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ON NEARLY EUCLIDEAN DECOMPOSITION FOR SOME CLASSES OF BANACH SPACES

Stanisław Szarek and Nicole Tomczak-Jaegermann

Abstract

We introduce a new affine invariant of finite dimensional normed space. Using it we show that for some large classes of finite dimensional normed spaces there is a constant C such that every normed space X contains two subspaces E_1, E_2, orthogonal with respect to the suitable euclidean norm on X and satisfying $\mathrm{d}(E_i, l_2^{[\dim X/2]}) \leq C$.

0. Introduction

The motivation of this paper is to investigate which finite dimensional normed spaces admit Kashin's decomposition on nearly Euclidean orthogonal subspaces. Kashin discovered (cf. [7] and [11]) the following

(K) For arbitrary positive integer n the space l_1^{2n} contains two orthogonal (in the sense of l_1^{2n}) subspaces E_1, E_2 satisfying

$$\mathrm{d}(E_i, l_2^n) < 4e \quad \text{for } i = 1, 2.$$

The proof of (K) given in [11] depends only on the fact that

$$\mathrm{vol}\, B(l_1^n) \leq \left(\frac{2e}{\pi}\right)^{n/2} \cdot \mathrm{vol}\, B(l_2^n),$$

where $B(l_1^n)$ and $B(l_2^n)$ are the unit balls of l_1^n and l_2^n respectively. This suggests to investigate the volume ratio – an affine invariant of Minkowski spaces. We define the volume ratio of a finite dimensional normed space X by

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$$\mathrm{vr}(X) = (\mathrm{vol}\, B(X)/\mathrm{vol}\, \mathscr{E})^{1/k},$$

where $k = \dim X$ if X is a real space and $k = 2 \dim X$ in the complex case, $B(X)$ is the unit ball of X, $\mathscr{E}$ is the ellipsoid of maximal volume contained in $B(X)$. The relation between the volume ratio of a space X and the Kashin's decomposition for X is expressed by the following

(*) Let X be a real or complex normed space of the dimension $2n$. Then there exist two subspaces E_1, E_2 such that

$$\mathrm{d}(E_i, l_2^n) < 4\pi\, \mathrm{vr}(X)^2 \quad \text{for } i = 1, 2.$$

Moreover E_1, E_2 can be chosen to be orthogonal with respect to the Euclidean norm determined by the ellipsoid of maximal volume contained in the unit ball of X.

The proof of (*) is essentially contained in [11].

The concept of the volume ratio also allows to introduce new isomorphism invariants of finite dimensional Banach spaces similar to Kolomogorov capacity for Schwartz spaces. We present here two such invariants. The asymptotic volume growth (in symbols $\mathrm{avg}(Y)$) of a Banach space Y is defined by

$$\mathrm{avg}(Y) = \{(t_n)_{n=1}^{\infty} \in R^{\infty} \mid (t_n) = o(\mathrm{vr}(Y, n))\},$$

where $\mathrm{vr}(Y, n) = \sup\{\mathrm{vr}(E) \mid E \subset Y, \dim E = n\}$.

Let (E_n) be an increasing sequence of subspaces of a Banach space Y such that $\dim E_n = n$ $(n = 1, 2, \ldots)$ and UE_n dense in Y. The approximate volume ratio of Y with respect to (E_n) (in symbols $\mathrm{app\,vr}(Y; (E_n))$) is defined by $\mathrm{app\,vr}(Y; (E_n)) = \{(t_n) \in R^{\infty} \mid (t_n) = o(\mathrm{vr}(E_n))\}$.

The main aim of the present paper is to give the "good" estimate of the volume ratio of some classes of Banach spaces. Therefore these spaces admit the Kashin's decomposition on nearly Euclidean orthogonal subspaces.

The paper is organized as follows.

Section 1 has the preliminary character. In Section 2 we investigate the connection between the volume ratio and the cotype 2 constant of a Banach space. In particular we prove that for a finite dimensional normed space X the volume ratio $\mathrm{vr}(X)$ can be estimate from above in terms of the cotype 2 constant and the unconditional constant of X. We investigate also the cotype properties of a Banach space X for which $\mathrm{avg}(Y) = c_0$. In Section 3 we estimate the volume ratio of the tensor products $l_p^n \hat{\otimes} l_2^n$ for $1 \leq p \leq 2$ and of unitary ideals of operators acting in a Hilbert space.

After this paper has been written up, we learned that the results similar to our Theorem 2.1 and Proposition 2.2 had also been obtained by T.K. Carne.

The concepts of the volume ratio and of the related isomorphism invariants have been suggested to us by A. Pełczyński. We would like to thank him and also T. Figiel for many valuable conversations.

1. Preliminaries

Our notation is standard in the Banach space theory (cf. eg. [8]).

If X is a Banach space, a basis (e_i) in X is called unconditional, if there is a constant C so that for every $x = \Sigma_i x_i e_i \in X$ and every sequence (ϵ_i) with $|\epsilon_i| = 1$ one has

$$\left\| \sum_i \epsilon_i x_i e_i \right\| \leq C \left\| \sum_i x_i e_i \right\|.$$

The unconditional constant of a basis (e_i) is defined as $\mathrm{unc}(e_i) = \inf C$. A basis (e_i) is called unconditionally monotone if $\mathrm{unc}(e_i) = 1$. A basis (e_i) is called symmetric if for every $x = \Sigma_i x_i e_i \epsilon X$, every sequence (ϵ_i) with $|\epsilon_i| = 1$ and every permutation π of natural numbers one has

$$\left\| \sum_i \epsilon_i x_i e_{\pi(i)} \right\| = \left\| \sum_i x_i e_i \right\|.$$

We recall also the notion of type and cotype. Let $1 < p \leq 2 \leq q < \infty$. A Banach space X is said to be of type p (resp. cotype q) if there is a constant K^p (resp. K_q) so that for every finite sequence (x_i) in X one has

$$K^p \left(\int_0^1 \left\| \sum_i r_i(t) x_i \right\|^2 \mathrm{d}t \right)^{1/2} \leq \left(\sum_i \|x_i\|^p \right)^{1/p},$$

(resp.

$$K_q \left(\int_0^1 \left\| \sum_i r_i(t) x_i \right\|^2 \mathrm{d}t)^{1/2} \leq \left(\sum_i \|x_i\|^q \right)^{1/2} \right).$$

where $r_i(\cdot)$ denote the i-th Radenacher function on the interval $\langle 0, 1 \rangle$, i.e. $r_i(t) = \mathrm{sgn}(\sin 2^i \pi t)$. We define the type p constant of the space X as $K^p(X) = \sup K^p$ (resp. the cotype q constant as $K_q(X) = \inf K_q$).

A subset B of a Banach space E is called a body if it is the closed unit ball for an equivalent norm on E.

Let $B \subset R^n$ be a body. Let $\mathscr{E} \subset B$ be the ellipsoid of maximal volume contained in B. We define the volume ratio of B by

$$\mathrm{vr}(B) = (\mathrm{vol}\, B/\mathrm{vol}\, \mathscr{E})^{1/n}.$$

For a body $C \subset \mathbb{C}^n$ we define the (complex) volume ratio by

$$\mathrm{vr}(\tilde{B}) = (\mathrm{vol}\, \tilde{B}/\mathrm{vol}\, \mathscr{E})^{1/2n},$$

(here by $\mathrm{vol}(\tilde{B})$ we mean the volume of $\tilde{B}$ regarded as $2n$-dimensional real body).

Remark: Since all translations invariant measures (i.e. volumes) on R^k are proportional, the ellipsoid of maximal volume and the volume ratio of a body do not depend on a choice of a particular volume.

If X is a Banach space the unit ball in X will be denoted by $B(X)$. For finite dimensional normed space X we will use the notation $\mathrm{vr}(X)$ instead of $\mathrm{vr}(B(X))$ ($\widetilde{\mathrm{vr}}(X)$ in the complex case). It is easy to see (cf. e.g. Proposition 1.3) that if X is a real Banach space and $\tilde{X}$ is its complexification, then

$$2^{-1}\, \mathrm{vr}(X) \le \widetilde{\mathrm{vr}}(\tilde{X}) \le 4\, \mathrm{vr}(X).$$

For a Banach space X we define the asymptotic volume growth of X by

$$\mathrm{avg}(X) = \{(t_n) \in R^\infty \mid (t_n) = o(\mathrm{vr}(X, n))\},$$

where $\mathrm{vr}(X, n) = \sup\{\mathrm{vr}(E) \mid E \subset X, \dim E = n\}$.

In the sequel a slight modification of the notion of δ-net will be very useful. Let B_1, B_2 be bodies. We say that a set $\mathscr{N}$ is a δB_2-net for B_1 if $B_1 \subset \delta B_2 + \mathscr{N}$. We will use the following elementary lemma.

Lemma 1.1: (i) *If $B \subset R^n$ ($\mathbb{C}^n$) is a body, $\mathscr{E} \subset B$ is the ellipsoid of maximal volume contained in B, then there is a $\delta\mathscr{E}$-net for B of cardinality less than* $[(2+\delta)/\delta]^n\ \mathrm{vr}(B)^n([2+\delta)/\delta]^{2n}\ \widetilde{\mathrm{vr}}(B)^{2n}$ *in the complex case).*

(ii) *If B is a body, $\tilde{\mathscr{E}}$ is any ellipsoid and there is a $\delta\tilde{\mathscr{E}}$-net of cardinality k, then* $\mathrm{vr}(B) \le \delta r k^{1/n}$, *where $r > 0$ is such that $\tilde{\mathscr{E}} \subset rB$, (in the complex case one has* $\widetilde{\mathrm{vr}}(B) \le \delta r k^{1/2n}$).

PROOF: We will prove the both statements in the real case only. The modification in the complex case is obvious.

(i) It is a modification of well-known argument (cf. [2]), Lemma 2.4). Let $\{x_i\}_{i=1}^m$ be the maximal set in B such that $x_i \notin x_j + \delta\mathscr{E}$ for every $1 \leq i \leq j \leq m$ (then also $x_j \notin x_i + \delta\mathscr{E}$). The maximality of $\{x_i\}_{i=1}^m$ implies that it forms a $\delta\mathscr{E}$-net for B. The sets $x_i + \frac{1}{2}\delta\mathscr{E}$ are all disjoint and, since $\mathscr{E} \subset B$, are contained in $(1+\frac{1}{2}\delta)B$. By comparing the volumes we get that $m(\delta/2)^n \operatorname{vol}\mathscr{E} \leq (1+\delta/2)^n \operatorname{vol} B$, thus $m \leq [(2+\delta)/\delta]^n \operatorname{vr}(B)^n$.

(ii) Indeed, $B \subset \bigcup_{i=1}^m x_i + \delta\tilde{\mathscr{E}}$, hence $\operatorname{vol} B \leq k\delta^n \operatorname{vol}\tilde{\mathscr{E}}$. Consequently,

$$\operatorname{vr}(B) \leq (\operatorname{vol} B/\operatorname{vol} 1/r\tilde{\mathscr{E}})^{1/n} = r(\operatorname{vol} B/\operatorname{vol}\tilde{\mathscr{E}})^{1/n} \leq \delta r k^{1/n}. \quad \square$$

Let us recall the fact due to F. John [5] that if $B \subset R^n$ is a body then there exists the unique ellipsoid $\mathscr{E}'$ of minimal volume containing B and $B \supset n^{-1/2}\mathscr{E}'$. Since there is an obvious duality between the ellipsoid $\mathscr{E}' \supset B$ of minimal volume containing B and the ellipsoid $\mathscr{E} \subset B$ of maximal volume contained in B, the ellipsoid $\mathscr{E}$ is also unique and has the dual property: $\mathscr{E} \subset B \subset n^{1/2}\mathscr{E}$. This shows that for every body $B \subset R^n$ we have

$$\operatorname{vr}(B) \leq n^{1/2}. \tag{1.1}$$

We will also need the fact that

$$\operatorname{vr}(l_1^n) \leq \sqrt{\frac{2e}{\pi}}. \tag{1.2}$$

PROPOSITION 1.2: *Let $1 \leq p, q \leq 2$, let n, m be natural numbers. By $(\Sigma_{k=1}^n \oplus l_q^m)_p$ we denote the direct sum of n copies of l_q^m in the sense of l_p. Then*

$$\operatorname{vr}\left(\left(\sum_{k=1}^n \oplus l_q^m\right)_p\right) \leq \sqrt{\frac{2e}{\pi}}.$$

PROOF: It is obvious that if $1 \leq p, q \leq 2$, then the following inclusions hold

$$\begin{aligned} \mathscr{E}_1 = n^{1/2-1/p} m^{1/2-1/q} B(l_2^{nm}) \subset B\left(\left(\sum_{k=1}^n \oplus l_q^m\right)_p\right) \\ \subset n^{1-1/p} m^{1-1/q} B(l_1^{nm}) = B_1. \end{aligned} \tag{1.3}$$

Now let us observe that the ellipsoid $\mathscr{E}_1$ is just the ellipsoid of maximal volume contained in B_1. Thus (1.3) and (1.2) imply $\mathrm{vol}\, B((\Sigma_{k=1}^n \oplus l_q^m)_p) \leq \mathrm{vol}\, B_1 \leq (\sqrt{2e/\pi})^{nm}\, \mathrm{vol}\, \mathscr{E}_1$. Thus

$$\mathrm{vr}\left(\left(\sum_{k=1}^{n} \oplus l_q^m\right)_p\right) \leq \sqrt{\frac{2e}{\pi}},$$

what is the required estimate. □

PROPOSITION 1.3: *Let Y, Z be finite dimensional real normed spaces. Then*

$$\mathrm{vr}(Y)^\theta\, \mathrm{vr}(Z)^{1-\theta} \leq \mathrm{vr}((Y \oplus Z)_{l_\infty}) \leq \sqrt{2}\, \mathrm{vr}(Y)^\theta\, \mathrm{vr}(Z)^{1-\theta},$$

where $\theta = \dim Y/(\dim Y + \dim Z)$.

PROOF: Since we are interested in the comparison of volumes of $B((Y \oplus Z)_{l_\infty})$ and the suitable ellipsoid $\mathscr{E}$, without loss of generality we can assume that Y and Z are orthogonal with respect to the usual inner product in R^n. Then from the Fubini's theorem we get

$$\mathrm{vol}\, B((Y \oplus Z)_{l_\infty}) = \mathrm{vol}\, B(Y)\, \mathrm{vol}(Z).$$

To establish the left-hand side inequality let us consider the ellipsoid $\mathscr{E}$ of maximal volume contained in $B((Y \oplus Z)_{l_\infty})$. Since the ellipsoid $\mathscr{E}$ is unique it must be symmetric with respect to the subspaces $Y \oplus \{0\}$ and $\{0\} \oplus Z$. Let us denote $\mathscr{E}_Y = \mathscr{E} \cap Y \oplus \{0\}$ and $\mathscr{E}_Z = \mathscr{E} \cap \{0\} \oplus Z$, then one has $\mathscr{E} \subset (\mathscr{E}_Y \oplus \mathscr{E}_Z)_{l_\infty}$ and consequently $\mathrm{vol}\, \mathscr{E} \leq \mathrm{vol}\, \mathscr{E}_Y\, \mathrm{vol}\, \mathscr{E}_Z$. Thus

$$\begin{aligned}
&\mathrm{vr}((Y \oplus Z)_{l_\infty}) \\
&\quad = [\mathrm{vol}\, B((Y \oplus Z)_{l_\infty})/\mathrm{vol}\, \mathscr{E}]^{1/(\dim Y + \dim Z)} \\
&\quad \geq [\mathrm{vol}\, B(Y)\, \mathrm{vol}\, B(Z)/\mathrm{vol}\, \mathscr{E}_Y\, \mathrm{vol}\, \mathscr{E}_Z]^{1/(\dim Y + \dim Z)} \\
&\quad \geq [\mathrm{vr}(Y)^{\dim Y}\, \mathrm{vr}(Z)^{\dim Z}]^{1/(\dim Y + \dim Z)} \\
&\quad = \mathrm{vr}(Y)^\theta\, \mathrm{vr}(Z)^{1-\theta}.
\end{aligned}$$

To prove the right-hand side inequality let us denote by $\mathscr{E}_1 \subset B(Y)$ and $\mathscr{E}_2 \subset B(Z)$ the ellipsoids of maximal volume contained in $B(Y)$ and $B(Z)$ respectively. It is obvious that

$$(\mathscr{E}_1 \oplus \mathscr{E}_2)_{l_2} \subset B((Y \oplus Z)_{l_\infty}). \tag{1.4}$$

To estimate $\mathrm{vol}((\mathscr{E}_1 \oplus \mathscr{E}_2)_{l_2})$ from below let us observe that $(\mathscr{E}_1 \oplus \mathscr{E}_2)_{l_2} \subset (\sqrt{2})^{-1}(\mathscr{E}_1 \oplus \mathscr{E}_2)_{l_\infty}$. Thus, applying the Fubini's theorem once more, we obtain

$$\begin{aligned}\mathrm{vol}((\mathscr{E}_1 \oplus \mathscr{E}_2)_{l_2}) &\geq (\sqrt{2})^{-\dim Y - \dim Z}\, \mathrm{vol}((\mathscr{E}_1 \oplus \mathscr{E}_2)_{l_\infty}) \\ &= (\sqrt{2})^{-\dim Y - \dim Z}\, \mathrm{vol}\, \mathscr{E}_1\, \mathrm{vol}\, \mathscr{E}_2.\end{aligned}$$

Thus we can estimate

$$\begin{aligned}&\mathrm{vol}\, B((Y \oplus Z)_{l_\infty}) \\ &\quad = \mathrm{vol}\, B(Y)\, \mathrm{vol}\, B(Z) \\ &\quad = \mathrm{vr}(Y)^{\dim Y}\, \mathrm{vr}(Z)^{\dim Z}\, \mathrm{vol}\, \mathscr{E}_1\, \mathrm{vol}\, \mathscr{E}_2 \\ &\quad \leq [\sqrt{2}\, \mathrm{vr}(Y)^{\theta}\, \mathrm{vr}(Z)^{1-\theta}]^{\dim Y + \dim Z}\, \mathrm{vol}((\mathscr{E}_1 \oplus \mathscr{E}_2)_{l_2}).\end{aligned}$$

Hence the inclusion (1.4) completes the proof

$$\begin{aligned}&\mathrm{vr}((Y \oplus Z)_{l_\infty}) \\ &\quad \leq [\mathrm{vol}\, B((Y \oplus Z)_{l_\infty}) / \mathrm{vol}((\mathscr{E}_1 \oplus \mathscr{E}_2)_{l_2})]^{1/(\dim Y + \dim Z)} \\ &\quad \leq \sqrt{2}\, \mathrm{vr}(Y)^{\theta}\, \mathrm{vr}(Z)^{1-\theta}. \quad \square\end{aligned}$$

2. The connection with the notion of cotype 2

We do not know whether $\mathrm{vr}(X)$ can be estimated in terms of $K_2(X)$. One has however

THEOREM 2.1: *If X is a finite dimensional real or complex normed space with a basis (e_i), then*

$$\mathrm{vr}(X) \leq C K_2(X)\, \mathrm{unc}(e_i),$$

where C is a universal constant.

The proof of this theorem depends on some auxiliary results. The first one is rather well-known (cf. e.g. [1]).

Let X be a Banach space of cotype 2 with an unconditional basis (e_i). Then there exists an unconditionally monotone norm $|||\cdot|||$ on X such that the dual norm $|||\cdot|||_*$ on X^* satisfies

(i) $$\left|\left|\left| \sum_i (|x_i|^2 + |y_i|^2)^{1/2} e_i^* \right|\right|\right|_*^2$$

$$\leq \left|\left|\left| \sum_i x_i e_i^* \right|\right|\right|_*^2 + \left|\left|\left| \sum_i y_i e_i^* \right|\right|\right|_*^2,$$

and

(ii) $$\|x\| \leq |||x||| \leq CK_2(X)\,\mathrm{unc}(e_i)\|x\| \quad \text{for } x \in X,$$

where C is a universal constant.

A norm $|||\cdot|||_*$ satisfying the condition (i) above is called 2-convex. Using that we prove the following proposition.

PROPOSITION 2.2: *Let $(X, |||\cdot|||)$ be an n-dimensional real or complex normed space with an unconditionally monotone basis (e_i) and let the dual norm $|||\cdot|||_*$ on X^* be* 2*-convex (i.e. satisfies the condition* (i) *above).*

Then there is a sequence $\alpha_1, \ldots, \alpha_n$ of real numbers such that for every $x = \sum_{i=1}^n x_i e_i \in X$ one has

$$\sum_{i=1}^n |x_i| \leq \left|\left|\left|\sum_{i=1}^n \alpha_i x_i e_i\right|\right|\right| \leq A\sqrt{n}\left(\sum_{i=1}^n |x_i|^2\right)^{1/2}. \tag{2.1}$$

If a basis (e_i) is symmetric, then $\alpha_i = n/|||\sum_{i=1}^n e_i|||$ for $i = 1, \ldots, n$ and $A = 1$; in general $A = 1$ in the real case and $A = \sqrt{2}$ in the complex case.

(Some generalization of this for arbitrary Banach lattices will appear also in [6]).

PROOF: We will need the following lemma:

LEMMA 2.3 ([4]): *Let E be an n-dimensional real normed space with an unconditionally monotone basis (f_k), let E^* denote its dual and let (f_k^*) be the dual basis in E^*. Then for every sequence $a_1, \ldots, a_n$ of real numbers there are sequences $s_1, \ldots, s_n$ and $t_1, \ldots, t_n$ such that*

$$a_k = s_k t_k \quad \text{for } k = 1, \ldots, n,$$
$$\left\|\sum_{k=1}^n s_k f_k\right\|_E \left\|\sum_{k=1}^n t_k f_k^*\right\|_{E^*} = \sum_{k=1}^n |a_k|.$$

We will prove the inequalities (2.1) in the real case only. The generalization to the complex case is obvious.

Our assumtpions yield that the function

$$(y_k) \to |||(y_k)|||_E = \left|\left|\left|\sum_{k=1}^n \sqrt{|y_k|}\, e_k\right|\right|\right|_*^2$$

is a norm on R^n. Let us denote $E = (R^n, |||\cdot|||_E)$ and let (f_k) be the standard unit vector basis in R^n. Now applying lemma for the space E and the sequence $a_k = 1/n$ ($k = 1, \ldots, n$) we can choose the sequences (s_k), (t_k) with $\|\Sigma_{k=1}^n s_k f_k\|_E = 1$ and $\|\Sigma_{k=1}^n t_k f_k^*\|_{E^*} = 1$ such that $s_k t_k = 1/n$ for $k = 1, \ldots, n$. Thus for every $(y_k) \in R^n$ we have

$$\sum_{k=1}^n |y_k| \leq \left\| \sum_{k=1}^n t_k^{-1} y_k f_k \right\|_E = n \left\| \sum_{k=1}^n s_k y_k f_k \right\|_E \leq n \sup_{1 \leq k \leq n} |y_k|.$$

Now, given sequence $(z_k) \in R^n$ let us apply the above inequality for $y_k = |z_k|^2$. Then from the definition of the norm in E we get

$$\left(\sum_{k=1}^n |z_k|^2 \right)^{1/2} \leq \left|\left|\left| \sum_{k=1}^n \beta_k z_k e_k \right|\right|\right| \leq \sqrt{n} \sup_{1 \leq k \leq n} |z_k|,$$

(here $\beta_k = 1/\sqrt{|t_k|}$).

By the duality the last inequality is clearly equivalent to (2.1) with $\alpha_k = \sqrt{n}/\beta_k = \sqrt{n|t_k|} = 1/\sqrt{|s_k|}$.

If a basis (e_i) in X is symmetric, the basis (f_k) in E is symmetric too, so

$$\left\| \sum_{k=1}^n f_k \right\|_E \left\| \sum_{k=1}^n f_k^* \right\|_{E^*} = n$$

and we can put

$$s_k = \left\| \sum_{i=1}^n f_i \right\|_E^{-1} \quad \text{and} \quad t_k = \left\| \sum_{i=1}^n f_i^* \right\|_E^{-1}$$

for $k = 1, \ldots, n$. It gives the desired expression for

$$\alpha = 1/\sqrt{|s_k|} = \left\| \sum_{i=1}^n f_i \right\|_E^{1/2} = \left|\left|\left| \sum_{i=1}^n e_i^* \right|\right|\right|_* = \left|\left|\left| \sum_{i=1}^n e_i \right|\right|\right|^{-1}$$

Now we are ready to prove Theorem 2.1.

PROOF OF THEOREM 2.1: Let $|||\cdot|||$ be the norm on X satisfying the conditions (i) and (ii). Proposition 2.2 implies that there is a sequence $\alpha_1, \ldots, \alpha_n$ of real numbers such that for every $y = \Sigma_i y_i e_i \in X$ one has

$$\text{(2.1')} \qquad \sum_i |\alpha_i^{-1} y_i| \leq |||y||| \leq A\sqrt{n} \left(\sum_{i=1}^n |\alpha_i^{-1} y_i|^2 \right)^{1/2}.$$

Let us consider the real case first. Let us define the ellipsoid $\bar{\mathscr{E}} = \{(y_i) \in R^n | \Sigma_{i=1}^n |\alpha_i^{-1} y_i|^2 \le 1/n\}$. Then the inequalities (2.1′) together with the property (ii) of the norm $|||\cdot|||$ give

$$\bar{\mathscr{E}} \subset B(X, \|\cdot\|) \subset CK_2(X) \operatorname{unc}(e_i) B_1,$$

where $B_1 = \{(y_i) \in R^n | \Sigma_{i=1}^n |\alpha_i^{-1} y_i| \le 1\}$. Thus, to obtain the desired estimate, it suffices to observe that from (1.2) we have

$$[\operatorname{vol} CK_2(X) \operatorname{unc}(e_i) B_1 / \operatorname{vol} \bar{\mathscr{E}}]^{1/2} \le CK_2(X) \operatorname{unc}(e_i).$$

This completes the proof in the real case. In the complex case the proof is similar. □

In Section 3 we will need the following fact.

PROPOSITION 2.5: *Let X be an n-dimensional real or complex normed space with a symmetric basis (e_i) and let $\alpha = n/\|\Sigma_{i=1}^n e_i\|$.* Then for every $x = \Sigma_{i=1}^n x_i e_i \in X$ one has

$$\sum_{i=1}^n |x_i| \le \alpha \left\| \sum_{i=1}^n x_i e_i \right\| \le \sqrt{2e}\, \operatorname{vr}(X) \sqrt{n} \left(\sum_{i=1}^n |x_i|^2 \right)^{1/2}. \tag{2.2}$$

PROOF: The left-hand side inequality is very easy. For the suitable choice of ϵ_i, with $|\epsilon_i| = 1$ we have

$$\begin{aligned} \sum_{i=1}^n |x_i| &= \left\langle \sum_{i=1}^n \epsilon_i x_i e_i, \sum_{j=1}^n e_j^* \right\rangle \\ &\le \left\| \sum_{i=1}^n \epsilon_i x_i e_i \right\| \cdot \left\| \sum_{i=1}^n e_i^* \right\|_*, \end{aligned}$$

so, since the basis (e_i) is symmetric, we get

$$\sum_{i=1}^n |x_i| \le \left\| \sum_{i=1}^n \epsilon_i x_i e_i \right\| \left\| \sum_{i=1}^n e_i^* \right\|_* = \alpha \left\| \sum_{i=1}^n x_i e_i \right\|.$$

To show the right-hand side inequality let us observe first two facts:

(1) the cube $Q = \{x = \Sigma_{i=1}^n x_i e_i \epsilon X \mid \max_{i \le j \le n} |x_j| \le 1/\|\Sigma_{i=1}^n e_i\|\}$ is contained in the unit ball $B(X)$,

(2) the unique ellipsoid $\mathscr{E}$ of maximal volume contained in $B(X)$ is of the form $\mathscr{E} = \{x = \Sigma_{i=1}^n x_i e_i \epsilon X \mid \Sigma_{i=1}^n |x_i|^2 \le R^2\}$.

Now $\mathscr{E} \subset B(X)$ is equivalent to

$$\left\|\sum_{i=1}^{n} x_i e_i\right\| \le R^{-1}\left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$$

for every $x = \sum_{i=1}^{n} x_i e_i \epsilon X$. So, it remains to estimate from below the radius R of the euclidean ball of maximal volume contained in $B(X)$. To do this let us recall the well-known formulae

$$\operatorname{vol} Q = 2^n \left\|\sum_{i=1}^{n} e_i\right\|^{-n},$$
$$\operatorname{vol} \mathscr{E} \le (2e\pi)^{n/2} R^n n^{-n/2},$$

in the real case and

$$\operatorname{vol} Q = \pi^n \left\|\sum_{i=1}^{n} e_i\right\|^{-2n},$$
$$\operatorname{vol} \mathscr{E} \le (2e\pi)^{n/2} R^n n^{-n/2},$$

in the complex case. In the real case we have

$$2^n \left\|\sum_{i=1}^{n} e_i\right\|^{-n} = \operatorname{vol} Q \le \operatorname{vol} B(X) = \operatorname{vr}(X)^n \operatorname{vol} \mathscr{E}$$
$$\le \operatorname{vr}(X)^n (2e\pi)^{n/2} R^n n^{-n/2}.$$

So we get the estimate for R

$$R \ge \sqrt{\frac{2}{e}}\sqrt{n} / \left\|\sum_{i=1}^{n} e_i\right\| \operatorname{vr}(X) \ge \sqrt{2e}\sqrt{n} / \left\|\sum_{i=1}^{n} e_i\right\| \operatorname{vr}(X),$$

what completes the proof in the real case. In the complex case the proof is similar. □

Now we shall prove some facts about spaces X such that $\operatorname{avg}(X) = c_0$

PROPOSITION 2.4: *If E is a Banach space such that* $\operatorname{avg}(X) = c_0$ *then it is of cotype $2+\epsilon$ for every $\epsilon > 0$.*

PROOF: Observe that if $\operatorname{avg}(X) = c_0$ then $\sup_n \operatorname{vr}(X, n) < \infty$, hence for every finite dimensional subspace $E \subset X$ we have $\operatorname{vr}(E) \le \sup_n \operatorname{vr}(X, n) < \infty$. The result of Maurey and Pisier [10] states that if X is a Banach space and $q_0 = \inf\{q \mid X \text{ is of cotype } q\}$, then for every n there is a subspace $E_n \subset X$ with $\dim E_n = n$ and $d(E_n, l_{q_0}^n) \le 2$. For these spaces E_n we have $\mathrm{vr}(E_n) \geq \frac{1}{2}\mathrm{vr}(l_{q_0}^n) \geq c_{q_0} n^{1/2-1/q_0}$. Consequently $q_0 = 2$. □

As was kindly communicated to us by T. Figiel, the ϵ cannot be omitted in this statement. This is shown by the following example due to W. Johnson ([2], Example 5.3).

EXAMPLE 2.5: *There is a Banach space with an unconditional basis, which is not of cotype* 2 *and yet* $\mathrm{avg}(E) = c_0$.

PROOF: Let E be the space constructed in [2], which is of type 2 but not of cotype 2 and yet has the property: for every subspace XE with $\dim X = n$ there is a subspace YX with $\dim Y \geq n - \log n$ such that $d(Y, l_2^{\dim Y}) \leq 2$. We will show that $\mathrm{avg}(E) = c_0$.

Observe first that such a subspace Y is complemented in E (consequently in X). It follows from the result of Maurey [9] which says that if E and F are Banach spaces, E is of type 2, F is of cotype 2 and E_0 is a subspace of E, then for every operator $u: E_0 \to F$ there is an operator $\tilde{u}: E \to F$ such that $\tilde{u} \mid E_0 = u$ and $\|\tilde{u}\| \leq 2K^2(E)K_2(F)\|u\|$. Set $E_0 = F = Y$ and $u = id: Y \to Y$. Then $P = \tilde{u}: E \to Y$ is the required projection with $\|P\| \leq 4K^2(E)$ (because $K_2(Y) \leq d(Y, l_2^{\dim Y})K_2(l_2) \leq 2$).

Thus any subspace $X \subset E$ with $\dim X = n$ can be written as $X = Y \oplus Z$ with $\dim Z \geq \log n$, $d(Y, l_2^{\dim Y}) \leq 2$ and there exists the projections $P: X \to Y$ with $\|P\| \leq 4K^2(E)$ and $Q: X \to Z$ with $\|Q\| \leq 4K^2(E) + 1$. Then for every $x = y + z \in X$ we have

$$\max(\|y\|, \|z\|)/(4K^2(E) + 1) \leq \|x\| \leq 2 \max(\|y\|, \|z\|),$$

hence

$$2^{-1}B((Y \oplus Z)_{l_\infty}) \subset B(X) \subset (4K^2(E) + 1)B(Y \oplus Z)_{l_\infty}).$$

It easily implies that

$$\mathrm{vr}(X) \leq 2(4K^2(E) + 1)\,\mathrm{vr}((Y \oplus Z)_{l_\infty}).$$

To estimate $\mathrm{vr}((Y \oplus Z)_{l_\infty})$ from above we apply Proposition 1.2. We get $\mathrm{vr}((Y \oplus Z)_{l_\infty}) \leq \sqrt{2}\ \mathrm{vr}(Y)^\theta\ \mathrm{vr}(Z)^{1-\theta}$ with $\theta = \dim Y(\dim Y + \dim Z)^{-1}$.

Since $d(Y, l_2^{\dim Y}) \leq 2$, then $\mathrm{vr}(Y) \leq 2$. Also $(d, Z, l_2^{\dim Z}) \leq (\dim Z)^{1/2} \leq (\log n)^{1/2}$ and hence $\mathrm{vr}(Z) \leq (\log n)^{1/2}$. Moreover $\theta \leq 1$ and $1 - \theta \leq$

$\log n/n$. Putting all these estimates together we finally obtain

$$\mathrm{vr}((Y \oplus Z)_{l_\infty}) \leq \sqrt{2}\, 2(\log n)^{\log n/2n} \leq 4$$

This gives the estimate $\mathrm{vr}(X) \leq 8(4K^2(E) + 1)$ valid for arbitrary finite dimensional subspace XE. This proves that $\sup_n \mathrm{vr}(E, n) < \infty$, hence $\mathrm{avg}(E) = c_0$. □

3. Tensor products of the spaces l_p

In this section we will consider only complex Banach spaces. If E, F are Banach spaces, by $E \hat{\oplus} F$ (resp. $E \hat{\hat{\oplus}} F$) we will denote the completion of the algebraic tensor product $E \otimes F$ in the norm defined for $u \in E \otimes F$ by

$$\|u\|_{E \hat{\oplus} F} = \inf\left\{\sum_i \|e_i\| \|f_i\| \mid u = \sum_i e_i \otimes f_i\right\}$$

$$\text{(resp. } \|u\|_{E \hat{\oplus} F} = \sup\left\{\left|\sum_i e^*(e_i) f^*(f_i)\right| \mid e^* \in E^*,$$

$$\|e^*\| \leq 1, f^* \in F^*, \|f^*\| \leq 1\right\}.$$

A norm $\|\cdot\|_{\mathfrak{A}}$ on $l_2^n \otimes l_2^n$ is said to be unitarily invariant if for all unitary operators $U, V : l_2^n \to l_2^n$ one has $\|\Sigma_i Ue_i \otimes Vf_i\|_{\mathfrak{A}} = \|\Sigma_i e_i \otimes f_i\|_{\mathfrak{A}}$ for every $u = \Sigma_i e_i \otimes f_i \in l_2^n \otimes l_2^n$. The Banach space $(l_2^n \otimes l_2^n, \|\cdot\|_{\mathfrak{A}})$, denoted also by $\mathfrak{A}$, will be called a unitary ideal.

Let $\mathfrak{A}$ be a unitary ideal. It is well known (cf. [4]) that the formula

$$\|(t_i)\|_{l_{\mathfrak{A}}} = \left\|\sum_{i=1}^n t_i e_i \otimes e_i\right\|_{\mathfrak{A}}$$

defines a symmetric norm on $\mathbb{C}^n$ (here (e_i) is an arbitrary orthonormal system in l_2^n). We put $l_{\mathfrak{A}} = (\mathbb{C}^n, \|\cdot\|_{l_{\mathfrak{A}}})$. Conversely, given an n-dimensional symmetric space E we can construct a unitarily invariant norm $\|\cdot\|_{C_E}$ in the following way

$$\left\|\sum_k e_k \otimes f_k\right\|_{C_E} = \|(s_j)\|_E,$$

where $s_j = |\lambda_j(\Sigma_k (f_k, f_k) e_k \otimes e_k)^{1/2}$ (here $\lambda_1(u) \geq \lambda_2(u) \geq \cdots \geq \lambda_n(u)$

denotes the sequence of eigenvalues of the element u regarded as a linear operator in l_2^n).

Each of these operations is the invers of the other one, i.e. for every symmetric space E we have $E = l_{C_E}$ and for any unitary ideal $\mathfrak{A}$ we have $\mathfrak{A} = C_{l_{\mathfrak{A}}}$. Moreover, $(l_{\mathfrak{A}})^*$ is isometrically isomorphic to $l_{\mathfrak{A}^*}$.

If $E = l_2^n$, then the space C_E is usually called the space of Hilbert-Schmidt operators and will be denoted by $HS(l_2^n)$, or simply by HS. For the norm $\|\cdot\|_{HS}$ we have the formula

$$\Big\|\sum_{i,j} a_{ij} e_i \otimes f_j\Big\|_{HS} = \Big(\sum_{i,j} |a_{ij}|^2\Big)^{1/2},$$

where (e_i) and (f_j) are arbitrary orthonormal systems in l_2^n.

Let us notice that $C_{l_1^n} = l_2^n \hat{\bigoplus} l_2^n$ and $C_{l_\infty^n} = l_2^n \hat{\hat{\otimes}} l_2^n$.

Let us observe also that for every unitary ideal $\mathfrak{A}$ the ellipsoid $\mathscr{E}$ of maximal volume contained in $B(\mathfrak{A})$ is of the form $rB(HS)$. This follows from the uniqueness of the ellipsoid $\mathscr{E}$ and the fact that unitary operators in l_2^n induce linear isometries of $\mathfrak{A}$ and hence $\mathscr{E}$ must be unitarily invariant.

PROPOSITION 3.1: $\widetilde{\mathrm{vr}}(l_2^n \hat{\otimes} l_2^n) \leq 32000$.

PROOF: We shall show that there is a $8/\sqrt{n}\, B(HS)$-net for the unit ball $B(l_2^n \hat{\otimes} l_2^n)$ with the cardinality smaller than $(4000)^{2n^2}$. On the other hand the inclusion of balls in symmetric spaces $1/\sqrt{n}\, B(l_2^n) \subset B(l_1^n)$ implies the similar inclusion of balls of unitary ideals $1/\sqrt{n}\, B(HS) \subset B(l_2^n \hat{\otimes} l_2^n)$. Thus, Lemma 1.1 (ii) gives the desired estimate for the volume ratio.

We need the following lemma.

LEMMA: *Let $\mathcal{O}_n$ denotes the set of all orthonormal bases in l_2^n. For every sequence $\epsilon_1, \ldots, \epsilon_n$ with $0 < \epsilon_k < 1$ $(k = 1, 2, \ldots, n)$ there is a set $\mathcal{N} \subset \mathcal{O}_n$ having the properties*

(1) *for every $(e_k) \in \mathcal{O}_n$ there is $(f_k) \in \mathcal{N}$ such that*

$$\max_{1 \leq k \leq n} \|e_k - f_k\|/\epsilon_k \leq 1,$$

(2) card $\mathcal{N} \leq 5^{2n^2}(\epsilon_1, \ldots, \epsilon_n)^{-2n}$.

PROOF OF THE LEMMA: For every $k = 1, \ldots, n$ let us choose

$\frac{1}{2}\epsilon_k B(l_2^n)$-net N_k for the unit ball $B(l_2^n)$ with card $N_k \leq (5/\epsilon_k)^{2n}$ (it is possible by Lemma 1.1(i)). Set $\tilde{N} = N_1 \times \cdots \times N_n \subset B(l_2^n) \times \cdots \times B(l_2^n)$. Hence card $\tilde{N} \leq 5^{2n^2}(\epsilon_1, \ldots, \epsilon_n)^{-2n}$. Then for every $\varphi = (\varphi_k) \in \tilde{N}$ let us pick one orthonormal system f_φ in the ball $\{\psi = (\psi_k) \mid \max_{1\leq k\leq n} \|\psi_k - \varphi_k\|/\epsilon_k \leq \frac{1}{2}\}$, if such a system f_φ exists. Let us define

$$\mathcal{N} = \{f \in \mathcal{O}_n \mid f = f_\varphi \text{ for some } \varphi \in \tilde{N}\}.$$

Since card $\mathcal{N} \leq$ card $\tilde{N}$ the condition (i) is satisfied. Moreover,

$$\mathcal{O}_n \subset \bigcup_{f\in\mathcal{N}} \left\{\psi = (\psi_k) \mid \max_{1\leq k\leq n} \|\psi_k - f_k\|/\epsilon_k \leq 1\right\},$$

so the condition (2) is satisfied too.

In order to constract the nice $8/\sqrt{n}\, B(HS)$-net for the ball $B(l_2^n \hat{\otimes} l_2^n)$ let us consider first the set $\mathfrak{J}$ of all sequences $I = (I_1, \ldots, I_m)$ of subsets of the set $\{1, \ldots, n\}$ with $m = [\log_2 n] + 1$, (and hence $2^{m-1} \leq n < 2^m$) having the following properties

(i) $$\bigcup_{k=1}^{m} I_k = \{1, \ldots, n\}, \quad I_k \cap I_j = \emptyset \quad \text{for } k \neq j,$$

(ii) $$\sum_{k=1}^{m} 2^{-k} \operatorname{card} I_k \leq 2.$$

Obviously, card $\mathfrak{J} \leq 2^{n^2}$.

Next, given a partition $\mathfrak{J} = \{I_1, \ldots, I_m\} \in \mathfrak{J}$ let us define the sequence $\epsilon_1, \ldots, \epsilon_n$ by

$$\epsilon_j = (2^k/n)^{1/2} \quad \text{for } j \in I_k, \quad k = 1, \ldots, m,$$

and for this sequence (ϵ_j) let us construct, as in the lemma, a set $N_I \in \mathcal{O}_n$ of orthonormal systems in l_2^n. Then the properties (1) and (2) imply

(1′) for every $(e_k) \in \mathcal{O}_n$ there is $(f_k) \in N_I$ such that for every $k = 1, \ldots, m$ one has

$$\|e_j - f_j\| \leq (2^k/n)^{1/2} \quad \text{if } j \in I_k,$$

(2′) card $N_I \leq 8^{2n^2}$.

The property (1′) is obvious; to show (2′) let us observe that by (2) we get

$$\begin{aligned}\operatorname{card} N_I &\le 5^{2n^2}\Big(\prod_{k=1}^{m} (n/2^k)^{\operatorname{card} I_k/2}\Big)^{2n}\\ &= 5^{2n^2}\Big(\prod_{k=1}^{m} ((n/2^k)^{2^{k-1}/n})^{\operatorname{card} I_k/2^k}\Big)^{2n^2}\end{aligned}$$

So the inequality $t^{1/2t} \le e^{1/2e}$ combined with the property (ii) of the partition I gives (2′).

For the unit ball $B(l_1^n)$ let us choose a $1/\sqrt{n}\,B(l_2^n)$-net M_1 with $\operatorname{card} M_1 \le (32e/\pi)^n$. It is possible because $\operatorname{vr}(l_1^n) \le 4\sqrt{2e/\pi}$. Next, let us define $M_2 \subset B(l_2^n \hat{\otimes} l_2^n)$ by

$$M_2 = \Big\{ w \in B(l_2^n \hat{\otimes} l_2^n),\ w = \sum_j \xi_j f_j \otimes f_j \text{ with } (\xi_j) \in M_1 \text{ and } (f_j) \in \bigcup_{I^\epsilon} N_I \Big\}.$$

Finally we set

$$M = \{ v \in l_2^n \hat{\otimes} l_2^n \mid v = v_1 + \mathrm{i} v_2 \text{ with } v_1, v_2 \in M_2 \}.$$

We will show that M is a desired net for the ball $B(l_2^n \hat{\otimes} l_2^n)$. To do this let us observe first that putting together the estimates for card $\mathfrak{I}$, card N_I and card M_1 we get the estimate for card M

$$\begin{aligned}\operatorname{card} M &\le (\operatorname{card} M_2)^2 \le (\operatorname{card} \mathfrak{I} \cdot \operatorname{card} N_I \cdot \operatorname{card} M_1)^2\\ &\le (2^{n^2} 8^{2n^2} (2\sqrt{n}+1)^{2n} (32e/\pi)^n)^2 \le (4000)^{2n^2}.\end{aligned}$$

Thus, to complete the proof, it remains to show that M forms a $8/\sqrt{n}\,B(HS)$-net for $B(l_2^n \hat{\otimes} l_2^n)$. Let us define

$$B_R(l_2^n \hat{\otimes} l_2^n) = \{ u \in B(l_2^n \hat{\otimes} l_2^n) \mid u \text{ is selfadjoint} \}.$$

Then

$$B(l_2^n \hat{\otimes} l_2^n) \subset B_R(l_2^n \hat{\otimes} l_2^n) + \mathrm{i} B_R(l_2^n \hat{\otimes} l_2^n),$$

hence it is enough to check that M_2 forms $4/\sqrt{n}\,B(HS)$-net for $B_R(l_2^n \hat{\otimes} l_2^n)$.

To see this let us fix $u \in B_R(l_2^n \hat{\otimes} l_2^n)$ and let us observe that u is of the form

$$u = \sum_{j=1}^{n} \lambda_j e_j \otimes e_j \quad \text{for some } (\lambda_j) \in R^n,\ (e_j) \in \mathcal{O}_n.$$

Moreover, $\|u\|_{l_2^n \hat{\otimes} l_2^n} = \Sigma_{j=1}^n |\lambda_j| \leq 1$. Next let us define the partition *I of the set* $\{1, \ldots, n\}$ by

$$I_k = \{j \mid 2^{-k} < |\lambda_j| \leq 2^{-k+1}\} \quad \text{for } k \leq m-1,$$
$$I_m = \{j \mid |\lambda_j| \leq 2^{-m+1}\}.$$

Then $I \in \mathcal{I}$. The verification of the property (i) is immediate, to show (ii) let us notice that

$$\sum_{k=1}^{m} 2^{-k} \text{ card } I_K \leq \sum_{j=1}^{n} |\lambda_j| + 2^{-m} \text{ card } I_m \leq 2.$$

So let us pick a sequence $(\xi_j) \in M_1$ such that

$$(*) \qquad \left(\sum_{j=1}^{n} |\xi_j - \lambda_j|^2\right)^{1/2} \leq 1/\sqrt{n}$$

and the orthonormal system $(f_j) \in N_I$ such that

$$(**) \qquad \|e_j - f_j\| \leq \sqrt{2k/n} \quad \text{for } j \in I_k,\ k = 1, \ldots, m.$$

Let us define $w = \Sigma_{j=1}^n \xi_j f_j \otimes f_j$. Then $w \in M_2$ and we will show that $\|u - w\|_{HF} \leq 4\sqrt{n}$. By the triangle inequality we have

$$\|u - w\|_{HS} = \left\|\sum_{j=1}^{n} \lambda_j e_j \otimes e_j - \sum_{j=1}^{n} \xi_j f_j \otimes f_j\right\|_{HS}$$
$$\leq \left\|\sum_{j=1}^{n} \lambda_j (e_j - f_j) \otimes e_j\right\|_{HS} + \left\|\sum_{j=1}^{n} \lambda_j f_j \otimes (e_j - f_j)\right\|_{HS}$$
$$\left\|\sum_{j=1}^{n} (\lambda_j - \xi_j) f_j \otimes f_j\right\|_{HS}.$$

Since (e_j) and (f_j) are orthonormal systems, the last norm is equal to $(\Sigma_{j=1}^n |\lambda_j - \xi_j|^2)^{1/2}$, and the first two norms are equal to $(\Sigma_{j=1}^n |\lambda_j|^2 \|e_j - f_j\|^2)^{1/2}$. Using (*) and the definition of the partition I, we get

$$\left(\sum_{j=1}^{n} |\lambda_j|^2 \|e_j - f_j\|^2\right)^{1/2} \leq \sum_{k=1}^{m} \sum_{j \in I_k} |\lambda_j| 2^{-k+1} 2^k / n^{1/2} = \left(2 \sum_{j=1}^{n} |\lambda_j| / n\right)^{1/2} = \sqrt{2/n}.$$

Thus combining the above inequality with (*) we finally get

$$\|u - w\|_{HS} \leq 2\sqrt{2}/\sqrt{n} + 1/\sqrt{n} < 4/\sqrt{n}.$$

This proves that M_2 forms a $4/\sqrt{n}B(HS)$-net for $B_R(l_2^n \hat{\otimes} l_2^n)$ and completes the proof of Proposition 3.1. □

For an arbitrary unitary ideal we have the following result.

PROPOSITION 3.2: $\widetilde{\mathrm{vr}}(\mathfrak{A}) \le (\sqrt{\pi e}/2^{1/4})\, 32000\, \widetilde{\mathrm{vr}}(l_{\mathfrak{A}})$.

PROOF: Let us apply the complex version of the inequality (2.2) for the symmetric space $l_{\mathfrak{A}}$. Thus for the unitary ideal we obtain

$$\alpha\sqrt{\pi e}/(2^{1/4}\sqrt{n}\ \mathrm{vr}(l_{\mathfrak{A}}))B(HS) \subset B(\mathfrak{A}) \subset B(l_2^n \hat{\otimes} l_2^n).$$

As it was mentioned in the beginning of this section, the ball $1/\sqrt{n}\,B(HS)$ is the ellipsoid of maximal volume contained in $B(l_2^n \hat{\otimes} l_2^n)$. Thus, the above inclusions and Proposition 3.1 imply

$$\begin{aligned}\mathrm{vol}\, B(\mathfrak{A}) &\le \alpha^{2n^2}\, \mathrm{vol}\, B(l_2^n \hat{\otimes} l_2^n)\\ &\le (\alpha\ 32000)^{2n^2}\, \mathrm{vol}(1/\sqrt{n}\, B(HS))\\ &\le (32000\ 2^{1/4}\, \widetilde{\mathrm{vr}}(l_{\mathfrak{A}})/\sqrt{\pi e})^{2n^2}\, \mathrm{vol}(\alpha\sqrt{\pi e}/2^{1/4}\sqrt{n}\ \widetilde{\mathrm{vr}}(l_{\mathfrak{A}})B(HS)),\end{aligned}$$

what we wanted to prove. □

Let us consider now the tensor products $l_p^n \hat{\otimes} l_q^n$ for arbitrary $1 \le p$, $q \le 2$. It seems possible that the following question has the positive answer.

PROBLEM 3.3: *Let* $1 \le p \le q \le 2$. *Does there exist a constant* C_q *(independent on* p*) such that* $\mathrm{vr}(l_p^n \hat{\oplus} l_q^n) \le C_q$?

We can only prove the estimate of this kind for $q = 2$.

PROPOSITION 3.4: *Let* $1 \le p \le 2$. *Then* $\mathrm{vr}(l_p^n \hat{\otimes} l_2^n) \le 32000$.

PROOF: If $1 \le p \le 2$ we have the following inclusions for the unit balls

$$B(l_1^n \hat{\otimes} l_2^n) \subset B(l_p^n \hat{\otimes} l_2^n) \subset B(l_2^n \hat{\otimes} l_2^n)$$

On the other hand one has $l_1^n \hat{\otimes} l_1^n = (\Sigma_{k=1}^n \oplus l_2^n)_1$ and $HS(l_2^n) = l_2^{n^2}$, so the inclusions (1.3) imply

$$1/\sqrt{n}\, B(HS) \subset B(l_p^n \hat{\otimes} l_2^n) \subset B(l_2^n \hat{\otimes} l_2^n).$$

Then by Proposition 3.1 we obtain

$$\begin{aligned}\mathrm{vr}(l_p^n \hat{\otimes} l_2^n) &\le (\mathrm{vol}\, B(l_p^n \hat{\otimes} l_2^n)/\mathrm{vol}\, 1/\sqrt{n}\, B(HS))^{1/2n^2}\\ &\le (\mathrm{vol}\, B(l_2^n \hat{\otimes} l_2^n)/\mathrm{vol}\, 1/\sqrt{n}\, B(HS))^{1/2n^2}\\ &= \mathrm{vr}(l_2^n \hat{\otimes} l_2^n) \le 32000. \quad \square\end{aligned}$$

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