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FAMILIES OF ALGEBRAIC CURVES WITH NODES

Allen Tannenbaum

1. Introduction

In this paper we discuss the following general question: Given a smooth projective surface F , and a smooth irreducible curve $X \subset F$, when in the complete linear system $|X|$ can we find integral curves with precisely δ nodes and no other singular points for $0 \leq \delta \leq p_a(X)$? Severi in [19] Anhang F treats this question for the case $F = \mathbb{P}^2$, and finds that for given degree d one can always find integral curves with precisely δ nodes and no other singular points for all $0 \leq \delta \leq (d-1)(d-2)/2$. In our previous paper [20] we discuss an analogous question involving the possible geometric genera of curves lying on smooth surfaces.

Our technique is to generalize Severi's proof in the plane to the case of smooth rational surfaces using methods from [7], [12], and [21]. We then give a general criterion (*Corollary* (2.14)) for answering the above question for smooth rational surfaces and use this to give a modern proof of Severi's result in \mathbb{P}^2 . We next apply our techniques to ruled rational surfaces and find that in \mathbb{P}^n ($n \geq 3$) there exist integral non-degenerate (i.e. lying in no hyperplane) curves of degree $d \geq n$ with precisely δ nodes and no other singular points for all δ from 0 until the Castelnuovo bound.

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Notation and terminology

(i) All our schemes will be defined over a fixed algebraically closed field k , $\text{char } k = 0$. By “point” of a scheme, unless stated otherwise, we mean “closed point”.

(ii) By *surface* we mean a smooth integral projective algebraic 2-dimensional scheme. By *curve on a surface* we mean an effective Cartier divisor. By *curve in \mathbb{P}^n* we mean an equidimensional closed subscheme of \mathbb{P}^n of dimension 1.

(iii) If F is a surface, by *family of curves of F over T* (T an algebraic k -scheme), we mean a relative effective Cartier divisor $\mathcal{D} \subset F \times T$ over T .

(iv) Given \mathfrak{F} a coherent sheaf on a scheme X , $h^i(\mathfrak{F}) = \dim H^i(X, \mathfrak{F})$.

(v) For X a projective scheme of dimension n , the *arithmetic genus* $p_a(X)$ is defined to be $p_a(X) = (-1)^n (\chi(\mathcal{O}_X) - 1)$ where

$$\chi(\mathcal{O}_X) = \sum_{i=0}^n (-1)^i h^i(\mathcal{O}_X).$$

(vi) For X a reduced projective curve, the *geometric genus* $g(X)$ is defined to be the arithmetic genus of the normalization of X .

(vii) By a *non-degenerate curve in \mathbb{P}^n* , we mean a reduced curve in \mathbb{P}^n not contained in any hyperplane of \mathbb{P}^n .

(viii) A *rational surface* is a surface birationally equivalent to \mathbb{P}^2 .

(ix) Let $f: X \rightarrow Y$ be a morphism of schemes. Then

$\Omega_{X/Y}$ = sheaf of relative differentials of X over Y

$\Theta_{X/Y} = \text{Hom}_{\mathcal{O}_X}(\Omega_{X/Y}, \mathcal{O}_X)$ (where ‘*Hom*’ denotes ‘sheaf hom’)
= sheaf of relative tangents of X over Y .

If $Y = \text{Spec } k$, then we let $\Omega_X = \Omega_{X/k}$ and $\Theta_X = \Theta_{X/k}$. See Hartshorne [6] pages 172–184 for details.

(x) If X is a projective scheme, we denote by ω_X its dualizing sheaf. See [6] pages 239–249.

(xi) If X is a surface or a smooth projective curve, we denote by K_X a canonical divisor.

(xii) As in Griffiths-Harris [5] pages 20–21, we use the term ‘generic’ in the following classical sense: if we are given a family of schemes parametrized by a variety (i.e. an integral separated scheme of finite type over k), then to say ‘a generic member of the family has a certain property’ means that ‘the property holds for all closed points in a dense Zariski open subset’.

(xiii) For other terminology we use the standard definitions of modern algebraic geometry. See e.g. [6].

1. Families of morphisms into schemes

In this section we review some of the work of Horikawa in [7] and [8], and Wahl in [21] which we will need in Section 2 and 3. See also Lindner [11]. We first recall the following definition from [7]:

DEFINITION (1.1): Let Y be a smooth projective algebraic k -scheme. Then a *family of morphisms into Y* is a quadruplet $(\mathcal{Y}, \varphi, \pi, T)$ where \mathcal{Y} and T are smooth algebraic k -schemes with T irreducible, $\pi: \mathcal{Y} \rightarrow T$ is a surjective smooth proper morphism, and $\varphi: \mathcal{Y} \rightarrow Y \times T$ is a morphism such that if $p: Y \times T \rightarrow T$ is the projection map, then $p \circ \varphi = \pi$. One can define the notion of equivalence of such families in the obvious way. See [7], page 373.

REMARKS (1.2): Let $f: X \rightarrow Y$ be a morphism of smooth projective algebraic k -schemes such that $\Omega_{X/Y}$ is a torsion sheaf. Then if $\Theta_X =$ tangent sheaf of X , and $\Theta_Y =$ tangent sheaf of Y , from [7] if we define $\tilde{N}_{X/Y} = f^* \Theta_Y / \Theta_X$ we have an exact sequence

$$0 \rightarrow \Theta_X \rightarrow f^* \Theta_Y \rightarrow \tilde{N}_{X/Y} \rightarrow 0.$$

Now let $(\mathcal{Y}, \varphi, \pi, T)$ be a family of morphisms into Y as in (1.1) and suppose that for some closed point $t_0 \in T$, the induced morphism $\varphi_{t_0}: \mathcal{Y}_{t_0} \rightarrow Y$ (\mathcal{Y}_{t_0} is the fiber of \mathcal{Y} over t_0) is equivalent to $f: X \rightarrow Y$. Then we have ([7] page 375) a *characteristic map*

$$\rho: (\text{Tangent space of } T \text{ at } t_0) \rightarrow H^0(X, \tilde{N}_{X/Y}).$$

Horikawa proves the following two facts ([7] page 376 and page 382):

(i) If the characteristic map ρ is surjective, then the family $(\mathcal{Y}, \varphi, \pi, T)$ is ‘complete’ at t_0 in the sense that if $(\mathcal{Y}', \varphi', \pi', T')$ is any other family of morphisms into Y such that for some closed point $t'_0 \in T'$ the induced morphism $\varphi'_{t'_0}: \mathcal{Y}'_{t'_0} \rightarrow Y$ is equivalent to $f: X \rightarrow Y$, then there exists an open neighborhood U of t'_0 in T' and a morphism $h: U \rightarrow T$ with $h(t'_0) = t_0$ such that the restriction of $(\mathcal{Y}', \varphi', \pi', T')$ to U is equivalent to the family of morphisms gotten by pulling back $(\mathcal{Y}, \varphi, \pi, T)$ to U via h .

(ii) Given $f: X \rightarrow Y$ as above, if $H^1(X, \tilde{N}_{X/Y}) = 0$, then a family of morphisms into Y $(\mathcal{Y}, \varphi, \pi, T)$ always exists such that for some closed

point $t_0 \in T$ we have that $\varphi_{t_0}: \mathcal{Y}_{t_0} \rightarrow Y$ is equivalent to $f: X \rightarrow Y$, and the characteristic map is bijective.

REMARKS (1.3): We now briefly review Wahl’s work in [21]. Let V be a non-singular variety, $D \subset V$ an effective Cartier divisor, and A a finite local artinian k -algebra. Then we define a functor H , from the category of finite local artinian k -algebras to the category of sets by

$$H(A) = \{\text{subschemes of } V \times_{\text{Spec } k} \text{Spec } A \text{ flat over } A, \text{ inducing } D \text{ on } V\}.$$

Let N_D be the normal sheaf of D in V . Then H is pro-representable, $H^{(k[\epsilon]/\epsilon^2)} = H^0(D, N_D)$, and smoothness is obstructed by elements in the image of the map $H^1(V, \mathcal{O}_V(D)) \rightarrow H^1(D, N_D)$. For details see Mumford [12] pages 157–160.

Next using the Lichtenbaum-Schlessinger cotangent complex [10], define

$$N'_D = \ker(N_D \rightarrow T^1(D/k, \mathcal{O}_D)).$$

Then if we define a functor from the category of finite local artinian k -algebras to the category of sets by

$$H'(A) = \{\text{liftings in } H(A) \text{ which are locally trivial deformations of } D \text{ in the Zariski topology}\}$$

we have that H' is pro-representable, $H'^{(k[\epsilon]/\epsilon^2)} = H^0(D, N'_D)$, and smoothness is obstructed by elements in $H^1(D, N'_D)$. For proofs, see [21] page 557.

We now have the following proposition (see also [11]):

PROPOSITION (1.4): *Let S be a smooth irreducible projective surface, $D \subset S$ a reduced curve with isolated singular points p_1, \dots, p_r such that each p_i is a union of non-singular branches. Then if $\pi: \tilde{D} \rightarrow D$ is the normalization of D and $i: D \hookrightarrow S$ the natural inclusion, let $\pi' = i \circ \pi: \tilde{D} \rightarrow S$. Then we have (notation as in (1.2) and (1.3)):*

- (i) $h^0(N'_D) \leq h^0(\tilde{N}_{\tilde{D}/S})$;
- (ii) if the p_i are nodes for all $1 \leq i \leq r$, then $H^j(D, N'_D) \cong H^j(\tilde{D}, \tilde{N}_{\tilde{D}/S})$ for all $j \geq 0$.

PROOF. First from the exact sequence

$$0 \rightarrow N'_D \rightarrow N_D \rightarrow T^1(D/k, \mathcal{O}_D) \rightarrow 0$$

it is immediate that $N'_D = J \cdot N_D$ where J is the Jacobian ideal (the first Fitting ideal of Ω_D ; see e.g. [14] page 251). Then if $C = \text{Ann}_{\mathcal{O}_D}(\pi_* \mathcal{O}_{\tilde{D}} \sigma_D)$ (C is the ‘conductor’) it is well known (see e.g. [15]) that we have an exact sequence

$$0 \rightarrow J \rightarrow C \rightarrow K \rightarrow 0$$

where K is supported on the singular points of D and moreover an easy calculation shows that $K_{p_i} = 0$ if p_i is a node. Therefore we have an exact sequence

$$(*) \quad 0 \rightarrow N'_D \rightarrow C \cdot N_D \rightarrow K \rightarrow 0$$

(where we identify K and $K \otimes N_D$ since N_D is invertible).

Next from Piene [14] page 261, we have that $\omega_{\tilde{D}} \cong \tilde{C} \otimes \pi^* \omega_D$ where $\tilde{C} = C \mathcal{O}_{\tilde{D}}$ and $\omega_{\tilde{D}}, \omega_D$ are the dualizing sheaves on \tilde{D}, D respectively. But we have from (1.2) and our hypotheses an exact sequence of locally free sheaves

$$0 \rightarrow \mathcal{O}_{\tilde{D}} \rightarrow \pi'^* \mathcal{O}_S \rightarrow \tilde{N}_{\tilde{D}|S} \rightarrow 0.$$

Thus taking the highest exterior powers of the members of this sequence we get that

$$\begin{aligned} \tilde{N}_{\tilde{D}|S} &\cong \omega_{\tilde{D}} \otimes \pi'^* \wedge^2 \mathcal{O}_S \\ &\cong \omega_{\tilde{D}} \otimes \pi'^* \mathcal{O}_S(-K_S) \\ &\cong \tilde{C} \otimes \pi^* \omega_D \otimes \pi'^* \mathcal{O}_S(-K_S) \\ &\cong \tilde{C} \otimes \pi^*(\omega_D \otimes i^* \mathcal{O}_S(-K_S)) \\ &\cong \tilde{C} \otimes \pi^* N_D \quad (\text{from the adjunction formula}). \end{aligned}$$

Hence we see that

$$\begin{aligned} \pi_* \tilde{N}_{\tilde{D}|S} &\cong \pi_*(\tilde{C} \otimes \pi^* N_D) \\ &\cong \pi_* \tilde{C} \otimes N_D \quad (\text{by the projection formula}) \\ &\cong C \cdot N_D. \end{aligned}$$

Now this means from (*) we have an exact sequence

$$0 \rightarrow N'_D \rightarrow \pi_* \tilde{N}_{\tilde{D}|S} \rightarrow K \rightarrow 0.$$

Finally noting that since π is an affine morphism we have that

$$H^i(D, \pi_* \tilde{N}_{\tilde{D}|S}) \cong H^i(\tilde{D}, \tilde{N}_{\tilde{D}|S})$$

for all $j \geq 0$, which completes the proof of the proposition. Q.E.D.

We conclude this section with some definitions and remarks generalizing our discussion in (1.3):

DEFINITION (1.5): Let S be a smooth irreducible projective surface, $D \subset S$ a reduced curve with isolated singularities p_1, \dots, p_r . Then if A is a finite local artinian k -algebra we say that a lifting

$$\bar{D} \in H(A) = \{\text{subschemes of } S \times_{\text{Spec } k} \text{Spec } A \text{ flat over } A \text{ inducing } D \text{ on } S\}$$

is *semi-locally trivial with respect to* p_1, \dots, p_m ($m \leq r$) if for every open affine $U \subset S$ such that U doesn't contain p_{m+1}, \dots, p_r , the lifting $\bar{D} \cap (U \times \text{Spec } A)$ defines a deformation of $D \cap U$ equivalent to the trivial deformation. See also [21]. We can then define a functor $H''(p_1, \dots, p_m)$ from the category of finite local artinian k -algebras to the category of sets by

$$H''(p_1, \dots, p_m)(A) = \{\text{semi-locally trivial liftings in } H(A) \text{ with respect to } p_1, \dots, p_m\}.$$

DEFINITION (1.6): Let $D \subset S$, p_1, \dots, p_r be as in (1.5). Choose an open affine cover $\{U_i\}_{1 \leq i \leq N}$ ($N > r$) of D such that $p_j \in U_i$ for $1 \leq j \leq r$ and $p_j \notin U_i$ for $i \neq j$ $1 \leq i \leq N$. Recall from [10] page 54, that in general for V a separated scheme of finite type over k , if V is smooth over k , we have $T^1(V/k, \mathcal{O}_V) = 0$. Hence using the sheaves N_D and N'_D defined in (1.3) we see that

$$(*) \quad N_D|_{U_i \cap U_j} = N'_D|_{U_i \cap U_j} \quad \text{for } i \neq j.$$

Next on each U_i $1 \leq i \leq N$ define a sheaf $N''_D(p_1, \dots, p_m)_i$ by

- (i) if $1 \leq i \leq m$, then $N''_D(p_1, \dots, p_m)_i = N'_D|_{U_i}$;
- (ii) if $m + 1 \leq i \leq N$, then $N''_D(p_1, \dots, p_m)_i = N_D|_{U_i}$.

Then from (*) we see that we may glue the $N''_D(p_1, \dots, p_m)_i$ to get a sheaf $N''_D(p_1, \dots, p_m)$ on D .

REMARKS (1.7): We use the notation of (1.6). First note that $N''_D(p_1, \dots, p_r) \cong N'_D$. Next recalling the definition of $H''(p_1, \dots, p_m)$ in (1.5), from [21] page 557 it is clear that $H''(p_1, \dots, p_m)$ is pro-representable, $H''(p_1, \dots, p_m)(k[\epsilon]/\epsilon^2) = H^0(D, N''_D(p_1, \dots, p_m))$ and that if $H^1(D, N''_D(p_1, \dots, p_m)) = 0$, then there are no obstructions to smoothness.

Finally note that for A a finite local artinian k -algebra, $H''(p_1, \dots, p_m)(A)$ consists of divisors $\bar{D} \subset S \times_{\text{Spec } k} \text{Spec } A$ which don't formally change the singularities p_1, \dots, p_m of D .

REMARK (1.8): Suppose our functors $H, H', H''(p_1, \dots, p_m)$ from (1.3) and (1.5) are pro-represented by the complete local k -algebras R_1, R_2, R_3 respectively. As we will see from (2.1) and (2.7) below (see also [21]) these functors are induced (in the sense of [18] page 271) from functors from $(\text{Schemes}/k)^0 \rightarrow (\text{Sets})$. Then if there are no obstructions to smoothness, so that each of the R_i is a ring of formal power series over k , then each of the functors is algebraisable, so that in particular there exist 'universal' smooth algebraic k -schemes X_i , closed points $x_i \in X_i$, and unique isomorphisms $\hat{\mathcal{O}}_{X_i, x_i} \cong R_i$ for $i = 1, 2, 3$. For details about this and about more general cases see Artin [1], [2] and Seshadri [18] pages 270–274.

2. Families of curves with nodes on rational algebraic surfaces

In this section we study the problem of finding integral curves lying on a fixed rational surface linearly equivalent to some smooth irreducible curve X , with δ nodes and no other singular points for $0 \leq \delta \leq p_a(X)$. Our method is to generalize the steps of Severi's proof in [19] Anhang F. See also Popp [16] for a modern treatment of the Severi theory in the plane and Mumford [22] (Appendix 2 to Chapter 8, pages 229–230) for a brief discussion of some of Severi's results. Needless to say, many of the key ideas are already in Severi.

Throughout this section F will denote a fixed smooth irreducible rational projective algebraic surface.

We begin with the following definition from [21] page 558:

DEFINITION (2.1). Let T be an algebraic k -scheme and $X \rightarrow T$ a flat morphism of finite type. Then we say $X \rightarrow T$ is a *formally locally trivial family of deformations* if for every closed point $t \in T$ and every $n > 0$ in the diagram

$$\begin{array}{ccccc}
 X & \longleftarrow & X_n & \longleftarrow & X_1 \\
 \downarrow & & \downarrow & & \downarrow \\
 T & \longleftarrow & \text{Spec } \hat{\mathcal{O}}_{T, t} / m_t^n & \longleftarrow & \text{Spec } k
 \end{array}$$

(where $m_i =$ maximal ideal of $\mathcal{O}_{T,t}$), X_n is a locally trivial deformation of X_1 in the Zariski topology.

We can now state our first lemma:

LEMMA (2.2): *Let $X \subset F$ be a smooth irreducible curve. Let $Y \in |X|$ be a reduced curve with precisely δ nodes and no other singular points such that $Y = Y_1 + \cdots + Y_r$, Y_i irreducible and $K_F \cdot Y_i < 0$ for all $1 \leq i \leq r$. Then there exists a smooth irreducible algebraic k -scheme $V_\delta(|X|; Y)$ of dimension $= \dim|X| - \delta$ parameterizing reduced curves in $|X|$ with precisely δ nodes and no other singular points which are flat deformations of Y in F and such that the deformations are formally locally trivial and may be taken over smooth irreducible parameter schemes.*

PROOF: Let $\pi: \tilde{Y} \rightarrow Y$ be the normalization of Y , $i: Y \hookrightarrow F$ the natural inclusion map, and $\pi' = i \circ \pi: \tilde{Y} \rightarrow F$. Then if \mathcal{O}_F is the tangent sheaf of F , $\mathcal{O}_{\tilde{Y}}$ the tangent sheaf of \tilde{Y} , as in (1.2) we have an exact sequence of locally free sheaves

$$(1) \quad 0 \rightarrow \mathcal{O}_{\tilde{Y}} \rightarrow \pi'^* \mathcal{O}_F \rightarrow \tilde{N}_{\tilde{Y}/F} \rightarrow 0.$$

By (1.2), (1.3), and (1.4) above the infinitesimal deformations of $Y \hookrightarrow F$ which don't formally change the singularities of Y correspond to the sections of $H^0(\tilde{Y}, \tilde{N}_{\tilde{Y}/F})$ and smoothness is obstructed by elements in $H^1(\tilde{Y}, \tilde{N}_{\tilde{Y}/F})$.

Next taking the highest exterior powers of the members of the exact sequence (1) we get that

$$\tilde{N}_{\tilde{Y}/F} \cong \pi'^* \mathcal{O}_F(-K_F) \otimes \mathcal{O}_{\tilde{Y}}(K_{\tilde{Y}}).$$

Thus we have that

$$\begin{aligned} \deg \tilde{N}_{\tilde{Y}/F} &= 2g(\tilde{Y}) - 2 - K_F \cdot Y \\ &= 2g(\tilde{Y}) - 2 - K_F \cdot X. \end{aligned}$$

But by Serre duality

$$h^1(\tilde{N}_{\tilde{Y}/F}) = h^0(\pi'^* \mathcal{O}_F(K_F)), \text{ and if}$$

$\tilde{Y}_i =$ normalization of Y_i for $1 \leq i \leq r$, we see that

$$\deg \pi'^* \mathcal{O}_F(K_F) |_{\tilde{Y}_i} = K_F \cdot Y_i < 0$$

which implies that $h^1(\tilde{N}_{\tilde{Y}|F}) = 0$. Hence there exists (see (1.2) or (1.8)) a smooth algebraic k -scheme $V_\delta(|X|; Y)$ parametrizing formally locally trivial deformations of Y in F of dimension

$$(2) \quad \begin{aligned} h^0(\tilde{N}_{\tilde{Y}|F}) &= 1 - g(\tilde{Y}) + \deg \tilde{N}_{\tilde{Y}|F} \\ &= g(\tilde{Y}) - K_F \cdot X - 1. \end{aligned}$$

Now we will compute $\dim|X|$. By Riemann-Roch for surfaces we have

$$(3) \quad h^0(\mathcal{O}_F(X)) - h^1(\mathcal{O}_F(X)) + h^2(\mathcal{O}_F(X)) = \frac{X \cdot (X - K_F)}{2} + 1 + h^2(\mathcal{O}_F).$$

We claim that $h^1(\mathcal{O}_F(X)) = 0$, $h^2(\mathcal{O}_F(X)) = h^2(\mathcal{O}_F)$. Indeed from the exact sequence

$$0 \rightarrow \mathcal{O}_F \rightarrow \mathcal{O}_F(X) \rightarrow \mathcal{O}_X \otimes \mathcal{O}_F(X) \rightarrow 0, \text{ using the fact that } h^1(\mathcal{O}_F) = 0,$$

we get an exact sequence

$$0 \rightarrow H^1(\mathcal{O}_F(X)) \rightarrow H^1(\mathcal{O}_X \otimes \mathcal{O}_F(X)) \rightarrow H^2(\mathcal{O}_F) \rightarrow H^2(\mathcal{O}_F(X)) \rightarrow 0.$$

But $\deg(\mathcal{O}_X \otimes \mathcal{O}_F(X)) = X^2$ and since

$$X^2 + K_F \cdot X = 2g(X) - 2 \text{ and } K_F \cdot X = X_F \cdot Y < 0, \text{ we see that } X^2 > 2g(X) - 2$$

so that $H^1(\mathcal{O}_X \otimes \mathcal{O}_F(X)) = 0$ which proves the claim. We then have by (3) that

$$\begin{aligned} H^0(\mathcal{O}_F(X)) &= X \cdot (X - K_F)/2 + 1 \\ &= X \cdot (X + K_F)/2 - X \cdot K_F + 1 \\ &= p_a(X) - X \cdot K_F. \end{aligned}$$

Thus $\dim|X| = p_a(X) - X \cdot K_F - 1$.

But since all the singular points of Y are nodes $p_a(X) - g(\tilde{Y}) = \delta$, so that by (2) we have

$$\dim V_\delta(|X|; Y) = \dim|X| - \delta. \quad \text{Q.E.D.}$$

REMARKS-NOTATION (2.3): (i) From now on we fix the notation and

hypotheses of LEMMA (2.2) with F, X, Y, Y_i ($K_F \cdot Y_i < 0$) $1 \leq i \leq r$, $\delta, V_\delta(|X|; Y)$ with the added condition $\delta > 0$.

(ii) If C is a curve in F , then by ' $C \in V_\delta(|X|; Y)$ ' we mean there exists some closed point in $V_\delta(|X|; Y)$ corresponding to C .

(iii) If $F = \mathbb{P}^2$, and $\deg X = d$, then from the proof of (2.2) above we see that $\dim V_\delta(|X|; Y) = 3d + g(\tilde{Y}) - 1$ and the degree of the 'characteristic series' is equal to $\deg \tilde{N}_{\tilde{Y}/F} = 3d + 2g(\tilde{Y}) - 2$ since $K_{\mathbb{P}^2}$ is linearly equivalent to -3ℓ where ℓ is a line. These formulae are also obtained by Severi in [19] page 317.

(iv) If one examines the proof of (2.2) above, it seems that we can weaken the hypothesis that F is rational and only assume that F is regular (i.e. $h^1(\mathcal{O}_F) = 0$). However, as pointed out by the referee, this generalization is illusory. Indeed, as noted by the referee, if one makes the assumption that $\dim|X| > 0$ (which follows from (2.3) (i) above), then the hypothesis that $K_F \cdot X < 0$ together with $h^1(\mathcal{O}_F) = 0$ implies that F is rational. For if $\dim|X| > 0$, then $X \cdot X \geq 0$ so that X has non-negative intersection with every effective divisor on F , which implies $h^0(mK_F) = 0$ for all $m > 0$. Then since $h^1(\mathcal{O}_F) = 0$ we have that F is rational.

(v) In the statement of LEMMA (2.2) we make the hypothesis that X is smooth, irreducible. Actually we claim that if we only take X to be irreducible and $X^2 > 2p_a(X) - 2$ (which follows from the hypothesis that $K_F \cdot X = K_F \cdot Y < 0$), then automatically a generic member of $|X|$ is smooth. To prove this we first recall some definitions from [13] and [20]:

DEFINITIONS (2.4): If L is a complete linear system on a smooth irreducible surface S and p is an infinitely near point over S so that $p \in S', \pi: S' \rightarrow S$ a birational morphism, we define the *proper transform* of L, \tilde{L} , to be $|\tilde{D}|$ where \tilde{D} is the proper transform on S' of a generic member of L . Then if p is a base point of \tilde{L} , we say that p is an *infinitely near base point* of L .

Moreover if D is any curve on S and p is an infinitely near point as above, then the *multiplicity of D at p* is the multiplicity of p on the proper transform of D . Finally if L' is any linear system on S and q is any point, ordinary or infinitely near, then the *multiplicity of L' at q* is the multiplicity of a generic member of L' at q .

We can now prove the claim of (2.3) (iv):

PROPOSITION (2.5): *Let $C \subset F$ be a reduced, irreducible curve such that $C^2 > 2p_a(C) - 2$ (F as in (2.2)). Then a generic member of $|C|$ is smooth.*

PROOF: Let $L = |C|$. If $\dim L = 0$, then it is easy to see that C must be a smooth rational curve with $C^2 = -1$ i.e. C is an exceptional curve of the first kind. So we assume $\dim L > 0$. We must then show L has no base points of multiplicity ≥ 2 . So suppose to the contrary that p is a base point of L with multiplicity $m_0 \geq 2$, and that the other base points of L (including the infinitely near ones) are p_1, \dots, p_r with multiplicities m_1, \dots, m_r respectively. Blow up the p_i and let \tilde{L} be the proper transform of L on the blown-up surface F' . Then note $\dim \tilde{L} = \dim L$.

Next using the fact that $h^1(\mathcal{O}_F) = 0$ from the exact sequence

$$(*) \quad 0 \rightarrow \mathcal{O}_F \rightarrow \mathcal{O}_F(C) \rightarrow \mathcal{O}_F(C) \otimes \mathcal{O}_C \rightarrow 0$$

we have that $\dim L = h^0(\mathcal{O}_F(C) \otimes \mathcal{O}_C)$. Then since $C^2 > 2p_a(C) - 2$ by Riemann-Roch we have $h^0(\mathcal{O}_F(C) \otimes \mathcal{O}_C) = 1 - p_a(C) + C^2 > p_a(C) - 1$ so that $\dim \tilde{L} = \dim L \geq p_a(C)$. Let $\tilde{C} \in \tilde{L}$ be a generic member. Note that \tilde{C} is smooth since \tilde{L} has no base points, and moreover since we've assumed that L has a base point of multiplicity ≥ 2 , we have that $p_a(C) > p_a(\tilde{C}) = g(\tilde{C})$. Now applying an analogous exact sequence as that of (*) to the blown-up surface F' , we see that

$$\dim \tilde{L} = h^0(\mathcal{O}_{F'}(\tilde{C}) \otimes \mathcal{O}_{\tilde{C}}) \geq p_a(C) > g(\tilde{C})$$

so that $h^1(\mathcal{O}_{F'}(\tilde{C}) \otimes \mathcal{O}_{\tilde{C}}) = 0$ and therefore

$$(**) \quad \begin{aligned} h^0(\mathcal{O}_{F'}(\tilde{C}) \otimes \mathcal{O}_{\tilde{C}}) &= 1 - g(\tilde{C}) + \tilde{C}^2 \\ &= 1 - p_a(C) + C^2 \end{aligned}$$

(since $\dim \tilde{L} = \dim L$). But

$$\tilde{C}^2 = C^2 - \sum_{i=0}^r m_i^2, \quad g(\tilde{C}) = p_a(C) - \sum_{i=0}^r \frac{m_i(m_i - 1)}{2}$$

which implies by (**) that

$$\sum_{i=0}^r \frac{m_i(m_i + 1)}{2} = 0$$

which means that $m_i = 0$ for all $0 \leq i \leq r$, a contradiction which proves the proposition. Q.E.D.

COROLLARY (2.6): *Notation as in (2.2). Then if $Y = Y_1 + \dots + Y_r$ has*

precisely r irreducible components a generic member of $V_\delta(|X|; Y)$ will also have precisely r irreducible components.

PROOF: It clearly suffices to assume $r = 2$, so that $Y = Y_1 + Y_2$, Y_i irreducible. Suppose that Y_i has precisely δ_i nodes ($i = 1, 2$), so that $\delta = \delta_1 + \delta_2 + Y_1 \cdot Y_2$. Moreover since Y_i is irreducible and $Y_i \cdot K_F < 0$, we have by PROPOSITION (2.5) that there exist smooth irreducible curves X_i such that $Y_i \in |X_i|$ for $i = 1, 2$. By LEMMA (2.2), there exist smooth irreducible parameter schemes $V_{\delta_i}(|X_i|; Y_i)$ (with the analogous properties defined in (2.2)) of dimensions $= \dim|X_i| - \delta_i$ ($i = 1, 2$). But

$$p_a(X) = p_a(X_1 + X_2) = p_a(X_1) + p_a(X_2) + X_1 \cdot X_2 - 1$$

and since we have from the proof of (2.2) that

$$\begin{aligned} \dim|X| &= p_a(X) - K_F \cdot X - 1 \\ \dim|X_i| &= p_a(X_i) - K_F \cdot X_i - 1 \quad (i = 1, 2) \end{aligned}$$

we see that

$$\dim|X| = \dim|X_1| + \dim|X_2| + X_1 \cdot X_2.$$

Then since $X_1 \cdot X_2 = Y_1 \cdot Y_2$ we get that

$$\dim V_\delta(|X|; Y) = \dim V_{\delta_1}(|X_1|; Y_1) + \dim V_{\delta_2}(|X_2|; Y_2)$$

which implies the required result.

Q.E.D.

DEFINITION (2.7). Let F' be a smooth irreducible projective surface, T an algebraic k -scheme, and $\mathcal{C} \subset F' \times T$ an effective relative Cartier divisor (so that $\mathcal{C} \rightarrow T$ is a flat family of curves on F'). Suppose for $0 \in T$ a closed point, $C \stackrel{\text{def}}{=} \mathcal{C}_0$ (\mathcal{C}_0 is the fiber of \mathcal{C} over 0) is a reduced curve having isolated singular points p_1, \dots, p_s . For each $n > 0$ consider the induced diagram

$$\begin{array}{ccccc} \mathcal{C} & \longleftarrow & \mathcal{C}_n & \longleftarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ T & \longleftarrow & \text{Spec } \mathcal{O}_{T,0}/m_0^n & \longleftarrow & \text{Spec } k \end{array}$$

where $m_0 =$ maximal ideal in $\mathcal{O}_{T,0}$. Then we say that \mathcal{C} is a *formally semi-locally trivial family of deformations of C with respect to p_1, \dots, p_m* (for $m \leq s$) if for every open affine $U \subset F'$ such that $p_k \notin U$ for all $m+1 \leq k \leq s$, the induced deformation $\mathcal{C}_n \cap (U \times \text{Spec } \mathcal{O}_{T,0}/m_0^n)$ of $C \cap U$ is equivalent to the trivial deformation for each $n > 0$.

DEFINITION (2.8): Let T be a smooth irreducible curve of finite type over k and suppose that $\mathcal{Y} \subset F \times T$ is an effective relative Cartier divisor which as a flat family of curves over T has generic fibers $\mathcal{Y}_t (t \neq 0)$ reduced curves with precisely $\bar{\delta} < \delta$ nodes and no other singular points and special fiber $\mathcal{Y}_0 = Y$. (Our ‘ Y ’ and ‘ δ ’ are the same as in (2.2). By ‘generic’ we mean in the classical sense, as in (xii) of *Notation and Terminology* above.) Suppose moreover that if $p_1, \dots, p_{\bar{\delta}}$ are the nodes of Y , that the $\bar{\delta}$ nodes of Y , say $p_1, \dots, p_{\bar{\delta}}$, which are the specializations of the nodes of the \mathcal{Y}_t are such that \mathcal{Y} is a formally semi-locally trivial family of deformations of Y with respect to $p_1, \dots, p_{\bar{\delta}}$. Then $p_1, \dots, p_{\bar{\delta}}$ are called the *assigned nodes of Y relative to \mathcal{Y}* while the other $\delta - \bar{\delta}$ nodes are called *unassigned nodes relative to \mathcal{Y}* . When no confusion is possible, we will sometimes omit the explicit reference to \mathcal{Y} .

We now have the following lemma:

LEMMA (2.9): *Suppose there exists a flat family of curves $\mathcal{Y} \rightarrow T$ as described in (2.8). Then there exists a smooth irreducible algebraic k -scheme $V_{\bar{\delta}}(|X|; \mathcal{Y})$ parameterizing reduced curves in $|X|$ with precisely $\bar{\delta}$ nodes and no other singular points which are flat deformations of $\mathcal{Y}_t (t \in T, t \neq 0)$ in F and such that the deformations are formally locally trivial and may be taken over smooth irreducible parameter schemes. Moreover $\dim V_{\bar{\delta}}(|X|; \mathcal{Y}) = \dim |X| - \bar{\delta}$.*

PROOF: It is clear we need only verify that the hypotheses of Lemma (2.2) apply to a generic fiber $\mathcal{Y}_t (t \neq 0)$ of \mathcal{Y} . But suppose that $\mathcal{Y}_t = \bar{Y}_1 + \dots + \bar{Y}_{r'}$, $\bar{Y}_j (1 \leq j \leq r')$ irreducible components of \mathcal{Y}_t . Then for each \bar{Y}_j , there exists $Y' \subseteq Y$ (a sum of irreducible components of Y) such that Y' specializes to \bar{Y}_j . But since $K_F \cdot \bar{Y}_j = K_F \cdot Y' < 0$, we see the hypotheses of (2.2) do indeed apply to \mathcal{Y}_t . Q.E.D.

REMARK (2.10): We now come to the (as we will see) crucial question of when a family \mathcal{Y} as described in (2.8) exists. More specifically, we want to know when we can choose an arbitrary subset of $\bar{\delta} < \delta$ nodes of Y and find a family \mathcal{Y} (as in (2.8)) relative to which

these nodes will be assigned. The answer is given in the following proposition:

PROPOSITION (2.11): *Let $X \subset F$, $Y \in |X|$ be as in (2.2) so that Y is reduced with precisely δ nodes p_1, \dots, p_δ . Then for any $\bar{\delta} < \delta$ nodes of Y , say $p_1, \dots, p_{\bar{\delta}}$, there exists a flat family of curves $\mathfrak{Y} \subset F \times T$ as defined in (2.8) relative to which $p_1, \dots, p_{\bar{\delta}}$ are assigned and hence as in (2.9) we get a smooth parameter space $V_{\bar{\delta}}(|X|; \mathfrak{Y})$ of dimension $= \dim|X| - \bar{\delta}$.*

PROOF: First from (1.3), (1.4), and (2.2) we have that $h^0(N'_Y) = \dim V_\delta(|X|; Y) = \dim|X| - \delta$. Next from (1.6) we have an exact sequence

$$(*) \quad 0 \rightarrow N'_Y \rightarrow N''_Y(p_1, \dots, p_{\bar{\delta}}) \rightarrow K \rightarrow 0$$

where K is supported on $p_{\bar{\delta}+1}, \dots, p_\delta$. Then since $h^1(N'_Y) = 0$ (by (1.4) and (2.2)), from the long exact cohomology sequence associated to (*), we see that $h^1(N''_Y(p_1, \dots, p_{\bar{\delta}})) = 0$ and so

$$\begin{aligned} h^0(N''_Y(p_1, \dots, p_{\bar{\delta}})) &= h^0(N'_Y) + \delta - \bar{\delta} \\ &= \dim|X| - \bar{\delta}. \end{aligned}$$

The proposition now follows immediately from (1.7) and (1.8). Q.E.D.

DEFINITION (2.12): Let \mathfrak{Y} be as in (2.11). Then we say that Y is *virtually connected with respect to \mathfrak{Y}* (or more simply *virtually connected* if the reference to \mathfrak{Y} is clear) if for every expression $Y = Y'_1 + Y'_2$ of Y as a sum of effective divisors at least one of the points of the intersection of Y'_1 and Y'_2 is at an unassigned node.

We can now prove the following fundamental theorem (see also Severi [19] pages 322–327):

THEOREM (2.13): *Let \mathfrak{Y} be as in (2.11). Suppose for any $Y' \subseteq Y$, a connected sum of irreducible components of Y , that there exists $X' \subset F$ a smooth irreducible curve such that $Y' \in |X'|$. Then Y is virtually connected relative to \mathfrak{Y} if and only if a generic member of $V_{\bar{\delta}}(|X|; \mathfrak{Y})$ is irreducible.*

PROOF: First note that on an arbitrary smooth irreducible projective algebraic surface S , if $C \subset S$ is a smooth irreducible curve and

$C' \in |C|$, then C' is connected. (See e.g. [22] page 36, or [6] page 281).

We proceed by induction on the number of irreducible components r of Y . If $r = 1$, there is nothing to prove, so assume $r = 2$. Then $Y = Y_1 + Y_2$, Y_i irreducible. Suppose that precisely $\bar{\delta}_i$ of the assigned nodes of Y lie on Y_i and are nodes of Y_i , $i = 1, 2$. Then if $Y_1 \cdot Y_2 = j + j'$ where j of the points of intersection correspond to unassigned nodes of Y , and j' of the points of intersection correspond to assigned nodes of Y , we have $\bar{\delta} = \bar{\delta}_1 + \bar{\delta}_2 + j'$. By hypothesis (or in this case since the Y_i are irreducible and $K_F \cdot Y_i < 0$, by (2.5)) we have that there exist smooth irreducible curves X_i such that $Y_i \in |X_i|$, $i = 1, 2$. Moreover by (2.11) there exist for each $i = 1, 2$ a flat family of curves \mathfrak{Y}_i with special fiber Y_i relative to which the $\bar{\delta}_i$ nodes of Y_i , which were considered assigned nodes of Y relative to \mathfrak{Y} , are assigned nodes of Y_i relative to \mathfrak{Y}_i . Hence we have the corresponding smooth parameter schemes for $i = 1, 2$ $V_{\bar{\delta}_i}(|X_i|; \mathfrak{Y}_i)$ as in (2.9).

Let $\bar{p}_C = p_a(C)$ - (number of assigned nodes) where $C = Y, Y_1$, or Y_2 . Then we have $p_a(Y) = p_a(Y_1) + p_a(Y_2) + j + j' - 1$ which implies $\bar{p}_Y = \bar{p}_{Y_1} + \bar{p}_{Y_2} + j - 1$. Next by (2.2) and (2.9) we have

$$d \stackrel{\text{def}}{=} \dim V_{\bar{\delta}}(|X|; \mathfrak{Y}) = \bar{p}_Y - K_F \cdot X - 1$$

$$d_i \stackrel{\text{def}}{=} \dim V_{\bar{\delta}_i}(|X_i|; \mathfrak{Y}_i) = \bar{p}_{Y_i} - K_F \cdot X_i - 1 \quad (i = 1, 2).$$

Since $K_F \cdot X = K_F \cdot X_1 + K_F \cdot X_2$ ($X_1 + X_2 \in |X|$) we get that $d = d_1 + d_2 + j$. Now if $j = 0$ so that $d = d_1 + d_2$, it is clear a generic member of $V_{\bar{\delta}}(|X|; \mathfrak{Y})$ must be a sum $\bar{D}_1 + \bar{D}_2$ where $\bar{D}_i \in V_{\bar{\delta}_i}(|X_i|; \mathfrak{Y}_i)$. Conversely suppose $j > 0$ and suppose that a generic member $\bar{D} \in V_{\bar{\delta}}(|X|; \mathfrak{Y})$ is reducible, say $\bar{D} = \bar{D}_1 + \bar{D}_2$ where the \bar{D}_i are effective divisors. The since we have assumed Y has precisely two irreducible components, by (2.6) a generic member of $V_{\bar{\delta}}(|X|; Y)$ also has precisely two irreducible components. Thus there exists $D = D_1 + D_2 \in V_{\bar{\delta}}(|X|; Y)$, D_i irreducible ($i = 1, 2$), such that D_i is a specialization of \bar{D}_i (in the sense of (2.8)). But then by definition $\bar{D}_i \in V_{\bar{\delta}_i}(|X_i|; \mathfrak{Y}_i)$, which implies $d = d_1 + d_2$, a contradiction which completes the proof of the theorem for the case $r = 2$.

We may assume now by induction that the theorem is true with respect to any effective divisor $Y' \subseteq Y$, Y' a connected sum of k irreducible components of Y , $2 \leq k < r$, and we must show it is true for $k = r$, i.e. for Y .

Suppose first that Y is virtually connected. Then there exists some $Y' \subseteq Y$, a connected sum of $2 \leq k < r$ irreducible components of Y ,

which is virtually connected (with respect to the assigned and unassigned nodes of Y which are nodes of Y'). But by induction, Y' will be a specialization (in the sense of (2.8)) of some irreducible \bar{Y}' with nodes precisely the assigned nodes of Y' and no other singular points. Then $(Y - Y') + \bar{Y}'$ will be virtually connected and have fewer irreducible components than Y , and hence we may apply induction to $(Y - Y') + \bar{Y}'$ to get the required conclusion.

Conversely suppose Y isn't virtually connected. Then if there exists $Y' \subsetneq Y$ a connected sum of $2 \leq k < r$ irreducible components of Y which is virtually connected, Y' will be a specialization (again as in (2.8)) of some irreducible \bar{Y}' by induction. But then $(Y - Y') + \bar{Y}'$ is not virtually connected, and has fewer irreducible components than Y , so that we may complete the argument by induction. If no $Y' \subsetneq Y$ (a connected sum of $2 \leq k < r$ irreducible components) is virtually connected, then all the nodes of Y which are defined by the intersections of its irreducible components are assigned, and hence a similar proof as that of (2.6) (or from the proof for the case $r = 2$ above) shows a generic member of $V_{\bar{\delta}}(|X|; \mathfrak{Y})$ has the same number of irreducible components as Y . Q.E.D.

We thus have the following criterion for the solution of the problem discussed at the beginning of this section:

COROLLARY (2.14): *Same hypotheses as in (2.13). Suppose moreover $\delta > p_a(X)$ and for each $0 \leq \bar{\delta} \leq p_a(X)$ there exist $\bar{\delta}$ nodes of Y such that with respect to the associated families $\mathfrak{Y}_{\bar{\delta}}$ relative to which these nodes are assigned, Y is virtually connected. Then there exist integral curves in $|X|$ with precisely $\bar{\delta}$ nodes and no other singular points for all $0 \leq \bar{\delta} \leq p_a(X)$.*

PROOF Immediate from (2.13).

Q.E.D.

3. Families of curves with nodes in \mathbb{P}^n

In this section we apply COROLLARY (2.14) to obtain a proof of Severi's result about plane curves with nodes as well as showing that in \mathbb{P}^n ($n \geq 3$) for $d \geq n$, there exist integral non-degenerate curves of degree d with precisely δ nodes and no other singular points for all δ from 0 to the Castelnuovo bound (see our discussion in (3.2) below).

We begin with the following theorem of Severi (see [19] page 329):

THEOREM (3.1): *There exist integral plane curves of degree d with*

precisely δ nodes and no other singular points for all $0 \leq \delta \leq (d-1)(d-2)/2$.

PROOF: Let $Y = \ell_1 \cup \dots \cup \ell_d$ be the union of d generic lines $\ell_i \subset \mathbb{P}^2$ ($i = 1, \dots, d$) so that Y has $d(d-1)/2$ nodes. Let p_1, \dots, p_{d-1} be the points of intersection of ℓ_1 with the other $(d-1)$ lines. Then noting that $d(d-1)/2 - \delta \geq d-1$ for all $0 \leq \delta \leq (d-1)(d-2)/2$, choosing any $d(d-1)/2 - \delta$ nodes of Y including p_1, \dots, p_{d-1} we may find by (2.11) a flat family \mathcal{Y} relative to which these nodes will be unassigned. But then with this choice of unassigned nodes Y is virtually connected and hence we are done by (2.14). Q.E.D.

REMARKS (3.2): We now recall some basic facts about the Castelnuovo bound. For details see [4], [5], and [20].

Let $d \geq n \geq 3$ be integers and let $M = [(d-1)/(n-1)]$ be the greatest integer not exceeding $(d-1)/(n-1)$. Then Castelnuovo in [4] proves that $M/2(2d - (M+1)(n-1) - 2)$ is the maximal possible genus for nondegenerate (i.e. lying in no hyperplane) smooth irreducible curves of degree d in \mathbb{P}^n . Moreover this bound, 'the Castelnuovo bound', is realizable and for $d > 2n$ all such curves will lie in rational scrolls of degree $n-1$ in \mathbb{P}^n not contained in any hyperplane or if $n = 5$, the Veronese surface. For a modern proof of these facts see Griffiths-Harris [5] pages 251–253 and pages 527–533.

Now for $d \geq n \geq 3$ and $M = [(d-1)/(n-1)]$ as above, let $m \geq 1$ be an integer such that $(n-1)/2 \geq m \geq n - d/(M+1) - 1$. Set $e = n - 2m - 1$ so that $e \geq 0$, and note that $d - (n-m-1)(M+1) \geq 0$, and if $d - (n-m-1)(M+1) = 0$, then $e > 0$. Let $F_e = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(e) \otimes \mathcal{O}_{\mathbb{P}^1})$ be a ruled rational surface, and let B and f be generators of the Neron-Severi group of F_e with $B^2 = e$, $B \cdot f = 1$, $f^2 = 0$. Then from [6] page 380 if $X \in |(M+1)B + (d - (n-m-1)(M+1))f|$ is a generic member, X will be irreducible and non-singular. Moreover $X \cdot (B + mf) = d$, and $p_a(X) = M/2(2d - (M+1)(n-1) - 2)$. From [6] $B + mf$ is a very ample divisor on F_e , and the image of F_e under $|B + mf|$ is a smooth rational scroll of degree $n-1$ in \mathbb{P}^n . Thus the image of X under $|B + mf|$ will be a smooth irreducible non-degenerate curve of degree d lying in \mathbb{P}^n with the maximal possible genus defined by the Castelnuovo bound.

Finally note that if $n \geq 3$ is odd, then $F_0 = \mathbb{P}^1 \times \mathbb{P}^1$ may be embedded in \mathbb{P}^n as a smooth rational scroll of degree $n-1$ not contained in any hyperplane by the very ample divisor $B + ((n-1)/2)f$. If $n \geq 4$ is even, then F_1 , which is isomorphic to \mathbb{P}^2 blown up at one point, may be embedded in \mathbb{P}^n as a smooth rational scroll of degree $n-1$ not contained in any hyperplane by the very ample divisor $B + ((n-2)/2)f$.

For our result (3.3) below it will be enough to consider curves with the maximal genus lying on such surfaces. For details about ruled rational surfaces see Hartshorne [6] pages 369–383.

THEOREM (3.3): *There exist integral non-degenerate curves of degree $d \geq n \geq 3$ in \mathbb{P}^n with precisely δ nodes (and no other singular points) for all δ from 0 to the Castelnuovo bound.*

PROOF: We use the notation of (3.2). Let $Y \subset F_e$ be a curve consisting of the union of $(M+1)$ generic curves each linearly equivalent to B , and $d - (n - m - 1)(M+1)$ generic curves each linearly equivalent to f . Then Y has precisely

$$\lambda_1 \stackrel{\text{def}}{=} (M+1)(d - (n - m - 1)(M+1)) + \frac{M(M+1)e}{2}$$

nodes and no other singular points. Moreover since K_{F_e} is numerically equivalent to $-2B + (e-2)f$ we have for every irreducible component Y_i of Y that $K_{F_e} \cdot Y_i < 0$, then there exists $X' \subset F_e$ a smooth irreducible curve such that $Y' \in |X'|$. See [6] page 380. We divide the cases for n odd and n even.

Case (i): n odd. Then let $m = (n-1)/2$ (notation of (3.2)) so that $e = n - 2m - 1 = 0$. In Y choose some irreducible component B_1 linearly equivalent to B , and some irreducible component f_1 linearly equivalent to f , and suppose B_1 and f_1 intersect at the point p . Then $Y - (B_1 + f_1)$ and $B_1 + f_1$ intersect in

$$\lambda_2 \stackrel{\text{def}}{=} d - (n - m - 1)(M+1) + (M-1) \text{ points } p_i \text{ for } 1 \leq i \leq \lambda_2.$$

Next note that

$$\lambda_1 - (\lambda_2 + 1) = \frac{M}{2} (2d - (n-1)(M+1) - 2).$$

Hence if we regard any $\bar{\delta}$ of the nodes of Y as unassigned for $\lambda_2 + 1 \leq \bar{\delta} \leq \lambda_1$ including $p, p_1, \dots, p_{\lambda_2}$, then relative to the associated flat family \mathfrak{Y} , Y is virtually connected, and hence we are done by COROLLARY (2.14).

Case (ii): n even. Let $m = (n-2)/2$, so that $e = 1$. In Y choose some irreducible component B_1 linearly equivalent to B , and note that $Y - B_1$ and B_1 intersect in

$$\lambda_3 \stackrel{\text{def}}{=} M + d - (n - m - 1)(M + 1) \text{ points } q_i \text{ for } 1 \leq i \leq \lambda_3.$$

Then

$$\lambda_1 - \lambda_3 = \frac{M}{2} (2d - (n - 1)(M + 1) - 2).$$

Hence if we regard any $\bar{\delta}$ nodes of Y as unassigned for $\lambda_3 \leq \bar{\delta} \leq \lambda_1$ including $q_1, \dots, q_{\lambda_3}$, then Y is virtually connected relative to the associated first family, and once more we can complete the proof by (2.14). Q.E.D.

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