12, Proposition 21] shows that projection to the first factor induces an isomorphism $p^{-1}(x) \to f'^{-1}(f(x)) - \{x\}$ where $f' = f|_T : T \to Y$. (Observe that $p^{-1}(x) = (T \times \{x\}) \cap D = h'^{-1}(\Delta_Y)$, where h' is the restriction of h to $T \times \{x\} - \{(x, x)\}$.) Applying this in the case $T = S_1^{(q)}(f)$ and using Theorem (4.3), we obtain the result. Q.E.D.

PROPOSITION (6.2): *If $Z = Z(f)$ is smooth, if $g: Z \to X$ is finite, and if $Z \to g(Z) = M_2$ is birational, then the points of N_2 are precisely the pinch points of M_2.*

Let W be a variety whose normalization is nonsingular. We define a *pinch point* of W to be a point $x \in W$ such that one (or more) of the branches of W at x is singular. In other words, $\hat{\mathcal{O}}_{W,x}/P$ is not a regular local ring for some minimal prime ideal $P \subset \hat{\mathcal{O}}_{W,x}$. Thus, Proposition (6.2) is an immediate consequence of [15, Proposition 3].

The case where $x \notin S_1^{(q+1)}(f)$ may be the most interesting case of Proposition (6.1). In any event, one can consider an irreducible component V of $S_1^{(q)}(g)$ such that $V \not\subset E$. Proposition (6.1) provides a way to determine whether g' maps V birationally onto its image.

THEOREM (6.3): *Let $f: X^r \to Y^m$ be a morphism of nonsingular projective varieties, and assume that $Z = Z(f)$ is smooth and of pure dimension $2r - m$. If $N = N_2(f)$ is empty or of pure codimension $2(m-r)+1$ in X, if $\dim S_i(r) \leqq 3r - 2m - i - 1$ for all $i \geqq 2$, and if $g: Z \to X$ maps $S_1(g)$ birationally onto its image, then*

$$[N_2] = f^* f_* c_{m-r+1}(\nu_f) - 2c_{m-r}(\nu_f) c_{m-r+1}(\nu_f) - \sum_{j=1}^{m-r} 2^j c_{m-r-j}(\nu_f) c_{m-r+1+j}(\nu_f)$$

in $\mathcal{A}(X)$.

PROOF: As in the proof of Theorem (5.6) we show that every irreducible component of $S_1(g)$ has dimension $3r-2m-1$. This implies that $S_1(g)$, if nonempty, has the "correct" codimension in Z, namely $m-r+1=\dim(X)-\dim(Z)+1$. Thus, Lemma (5.2) implies that $[S_1(g)]=c_{m-r+1}(\nu_g)$ and therefore $[N_2]=g_*c_{m-r+1}(\nu_g)$.

In evaluating this expression, we will use the methods and notation of Section 5. First of all, $(1+e)^{-1}=1-e(1+e)^{-1}$ so that

$$\text{(i)}\qquad c_{m-r+1}(\nu_g)=\theta^*c_{m-r+1}(\pi^*p_2^*\tau_X-\pi^*p_1^*\tau_X+I\otimes\pi^*p_2^*\nu_f)-c_{m-r}(\nu_g)\theta^*(e),$$

by Proposition (5.5). Now $c_1(I)=-e$ and $c_i(\pi^*p_2^*\tau_X-\pi^*p_1^*\tau_X)e=0$ for $i\geqq 1$. (Apply the projection formula to $j:E\to(X\times X)'$ and observe that $j^*\pi^*p_\ell^*F=\rho^*F$, $\ell=1,2$, for any coherent sheaf F on X.) Therefore, Lemma (5.1) implies

$$\begin{aligned}&c_{m-r+1}(\pi^*p_2^*\tau_X-\pi^*p_1^*\tau_X+I\otimes\pi^*p_2^*\nu_f)\\&=c_{m-r+1}(\pi^*p_2^*\tau_X-\pi^*p_1^*\tau_X+\pi^*p_2^*\nu_f)\\&=c_{m-r+1}(\pi^*p_2^*f^*\tau_Y-\pi^*p_1^*\tau_X).\end{aligned}$$

Together with (i), this gives:

$$\text{(ii)}\qquad c_{m-r+1}(\nu_g)=\theta^*c_{m-r+1}(\pi^*p_2^*f^*\pi_Y-\pi^*p_1^*\tau_X)-c_{m-r}(\nu_g)\theta^*(e).$$

By Lemma (5.6.1) and Lemma (5.6.3) we have:

$$\text{(iii)}\qquad g_*c_{m-r}(\nu_g)\theta^*(e)=\sum_{j=0}^{m-r}2^jc_{m-r+1+j}(\nu_f)c_{m-r-j}(\nu_f).$$

LEMMA (6.3.1): *If* $a\in\mathcal{A}(X)$, *then*

$$p_{2*}\pi_*([Z]\pi^*p_1^*a)=f^*f_*(a)-ac_{m-r}(\nu_f).$$

PROOF: By [11, formula (V, 56)], we have

$$[Z]=\pi^*(f\times f)^*[\Delta_Y]-j_*\sum_{\ell=1}^{m}c_{m-\ell}(\rho^*f^*\tau_Y)\xi^{\ell-1}.$$

Moreover, [11, formula (V, 78)] gives

$$p_{2*}\pi_*(\pi^*(f\times f)^*[\Delta_Y]\pi^*p_1^*(a))=f^*f_*(a),$$

which explains the first term. As for the second term, we apply the projection formula to j and then to ρ and obtain

$$\begin{aligned}
&p_{2*}\pi_*\left(j_*\left(\sum_{\ell=1}^{m} c_{m-\ell}(\rho^* f^* \tau_Y)\xi^{\ell-1}\right)\pi^* p_1^*(a)\right)\\
&= \rho_*\left(\sum_{\ell=1}^{m} c_{m-\ell}(\rho^* f^* \tau_Y)\xi^{\ell-1} j^* \pi^* p_1^*(a)\right)\\
&= \rho_*\left(\sum_{\ell=1}^{m} \rho^*(c_{m-\ell}(f^*\tau_Y)a)\xi^{\ell-1}\right)\\
&= \sum_{\ell=1}^{m} c_{m-\ell}(f^*\tau_Y)\rho_*(\xi^{\ell-1})a\\
&= c_{m-r}(\nu_f)\cdot a.
\end{aligned}$$

Q.E.D.

We now apply the projection formula to $\theta: Z \to (X \times X)'$ and to $p_2\pi$. This gives:

$$\begin{aligned}
&g_*\theta^* c_{m-r+1}(\pi^* p_2^* f^* \tau_Y - \pi^* p_1^* \tau_X)\\
&= p_{2*}\pi_*([Z] c_{m-r+1}(\pi^* p_2^* f^* \tau_Y - \pi^* p_1^* \tau_X))\\
&= p_{2*}\pi_*\left([Z]\sum_{i=0}^{m-r+1} \pi^* p_2^* c_{m-r+1-i}(f^*\tau_Y)\pi^* p_1^* s_i(\tau_X)\right)\\
&= \sum_{i=0}^{m-r+1} c_{m-r+1-i}(f^*\tau_Y) p_{2*}\pi_*([Z]\pi^* p_1^* s_i(\tau_X)).
\end{aligned}$$

Thus, we can apply Lemma (6.3.1) to show that

$$\begin{aligned}
g_*\theta^* c_{m-r+1}(\pi^* p_2^* f^* \tau_Y - \pi^* p_1^* \tau_X) = &\sum_{i=0}^{m-r+1} c_{m-r+1-i}(f^*\tau_Y) f^* f_* s_i(\tau_X)\\
&- c_{m-r+1}(\nu_f) c_{m-r}(\nu_f).
\end{aligned}$$

Applying the projection formula (this time to $f: X \to Y$) we can replace this expression with:

$$\begin{aligned}
\text{(iv)} g_*\theta^* c_{m-r+1}(\pi^* p_2^* f^* \tau_Y - \pi^* p_1^* \tau_X)\\
= f^* f_* c_{m-r+1}(\nu_f) - c_{m-r}(\nu_f) c_{m-r+1}(\nu_f).
\end{aligned}$$

Putting (ii), (iii), and (iv) together, we obtain the desired formula.

Q.E.D.

When $m = r + 1$, Theorem (6.3) gives the formula

$$[N_2] = f^* f_* c_2(\nu_f) - 2c_1(\nu_f)c_2(\nu_f) - 2c_3(\nu_f).$$

If $Y = \mathbb{P}^{r+1}$ and $h \in \mathcal{A}(\mathbb{P}^{r+1})$ is the class of a hyperplane, then

$$f_*(af^*(h^j)) = \deg(f_*(a))h^{i+j+1}$$

for any $a \in \mathcal{A}_i(X)$. The degree is calculated in $\mathcal{A}(\mathbb{P}^{r+1})$. Thus $f_*(a) \in \mathcal{A}_{i+1}(\mathbb{P}^{r+1})$ and $\deg(f_*(a))$ is the degree of the zero-dimensional class $f_*(a)h^{r-i} \in \mathcal{A}_{r+1}(\mathbb{P}^{r+1})$. Thus, we obtain:

$$\begin{aligned}(6.4)\quad [N_2] = &\left\{\binom{r+2}{2} \deg f_*[X] + (r+2)\deg f_*(s_1(X))\right.\\ &\left. + \deg f_* s_2(X)\right\} f^*(h^3)\\ &-\tfrac{2}{3}(r+1)(r+2)(2r+3)f^*(h^3) - 2(r+2)(2r+3)s_1(X)f^*(h^2)\\ &-2(r+2)(s_1(X)^2 + 2s_2(X))f^*(h) - 2(s_1(X)s_2(X) + s_3(X)),\end{aligned}$$

where the notation is as in (5.7) *and* (5.8).

When $r = 3$, a generic map $f: X \to \mathbb{P}^4$ has finitely many stationary points. σ = degree $[N_2]$ is the number of these, counted with suitable multiplicities. One could also express the number of stationary points in terms of the *elementary projective characters* of $f(X) \subset \mathbb{P}^4$, defined as in [19, p. 2]. This does not seem to have been done, but Roth has studied relations between σ and other projective characters [18, Section 5] and gave a formula for the number of stationary points in the case where $f(X)$ is ruled by planes. (See [17].) We will now study this case.

EXAMPLE (6.5): Let C be a nonsingular projective curve of genus g and let $X = \mathbb{P}(F)$, where F is a locally free sheaf of rank 3 on C. Suppose that $f: X \to \mathbb{P}^4$ maps X birationally onto $f(X) \subset \mathbb{P}^4$ and sends the fibres of the structural map $\varphi: X = \mathbb{P}(F) \to C$ onto planes in $\mathbb{P}^4$.

If we arrange things as in [11, Chapter III, Section B], we may assume that $\mathcal{O}_{X/C}(1) \cong f^*\mathcal{O}_{\mathbb{P}^4}(1)$, where $\mathcal{O}_{X/C}(1)$ is the tautological quotient line bundle of $\varphi^*(F)$ on $X = \mathbb{P}(F)$. Then the basic structural equation of the $\mathcal{A}(C)$-algebra $\mathcal{A}(X)$ is

$$f^*(h^3) = ef^*(h^2),$$

where $e = \varphi^* c_1(F)$. Moreover there is an exact sequence:

$$0 \to \mathcal{O}_X \to \varphi^*(F^*) \otimes \mathcal{O}(1) \to \nu_{X/C} \to 0.$$

[See [1, p. 11].) Since $\varphi: X \to C$ is smooth, $\nu_{X/C}$ is a bundle and $c(X) = (1-\gamma)c(\nu_{X/C})$, where $\gamma = \varphi^*(K_C)$. Thus:

$$c_1(X) = 3f^*(h) - e - \gamma$$
$$c_2(X) = 3f^*(h^2) - (2e + 3\gamma)f^*(h)$$
$$c_3(X) = -3\gamma f^*(h^2).$$

Noting that $\deg f_*(\gamma) = 2g - 2$, we obtain:

$$[N_2] = (2d - 10 + 2(2g-2))f^*(h^3) - 12\gamma f^*(h)$$

where $d = \deg f_*[X]$. Setting $\sigma = \deg[N_2]$, we obtain

$$\sigma = 2d^2 - 10d + (2d-12)(2g-2)$$

or equivalently

$$\sigma = 2d(a-d) - 12(a-d) + 2d,$$

where $a = 2g - 2 + 2d$ is the *class* of a plane section of X. This agrees with Roth's result [17, formula (12)].

REMARK (6.6): The methods of Sections 5 and 6 can be used to verify the triple point formula given in [11, footnote on p. 389]. In particular, Lemma (5.3) and Proposition (5.5) are useful. I have been informed that Kleiman has independently obtained results similar to Lemma (5.3) and Proposition (5.5). (His knowledge of the aforementioned triple point formula before he knew about my work would seem to indisputably corroborate that information.) I understand that Kleiman's results will appear in a forthcoming work entitled "Multiple point formulas".

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Department of Mathematics
University of Minnesota
Minneapolis, Minnesota 55455, USA