STEVEN L. KLEIMAN

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RELATIVE DUALITY FOR QUASI-COHERENT SHEAVES

Steven L. Kleiman

Introduction

In practice it is useful to know when the relative duality map

$$D^m : \underline{\mathrm{Ext}}_f^m(F, \omega_f \otimes_Y N) \to \underline{\mathrm{Hom}}_Y(R^{r-m}f_*F, N)$$

is an isomorphism, where $f: X \to Y$ is a flat, locally projective, finitely presentable map whose fibers $X(y)$ are pure r-dimensional, F is a quasi-coherent sheaf on X, and N is one on Y, and where $\underline{\mathrm{Ext}}_f^m$ denotes the mth derived function of $f_* \underline{\mathrm{Hom}}_X$. (The notion $\underline{\mathrm{Ext}}_f$ was used by Grothendieck in his Bourbaki talk on the Hilbert scheme, no. 221, p. 4, May 1961, and it may have originated there.) Theorem (21) below deals with the following criterion: For f flat and lfp (locally finitely presentable), D^m is an isomorphism for all N if and only if $R^{r-m}f_*F$ commutes with base-change. It is proved that the criterion is valid for $0 \leq m \leq r$ if and only if the $X(y)$ are Cohen-Macaulay. Below, moreover, all the preliminary duality theory is developed from the beginning.

General Grothendieck duality theory ([10]; see also [4,I,2, pp. 161–2]) seems to require more to yield that D^m is an isomorphism for all N, namely, that all the R^qf_*F be locally free. (Thus, for example, this article would be a sounder reference than [4] to appeal to in the proof of the lemma on p. 28 of 'Enriques' classification of surfaces in char. p. Part II' by Bombieri and Mumford in *Complex Analysis and Algebraic Geometry*, Cambridge University Press (1977).) Moreover, the approach below is simpler and more naive than that of the general theory; it is largely Grothendieck's original approach [6], but with some modifications inspired by Deligne's work [10, Appendix], and it

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does not involve the derived category. While some preliminary results can be derived from the general theory, there is no advantage in doing so; little or no more work is required in obtaining them from scratch.

In general, a dualizing sheaf ω_f does not exist, only a left-exact functor $f^!N$ from quasi-coherent sheaves N on Y to those on X. By definition, ω_f exists when $f^!N$ has a tensor-product form, $f^!N = \omega_f \otimes N$. (The form is required to be preserved by open restriction.) For example, ω_f exists when Y is the spectrum of a field because every sheaf on Y is free. The functor $f^!N$ is accompanied by a map of functors $t_fN : R^rf_*f^!N \to N$ and the pair $(f^!, t_f)$ is a right adjoint to R^rf_*.

Locally, X may be viewed as a closed Y-subscheme of a suitable projective space $P = \mathbb{P}^e_Y$ and then there is a canonical isomorphism,

$$f^!N = \underline{\mathrm{Ext}}^{e-r}_P(O_X, O_P(-e-1) \otimes_Y N).$$

This isomorphism could be made the basis of a local construction of $f^!N$. However, it is slicker and more general to use the 'special adjoint functor' theorem [12, Thm. 2, p. 125]. This is done below in Part I. The method was inspired by Deligne's work [10, Appendix], although Deligne's actual construction is different. It was, in turn, inspired by Verdier's work [15] and it requires Y to be locally noetherian so that there are enough quasi-coherent injectives.

Part I also treats of the basic properties of a 'dualizing pair' $(f^!, t_f)$, including notably some results about base-change. One such result (Thm. (5, i)) asserts that the formation of $(f^!, t_f)$ commutes with flat base-change (Thm. (5, i)). This result is similar to Verdier's main theorem [16, Thm. 2, p. 394], but it is much easier to prove. It is used mostly for open embeddings and permits in many situations a reduction to the case of an affine base-change. Another result (Thm. (5, iii)) deals with that case of an affine base-change $g : Y' \to Y$; it asserts the formula,

$$(f_{Y'})^!N' = (f^!g_*N')^\sim.$$

Finally, a third result (Prop. (9)) asserts that if a dualizing sheaf ω_f exists, then it is flat, it continues to exist after any base-change, and its formation commutes with base-change.

Part II contains the main result (Thm. (20)), the criterion for D^m to be an isomorphism. For m in the restricted range $0 \le m \le n$ for $n < r$, the condition that the $X(y)$ be Cohen-Macaulay is too strong; the appropriate condition is, as one would hope, the vanishing of $H^{r-q}(O_{X(y)}(-p))$ for $q = 1, \ldots, n$ and $p \ge p_0(y)$, or equivalently, the vanishing of certain Ext's. The proof involves in an essential way nearly everything before it in both Parts I and II. Another result (Thm. (15)) in Part II gives the rather useful formula,

$$g^!N = (\underline{\mathrm{Ext}}_X^{r-s}(h_*O_Z, f^!N))^{\sim};$$

here $h: Z \to X$ is a finite and finitely presentable map such that the composition $g = f \circ h: Z \to Y$ is flat with pure s-dimensional fibers, and X/Y is required to satisfy an appropriate condition.

A third result (Prop. (22)) in Part II asserts that if the $X(y)$ are Cohen-Macaulay, then the restriction $\omega_f \mid V$, where $V \subset X$ is the smooth locus of f, is canonically isomorphic to the sheaf $\det(\Omega^1_{V/Y})$. The proof is an adaptation of Verdier's neat proof of a similar result [16, Thm. 3, p. 397]; it takes advantage of the flexibility of relative duality theory and does not involve a prior computation of $\det(\Omega^1)$ on projective space. In Example (25), relative duality is used to derive from the infinitesimal study of the Hilbert scheme a generalization of Mattuck's formula for Ω^1 of the symmetric product of a smooth curve. The article closes with a brief discussion in Remark (26) of some refinements.

Acknowledgements

Allen Altman inspired this work. He insisted that the time was right for it, and he helped discover the main result, the criterion for D^m to be an isomorphism. As fate would have it, he did not join in the subsequent workout.

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I. Dualizing pairs

Fix a proper, finitely presentable morphism of schemes, $f: X \to Y$. Fix an integer r such that the condition, $\dim(X(y)) \leq r$, holds for all $y \in Y$, where $X(y)$ denotes the fiber $X \otimes k(y)$ over y.

DEFINITION (1): An *r-dualizing pair* $(f^!, t_f)$ consists of a covariant functor $f^!$ from the category of quasi-coherent sheaves on Y to the category of quasi-coherent sheaves on X and a map of functors,

$t_f:(R'f_*)f^! \to \mathrm{id}$, inducing a (bifunctorial) isomorphism of (quasi-coherent) sheaves on Y,

$$f_*\underline{\mathrm{Hom}}_X(F, f^!N) \xrightarrow{\sim} \underline{\mathrm{Hom}}_Y(R'f_*F, N), \tag{1.1}$$

for each quasi-coherent sheaf F on X and each quasi-coherent sheaf N on Y.

PROPOSITION (2): *Let* $(f^!, t_f)$ *be an r-dualizing pair.*

(i) t_f *makes* $f^!$ *a right adjoint to* $R'f_*$; *that is,* t_f *induces a (bifunctorial) isomorphism of groups,*

$$\mathrm{Hom}_X(F, f^!N) \xrightarrow{\sim} \mathrm{Hom}_Y(R'f_*F, N). \tag{2.1}$$

(ii) $(f^!, t_f)$ *is determined up to unique isomorphism; in fact, for a fixed N the pair* $(f^!N, t_fN)$ *is determined up to unique isomorphism by* (1.1) *or by* (2.1) *with F variable.*

(iii) $f^!$ *is left exact and it commutes with arbitrary (small) inverse limits.*

(iv) $f^!$ *commutes with (small) pseudo-filtered direct limits.*

PROOF: Assertion (i) comes by taking global sections in (1.1). Assertions (ii) and (iii) are well-known, simple formal consequences of (i) [EGA 0_{I}, 1.5, pp. 38–41].

To prove (iv) let (N_α) be a pseudo-filtered direct system of quasi-coherent sheaves on Y, and consider the canonical map u from $\varinjlim f^!N_\alpha$ into $f^! \varinjlim N_\alpha$. We have to prove u is an isomorphism. The question is local, so we may assume Y is affine. Then X is quasi-compact and separated. Now, u will be an isomorphism if the induced map $\mathrm{Hom}_X(F, u)$ is an isomorphism for each quasi-coherent sheaf F on X. In fact, since a quasi-coherent sheaf on a quasi-compact and quasi-separated scheme is a (filtered) direct limit of lfp (locally finitely presentable) sheaves [EGA I, 6.9.12, p. 320], we may take F to be lfp. Then $\mathrm{Hom}_X(F, -)$ commutes with pseudo-filtered direct limits because $\underline{\mathrm{Hom}}_X(F, -)$ clearly does and f_* does. (To obtain the commuting of f_* and limits, it is easy to adapt the proof of [5, Thm. 3.10.1, p. 162].) Hence in view of (2.1) it suffices to prove that the canonical map,

$$\varinjlim \mathrm{Hom}_Y(R'f_*F, N_\alpha) \to \mathrm{Hom}_Y(R'f_*F, \varinjlim N_\alpha),$$

is an isomorphism. It is an isomorphism, however, because $R'f_*F$ is lfp by (3, iv) below.

LEMMA (3): *Let F be a quasi-coherent sheaf on X.*
(i) $R^q f_* F = 0$ *holds for* $q > r$.
(ii) $R^r f_* F$ *is right exact in F.*
(iii) $R^r f_* F$ *commutes with base-change.*
(iv) $R^r f_* F$ *is lfp (locally finitely presentable) if F is.*

PROOF: The questions are local on Y, so we may assume Y is affine.

Assertion (i) obviously implies (ii). If (i) and so (ii) hold, since $F \otimes_Y N$ is quasi-coherent for any quasi-coherent N on Y, then $R^q f_*(F \otimes N)$ is right exact in N for $q \geq r$; hence the following assertion, which includes (iii), also holds:

(iii′) $R^q f_* F$ commutes with base-change for $q \geq r$.

For (i) as well as (iv), we may assume F is lfp. Indeed, since Y is affine, X is quasi-compact and separated; so F is a filtered direct limit of lfp sheaves F_α by [EGA I, 6.9.12, p. 320], and $R^q f_* F$ is equal to $\varinjlim R^q f_* F_\alpha$ for any q. (For the latter, the proof of [5, Thm. 4.12.1, p. 194] is easily adapted.)

There exist a noetherian scheme Y_0, a proper map $f_0 : X_0 \to Y_0$, an lfp sheaf F_0 on X_0, and a map $Y \to Y_0$ such that f and F come from f_0 and F_0 by base-change [EGA IV_3, 8]. Since Y_0 is noetherian, (i) and (iv) hold for f_0 and F_0 by [EGA III_1, 4.2.2, p. 130, and 3.2.1, p. 116]. Hence (iii′) holds for f_0 and F_0. Therefore (i) and (iv) hold for f and F.

THEOREM (4) (Existence): *An r-dualizing pair exists.*

PROOF: Suppose first that Y is affine. We now construct a right adjoint to $R^r f_*$ by verifying the hypotheses of the 'special adjoint functor' theorem [12, Thm. 2, p. 125, in dual formulation]. First, the category of quasi-coherent sheaves on any scheme has arbitrary (small) direct limits [EGA I, 2.2.2, iv, p. 217] and it obviously has small hom-sets. Next, $R^r f_*$ preserves all (small) direct limits and all pushouts of families of epimorphisms because it is right-exact by (3, ii) and because it preserves all (small) filtered direct limits since f is quasi-compact and quasi-separated.

Finally, we have to find a small generating set Q for the category of quasi-coherent sheaves on X. Let Q be the (small) set of all quasi-coherent quotients of all the sheaves of the form $\bigoplus_U (O_U^{\oplus n(U)})$ where U runs through all the open subsets of X, O_U denotes the extension by zero of the restriction $O_X \mid U$, and $n(U)$ is a positive integer depending on U. Obviously, every lfg (locally finitely generated) quasi-coherent sheaf on X is isomorphic to one in Q. Now, every quasi-coherent sheaf is a direct limit of its lfg quasi-coherent subsheaves [EGA I, 6.9.9, p. 319] because X is quasi-compact and quasi-separated as Y is affine. Therefore, Q is a small generating set, as desired.

We now have a pair $(f^!, t_f)$ such that (2.1) holds for each quasi-coherent F and N. If F is lfp, then (1.1) also holds for each quasi-coherent N by [EGA I, 1.3.12, ii, p. 202] because Y is affine and $R^r f_* F$ is lfp (3, iv). Therefore, (1.1) holds for each quasi-coherent F and N, because every quasi-coherent sheaf on X is a filtered direct limit of lfp sheaves [EGA I, 6.9.12, p. 320] and $R^r f_*$, so both sides of (1.1), preserve such limits, since f and X are quasi-compact and quasi-separated. Thus an r-dualizing pair exists when Y is affine.

In general, Y has a basis consisting of its affine open subschemes U, and we now prove that the r-dualizing pairs for the various restrictions $f \mid U$ patch together. Let N be a quasi-coherent sheaf on Y. To prove that the various pairs of sheaves $(f \mid U)^!(N \mid U)$ and maps $t_{f \mid U}(N \mid \mathrm{U})$ define a global sheaf $f^!N$ and global map $t_f N$, for which, obviously, (1.1) will automatically hold, it clearly suffices to show that whenever U contains V then the restriction of the pair for U is canonically isomorphic to the pair for V. To show this, by uniqueness (2, ii) it suffices to prove that the restriction of $t_{f|U}(N \mid U)$ induces an isomorphism,

$$(f \mid V)_* \, \underline{\mathrm{Hom}}_{f^{-1}V}(G, (f \mid U)^!(N \mid U) \,|\, f^{-1}V) \xrightarrow{\sim} \underline{\mathrm{Hom}}_V(R^r(f \mid V)_* G, N \mid V),$$

for each quasi-coherent sheaf G on $f^{-1}V$. Now, this would-be isomorphism is the restriction of the isomorphism,

$$(f \mid U)_* \, \underline{\mathrm{Hom}}_{f^{-1}U}(j_* G, (f \mid U)^!(N \mid U)) \xrightarrow{\sim} \underline{\mathrm{Hom}}_U(R^r(f \mid U)_* j_* G, N \mid U),$$

where $j : f^{-1}V \to f^{-1}U$ is the inclusion. Note that j is affine, so $j_* G$ is quasi-coherent.

THEOREM (5) (Behavior under base-change): *Consider a cartesian square,*

(5.1)
$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\uparrow & \square & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array} .$$

Let $(f^!, t_f)$ and $(f'^!, t_{f'})$ be r-dualizing pairs for f and f'. Then the following formulas (or functorial, canonical isomorphisms) in the *quasi-coherent sheaves N on Y and N′ on Y′ hold under the prescribed conditions on g*:

(i) *if g is flat,*

$$f'^! g^* N = g'^* f^! N, \quad t_{f'} g^* N = g^* t_f N.$$

(ii) *if g is quasi-compact and quasi-separated,*

$$g'_* f'^! N' = f^! g_* N', \quad (g_* t_{f'} N') \circ (b f'^! N') = t_f g_* N',$$

where $b : R^r f_* g'_* \to g_* R^r f'_*$ *is the canonical map. (It is the composition of the edge homomorphisms of the two Leray spectral sequences.)*

(iii) *if g is affine,*

$$f'^! N' = (f^! g_* N')^\sim, \quad t_{f'} N' = (t_f g_* N')^\sim.$$

Proof: (i) If g is an open immersion, then the assertion follows directly from the construction of a dualizing pair, particularly the last paragraph of the proof of (4), and from the uniqueness (2, ii).

In general, by uniqueness (2, ii) it suffices to prove that the map $g^* t_f N$ induces an isomorphism,

$$f_*' \underline{\mathrm{Hom}}_{X'}(F', g'^* f^! N) \xrightarrow{\sim} \underline{\mathrm{Hom}}_{Y'}(R^r f'_* F', g^* N), \tag{5.2}$$

for each quasi-coherent sheaf F' on X'. The question is local on Y and Y', and the formation of $(f^! N, t_f N)$ commutes with open restriction by the preceding paragraph. Hence, we may assume Y and Y' are affine.

Suppose first that F' has the form $g'^* F$ where F is a quasi-coherent sheaf on X. Then (5.2) is the pullback of the isomorphism (1.1), because Hom's and higher direct images commute with flat base-change [EGA III_1, 1.14.15, p. 92].

It now suffices to find a presentation of the form,

$$g'^* H \to g'^* G \to F' \to 0,$$

where G and H are quasi-coherent sheaves on X, because both sides of (5.1) are left exact in F' since $R^r f'_*$ is right exact (3, ii). Set $G = g'_* F'$. It is quasi-coherent and the canonical map $\sigma : g'^* G \to F'$ is surjective in view of [EGA I, 1.7.1, p. 213] because g' is affine. Since $K = \mathrm{Ker}(\sigma)$ is quasi-coherent, similarly we may set $H = g'_* K$.

(ii) By uniqueness (2, ii) it suffices to prove that $(g_* t_{f'} N') \circ (bf'^! N')$ induces an isomorphism,

$$\mathrm{Hom}_X(F, g'_* f'^! N') \xrightarrow{\sim} \mathrm{Hom}_Y(R^r f_* F, g_* N'),$$

for each quasi-coherent sheaf F on X. By adjunction therefore it suffices to prove that the composition,

$$\mathrm{Hom}_{X'}(g'^* F, f'^! N') \xrightarrow{u} \mathrm{Hom}_{Y'}(R^r f'_* g'^* F, N') \xrightarrow{v} \mathrm{Hom}_{Y'}(g^* R^r f_* F, N')$$

is an isomorphism, where u is induced by $t_{f'} N'$ and v is induced by the adjoint $b^\#$. Now, u is an isomorphism by (2, i) and v is an isomorphism because $b^\#$ is by (3, iii).

(iii) When g is affine, (iii) and (ii) are equivalent because 'tilde' and direct image are essential inverses [EGA I, 9.2, p. 361] and because b is an isomorphism [EGA III_1, 1.4.14, p. 92].

Definition (6): An *r-dualizing sheaf* ω_f or $\omega_{X/Y}$ is a sheaf on X for which there exists, for each open subscheme U of Y and each quasi-coherent sheaf N on U, a canonical isomorphism,

$$(f \mid U)^! N = \omega_f \otimes_Y N,$$

which is functorial in N and commutes with open restriction.

Example (7): (i) If Y is the spectrum of a field then an r-dualizing sheaf ω_f exists because every sheaf on Y is free.

(ii) If X is the projective r-space $\mathbb{P}^r_Y$ and f is the structure map, then an r-dualizing sheaf ω_f exists and is isomorphic to $O_X(-r-1)$.

Indeed, any quasi-coherent sheaf G on X is equal to $(\bigoplus_p f_* G(p))^\sim$ by [EGA II, 2.6.5, p. 38]. Hence taking $O_X(-p)$ for F in (1.1) yields the following formula:

$$f^! N = (\bigoplus_p \underline{\mathrm{Hom}}_S(R^r f_* O_x(-p), N))^\sim .$$

By Serre's explicit computation [EGA III_1, 2.1.12, p. 98], the sheaves $R^r f_* O_X(-p)$ are free and finitely generated; hence, the formula may be rewritten as follows:

(7.1.) $$f^! N = (\bigoplus_p \underline{\mathrm{Hom}}_S(R^r f_* O_X(-p), O_S))^\sim \otimes_Y N.$$

This formula is obviously compatible with open restriction. Thus ω_f exists. Finally Serre's same computation shows that the graded module in (7.1) is isomorphic (a priori noncanonically) to the module, $\bigoplus f_* O_X(-r-1)(p)$. Hence ω_f is isomorphic to $O_X(-r-1)$.

PROPOSITION (8): *The following statements are equivalent:*

(i) *An r-dualizing sheaf ω_f exists.*

(ii) *For each open subscheme U of Y in a basis, $(f \mid U)^!$ is right exact.*

(iii) *For each open subscheme U of Y, the functor $(f \mid U)^!$ is right exact.*

(iv) *For each open subscheme U of Y, the functor $(f \mid U)^!$ is exact.*

PROOF: Obviously, (i) implies (ii). And (ii) implies (iii) because each $(f \mid U)^!$ commutes with open restriction (5, i). Each $(f \mid U)^!$ is left exact (2, iii), so (iii) implies (iv). Obviously, (iv) implies (iii). Finally, (iii) implies (i) with $\omega_f = f^! O_Y$ formally [EGA III_2, 7.2.5, p. 46] because each $(f \mid U)^!$ preserves all (small) direct sums (2, iv) and commutes with open restriction (5, i).

PROPOSITION (9): *Assume an r-dualizing sheaf ω_f exists.*

(i) *There are formulas, or canonical isomorphisms,*

$$\omega_f = f^! O_Y \quad \text{and} \quad t_f N = (t_f O_Y) \otimes_Y N;$$

the second is functorial in N and commutes with open restriction.

(ii) *ω_f is flat.*

(iii) *In any base-change diagram (5.1), the map f' also admits an r-dualizing sheaf $\omega_{f'}$, and in fact, there is a canonical isomorphism, $\omega_{f'} = g'^* \omega_f$.*

PROOF: (i) The first formula is obvious, and the second holds formally because $R^r f_* f^!$ is right exact and preserves all (small) direct sums. The second commutes with open restriction because $R^r f_* f^!$ and t_f do (5, i).

(ii) ω_f is flat because of (8, (i) $\Rightarrow$ (iv)).

(iii) The question is local on Y and Y' by virtue of (5, i) and (8, (ii) $\Rightarrow$ (i)), so we may assume Y and Y' are affine. Then the assertion is obvious in view of (5, iii).

II. Higher duality

Fix a flat, locally projective, finitely presentable morphism of schemes, $f: X \to Y$. Assume that all the fibers $X(y)$ are equidimensional of the same dimension r.

DEFINITION (10): We shall say that *nth order duality* ($n \geq 1$) holds on X/Y if the following conditions are met:

(i) An r-dualizing sheaf ω_f exists and it is lfp (locally finitely presentable).

(ii) Let F and N be quasi-coherent sheaves on X and Y, and consider the bifunctorial map on Y

$$D^m = D^m(F, N): \underline{\mathrm{Ext}}^m_f(F, \omega_f \otimes_Y N) \to \underline{\mathrm{Hom}}_Y(R^{r-m}f_*F, N) \quad (m \geq 0)$$

induced by the sheaved Yoneda pairing and the map, $(t_f O_Y) \otimes_Y N$.

(a) Then D^m is an isomorphism for all $m \leq n$ if N is injective for the category of quasi-coherent sheaves on Y.

(b) Fix any F that is flat and lfp and fix any m with $m \leq n$. Then D^m is an isomorphism for any N if $R^{r-m}f_*F$ commutes with base-change. Conversely, $R^m f_*F$ commutes with base-change if $D^m(F \mid f^{-1}U, N')$ is injective for any affine, open subscheme U of Y and any quasi-coherent sheaf N' on U.

Moreover, condition (ii) is required to hold when Y is replaced by any open subscheme and f by the restriction.

We shall say that *full duality* holds rather than rth order duality holds.

PROPOSITION (11): (i) *Assume an r-dualizing sheaf ω_f exists and is lfp. Then nth order duality holds on X/Y if for each point y of Y there exist an open neighborhood U of y, a relatively very ample sheaf $O(1)$ on $f^{-1}U$ and an integer p_0 such that*

(a) *$R^r(f \mid U)_*O(-p)$ is locally free for $p \geq p_0$, and*

(b) *$R^{r-q}(f \mid U)_*O(-p) = 0$ holds for $p \geq p_0$ and $q = 1, \ldots, n$.*

(ii) *If nth order duality holds on X/Y, then for each quasi-compact open subscheme U of Y and for each relatively very ample sheaf $O(1)$ on $f^{-1}U$ there exists an integer p_0 such that (i, a) and (i, b) hold.*

(iii) *If nth order duality holds on X/Y, then nth order duality holds on any X'/Y' obtained form X/Y by a base-change $g: Y' \to Y$.*

PROOF: (i) The proof proceeds by induction on n. We have to check (10, ii) with $m = n$. The checking may be done locally on Y, so we may assume that Y is affine and that (a) and (b) hold with Y for U. Moreover, since Y is now affine and since ω_f is lfp, if we increase p_0 if necessary, we may assume (by reduction to the noetherian case [EGA IV_3, §8, §11] and by [14, (i), p. 58]) that the following relation holds:

$$(11.1)\quad H^q(X(y), \omega_f(y)(p)) = 0 \quad q \geq 1, \quad \text{for} \quad p \geq p_0 \quad \text{and for} \quad y \in Y.$$

For the F given in (10, ii), construct a surjection, $u: E \to F$, with E a direct sum of $O(-p)$'s for various $p \geq p_0$; use a finite sum if F is lfp. Set $G = \text{Ker}(u)$. If F is lfp and flat, then G is lfp [EGA IV_3, 11.3.9.1, p. 137] and flat as f is flat.

For the N given in (10, ii) consider the following commutative diagram:

$$(11.2)\quad \begin{array}{ccccc} \underline{\text{Ext}}_f^{n-1}(E, \omega \otimes N) & \to & \underline{\text{Ext}}_f^{n-1}(G, \omega \otimes N) & \to & \underline{\text{Ext}}_f^{n}(F, \omega \otimes N) \to 0 \\ \downarrow & & \downarrow & & \downarrow \\ \underline{\text{Hom}}_Y(R^{r-n+1}f_*E, N) & \to & \underline{\text{Hom}}_Y(R^{r-n+1}f_*G, N) & \to & \underline{\text{Hom}}_Y(R^{r-n}f_*F, N) \to 0, \end{array}$$

where the bottom line arises from part of the long exact sequence,

$$(11.3)\quad R^{r-n}f_*E \to R^{r-n}f_*F \to R^{r-n+1}f_*G \to R^{r-n+1}f_*E \to R^{r-n+1}f_*F.$$

The top line of (11.2) is exact because we have

$$(11.4)\quad \underline{\text{Ext}}_f^q(E, \omega \otimes N) = \bigoplus R^q f_*(\omega_f(p) \otimes N) = 0 \quad \text{for } q \geq 1$$

by the local-to-global spectral sequence for Ext's and by (11.1) and base-change theory.

Assume $n \geq 2$ first. Then the top left term in (11.2) is equal to 0 by (11.4). Moreover, by (b) the first and third terms in (11.3) are 0, so the second map in the bottom line of (11.2) is an isomorphism. Therefore (10, ii) with $m = n$ holds for F and N because (10, ii) with $m = n - 1$ holds for G and N by induction.

Assume $n = 1$ now. Suppose N is injective. Then the bottom line of (11.2) is exact because the first term of (11.3) is 0 by (b). Therefore, the third vertical map in (11.2) is an isomorphism because the first two are. Thus (10, ii, a) holds.

Suppose F is flat and lfp and $R^{r-1}f_*F$ commutes with base-change. Then $R^{r-1}f_*(F \otimes M)$ is right exact in M. Hence R^rf_*F is locally free because it is lfp and commutes with base-change (3). Therefore, the last map in (11.3), call it v, is locally split (in fact, split as Y is affine), because v is surjective since $R^{r+1}f_*G$ is 0 by (3, i). Now, the next-to-last term R^rf_*E is locally free by (a). Therefore, the kernel K of v is locally free. So, the second map in (11.3) is locally split, because it is injective since $R^{r-1}f_*E$ is 0 by (b). Consequently, the bottom line in (11.2) is exact. Hence the third vertical map in (11.2) is an isomorphism because the first two are. Thus the first part of (10, ii, b) holds.

Suppose $D^1(F, N)$ is injective for all quasi-coherent sheaves N on Y. Then the bottom line of (11.2) is exact in the middle for all such N, because the first two vertical maps are isomorphisms. It follows formally by taking for N the kernel K of the map v from R^rf_*E to R^rf_*F (the last map in (11.3)) that the inclusion map of K into R^rf_*E is split. Now, v is surjective since $R^{r+1}f_*G$ is 0 by (3, i). Hence v itself is split. Since R^rf_*E is locally free by (a), and R^rf_*F is lfp by (3, iv), therefore R^rf_*F is locally free. Consequently, since R^rf_*F commutes with base-change (3, iii), so does $R^{r-1}f_*F$. Thus also the second part of (10, ii, b) holds.

Finally, condition (ii) is required to hold also after Y is replaced by any open subscheme; it does because the hypotheses are obviously stable under open restriction.

(ii) Since U is quasi-compact and ω_f is flat, there is (as above) a p_0 such that the following relation holds:

$$R^q(f \mid U)_*(\omega_f \otimes N) = 0 \quad \text{for } q \geq 1, \quad \text{for } p \geq p_0, \tag{11.5}$$

and for any quasi-coherent sheaf N on U.

Fix $p \geq p_0$ and consider the map $D^q(O(-p), N)$ in the following form:

$$R^q(f \mid U)_*(\omega_f \otimes N) \to \underline{\mathrm{Hom}}_U(R^{r-q}(f \mid U)_*O(-p), N). \tag{11.6}$$

First take $q = 0$. Then (11.6) is an isomorphism, and it follows from (11.5) that its source is an exact functor in N. Since $R^r(f \mid U)_*O(-p)$ is lfp (3, iv), it is therefore locally free. Thus (i, a) holds. Moreover, $R^r(f \mid U)_*O(-p)$ commutes with base-change (3, iii).

Take $1 \leq q \leq n$ and suppose $R^{r-q+1}(f \mid U)_*O(-p)$ is locally free and commutes with base-change. Then $R^{r-q}(f \mid U)_*O(-p)$ also commutes with base-change by base-change theory. Hence, since nth order duality holds by hypothesis, (11.6) is an isomorphism. The source is equal to 0 by (11.5). Since N is arbitrary, therefore $R^{r-q}(f \mid U)_*O(-p)$ is equal to 0. Thus (i, b) follows by induction on q.

(iii) Let $f' : X' \to Y'$ denote $f \times Y'$. Then $\omega_{f'}$ exists and is lfp in view of (9, iii) because ω_f is so. Now, choose a covering of Y by quasi-compact open subschemes U such that $f^{-1}U$ admits a relatively ample sheaf $O(1)$. By (ii), for each U there exists an integer p_0 such that (i, a) and (i, b) hold. Moreover, the proof of (ii) yielded that each sheaf $R^{r-q}(f \mid U)_* O(-p)$ for $q = 0, \ldots, n$ commutes with base-change. Therefore (i, a) and (i, b) remain valid with f' for f, with $g^{-1}U$ for U, and with $g^*O(1)$ for $O(1)$. Hence nth order duality holds on X'/Y' by (i).

Example (12): For duality holds on X/Y for $X = \mathbb{P}(Q)$ where Q is a locally free sheaf of rank $r+1$ on Y. Indeed, an r-dualizing sheaf exists and is lfp by (7, ii) because this is a local matter. Finally, (11, i) holds with Y for U and r for p_0 by Serre's explicit computation [EGA III_1, 2.1.12, p. 98].

Remark (13): Condition (10, ii, b) cannot be usefully changed by replacing F by $F \otimes_Y M$ where M is an lfp sheaf on Y. Indeed, suppose that Y is affine and that the conditions (11, i, a) and (11, i, b) hold for $n = 1$ with Y for U. Take $F = O(-p_0)$, fix M and suppose that $D^1(F \otimes M, N)$ is an isomorphism for all N. We are going to prove that M is locally free.

Construct a surjection $M_0 \to M$, where M_0 is free and finitely generated, and let M_1 denote the kernel. The sequence,

$$(13.1) \quad 0 \to (R^r f_* F) \otimes M_1 \to (R^r f_* F) \otimes M_0 \to (R^r f_* F) \otimes M \to 0,$$

is exact because $R^r f_* F$ is locally free by hypothesis, and if (13.1) is split, then M is locally free as M is lfp. Consider the following diagram, with exact top line:

$$\begin{array}{ccccc}
\mathrm{Hom}_X(F \otimes M_0, \omega_f \otimes N) & \to & \mathrm{Hom}_X(F \otimes M_1, \omega_f \otimes N) & \to & \mathrm{Ext}^1_X(F \otimes M, \omega_f \otimes N) \\
\downarrow \wr\simeq & & \downarrow \simeq & & \\
\mathrm{Hom}_S(R^r f_* F \otimes M_0, N) & \to & \mathrm{Hom}_S(R^r f_* F \otimes M_1, N) & &
\end{array}$$

The two vertical maps are isomorphisms because ω_f is an r-dualizing sheaf. The third term at the top is equal to 0 because by hypothesis $D^1(F \otimes M, N)$ is an isomorphism and $R^{r-1} f_* F$ is equal to 0. Hence the bottom map is surjective. Since N is arbitrary, (13.1) is split, and so M is locally free.

In particular, therefore there is no use in dropping the requirement that F be flat in (10, ii, b). (The flatness enters into the proof of (11, i) in concluding that $R^r f_* F$ is locally free if and only if $R^{r-1} f_* F$ commutes with base-change.)

LEMMA (14): *Assume nth order duality holds on X/Y. Let F and N be quasi-coherent sheaves on X and Y. Assume either that F is flat and lfp or that N is injective for the category of quasi-coherent sheaves on X. Then the following relation holds:*

$$\underline{\mathrm{Ext}}^q_X(F, f^! N) = 0 \quad \text{for} \quad q \leq \min(n, r-d-1)$$
$$\text{with} \quad d = \max_{y \in Y} \dim \operatorname{Supp} F(y).$$

PROOF: The question is local, so we may assume X admits a relatively very ample sheaf $O(1)$. The proof proceeds by induction on q. The assertion is trivial for $q < 0$. For $q \geq 0$, the induction hypothesis implies that the local-to-global spectral sequence degenerates sufficiently to yield the following relation for any p:

$$f_* \underline{\mathrm{Ext}}^q_X(F(-p), f^! N) = \underline{\mathrm{Ext}}^q_f(F(-p), f^! N).$$

By duality, the right hand side is equal to 0, because $R^{r-q} f_* F(-p)$ is equal to 0 by (3, i). Now, there is an obvious canonical isomorphism,

$$\underline{\mathrm{Ext}}^q_X(F, f^! N) \otimes O(p) = \underline{\mathrm{Ext}}^q_f(F(-p), f^! N).$$

Therefore, since p is arbitrary, the assertion holds.

THEOREM (15): *Let $h: Z \to X$ be a finite and finitely presentable map such that the composition $g = f \circ h : Z \to Y$ is flat and all its fibers are equidimensional of the same dimension s. Assume $(r-s)$th order duality holds on X/Y. Then there is a canonical isomorphism of functors in N,*

$$g^! N \rightleftarrows (\underline{\mathrm{Ext}}^{r-s}_X(h_* O_Z, f^! N))^{\sim}. \tag{15.1}$$

PROOF: For any quasi-coherent sheaf G on Z, the sheaf,

$$g_* \, \underline{\mathrm{Hom}}_Z(G, \underline{\mathrm{Ext}}^{r-s}_X(h_* O_Z, f^! N)^{\sim}),$$

is clearly equal to the sheaf,

$$f_* \, \underline{\mathrm{Hom}}_{h_* O_Z}(h_* G, \underline{\mathrm{Ext}}^{r-s}_X(h_* O_Z, f^! N)). \tag{15.2}$$

Since h_*O_Z is a sheaf of O_X-algebras, there is an obvious change-of-rings spectral sequence. It degenerates by (14), which applies because h_*O_Z is flat and lfp in view of the hypotheses. So (15.2) is equal to the source of the map,

$$(15.3)\qquad D^{r-s}:\underline{\mathrm{Ext}}_f^{r-s}(h_*G, f^!N)\to\underline{\mathrm{Hom}}_Y(R^sf_*(h_*G), N).$$

Note that the target's first argument is equal to R^sg_*G because h is affine; hence by duality on Z/Y the target is equal to $g_*\,\mathrm{Hom}_Z(G, g^!N)$. Therefore by Yoneda's lemma the desired map (15.1) exists.

Checking that (15.1) is an isomorphism is a local matter. So we may assume f admits a relatively very ample sheaf $O(1)$, because it does locally. Then setting $G=h^*O(p)$ in (15.3) renders (15.3) an isomorphism, because then h_*G is flat and lfp and because $R^sf_*h_*G$ is equal to R^sg_*G, so commutes with base-change by (3, iii). It follows that tensoring both sides of (15.1) with $h^*O(p)$ for any p and then applying g_* yields an isomorphism. Since $h^*O(1)$ is relatively very ample for g as h is finite, therefore (15.1) is an isomorphism.

REMARK (16): Alternatively, the proof of (15) may be completed as follows. We may assume Y is affine. Then there is an exact sequence $0\to N\to N_1\to N_2$ in which N_1 and N_2 are injectives for the category of quasi-coherent sheaves on Y. Both sides of (15.1) are left exact in N; the right hand side is because $f^!$ is exact in N and because of (14) with N_1/N for N. Therefore (15.1) is an isomorphism, because it is when N is injective as (15.3) is then.

REMARK (17): If $r=s$ holds in (15), the proof simplifies; then, (14) is unnecessary and (15.3) is automatically an isomorphism for all G and N. Consequently, the hypotheses may be relaxed and the following result obtained: Let $f:X\to Y$ be a proper, finitely presentable map and $h:Z\to X$ a finite, finitely presentable map. Set $g=f\circ h$. Then there is a canonical isomorphism of functors in N,

$$g^!N \xleftarrow{\sim} (\mathrm{Hom}_X(h_*O_Z, f^!N)^{\sim}.$$

Moreover, therefore, the implicit hypothesis that f be locally projective is superfluous in the next result, (18).

COROLLARY (18): *Assume the hypotheses of* (15). *If h is flat, then an r-dualizing sheaf ω_f exists. If ω_g exists and if ω_f is invertible, then ω_f is given by the following formula*:

$$\omega_g = C \otimes h^*\omega_f, \quad \text{with } C = \underline{\mathrm{Hom}}_X(h_*O_Z, O_X)^\sim.$$

PROOF: Obvious.

COROLLARY (19): *Assume the hypotheses of* (15). *If Z is a relative complete intersection in X with normal sheaf ν, then an s-dualizing sheaf ω_f exists and is given by the following formula*:

$$\omega_g = \det(\nu) \otimes \omega_f.$$

PROOF: The fundamental local isomorphism [6, Prop. 4, p. 7; or 10, III, 7.2, p. 179; or 1, I, 4.5, p. 13] yields the formula,

$$\underline{\mathrm{Ext}}_X^{r-s}(O_Z, f^!N) = \det(\nu) \otimes f^!N.$$

Since ν is locally free, the right hand side is exact. So ω_g exists by (8), as all commutes with open restriction, and ω_g is given by the desired formula.

THEOREM (20): *For $n \geq 1$, the following statements are equivalent*:

(i) *nth order duality holds on the fiber $X(y)/k(y)$ for each $y \in Y$.*

(ii) *$H^{r-q}(O_{X(y)}(-p)) = 0$ holds for $q = 1, \ldots, n$, for $p \geq p_0(y)$, for any* (*resp. for some*) *very ample sheaf $O_{X(y)}(1)$, for each $y \in Y$.*

(iii) *$\underline{\mathrm{Ext}}_{P(y)}^{e-r+q}(O_{X(y)}, \omega_{P(y)/k(y)}) = 0$ holds for $q = 1, \ldots, n$, for any* (*resp. for some*) *embedding of $X(y)$ in a projective e-space $P(y)$, for each $y \in Y$.*

(iv) *nth order duality holds on X/Y.*

PROOF: (iv) $\Rightarrow$ (i) $\Leftrightarrow$ (ii). These implications hold by (11, iii and ii).

(ii) $\Leftrightarrow$ (iii). (See [1, (5.4), p. 79]). In brief, it is easy to see that the local-to-global spectral sequence degenerates for $p \gg 0$ and yields the relation,

$$H^0(\underline{\mathrm{Ext}}_{P(y)}^{e-r+q}(O_{X(y)}, \omega_{P(y)/k(y)})(p)) = \mathrm{Ext}_{P(y)}^{e-r+q}(O_{X(y)}(-p), \omega_{P(y)/k(y)}).$$

It follows that (ii) and (iii) are equivalent because full duality holds on $P(y)/k(y)$ by (12).

(iii) $\Rightarrow$ (iv). The issue is clearly local on Y. So we may assume that X is a closed Y-subscheme of a suitable projective e-space $P = \mathbf{P}_Y^e$. Full duality holds on P/Y by (12), so by (15) the following formula holds:

(20.1) $$f^! N = \underline{\mathrm{Ext}}_P^{e-r}(O_X, \omega_{P/Y} \otimes N).$$

Full duality also holds on the fibers $P(y)/k(y)$, so by (14) the Ext in (iii) is equal to 0 for $q \leq -1$ also. Hence by base-change theory for Ext's [2, (1.10)], the following relation holds:

(20.2) $$\underline{\mathrm{Ext}}_P^{e-r+q}(O_X, \omega_{P/Y} \otimes N) = 0 \quad \text{for } q \neq 0, q \leq n.$$

Moreover, for $q = 0$, the Ext in (20.2) is right exact in N and it is lfp for $N = O_Y$. Therefore, in view of (20.1), an r-dualizing sheaf exists by (8, i) and is lfp by (9, i), for Y may be replaced by an arbitrary open subscheme.

In view of (20.2) the change-of-rings spectral sequence degenerates and yields the following relation for $m \leq n$:

$$\underline{\mathrm{Ext}}_f^m(F, \underline{\mathrm{Ext}}_{O_P}^{e-r}(O_X, \omega_{P/Y} \otimes N)) = \underline{\mathrm{Ext}}_{P/Y}^m(F, \omega_{P/Y} \otimes N).$$

Therefore, nth order duality holds on X/Y because it does on P/Y.

THEOREM (21): *Full duality holds on X/Y if and only if each fiber $X(y)$ is Cohen-Macaulay.*

PROOF: The assertion is an immediate consequence of (20, iii $\Leftrightarrow$ iv), of (7, ii), and of local algebra [1, 5.22, p. 66].

PROPOSITION (22): *Suppose each fiber $X(y)$ is Cohen-Macaulay, and let $V \subset X$ be the smooth locus of f. Then there is a canonical isomorphism,*

$$\omega_f \mid V = \det \Omega^1_{V/Y}.$$

PROOF: Consider the graph $Z \subset X \times V$ of the inclusion of V into X. It is a relative complete intersection over V [EGA IV$_4$, 17.12.3] with $\Omega^1_{V/Y}$ as conormal sheaf [EGA IV$_4$, 16.3.1]. Hence by (19) there is a formula,

$$\omega_{Z/V} = (\det \Omega^1_{V/Y})^{-1} \otimes \omega_{X \times V/V},$$

because full duality holds on $X \times V/V$. Now, $\omega_{Z/V}$ is obviously equal to O_V, and $\omega_{X \times V/V}$ is equal to the pullback of $\omega_{X/Y}$ by (9, iii). Therefore the asserted formula holds.

COROLLARY (23): *If f is a local complete intersection map with virtual tangent bundle τ* [SGA 6, VII, 1, 2, pp. 466–481 *and* 0, 4.4, pp. 11–12] *then full duality holds and ω_f is given by the formula,*

$$\omega_f = (\det \tau)^{-1}.$$

PROOF: Full duality holds by (21). The formula may be checked locally, so we may assume X is a closed Y-subscheme of a projective space P. Then τ is represented by the difference, $(\tau_{P/Y} \mid X) - \nu_{X/P}$, where $\tau_{P/Y}$ denotes the tangent sheaf of P/Y and $\nu_{X/P}$ denotes the normal sheaf of X in P. Hence the desired formula follows from (19) and (22).

COROLLARY (24): *If X and Y are smooth over a base scheme S, then full duality holds on X/Y and ω_f is given by the following formula:*

$$\omega_f = (\det \Omega^1_{X/S}) \otimes f^*(\det \Omega^1_{Y/S})^{-1}. \tag{24.1}$$

PROOF: The map $f: X \to Y$ factors into the composition of the graph map $g: X \to X \times_S Y$ and the projection $h: X \times_S Y \to Y$. The map g is a regular embedding [EGA IV_4, 17.12.3] and its conormal sheaf is equal to $f^*\Omega^1_{Y/S}$ because g is a base-change of the diagonal map $Y \to Y \times_S Y$. The projection h is smooth and its sheaf of differentials is the pullback of that of X/S. Thus f is a local complete intersection map, whose virtual cotangent sheaf is represented by the difference, $\Omega^1_{X/S} - f^*\Omega^1_{Y/S}$. Hence the assertion results from (23) or directly from (21), (19) and (22).

EXAMPLE (25): Suppose X/Y is a family of Cohen-Macaulay curves. Let Z denote the scheme of divisors of degree n on X/Y. It is a smooth Y-scheme of relative dimension n and we seek a formula for $\Omega^1_{Z/Y}$ in terms of $\omega_{X/Y}$.

Let $W \subset X \times_Y Z$ denote the universal family of divisors. Let ν denote the normal sheaf of W in $X \times_Y Z$ and let $p_X: W \to X$ and $p_Z: W \to Z$ denote the projections. Then by general principles [7, pp. 22, 23, or SGA 3, p. 130] there is a formula,

$$\Omega^1_{Z/Y} = \underline{\mathrm{Hom}}_Z(p_{Z*}\nu, O_Z).$$

Applying duality on W/Z turns this formula into the following one:

$$\Omega^1_{Z/Y} = p_{Z*}\, \underline{\mathrm{Hom}}_W(\nu, \omega_{W/Z}).$$

By (19) and (9, iii) the sheaf $\omega_{W/Z}$ is equal to $\nu \otimes p_X{}^*\omega_{X/Y}$. Since W is a divisor, ν is invertible. Therefore the preceding formula yields the following one:

$$\Omega^1_{Z/Y} = p_{Z*}p_X{}^*\omega_{X/Y}.$$

This formula for the case in which X is a smooth curve over a field was given by Mattuck in [13, Formula (2) and Prop. 1, p. 781].

Remark (26): (i) For $n \geq 1$, there is an open, retrocompact subscheme U of Y (possibly empty) such that nth order duality holds on an X'/Y' obtained via a base-change map $g: Y' \to Y$ if and only if g factors through U. Indeed, (20, iii) is an open, retrocompact condition by the proof of [EGA IV_3, 12.3.4].

(ii) If Y is locally noetherian, then for a dualizing sheaf to exist it is enough for there to be a canonical isomorphism of global functors, $f^! N = \omega_f \otimes N$, because a quasi-coherent sheaf on an open subscheme of Y always extends over all of Y by [EGA I, 6.9.2, i, p. 317].

(iii) Condition (10, ii, a) came up only three times – in the proof of (11, i), where it was verified, in Lemma (14) and in Remark (16). Notably, it was not involved in the proof of (11, ii). Hence, it is a formal consequence of (10, ii, b).

(iv) In (11, i) the hypothesis that ω_f exist and be lfp can be eliminated. The existence of ω_f follows from the hypothesis (11, i, a) by the reasoning in (7, ii). The finiteness of ω_f is more difficult to prove, see (v) below.

(v) The sheaf $f^! N$ is lfp if N is, in two cases: (a) Y is locally noetherian, (b) ω_f exists. In case (a), reasoning as in the first part of the proof of (20, iii $\Rightarrow$ iv) we are reduced to observing that the Ext in (20.1) is lfp because Y is locally noetherian. In case (b), we have $f^! N = \omega_f \otimes N$, and ω_f is lfp by (vi) below.

(vi) Fix $y \in Y$. Suppose the natural map, $(f^! O_Y) \otimes k(y) \to f(y)^! k(y)$, which exists in view of (5, ii), is surjective. Then there exists an open neighborhood U of y such that $\omega_f \mid U$ exists and is lfp; in fact, there exists a noetherian scheme U_ν, a flat and projective map $f_\nu : X_\nu \to U_\nu$ such that ω_{f_ν} exists, and a map $U \to U_\nu$ such that $f \mid U = f_\nu \times U$.

Indeed, reasoning as in the first part of the proof of (20, iii $\Rightarrow$ iv) we are reduced to verifying a corresponding statement about Ext's. The latter can be proved using the ideas of [2, (1.9, i)], of [EGA IV_3, §12.3], and of [11, Appendix].

(vii) Suppose that Y is a proper, finitely presentable S-scheme, say, with structure map $g: Y \to S$, and that the fibers have dimension at most s. Then Formula (2.4.1) can be generalized to various extents. First, since $R^{r+s}(gf)_*$ is equal to $R^s g_* R^r f_*$ by the Leray spectral sequence and Lemma (3), and since adjunction commutes with composition, the following formulas follow formally:

$$(g \circ f)^! = f^! \circ g^! \quad \text{and} \quad t_{g \circ f} = t_g \circ R^s g_* t_f.$$

Hence, if f and g admit dualizing sheaves, then $g \circ f$ admits one too and it is given by the formula,

$$\omega_{g \circ f} = \omega_f \otimes f^* \omega_g.$$

Suppose now that ω_g is invertible; it is, for example, when g has Gorenstein fibers, because ω_g is flat and commutes with base-change (9, ii, iii). Then the preceding formula is equivalent to the following one:

$$\omega_f = \omega_{g \circ f} \otimes f^* \omega_g^{-1}.$$

Finally, if g (resp. $g \circ f$) is smooth and locally projective, then ω_g (resp. $\omega_{g \circ f}$) may be replaced by $\det \Omega^1_{Y/S}$ (resp. $\det \Omega^1_{X/S}$) by (22). In particular, Formula (24.1) is recovered via an alternate route. (There was no need here for f to be flat or locally projective.)

(viii) The functor $f^!$ is local on X in the following sense: Let $f_1: X_1 \to Y$ and $f_2: X_2 \to Y$ be proper, finitely presentable maps with $\dim X_i(y) \leq r$ such that there exists a Y-scheme U, an open Y-embedding of U into X_1, and one into X_2. Then there exists a functorial, canonical isomorphism,

$$f_2^! N \mid U = f_1^! N \mid U. \tag{26.1}$$

Here is an outline of a proof inspired to some extent by [10, Appendix]. We may assume Y is affine, and by [EGA IV$_3$, §8] and by (5, iii), noetherian. We may replace X_2 by the closure of U in $X_1 \times_Y X_2$ and so assume that X_2 is the closure of U and that f_2 factors through a map $h: X_2 \to X_1$ such that U is equal to $h^{-1}U$. The set of $x \in X_1(y)$ with $\dim h^{-1}(x) \geq d$ has dimension at most $r - d - 1$; hence by (3, i) the Leray spectral sequence degenerates sufficiently to yield the formula,

$$R^r f_{2*} F = R^r f_{1*} h_* F$$

for any quasi-coherent sheaf F on X_2. So, the duality isomorphism (2.1) yields the relation,

$$\mathrm{Hom}_{X_2}(F, f_2^!N) = \mathrm{Hom}_{X_1}(h_*F, f_1^!N).$$

Take $F = J^n$, where J is the ideal of $X_2 - U$, and pass to the limit over n. Since h_*J defines a subscheme supported on $X_1 - U$, the limit is equal to the following relation by [EGA I, 6.9.17, p. 323]:

$$\mathrm{Hom}_U(O_U, f_2^!N \mid U) = \mathrm{Hom}_U(O_U, f_1^!N \mid U).$$

Since we may assume U is affine, we obtain the desired relation (26.1).

(ix) A theory of residues can be developed in the spirit of this work following the outline in [16, pp. 398–400].

REFERENCES

[1] A. ALTMAN and S. KLEIMAN: *Introduction to Grothendieck Duality Theory.* Lecture Notes in Math. 146, Springer (1970).

[2] A. ALTMAN and S. KLEIMAN: "Compactifying the Picard scheme", (to appear in *Adv. Math.*).

[SGA 6] P. BERTHELOT, et alii, Theorie des Intersections et Théorème de Riemann-Roch, Lecture Notes in Math. 225, Springer (1971).

[4] P. DELIGNE and M. RAPOPORT: "Les Schémas de Modules de Courbes Elliptiques", in *Modular Functions of One Variable* II, Lecture Notes in Math 349, Springer (1973).

[SGA 3] M. DEMAZURE and A. GROTHENDIECK, *Schémas en Groupes I,* Lecture Notes in Math. 151, Springer (1970).

[5] R. GODEMENT, *Topologie Algébrique et Théorie des Faisceaux*, Hermann, Paris (1958).

[6] A. GROTHENDIECK, Théorème de dualité pour les faisceaux algébriques cohérents", Seminaire Bourbaki, 149 (May 1957).

[7] A. GROTHENDIECK, "Technique de descente et théorèmes d'existence en géométrie algébrique IV. Les schémas de Hilbert", Séminaire Bourbaki 221 (May 1961).

[EGA O_I, I] A. GROTHENDIECK and J. DIEUDONNÉ: *Eléments de Géométrie Algébrique I,* Grundlehren der math. Wissenschaften 166, Springer (1971).

[EGA II-IV_4] A. GROTHENDIECK and J. DIEUDONNÉ: *Eléments de Géométrie Algébrique,* Publ. Math. I.H.E.S., Nos. 8, 11, 17, 20, 24, 28, 32 (1961, '61, '63, '64, '65, '66, '67).

[10] R. HARTSHORNE: *Residues and Duality,* Lecture Notes in Math. 20, Springer (1966).

[11] K. LØNSTED and S. KLEIMAN: "Basics on Families of Hyperelliptic Curves", *Compositio Math., 38(1)* (1979) 83–111.

[12] S. MACLANE: *Categories for the Working Mathematician,* Graduate Texts in Math. 5, Springer (1971).

[13] A. MATTUCK: "Secant Bundles on Symmetric Products", American Journal Math., LXXXVII, 4 (1965), 779–797.

[14] D. MUMFORD: *Lectures on Curves on an Algebraic Surface,* Annals of Math. Studies No. 59, Princeton Univ. Press (1966).

[15] J.-L. VERDIER: "Duality dans la cohomologie des espaces localement compacts", Séminaire Bourbaki, 300 (Nov. 1965).
[16] J.-L. VERDIER: "Base change for twisted inverse image of coherent sheaves", in *Algebraic Geometry*, Bombay 1968, Oxford (1969), 393–408.

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Department of Mathematics
M.I.T., Room 2-278
Cambridge, MA 02139
U.S.A.