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S. Greco and C. Traverso

Summary

We give several results on seminormal schemes with particular concern to the study of their singularities. After showing that seminormality descends by faithful flatness and can be checked at the points of depth 1, we show that ordinary hypersurface singularities are seminormal (in characteristic zero) and that seminormality is preserved by a wide class of flat morphisms. In particular a sufficiently good seminormal scheme is analytically seminormal. We show also that a complex analytic space is seminormal iff it is weakly normal according to Andreotti-Norguet, and we characterize the seminormal varieties which are S_2 and Gorenstein in codimension 1 by giving an explicit description of their singularities in a "large" open set.

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0. Introduction

The main purpose of this paper is to give some information on seninormal (SN for short) schemes, with special concern to their singularities.

The notion of seminormality we deal with is the one given in [31], and coincides in characteristic zero with the one introduced by Andreotti-Bombieri [1]. The definition is recalled in section 1 along with some known results. From these we deduce that seminormality descends by faithful flatness (Th. 1.6, Cor. 1.7).

In section 2 we give some criteria for seminormality. The essential point, is that this property can be checked at the points of depth 1, that is in codimension 1 if property S_2 is verified (Th. 2.6, Cor. 2.7). From this it follows that if a scheme X is SN, then all the components of X which are S_2 are SN (Cor. 2.9); this may be false without S_2 (Example 2.11).

In Section 3 we show that a projective hypersurface with ordinary singularities only (that is a generic projection of a projective irreducible smooth variety) is SN provided the ground field has characteristic zero (Th. 3.7). We do not know whether this is true in positive characteristic; however by using a result by J. Roberts [27] we can show that any projective reduced irreducible variety is birationally equivalent to a SN projective hypersurface (Th. 3.5).

In Sections 4 and 5 conditions are given for the permanence of seminormality under base change. In particular we show that "seminormal" morphisms preserve seminormality (Th. 5.8), and that a sufficiently good (e.g. excellent) local ring is SN if and only if its completion is such (Cor. 5.3). From this it follows that a complex algebraic variety is SN if and only if its corresponding analytic space is such (Cor. 5.4).

In Section 6 we show that for a complex analytic space our algebraic definition of seminormality coincides with the notion of weak normality introduced by Andreotti-Norguet [2] (Th. 6.12). In order to do this we have to study in some detail the ring of holomorphic functions on a compact semianalytic Stein subset (6.1 to 6.8); in particular we show, by the aid of Matsumura's Jacobian criterion (see [36]) that such a ring is excellent (Cor. 6.8).

In Section 7 we show that under suitable conditions a SN scheme can be normalized by blowing up its conductor sheaf. This is used in the next section to characterize SN Gorenstein local rings of dimension 1.

In Section 9 the preceding results are used to characterize the SN varieties which are S_2 and Gorenstein in codimension 1 (e.g. Gorenstein varieties and in particular, locally complete intersections). These characterizations include the ones announced in [12]. Among other things we show that such varieties are locally hypersurfaces outside of a suitable closed subset of codimension > 1, and we give explicit analytic equations at the points of this open set. For example if the ground field is algebraically closed and of characteristic $\neq 2$, this equation at all singular closed points is always of the form $X_1X_2 = 0$ (Th. 9.10).

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1. Preliminaries and descent of seminormality by faithful flatness

All rings are assumed to be commutative and noetherian

In this section we recall some elementary facts on seminormal rings and we show that seminormality descends by faithful flatness.

1.1. Definition: A ring homomorphism $f: A \to B$ is a *quasi-isomorphism* if the following equivalent conditions hold:

(a) $(B \otimes_A k(\mathfrak{p}))_{\text{red}} = k(\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Spec}(A)$;

(b) the induced map on the spectra is bijective and with trivial residue field extensions.

1.2. Definition: Let B be a finite overring of the ring A (that is: A is a subring of B and B is finitely generated as an A-module). The *seminormalization of A in B* is the largest subring of B containing A which is (canonically) quasi-isomorphic to A. It always exists (see [31]) and is denoted by ${}^{+}_{B}A$.
If $A = {}^{+}_{B}A$ we say that A is *seminormal in B* (SN for short). The ring ${}^{+}_{B}A$ is always SN in B (l.cit.).

1.3. Definition: A ring A is a *Mori ring* if it is reduced and its integral closure $\bar{A}$ (in its total ring of fractions) is finite over A (see [15]).

If A is a Mori ring the *seminormalization* of A is ${}^{+}_{\bar{A}}A$ and is denoted by ${}^{+}A$. If $A = {}^{+}A$, the ring A is said to be *seminormal* (SN). Clearly ${}^{+}A$ is always SN. Observe that a SN ring is, by definition, a Mori ring which is noetherian. For more general definitions and relations with the Picard group see [1], [22], [31].

1.4. PROPOSITION: *Let B be a finite overring of A, and let $C = {}^{+}_{B}A$. Then*:

(a) *if $A \neq C$ the ring $C/\mathrm{Ann}_A(C/A)$ is not reduced*;

(b) *the ring $B/\mathrm{Ann}_C(B/C)$ is reduced.*

In particular if A is SN *in B, the ring $B/\mathrm{Ann}_A(B/A)$ is reduced.*

PROOF: See [31], 1.3 and 1.7.

1.5. PROPOSITION: *Let B be a finite overring of A and C a finite overring of B. We have*:

(a) *If A is* SN *in B and B is* SN *in C, then A is* SN *in C*;

(b) *if A is* SN *in C, then A is* SN *in B.*

PROOF: It follows easily from the definitions.

1.6. THEOREM: *Let $f: A \to A'$ be a faithfully flat (FF) ring homomorphism. Let B be a finite overring of A and put $B' = A' \otimes_A B$. Then if A' is* SN *in B', A is* SN *in B.*

PROOF: Let $C = {}^{+}_{B}A$ and put $C' = A' \otimes_A C$. By flatness C' is a subring of B' containing A'. Thus A' is SN in C' by 1.5. Let $\mathfrak{a} = \mathrm{Ann}_A(C/A)$, and put $\mathfrak{a}' = \mathfrak{a}A'$. Then $\mathfrak{a}' = \mathrm{Ann}_{A'}(C'/A')$ ([6], p. 40, Cor. 2). Then if $C \neq A$ we have a contradiction by 1.4, since $\mathfrak{a}' \cap \mathrm{A} = \mathfrak{a}$ by FF.

1.7. COROLLARY: *Let $f: A \to A'$ be a* FF *ring homomorphism. Then A is* SN *if A' is such.*

PROOF: Clearly A is reduced. Moreover by flatness we have that $\bar{A} \otimes_A A'$ has the same total ring of fractions as A'. Hence it is contained in the integral closure of A', which is finite by assumption. Then $\bar{A}$ is finite over A by FF ([6], p. 52, Prop. 11). The conclusion follows easily by 1.6 and 1.5(b).

1.8. COROLLARY: *Let A be a local ring. If the completion of A is* SN, *then A is* SN.

2. Criteria of seminormality

We give several criteria which can be used to check seminormality. In particular we show that seminormality is a local property, and that it can be checked at the primes of depth 1 – hence in codimension 1 if property S_2 is verified. From this it follows that if A is SN, any component of A which is S_2 is SN. A counterexample shows that this is false without S_2.

2.1. PROPOSITION: *Let B be a finite overring of A and put $C = {}^{+}_{B}A$. Let S be a multiplicative subset of A. Then*:

(a) *$S^{-1}C$ is the seminormalization of $S^{-1}A$ in $S^{-1}B$;*
(b) *if A in* SN *is B, then $S^{-1}A$ is* SN *in $S^{-1}B$.*

PROOF: By [31], 2.2 we have (b). From (b) it follows that $S^{-1}C$ is SN in $S^{-1}B$. Moreover it is easy to check that the inclusion $S^{-1}A \to S^{-1}C$ is a quasi-isomorphism, and (a) is proved.

2.2. COROLLARY: *Let A be a ring, and let S be a multiplicative subset of A. Then if A is* SN, *$S^{-1}A$ in such. Moreover if A is Mori and $C = {}^{+}A$, then $S^{-1}C = {}^{+}(S^{-1}A)$.*

2.3. COROLLARY: *Let A be a Mori ring* (*see* 1.3). *Then the subset of* Spec(A) *consisting of the primes $\mathfrak{p}$ such that $A_\mathfrak{p}$ is* SN *is open and nonempty.*

PROOF: Let $C = {}^{+}A$ and let $\mathfrak{a} = \mathrm{Ann}_A(C/A)$. Since C is a finite A-module we have $A_\mathfrak{p} = C_\mathfrak{p}$ if and only if $\mathfrak{a} \not\subset \mathfrak{p}$ ([6], p. 133, Prop. 17). By 2.1 $A_\mathfrak{p}$ is SN if and only if it coincides with $C_\mathfrak{p}$, and the conclusion follows, because $\mathfrak{a}$ contains a nonzero divisor. The converse of 2.1(b) is clearly false. However we have:

2.4. PROPOSITION: *Let A, B, S be as in* 2.1, *and assume that the square*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ S^{-1}A & \longrightarrow & S^{-1}B \end{array}$$

is a pull-back (*e.g. B is a subring of $S^{-1}B$ and $A = B \cap (S^{-1}A)$*). *Then if $S^{-1}A$ is* SN *in $S^{-1}B$ it follows that A is* SN *in B.*

PROOF: Put $C = {}^{+}_{B}A$. By 2.1 it follows that $S^{-1}C = S^{-1}A$. Hence by using the universal property of the pull-back we see that the embedding $A \to C$ has a right inverse $C \to A$, which must be injective. Thus $A = C$.

2.5. PROPOSITION: *Let B be a finite overring of A. Then the following are equivalent*:

(i) *A is* SN *in B*;

(ii) $\mathrm{nil}(B) \subset A$ *and* A_{red} *is* SN *in* B_{red};

(iii) *for any ideal $\mathfrak{b}$ of B contained in A the ring $A/\mathfrak{b}$ is* SN *in $B/\mathfrak{b}$*;

(iv) *$A/\mathfrak{b}$ is* SN *in $B/\mathfrak{b}$ for some ideal $\mathfrak{b}$* of *B contained in A.*

If moreover we have

(*) *every regular element of A is regular in B*

then the above conditions are equivalent to:

(v) *A is* SN *in $K \cap B$ (where K is the total ring of fractions of A, and the intersection is taken in the total ring of fractions of B).*

PROOF: Any ideal $\mathfrak{b}$ of B contained in A is contained in $\mathrm{Ann}_A(B/A)$. Hence $A_\mathfrak{p} = B_\mathfrak{p}$ for any prime ideal of A not containing $\mathfrak{b}$ ([6], p. 133, Prop. 17). It follows that a ring C between A and B is quasi-isomorphic to A if and only if $C/\mathfrak{b}$ is quasi-isomorphic to $A/\mathfrak{b}$. Now the equivalence of (i), (iii), (iv) is clear, as well as the implication (ii) $\to$ (iv). Finally if $C = A + \mathrm{nil}(B)$ it is easy to see that A and C are quasi isomorphic, whence (i) implies (ii).
If (*) is satisfied then $K \cap B$ is SN in B by 2.4 and (v) $\leftrightarrow$ (i) by 1.5.

2.6. THEOREM: *Let B be a finite overring of the ring A. Then*:

(a) *The following are equivalent*:

(i) *A is* SN *in B*;

(ii) *$A_\mathfrak{p}$ is* SN *in $B_\mathfrak{p}$ for all* $\mathfrak{p} \in \mathrm{Spec}(A)$;

(iii) *$A_\mathfrak{m}$ is* SN *in $B_\mathfrak{m}$ for all maximal ideals* $\mathfrak{m}$ *of A*;

(iv) *$A_\mathfrak{p}$ is* SN *in $B_\mathfrak{p}$ for all* $\mathfrak{p} \in \mathrm{Ass}_A(B/A)$;

(b) *If* (*) *of* 2.5 *holds the above are equivalent to*:

(v) *$A_\mathfrak{p}$ is* SN *in $B_\mathfrak{p}$ whenever depth* $A_\mathfrak{p} = 1$.

(c) *If moreover A is S_2 the above are equivalent to*:

(vi) *$A_\mathfrak{p}$ is* SN *in $B_\mathfrak{p}$* whenever $\mathfrak{p}$ *has height* 1.

PROOF: By 2.1 (i) implies (ii). Clearly (ii) implies (iii), (iv), (v), and an easy argument shows that (iii) implies (i). Assume (iv), and put $C = {}^{+}_{B}A$. If $C \neq A$ let $\mathfrak{p} \in \mathrm{Ass}_A(C/A)$. By 2.1 $C_\mathfrak{p}$ is the seminormalization of $A_\mathfrak{p}$ in $B_\mathfrak{p}$; hence $A_\mathfrak{p} = C_\mathfrak{p}$ by assumption, a contradiction. Thus $A = C$ and (iv) implies (i).

Assume now that (*) of 2.5 holds. Then (i) is equivalent to: A is SN in $C = K \cap B$ where K is the total ring of fractions of A (see 2.5). Let $\mathfrak{p} \in \mathrm{Ass}_A(C/A)$. Then $\mathfrak{p} = (x):(y)$ where x, y are in A and x is not a zero-divisor. Hence $\mathfrak{p} \in \mathrm{Ass}_A(A/Ax)$ and this implies that depth $A_{\mathfrak{p}} = 1$. Then A is SN in C by the equivalence of (i) and (iv) above. Thus (v) implies (i).
The last statement is immediate from the definition of S_2.

2.7. Corollary: *Let A be a Mori ring with integral closure $\bar{A}$. Then the following are equivalent*:

(i) *A is* SN;
(ii) *$A_{\mathfrak{m}}$ is* SN *for any maximal ideal $\mathfrak{m}$ of A*;
(iii) *$A_{\mathfrak{p}}$ is* SN *for any $\mathfrak{p} \in \mathrm{Spec}(A)$*;
(iv) *$A_{\mathfrak{p}}$ is* SN *for all $\mathfrak{p} \in \mathrm{Ass}_A(\bar{A}/A)$*;
(v) *$A_{\mathfrak{p}}$ is* SN *whenever depth $A_{\mathfrak{p}} = 1$.*

If moreover A is S_2 the above are equivalent to:

(vi) *$A_{\mathfrak{p}}$ is* SN *whenever* dim $A_{\mathfrak{p}} = 1$;
(vii) *The ring $\bar{A}/\mathfrak{m}$ is reduced, where $\mathfrak{b}$ is the conductor.*

Proof: The equivalence of (vi) and (vii) follows by 1.4 and the fact that the conductor localizes. The remaining equivalences follow by 2.6.

2.8. Corollary: *Let A be a* SN *ring, and let C be a subring of $\bar{A}$ containing A. If C is S_2, then it is* SN.

Proof: Assume first that A is a local ring of dimension 1 and maximal ideal $\mathfrak{m}$. By 1.4 we have $\mathfrak{m} = \mathrm{rad}\, \bar{A}$ whence $\mathfrak{m} = \mathrm{rad}\, C$. It follows that $\mathfrak{m}$ is the conductor of C, so that C is SN by 2.7(vii). The general case follows by 2.7(vi).

2.9. Corollary: *Let A be a* SN *ring and let $\mathfrak{p}$ be a minimal prime of A. If $C = A/\mathfrak{p}$ is S_2, then it is* SN. *In particular if* dim $A = 1$ *any such C is* SN.

Proof: Let $\mathfrak{p} = \mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the minimal primes of A, and put $A_i = A/\mathfrak{p}_i$, and $D = A_1 \times \bar{A}_2 \times \cdots \times \bar{A}_n$. Then we have $A \subset D \subset \bar{A}$ and D is S_2. Thus D is SN by 2.8 and hence $C = A_1$ is SN as well.

2.10. Remarks:

(i) The equivalence of (i) and (vii) of 2.6 follows also from [31] and [4], Prop. 6.1.

(ii) Corollary 2.9 is false without S_2 as the following example shows:

2.11. EXAMPLE: *There is a* SN *ring D such that $D/\mathfrak{p}$ is not* SN *for a minimal prime $\mathfrak{p}$.*

D is the ring of the affine variety V obtained as follows: let $\bar{V}$ be the disjoint union of two affine planes, and let V be the variety obtained by identifying the points $(t, 0)$ of the first plane to the points (t^2, t^3) of the second plane. Since the line $(t, 0)$ of the first plane is identified to a plane cuspidal curve, it follows that V is the union of the first plane, and of another component, consisting of the second plane which has acquired one singular point in the process. One can show that V is SN, while the singular component is clearly not SN, since it is homeomorphic to its normalization, which is the second plane. Now we give explicit computations. Let

$$A = k[X, Y, U, V, E, F]/I = k[x, y, u, v, e, f]$$

where

$$I = (U, V, E-1, F) \cap (X, Y, E, F-1)$$

(Thus $A \cong k[X, Y] \times k[U, V]$).

Let $J = (y, u^2 - v^3) \subset A$, and put $B = A/J$, so that B is the coordinate ring of the disjoint union of a line and a plane cuspidal cubic.

Put $C = k[R, S]/(R^2 - S^3) = k[r, s]$ and let $\phi: C \to B$ be the k-homomorphism defined by

$$\phi(r) = x^3 + u \bmod J$$
$$\phi(s) = x^2 + v \bmod J$$

Let $\pi: A \to B$ be the canonical homomorphism, and let D be the pull-back of ϕ and π.

One can check that D is a subring of A, and precisely

$$D = k[x^3 + u, x^2 + v, y, u^2 - v^3, xy] \subset A$$

Since ϕ is injective and $C = K \cap B$ where K is the total ring of fractions of C, it follows that C is SN in B by 2.5, whence D is SN in A by 2.4 (see 4.3 below for details). Since A in the normalization of D, it follows then that D is SN.

Observe now that $A' = A/fA$ is isomorphic to $k[X, Y]$, whence $\mathfrak{P} = fA$ is a minimal prime of A. Put $\mathfrak{p} = D \cap \mathfrak{P}$. Then $\mathfrak{p}$ is a minimal prime of D, and if $D' = D/\mathfrak{p}$ one can see that there is a canonical isomorphism

$$D' \cong k[X^2, X^3, Y, XY] \subset k[X, Y] = A'.$$

Hence A' is the normalization of D'. Let $\mathfrak{b} = X^2A$. Then $\mathfrak{b} \subset D'$ and $A'/\mathfrak{b}$ is not reduced, while $D'/\mathfrak{b}$ is reduced. Then by 2.5 it follows that $D'/\mathfrak{b}$ is not SN in $A'/\mathfrak{b}$, whence, by 2.5 again, D' is not SN.

2.12. Remark: In the above example one can see that D' is not S_2, which agrees with 2.9.

One can see also that D is not S_2. It might be interesting to see how the irreducible components of a SN S_2 ring are like.

3. Seminormal S_2 schemes and generic projections

After some basic definitions we show that any irreducible projective variety over an algebraically closed field is birationally equivalent to a SN hypersurface, and, in characteristic zero, that any hypersurface which is a generic projection of a nonsingular irreducible projective variety is SN.

In the following "scheme" means "locally noetherian scheme" (not necessarily separated).

3.1. Definition: A scheme X is said to be a *Mori scheme* if it is reduced and its normalization is finite over X. This is equivalent to: X has an affine cover whose rings are (noetherian and) Mori.

A ring A is Mori if and only if Spec(A) is a Mori scheme. An excellent scheme (in particular an algebraic scheme over a field) which is reduced is Mori ([18], IV_2.7.8.5 and 7.8.6).

3.2. Definition: A scheme X is SN if it is Mori and $\mathcal{O}_{X,x}$ is SN for all $x \in X$.

3.3. Proposition: *Let X be a scheme. Then the following are equivalent*:

(i) *X is* SN;

(ii) *every open subscheme of X is* SN;

(iii) *X can be covered by affine noetherian open* SN *subschemes*;

(iv) *for any open affine noetherian subscheme* U *of* X *the ring* $\Gamma(U, \mathcal{O}_X)$ *is* SN.

If moreover X *is* S_2 *the above are equivalent to*:

(v) *there is a closed subscheme* Z *of* X *whose codimension is* ≥ 2, *and such that* $X - Z$ *is* SN;

(vi) X *is Mori and* $\mathcal{O}_{X,x}$ *is* SN *whenever it has dimension* 1.

PROOF: The equivalence of (i), (ii) and (iii) is obvious, and (ii) is equivalent to (iv) by 2.7. Clearly (i) implies (v). If (v) holds X is a Mori scheme since by S_2 $\mathcal{O}_X = j_*(\mathcal{O}_{X-Z})$, where $j: X - Z \to X$ is the canonical map ([18], IV_4.21.13.4). Moreover if $\dim \mathcal{O}_{X,x} = 1$ then x cannot be in Z; thus (v) implies (vi) if X is S_2. Finally (vi) implies (i) by 2.7.

3.4. PROPOSITION: *Let* k *be a field, and let* X *be an algebraic* k*-scheme. Assume* X *is* S_2. *Then the following are equivalent*:

(i) X *is* SN;

(vii) *There is an open* $U \subset X$ *with* $\mathrm{codim}(X - U) \geq 2$ *and such that* $\mathcal{O}_{X,x}$ *is* SN *for all closed points of* X *which are in* U.

PROOF: By 3.3 (i)$\to$(vii). Conversely since X is a Jacobson scheme, every point of U is a generalization of some closed point of X contained in U ([17], p. 307, I.6.4). Hence U is SN by 2.1, and X is SN by 3.3.

3.5. THEOREM: *Let* Y *be an* r*-dimensional irreducible reduced projective variety over an algebraically closed field* k. *Then* Y *is birationally equivalent to a* SN *hypersurface* $X \subset \mathbf{P}_k^{r+1}$.

If moreover Y *is normal one can choose* X *such that there is a finite birational morphism* $p: Y \to X$.

PROOF: We may assume Y is normal. Apply then Theorem 1 of [27] with $m = r + 1$: we get a hypersurface $X \subset \mathbf{P}_k^{r+1}$ and a finite birational morphism $p: Y \to X$. Then Y is the normalization of X. Moreover there are two closed subvarieties V, W of X such that:

(a) V has pure codimension 1 and W has pure codimension 2;

(b) $\mathrm{Sing}(X) \subset V \cup p(\mathrm{Sing}(Y))$;

(c) there is an open dense subset U' of $V - W$ such that for any closed point x of X contained in U' the ring $\hat{\mathcal{O}}_x$ is isomorphic to $k[[X_1, \ldots, X_n]]/(X_1X_2)$. (apply l.cit. (iii) and (ii) with $i = 2, 3$).

Put $Z' = V - U'$, $Z'' = p(\mathrm{Sing}(Y))$ and $Z = Z' \cup Z''$. It is easy to see that Z has codimension ≥ 2.

If $U = X - Z$ and $x \in U$ is a closed singular point of X, we have $(\mathcal{O}_x)^\wedge = k[[X_1, \ldots, X_n]]/(X_1X_2)$. A straightforward computation shows that $(\mathcal{O}_x)^\wedge$ is SN (e.g. it is obtained from the normalization by glueing over $\mathfrak{p} = (X_1, X_2)/(X_1X_2)$). Hence $\mathcal{O}_x$ is SN by 1.8, and since X is a hypersurface the conclusion follows by 3.4.

3.6. REMARK: From the proof of [27], Th. 1 one can see that, when Y is normal, the morphism $p: Y \to X$ of 3.5 is the composition of a Veronese embedding of degree 2 and of a generic projection.

We do not know whether in positive characteristic, a generic projection only is sufficient to obtain a SN X. However we have the following

3.7. THEOREM: *Let $k = \mathbf{C}$ and let $X' \subset \mathbf{P}_k^n$ be a smooth irreducible algebraic variety of dimension d. Then the generic projection X in $\mathbf{P}_k^{d+1}$ is* SN.

This result is due to Salmon [29] for $d = 1$ and to Bombieri [5] for $d = 2$.

In order to prove 3.7 we need some preliminaries.

3.8. DEFINITION: Let X be an algebraic k-scheme (k algebraically closed) and let $Y \subset X$ be a closed irreducible subscheme of codimension 1. Y is said to be *bihyperplanar* if there is a dense open $U \subset Y$ such that $\mathrm{gr}(\mathcal{O}_{X,x}) \cong k[X_1, \ldots, X_n]/(X_1X_2)$ for all closed points x of X contained in U (this means that the tangent cone of X at "almost all" its closed points belonging to Y is the union of two distinct hyperplanes). The following result is classical (see e.g. [10]):

3.9. THEOREM: *Let X be as in* 3.7. *Then the singular locus of X is a union of closed irreducible subvarieties of codimension* 1 *which are bihyperplanar.*

The proof of 3.7 is a consequence of 3.9 and of the following Lemma.

3.10. LEMMA *Let A be a local ring containing a perfect coefficient field k. Assume* $\mathrm{gr}(A) \cong k[X_1, \ldots, X_n]/(X_1X_2)$. *Then A is* SN.

PROOF: Since $\mathrm{gr}(A) = \mathrm{gr}(\hat{A})$, by 1.8 we may assume A is complete. Then by the Cohen structure Theorem and by [20], p. 190, Lemma 6 we have $A = k[[X_1, \ldots, X_n]]/(f)$, where $f = X_1X_2 + g$, with g of order at least 3.

By the jacobian criterion of Nagata (see e.g. [18], 0_{IV}.22.7.3) the singular locus of Spec(A) is $V(I/f)$ where I is the ideal generated by f and its partial derivatives. It is easy to see that I contains $Y_1 = X_1 + (\partial g/\partial X_2)$ and $Y_2 = X_2 + (\partial g/\partial X_2)$, and that Y_1, Y_2 are linearly independent modulo $(X_1, \ldots, X_n)^2$. Hence I contains the prime ideal (of height 2) $(Y_1, Y_2) = \mathfrak{P}$. Thus either $I = \mathfrak{P}$, or I has height ≥ 3. In the second case A is normal by the Krull-Serre criterion. Hence we may assume $I = \mathfrak{P}$, and after a change of variables we may assume

(1) $f \in (X_1, X_2) = \mathfrak{P}$

(2) $\mathfrak{p} = \mathfrak{P}/(f)$ is the only singular prime of height 1 in A.

Therefore by (2) and 2.7 it is sufficient to show that $A_\mathfrak{p}$ is SN. Let $B = (A_\mathfrak{p})$. It is easy to see that the canonical embedding $k[[X_3, \ldots, X_n]] \to B$ factors uniquely through $K = k((X_3, \ldots, X_n))$, and that K is therefore a coefficient field of B. Thus we have $B = K[[X_1, X_2]]/fK[[X_1, X_2]]$. Let $\varphi = aX_1^2 + bX_1X_2 + cX_2^2$ be the leading form of f in $K[[X_1, X_2]]$, where $a, b, c \in k[[X_3, \ldots, X_n]]$ and $a(0) = c(0) = 0$, $b(0) = 1$. By the Hensel Lemma ([7], p. 84, Th. 1) applied to $R(T) = a + bT + cT^2$, φ is the product of two linear forms which are linearly independent over K. Thus with a suitable change of coordinates we may assume

$$f = X_1X_2 + f_3 + \cdots \quad (\text{in } K[[X_1, X_2]]).$$

By induction one can find forms $g_2, h_2, g_3, h_3, \ldots$ (of degree equal to the index) in X_1, X_2 so that

$$f = (X_1 + g_2 + g_3 + \cdots)(X_2 + h_2 + h_3 + \cdots).$$

Hence after a new change of coordinates one can assume $f = X_1X_2$ (in $K[[X_1, X_2]]$), and a straightforward computation shows now that B is SN. Hence $A_\mathfrak{p}$ is SN by 1.8, and the conclusion follows.

Proof of 3.7: By applying 3.9 and 3.10 one sees easily that there is a closed $Z \subset X$ of codimension ≥ 2 such that $X - Y$ is seminormal at each closed point. Since X is a hypersurface it follows that X is SN by 3.4.

4. A base change theorem

We show that the property "A is SN in B" is preserved by any reduced base change. Applications shall be given in the next section.

Recall that a ring homomorphism $A \to A'$ is reduced (resp. normal, regular, etc.) if it is flat and its fibers are geometrically reduced (resp. normal, etc.). For details see e.g. [14] or [26] or [18]. IV_2, p. 192 and following.

4.1. THEOREM: *Let B be a finite overring of A and let $A \to A'$ be a reduced ring homomorphism. Assume A is* SN *in B. Then A' is* SN *in $B' = A' \otimes_A B$.*

Before proving the theorem we give two lemmas.

4.2. LEMMA: *Let R be a ring and let*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle u} \\ C & \xrightarrow[v]{} & D \end{array} \tag{1}$$

be a pull-back of R-algebras. Let S be a flat R-algebra. Then the diagram obtained by tensoring (1) *by S over R is a pull-back.*

PROOF Clearly (1) is a pull-back if and only if the sequence of R-modules

$$0 \longrightarrow A \xrightarrow{\phi} B \oplus C \xrightarrow{\psi} D$$

is exact, where $\phi(a) = (f(a), g(a))$ and $\psi(b, c) = u(b) - v(c)$. The conclusion follows by the definition of flatness.

4.3. LEMMA: *Let* (1) *be a pull-back, and assume that the horizontal arrows are injective and finite, and that the vertical ones are surjective. Then if C is* SN *in D, it follows that A is* SN *in B.*

PROOF: Let $\mathfrak{b} = \operatorname{Ker} u$. Then it is easy to show that $\mathfrak{b} \subset A$ and that $A/\mathfrak{b} = C$. The conclusion follows then by 2.5.

PROOF OF 4.1: By [31], Th. 2.1, we may assume that A is obtained from B by "glueing" over a $\mathfrak{p} \in \operatorname{Spec}(A)$. This means that we have a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A_{\mathfrak{p}} & \longrightarrow & B_{\mathfrak{p}} \\ \downarrow & & \downarrow \\ k = k(\mathfrak{p}) & \longrightarrow & (k(\mathfrak{p}) \otimes_A B)_{\text{red}} = C \end{array}$$

where all the squares are pull-back's, and k is SN in C. Tensoring by A' over A we get the following commutative diagram, where all the squares are pull-back's (by 4.2).

$$\begin{array}{ccc} A' & \longrightarrow & B' \\ \downarrow & & \downarrow \\ A' \otimes A_{\mathfrak{p}} & \longrightarrow & A' \otimes B'_{\mathfrak{p}} \\ \downarrow & & \downarrow \\ A' \otimes k & \longrightarrow & A' \otimes C \end{array}$$

Since the second row is a localization of the first, by 2.4 it is sufficient to show that $A' \otimes A_{\mathfrak{p}}$ is SN in $A' \otimes B_{\mathfrak{p}}$. Hence by 4.3 it is sufficient to show that $A' \otimes k$ is SN in $A' \otimes C$. Moreover the canonical homomorphism $k \to A' \otimes k$ is reduced ([18], IV_2.6.8.3) and $A' \otimes C = (A' \otimes k) \otimes_k C$. The conclusion follows then by the next Lemma.

4.4. Lemma: *Let k be a field, $C \neq 0$ a finite reduced k-algebra and $k \to R$ a reduced homomorphism. Put $S = R \otimes_k C$, and assume k is* SN *in C. Then R is* SN *in S.*

Proof: The rings R and S are reduced ([18], IV_2.6.8.2 and [26], 21.E). Moreover S is FF over R, since C is FF over k. Hence if K is the total ring of fractions of R we have: $R = K \cap S$. The conclusion follows then by 2.5(v).

4.5. Corollary: *Let A, B, A' be as in* 4.1. *Put $C = {}^{+}_{B}A$ and $C' = A' \otimes_A C$. Then $C' = {}^{+}_{B'}A'$.*

Proof: Since $A \to A'$ is reduced it follows that $C \to C'$ is reduced. Moreover $B' = A' \otimes_A B = (A' \otimes_A C) \otimes_C B$. Thus we can apply 4.1 to show that C' is SN in B'. Moreover if $\mathfrak{P} \in \operatorname{Spec}(A')$ and $\mathfrak{p}$ is the contraction of $\mathfrak{P}$ to A we have:

$$(C' \otimes_{A'} k(\mathfrak{P}))_{\text{red}} = (C \otimes_A k(\mathfrak{P}))_{\text{red}} = ((C \otimes_A k(\mathfrak{p})) \otimes_{k(\mathfrak{p})} k(\mathfrak{P}))_{\text{red}} = k(\mathfrak{P})$$

because $A \to C$ is a quasi-isomorphism (see 1.1). Thus A' and C' are quasi-isomorphic and the conclusion follows.

5. Permanence of seminormality

We apply the base change theorem 4.1 to the seminormalization. In particular we show that normal homomorphisms preserve seminormality and, under some assumptions, that this is true for reduced homomorphisms with SN generic fibers. We give also results on seminormality of products and on the analytic seminormality.

5.1. PROPOSITION: *Let $f: A \to C$ be a reduced ring homomorphism having normal generic fibers (e.g. f is normal). Then:*

(a) *if A is* SN *then C is* SN;

(b) ${}^{+}A \otimes_A C = {}^{+}C$.

PROOF: By [15], 3.1 we have $\bar{C} = \bar{A} \otimes_A C$. The conclusion follows by 4.1 and 4.5.

Applications of the above result can be given by using the following proposition.

5.2. PROPOSITION: *The canonical homomorphism $f: A \to C$ is normal in the following cases:*

(a) *A is any ring and $C = A[X_1, \ldots, X_n]$;*

(b) *A is an algebra over a field k and $C = k' \otimes_k A$ where k' is a separable field extension of k*[1];

(c) *the formal fibers of A are geometrically normal (e.g. A is excellent) and C is an ideal-adic completion of A;*

(d) *A is as in (c) and C is a ring of restricted power series over A with respect to an ideal-adic topology (see* [7] *p.* 79, *or* [28], *or* [14], *p.* 18);

(e) *C is the henselization of A with respect to an ideal of A (see* [11] *or* [15]).

(f) *A is a local ring and C is a strict henselization of A* ([18], IV_4.18.8);

(g) *Let k be a complete valued non-archimedean field, D the ring of restricted power series in n variables over k (see e.g.* [16]) *B the subring of D consisting of all polynomials, $\mathfrak{b}$ an ideal of B, $A = B/\mathfrak{b}$ and $C = D/\mathfrak{b}D$;*

(h) *A is a local ring with geometrically normal formal fibers and C is a local ring which is a formally smooth A-algebra (for the discrete topologies, see* [18], O_{IV}.19).

[1] Remind that we are assuming C is noetherian. This is true if C is essentially of finite type over A and/or k' is finitely generated over k.

PROOF: (a) is obvious; (b) and (c) follow by [18], IV_2.7.4. To prove (d) recall that C is an ideal-adic completion of a polynomial ring ([14, 3.6 or [28], prop. 1). Moreover the assumption on the formal fibers is preserved by polynomial extensions ([18], IV_2.7.4.4) and the composition of two normal homomorphisms is normal ([18], IV_2.6.8.3). Then apply (a) and (c).

In cases (e), (f), C is a direct limit of étale A-algebras ([11], cor. to Lemma 3 and [18] IV_4.18.8.6), whence f is regular. For case (g) see [16], Prop. 7; finally the conclusion in case (h) follows by [13], 1.3.

5.3. COROLLARY: *Let A be a local excellent ring. Then A is* SN *if and only if its completion is* SN.

PROOF: It follows from 5.1, 5.2 and 1.8.

5.4. COROLLARY: *Let A be the local ring at a point x of a complex algebraic variety, and let B be the local ring at x of the corresponding analytic space. Then A is* SN *if and only if B is such.*

PROOF: By [30] B is noetherian and has the same completion as A. Hence B is faithfully flat over A. Moreover A is excellent and the conclusion follows by 5.3 and 1.7.

5.5. PROPOSITION: *Let A be a ring and let $C = A[[X]]$. Then A is* SN *if and only if C is* SN.

PROOF: C is FF over A ([14], 4.11). Hence if C is SN, A is such by 1.7. The converse follows from 5.1 and the next lemma.

5.6. LEMMA: *Let A be a ring and let $f: A \to C = A[[X]]$ be the canonical embedding. Then*:

(a) *f is integral* (*hence reduced*);

(b) *if A is Mori, the generic fibers of f are normal.*

PROOF: Let $\mathfrak{p} \in \operatorname{Spec}(A)$ and let K be a finite extension of $k(\mathfrak{p})$. Let A' be a finite A-algebra whose quotient field is K. Then $K \otimes_A A[[X]] = (K \otimes_{A'} A') \otimes_A A[[X]] = K \otimes_{A'} A'[[X]]$. Thus $K \otimes_A A[[X]]$ is a ring of fractions of $A'[[X]]$, and hence is a domain. It follows that the fibers of f are geometrically integral, and, a fortiori, f is reduced. To prove (b) observe that since $\bar{A}$ is finite over A we have $\bar{A} \otimes_A A[[X]] = \bar{A}[[X]]$, and since the latter is normal ([8], p. 20, Prop. 14), the conclusion follows.

5.7. Corollary: *Let k be a field and let X be a k-scheme. Let k' be a separable field extension of k, and put $X' = k' \otimes X$.*

Assume X' is locally noetherian. Then X' is SN *if and only if X is* SN.

Proof: It is an immediate consequence of 5.1, 5.2(b) and 1.7.

5.8. Theorem: *Let $f: A \to B$ be a reduced ring homomorphism, and assume A is* SN. *Then B is* SN *if and only if B is Mori and the generic fibers of f are* SN.

Proof: By 4.1 B is SN in $\bar{A} \otimes B = B'$. Thus it is sufficient to show that B' is SN. If K is the total ring of fractions of A we have $K \otimes_{\bar{A}} (\bar{A} \otimes_A B) = K \otimes_A B$, whence the generic fibers of $\bar{A} \to B'$ are SN. Moreover it is clear that $\bar{A} \to B'$ is reduced, and that B' is Mori. Thus we may assume A is normal, and $B = B'$.

Let now $\mathfrak{P} \in \operatorname{Spec}(B)$, and let $\mathfrak{p} = f^{-1}(\mathfrak{P})$. We assume depth $B_{\mathfrak{P}} = 1$, and we show that $B_{\mathfrak{P}}$ is SN. The conclusion will follow by 2.7. If $\mathfrak{p}$ is a minimal prime, $B_{\mathfrak{P}}$ is a localization of a generic fiber, and there is nothing to prove.

If dim $A_{\mathfrak{p}} \geq 1$, then depth $A_{\mathfrak{p}} \geq 1$, and since by flatness depth $A_{\mathfrak{p}} +$ depth $B_{\mathfrak{P}}/\mathfrak{p}B_{\mathfrak{P}} =$ depth $B_{\mathfrak{P}} = 1$ ([26], (21.B)), we must have depth $B_{\mathfrak{P}}/\mathfrak{p}B_{\mathfrak{P}} = 0$, that is $\mathfrak{p}B_{\mathfrak{P}} = \mathfrak{P}B_{\mathfrak{P}}$ as f is reduced. Moreover depth $A_{\mathfrak{p}} = 1$, and since A is normal $A_{\mathfrak{p}}$ is a DVR. Thus $B_{\mathfrak{P}}$ is also a DVR and a fortiori SN.

5.9. Corollary: *Let k be a perfect field and let X, Y be two seminormal k-schemes. Assume Y is locally of finite type over k. Then $X \otimes_k Y$ is* SN.

Proof: The question is local on both X and Y. Thus we may assume $X = \operatorname{Spec}(A)$, $Y = \operatorname{Spec}(B)$ and B of finite type over k. The canonical map $k \to B$ is reduced, since k is perfect, and hence the same holds for $A \to A \otimes_k B$ (see [18], IV_2.6.8 and IV_2.7.3). Moreover if $\mathfrak{p} \in \operatorname{Spec}(A)$ we have $k(\mathfrak{p}) \otimes_A (A \otimes_k B) = k(\mathfrak{p}) \otimes_k B$ which is SN by 5.7. Thus by 5.8 it is sufficient to show that $A \otimes_k B$ is Mori.

By the previous argument applied to $\bar{A}$ and $\bar{B}$ we see that $\bar{A} \to \bar{A} \otimes_k \bar{B}$ is reduced and has normal generic fibers. Thus by [15], Prop. 1, we have that $\bar{A} \otimes_k \bar{B}$ is normal. Moreover if K, L are the total rings of fractions of A, B respectively we have $A \otimes_k B \subset \bar{A} \otimes_k \bar{B} \subset K \otimes_k L$, and these three rings have the same total ring of fractions. Thus $\bar{A} \otimes_k \bar{B}$ is the integral closure of $A \otimes_k B$ and the conclusion follows.

5.10. REMARKS:

(i) For any ring A put: $\mathrm{NPic}(A)$ = the cokernel of the canonical homomorphism $\mathrm{Pic}(A) \to \mathrm{Pic}(A[T])$ where T is an indeterminate. By [31] a Mori ring is SN if and only if $\mathrm{NPic}(A) = 0$. Thus 5.1 and 5.8 assert that if a ring homomorphism $A \to B$ satisfies certain conditions, then $\mathrm{NPic}(A) = 0$ implies $\mathrm{NPic}(B) = 0$. Thus the following question arises: when is the induced group homomorphism $\mathrm{NPic}(A) \to \mathrm{NPic}(B)$ surjective?

(ii) Corollary 5.7 is false if k' is not assumed to be separable over k. Indeed $k' \otimes_k A$ might be non-reduced. For a counterexample where $k' \otimes_k A$ is a domain see 8.6 below.

6. Seminormality of analytic spaces

In this section we compare seminormality and weak normality for complex analytic spaces. We assume that the reader is familiar with the basic facts about complex analytic spaces, including normalization and Stein spaces (see e.g. [33], [35], [37]).

Let X be a reduced complex analytic space, and let $p: \bar{X} \to X$ be its normalization. A *weak normalization* of X is a complex analytic space X^* together with a holomorphic bijective map $X^* \to X$, which is universal with respect to this property (roughly speaking X^* is the "largest" analytic space homeomorphic to X). If $X = X^*$ we say that X is *weakly normal* (see Andreotti-Norguet [2]).

The weak normalization can be obtained by "enriching" the structure sheaf of X. This can be done in two equivalent ways, as shown in [2]: thus for any open $U \subset X$ we have:

$$\begin{aligned}\Gamma(U, \mathcal{O}_{X^*}) &= \{f \in \Gamma(p^{-1}(U), \mathcal{O}_{\bar{X}} \mid f(x) = f(y) \text{ whenever } p(x) = p(y)\} \\ &= \{\text{meromorphic functions on } U \text{ which are also continuous}\}\end{aligned}$$

The main result of this section asserts that ${}^+(\mathcal{O}_{X,x}) = \mathcal{O}_{X^*,x}$ for all x of a reduced complex analytic space X (Th. 6.12); from this it follows that seminormalization and weak normalization of a complex algebraic variety coincide analytically (Cor. 6.14).

In order to prove our results we need several preliminary facts on the ring of holomorphic functions over a compact semianalytic Stein subset (6.1 to 6.9).

Recall that a subset D of a complex analytic space X is said to be *semianalytic* if it is given, locally, by analytic equations and real analytic inequalities (see [34]). Such a subset is said to be *Stein* if it has a fundamental system of neighborhoods which are Stein spaces ([33], [35]) with the induced structure.

If $\mathscr{F}$ is a sheaf on X we put $\mathscr{F}_D = \varinjlim \Gamma(U, \mathscr{F})$ where U ranges in the set of neighborhoods of D.

In the following X is a complex analytic space, D is a compact Stein semianalytic subset of X, and A is the ring $\mathcal{O}_{X,D}$.

The ring A is the main tool in proving our results, and we shall study it in some detail. In particular we compare the localizations of A with the rings $\mathcal{O}_{X,x}$ ($x \in D$, see 6.2), we characterize Spec(A) in terms of the D-germs (Prop. 6.5), and we show that A is excellent (Cor. 6.8).

We recall that by a theorem of Frisch ([34], 1.9) *the ring A is noetherian.* We shall use this fact freely.

6.1. LEMMA: *Let X, D, A be as above, and let I be an ideal of A. Then $I \neq A$ if and only if it has a zero in D.*

PROOF: Since A is noetherian we have $I = (f_1, \ldots, f_n)$. We can extend $f_1, \ldots, f_n$ to a Stein neighborhood U of D, and if I has no zeros in D we may assume that $f_1, \ldots, f_n$ have no common zero in U. Then we have $1 = \Sigma f_i g_i$ for suitable g_i's in $\Gamma(U, \mathcal{O}_X)$ ([35], p. 244, Cor. 16) whence $1 \in I$ and $A = I$. The converse is obvious.

6.2. PROPOSITION: *Let X, D, A be as above. Then*

(i) *the map $\varphi : x \mapsto \mathfrak{m}_x = \{f \in A \mid f(x) = 0\}$ is a bijection between D and* max(A);

(ii) *for each $x \in D$ there is a canonical map $\psi : A_{\mathfrak{m}_x} \to \mathcal{O}_{X,x}$ which is faithfully flat* (*hence injective*);

(iii) *The above embedding induces an isomorphism on the completions.*

PROOF: Since global holomorphic functions on a Stein space separate points, it follows that the elements of A separate the points of D, whence φ is injective. Surjectivity follows from 6.1, and (i) is proved.

Let now $x \in D$ and put $R = A_{\mathfrak{m}_x}$, $S = \mathcal{O}_{X,x}$. The canonical homomorphism $\rho : A \to S$ factors through R, and this gives ψ. In order to prove (ii) it is sufficient to show that ρ is flat, i.e. that any solution in S of the linear system

$$\sum_{j=1}^{n} c_{ij} Y_j = 0 \qquad c_{ij} \in A; \quad i = 1, \ldots, m$$

is a linear combination, with coefficients in S, of solutions in A ([6], p. 44, Cor. 2).

For this extend the c_{ij}'s to some Stein neighborhood U of D. Then the sheaf of relations of the vectors $\underline{c}_j = (c_{lj}, \ldots, c_{mj})$, (considered as sections of $\mathcal{O}_X^m$) is coherent, and hence is generated by sections over U (Theorem A). The conclusion follows easily. This proves (ii).

Let now $\mathfrak{M}, \mathfrak{N}$ be the maximal ideals of R, S respectively, and let U be a Stein neighborhood of D. Let $z_1, \ldots, z_n \in \Gamma(U, \mathcal{O}_X)$ be such that $\mathfrak{N} = (z_1, \ldots, z_n)S$ ([35], p. 209, Def. 2). Put $I = (z_1, \ldots, z_n)R$. Then by (ii) we have $I = IS \cap R = \mathfrak{N} \cap R = \mathfrak{M}$ whence $\mathfrak{M}S = \mathfrak{N}$. From this we have $\mathfrak{M}^r S = \mathfrak{N}^r$ and by (ii) $\mathfrak{N}^r \cup R = \mathfrak{M}^r$ for all r, that is the canonical maps $R/\mathfrak{M}^r \to S/\mathfrak{N}^r$ are injective. Finally we have $R/\mathfrak{M} = \mathbf{C} = S/\mathfrak{N}$ and hence $R/\mathfrak{M}^r = S/\mathfrak{N}^r$ by [6], p. 105, Cor. 1. Taking inverse limits we have (iii), and the proof is complete.

Now we want to study Spec(A). For this we need the notion of D-germ.

6.3. Definition: Let X, D, A be as above. Two analytic subsets Y, Z of some open subsets of X are *D-equivalent* if they coincide in some neighborhood of D. The corresponding equivalence classes are called *D-germs*. If a D-germ is represented by the analytic set $Y \subset U \subset X$, we shall simply call it "the D-germ Y".

A D-germ Y determines the ideal $I(Y)$ of A consisting of those $f \in A$ which vanish identically on Y (the precise meaning of this being clear). Conversely if $J = (f_1, \ldots, f_n)$ is an ideal of A the equations $f_1 = \cdots = f_n = 0$ define an analytic set in some neighborhood of D, and hence a D-germ, which depends only on J and is denoted by $V(J)$.

6.4. Lemma: *With the above notations we have*: $I(V(J)) = \sqrt{J}$.

Proof: Let $f \in I(V(J))$. Extend f and $Y = V(J)$ to a Stein neighborhood of D. Let $x \in D$. Then the germ of Y at x is determined by $J\mathcal{O}_x$, whence $f_x \in \sqrt{J\mathcal{O}_{X,x}}$ by the analytic Nullstellensatz. The conclusion follows easily by 6.2(ii) and "passing from local to global" (e.g. [6], p. 112, Cor. 3).

6.5. Proposition: *Let X, D, A be as above. Then*:

(i) *There is a canonical bijection between the set of radical ideals of A and the set of D-germs;*

(ii) *The above bijection induces a bijection between* Spec(A) *and the set of irreducible D-germs.*

(iii) *If* $\mathfrak{p} \in \operatorname{Spec}(A)$ *corresponds to the D-germ Y then we have:*

$$A/\mathfrak{p} = \mathcal{O}_{Y,D\cap Y} \text{ and } k(\mathfrak{p}) = \mathcal{M}_{Y,Y\cap D}$$

(*where* $\mathcal{M}_Y$ *is the sheaf of meromorphic functions over* Y).

PROOF: The proof of (i) and (ii) can be worked out with standard techniques by using 6.4, while (iii) is an easy consequence of (ii) and of the next lemma.

6.6. LEMMA: *Let X be a complex analytic space, Y a complex analytic subspace of X and D a compact semianalytic Stein subset of X. Then:*

(i) $D \cap Y$ *is a compact semianalytic Stein subset of* Y;

(ii) *the restriction homomorphism* $\mathcal{O}_{X,D} \to \mathcal{O}_{Y,Y\cap D}$ *is surjective.*

PROOF: It is clear that $D \cap Y$ is compact and semianalytic. Moreover if $\{U_\alpha\}$ is a fundamental system of Stein neighborhoods of D, it follows that the $U_\alpha \cap Y$ form a fundamental system of neighborhoods of $D \cap Y$, which are Stein ([35], p. 210, (6)). Moreover for each α the restriction $\Gamma(U_\alpha, \mathcal{O}_X) \to \Gamma(U_\alpha \cup Y, \mathcal{O}_Y)$ is surjective ([35], p. 245, Th. 18) and the conclusion follows.

Now we want to show that the ring A is excellent. This, along with 6.2, allows us to compare seminormality of $A_{\mathfrak{m}_x}$ and $\mathcal{O}_{X,x}$.

6.7. LEMMA: *Let D be a compact semianalytic Stein subset of* $\mathbf{C}^n = X$, *and put* $A = \mathcal{O}_{X,D}$. *Then A is excellent* (*see* [26]).

PROOF: We use Matsumura's jacobian criterion ([36], Th. 2.7). By 6.2(iii) it follows easily that A is regular and that dim $A_\mathfrak{m} = n$ for all $\mathfrak{m} \in \max(A)$. Moreover $A/\mathfrak{m} = \mathbf{C}$ for all $\mathfrak{m} \in \max(A)$. Finally if $z_1, \ldots, z_n$ are coordinates in $\mathbf{C}^n$, it is easy to see that the partial derivations with respect to them induce $\mathbf{C}$-derivations $D_1, \ldots, D_n$ of A into itself, and that $D_i z_j = \delta_{ij}$. Thus the above mentioned criterion can be applied, and A is excellent.
(Remark: in [36], Th. 2.7 A is assumed to be a domain; but it is easy to see that this assumption is redundant).

Now let X be a complex Stein space, and let $x \in X$.

The *embedding dimension of X at x* is the (complex) dimension of the Zariski tangent space $\mathfrak{M}_{X,x}/(\mathfrak{M}_{X,x})^2$ of $\mathcal{O}_{X,x}$.

In [38] one proves (a stronger form of) the following:

THEOREM: *Let X be a complex Stein space and assume that its embedding dimension at every point is $\leq n$. Then X is isomorphic to a closed analytic subset $Y \subseteq \mathbf{C}^{2n+1}$.*

6.8. COROLLARY: *Let X be a complex analytic space, D a compact semianalytic Stein subset of X. Then $\mathcal{O}_{X,D}$ is excellent.*

PROOF: Let U be a Stein neighborhood of D, of bounded embedding dimension (always existing since D is compact, hence it is contained in a finite union of open affine subsets of X), and let $U \subseteq \mathbf{C}^N$ be a closed analytic immersion of U; then $\mathcal{O}_{X,D} = \mathcal{O}_{U,D}$ is a quotient of $\mathcal{O}_{\mathbf{C}^N,D}$ (see 6.6ii) which is excellent.

Now we study the preimage of a compact semianalytic Stein subset in the normalization.

6.9. LEMMA: *Let X be a reduced complex analytic space, and let $p: \bar{X} \to X$ be its normalization (see* [33] *or* [37]). *Let D be a compact semianalytic Stein subset of X. Then*

(i) *$p^{-1}(D)$ is a compact Stein semianalytic subset of $\bar{X}$;*

(ii) *$B = \mathcal{O}_{\bar{X},p^{-1}(D)}$ is the normalization of $A = \mathcal{O}_{X,D}$.*

PROOF: Since p is proper and analytic it is clear that $p^{-1}(D)$ is compact and semianalytic. Moreover it is easy to see that D has a fundamental system of relatively compact Stein neighborhoods $\{U_\alpha\}$. Since p is proper it is easy to see that $\{p^{-1}(U_\alpha)\}$ is a fundamental system of neighborhoods of $p^{-1}(D)$. Finally $p^{-1}(U_\alpha)$ is the normalization of U_α, and hence it is Stein ([33], p. 58, 1.13). This proves (i).

Moreover we have $B = \varinjlim \Gamma(p^{-1}(U_\alpha), \mathcal{O}_{\bar{X}})$, which easily implies $B \subset \mathcal{M}_{X,D}$, which is the total ring of fractions of A (by 6.5). Now we show that B is integral over A. For this let $f \in B$ and extend f to some $p^{-1}(U)$ where U is a Stein neighborhood of D. By the construction of $\bar{X}$ each $x \in U$ has a neighborhood V such that f restricted to $p^{-1}(V)$ is integral over $\Gamma(V, \mathcal{O}_X)$.

Since D is compact we may assume that there are an integer n and a finite open covering $U = \bigcup_{i=1}^r U_i$ of U such that f^n is a linear combination of $f^{n-1}, \ldots, f, 1$, on each U_i, with coefficients in $\Gamma(U_i, \mathcal{O}_X)$.

Let $\mathscr{R} \subset (p_*\mathcal{O}_{\bar{X}})_{|U}$ be the sheaf of relations of $1, \ldots, f^n$.

Since $p_*\mathcal{O}_{\bar{X}}$ is coherent ([37], ch. IV, Th. 7) also $\mathscr{R}$ is coherent, and hence is generated by its global sections. Then after refining the covering $\{U_i\}$ and restricting U we may assume that there are $(a_{i0}, \ldots, a_{in}) \in \Gamma(U, \mathscr{R})$ such that $a_{in}(x) \neq 0$ for all $x \in U_i$. Then we have: $\Sigma\, b_i a_{in} = 1$, $b_i \in \Gamma(U, \mathcal{O}_X)$ ([35], p. 244, Cor. 16), from which we deduce easily that f is integral over $\Gamma(U, \mathcal{O}_X)$, and that B is integral over A.

Finally since normality descends by faithful flatness, we have that B is normal by (i) and 6.2(ii). This proves (ii).

Now we are able to compare seminormalization and weak normalization.

6.10. LEMMA: *Let X be a reduced complex analytic space, let $p : \bar{X} \to X$ be its normalization and X^* its weak normalization. Let D, E be two compact Stein semi-analytic subsets of X, and let U be an open set such that $D \supset U \supset E$. Put $A = \mathcal{O}_{X,D}$, $B = \mathcal{O}_{X,E}$. Then the restrictions induce a commutative diagram*:

$$\begin{array}{ccccc}
\bar{A} & \xrightarrow{\alpha} & \Gamma(p^{-1}(U), \mathcal{O}_{\bar{X}}) & \xrightarrow{\beta} & \bar{B} \\
\uparrow & & \uparrow & & \uparrow \\
{}^+A & \longrightarrow & \Gamma(U, \mathcal{O}_{X^*}) & \longrightarrow & {}^+B \\
\uparrow & & \uparrow & & \uparrow \\
A & \xrightarrow{u} & \Gamma(U, \mathcal{O}_X) & \xrightarrow{v} & B
\end{array}$$

PROOF: The maps u and v are easily defined, and induce α and β by 6.9.

Let now $f \in {}^+A$. Then f induces a continuous function on D and hence on U. Moreover $f \in \bar{A}$ and hence its restriction to U is integral over $\Gamma(U, \mathcal{O}_X)$. This implies that $\alpha(f) \in \Gamma(U, \mathcal{O}_{X^*})$ ([2], Section 2, Prop. 1), whence $\alpha({}^+A) \subset \Gamma(U, \mathcal{O}_{X^*})$. Let now $f \in \Gamma(U, \mathcal{O}_{X^*})$. We want to prove the following:

(i) if $\mathfrak{p}_1, \mathfrak{p}_2 \in \operatorname{Spec}(\bar{B})$ and $\mathfrak{p}_1 \cap B = \mathfrak{p}_2 \cap B = \mathfrak{p}$, then $f \in \mathfrak{p}_1$ if and only if $f \in \mathfrak{p}_2$.

(ii) f induces in $k(\mathfrak{p}_1)$ an element of $k(\mathfrak{p}) \subset k(\mathfrak{p}_1)$.

This shows that $B \to \beta(\Gamma(U, \mathcal{O}_{X^*}))$ is a quasi-isomorphism, which implies $\beta(\Gamma(U, \mathcal{O}_{X^*})) \subset {}^+B$ (see 1.1 and 1.2).

Let Y_1, Y_2 be the $p^{-1}(E)$-germs corresponding to $\mathfrak{p}_1$ and $\mathfrak{p}_2$ (see 6.9(i) and 6.5(ii)) and let Y be the germ corresponding to $\mathfrak{p}$.

With an obvious meaning of the symbols, we have $p(Y_1) = Y = p(Y_2)$. Thus if $f \in \mathfrak{p}_1$ it follows that f vanishes identically on Y_1. Moreover it induces a continuous function on U, which vanishes identically on Y. Thus f vanishes identically on Y_2 and $f \in \mathfrak{p}_2$ by 6.4. This proves (i).

To prove (ii) observe that f induces a holomorphic function on Y considered as an analytic subset of X^*, and thus induces a meromorphic function on Y considered as an analytic subset of X. Then apply 6.5(iii).

6.11. Corollary: *A weakly normal complex analytic space X is* SN. (*that is*: $\mathcal{O}_{X,x}$ *is* SN *for all* $x \in X$).

Proof: Let $x \in X$ and let U be a neighborhood of x whose closure D is a compact Stein semianalytic subset of X.

Put $R = (\mathcal{O}_{X,D})_{\mathfrak{m}_x}$ and $S = \mathcal{O}_{X,x}$.

Applying 6.10 with $E = \{x\}$ and localising at $\mathfrak{m}_x$ we have: $R \subset {}^+R \subset S$. But ${}^+R$ is contained in the total ring of fractions K of R and S is faithfully flat over R by 6.2. Thus ${}^+R \subset K \cap S = R$, and R is SN.

Moreover R is excellent by 6.8 and $\hat{R} = \hat{S}$ by 6.2. Thus S is SN by 5.3 and 1.8. This completes the proof.

6.12. Theorem: *Let X be a reduced complex analytic space, and let X^* be its weak normalization. Then ${}^+(\mathcal{O}_{X,x}) = \mathcal{O}_{X^*,x}$ for all $x \in X$.*

Proof: Let $\{U_\alpha\}$ be a fundamental system of neighborhoods of x such that the closure D_α of each U_α is a compact Stein semianalytic subset of X. Applying 6.10 to $D_\alpha \supset U_\alpha \supset E = \{x\}$ and passing to the direct limits we have

$$\mathcal{O}_{X,x} \subset \mathcal{O}_{X^*,x} \subset {}^+\mathcal{O}_{X,x} \subset \overline{\mathcal{O}_{X,x}} = \mathcal{O}_{\bar{X},p^{-1}(x)}.$$

The conclusion follows by 6.11.

6.13. Corollary: *Let X be a complex algebraic variety and X^h the corresponding analytic space* (*see* [30]). *Then X is* SN *if and only if X^h is weakly normal.*

Proof: Apply 5.4 and 6.12.

6.14. Corollary: *Let X be a reduced complex algebraic variety.*

Then the analytic spaces $({}^{+}X)^{h}$ *and* $(X^{h})^{*}$ *are canonically isomorphic (that is: seminormalization and weak normalization of X coincide analytically).*

PROOF: By definition the canonical map ${}^{+}X \to X$ is bijective, and hence $({}^{+}X)^{h} \to X^{h}$ is also bijective. Then by the universal property of the weak normalization ([2], Section 1) we have a bijective holomorphic map $f:(X^{h})^{*} \to ({}^{+}X)^{h}$. By $({}^{+}X)^{H}$ is weakly normal by 6.13 and hence f is an isomorphism.

7. Normalization and blow-up of seminormal schemes

In this section we show that under suitable conditions a seminormal S_2 scheme can be normalized by blowing up the conductor.

We refer to [18].II.8.1 for general facts about the blow up. We shall state explicitly the following fact, which appears implicitly in l.cit.

7.1. PROPOSITION: *Let* X *be a scheme,* $\mathcal{J}$ *a coherent ideal of* $\mathcal{O}_X$, $f: Y \to X$ *the* X*-scheme obtained by blowing up* X *along* $\mathcal{J}$. *Then* $\mathcal{J}\mathcal{O}_Y$ *is an invertible ideal of* $\mathcal{O}_Y$. *Moreover if* $g: Z \to X$ *is an* X*-scheme and* $\mathcal{J}\mathcal{O}_Z$ *is invertible, there is a unique* X*-morphism* $Z \to Y$.

We denote by $\mathrm{gr}_{\mathfrak{a}}(A)$ the graded ring of A with respect to the ideal $\mathfrak{a}$, and we refer to [18].II for general facts about Proj of a graded ring. The main result of this section is the following.

7.2. THEOREM: *Let* A *be a Mori ring,* $\mathfrak{b}$ *the conductor of* A *and* $\mathfrak{a} = \sqrt{\mathfrak{b}}$. *Let* Y *be the blow up of* $X = \mathrm{Spec}(A)$ *along* $\mathfrak{a}$. *Let* $E = \mathrm{Proj}(\mathrm{gr}_{\mathfrak{a}}(A))$. *Assume further that* $\mathfrak{a}\bar{A}$ *is an invertible ideal of* $\bar{A}$. *Then*:

(a) *If* E *is reduced,* Y *is canonically isomorphic to* $\mathrm{Spec}(A)$, *and* E *is canonically isomorphic to* $\mathrm{Spec}(\bar{A}/\mathfrak{a}\bar{A})$;

(b) *If* $\bar{A}/\mathfrak{b}$ *is reduced, then* E *is reduced.*

PROOF: We show first that Y is affine. Let $Z = \mathrm{Spec}(\bar{A})$ and let $f: Y \to X$, $g: Z \to X$ be the canonical morphisms. By 7.1 there is a unique morphism $h: Z \to Y$ such that $f \circ h = g$. Since g is finite the fibers of h are finite. Since f is proper (hence separated) and g is proper it follows that h is proper ([18], II.5.4.3), and hence finite ([18], III.4.4.2). Let now $U = X - V(\mathfrak{a})$. Then $U = f^{-1}(U)$ and $U = g^{-1}(U)$ (canonically). Moreover U is dense in Y since its complement is defined by the invertible ideal $\mathfrak{a}\mathcal{O}_Y$. Thus h is dominant, hence surjective (a finite morphism is closed), and Y is then affine by [18], II.6.7.1.

Put $Y = \mathrm{Spec}(C)$. Since f and h are dominant we have canonical embeddings $A \subset C \subset \bar{A}$. Hence to prove (a) it is sufficient to show that C is normal. Let $x \in Y$. If $f(x) \in U$ then $\mathcal{O}_X$ is normal by our assumption on $\mathfrak{a}$. If $x \notin U$, then x belongs to $Y' = f^{-1}(V(\mathfrak{a}))$, whence $\mathcal{O}_{Y',x} = \mathcal{O}_{Y,x}/\mathfrak{a}\mathcal{O}_{Y,x}$. But Y' is canonically isomorphic to E ([18], IV_4.19.4.2), whence $\mathcal{O}_{Y',x}$ is reduced. Moreover $\mathfrak{a}\mathcal{O}_{Y,x}$ is generated by one regular element, and from this it follows that $\mathcal{O}_{Y,x}$ is either a DVR or a local ring of depth at least 2. Then Y and C are normal by Serre's criterion ([18], IV_2.5.8.6). This proves (a). To prove (b) observe that if $\bar{A}/\mathfrak{b}$ is reduced, then $\mathfrak{b} = \mathfrak{a}$ and thus $\mathfrak{a}$ is an ideal of $A, \bar{A}, C$ at the same time. Hence $C/\mathfrak{a}C$ is canonically embedded in $\bar{A}/\mathfrak{b}$ and hence it is reduced. The conclusion follows from the above mentioned isomorphism between E and $f^{-1}(V(\mathfrak{a}))$.

7.3. COROLLARY: *Let A be a* SN *ring and let $\mathfrak{b}$ be the conductor of A. If $\mathfrak{b}\bar{A}$ is invertible, then* $\mathrm{Spec}(\bar{A})$ *coincides with the scheme obtained by blowing up* $\mathrm{Spec}(A)$ *along $\mathfrak{b}$. Moreover* $\mathrm{Spec}(\bar{A}/\mathfrak{b})$ *is canonically isomorphic to* $\mathrm{Proj}(\mathrm{gr}_{\mathfrak{b}}(A))$.

PROOF: It is an immediate consequence of 1.4 and 7.2.

7.4. LEMMA: *Let A be a Mori ring, and let $\mathfrak{p}$ be a prime ideal associated to the conductor. Then* depth $A_{\mathfrak{p}} = 1$. *Hence if A is S_2, the conductor is unmixed of height* 1.

PROOF: By [7], p. 164, ex. 7b we have that $\mathfrak{p} \in \mathrm{Ass}_A(\bar{A}/A)$. Hence depth $A_{\mathfrak{p}} = 1$ (see proof of 2.7).

7.5. COROLLARY: *Let X be an S_2* SN *scheme, and let $\mathcal{B}$ be the conductor sheaf of X. Assume that the normalization of X is locally factorial (e.g. regular). Then X' is the scheme obtained by blowing up X along $\mathcal{B}$.*

PROOF: The question is local, so that we may assume $X = \mathrm{Spec}(A)$. Put $\mathfrak{b} = \Gamma(X, \mathcal{B})$. Then $\mathfrak{b}$ is the conductor of A and since A is S_2 it is unmixed of height 1 as an ideal of A (by 7.4). By Cohen-Seidenberg and 1.4 it follows that $\mathfrak{b}$ is unmixed of height 1 also as an ideal of $\bar{A}$. Since A is locally factorial it is easy to see that $\mathfrak{b}$ is an invertible ideal of $\bar{A}$ and the conclusion follows by 7.3.

7.6. REMARKS:

(i) We do not know whether a SN scheme can be normalized by a finite sequence of blow ups.

(ii) It might be interesting to characterize the algebraic varieties (or schemes) which can be normalized by a finite sequence of blow ups. Not all the varieties have this property as pointed out by H. Matsumura.

(iii) In the proof of 7.2 one sees that by blowing up a certain ideal of height 1 we get a finite scheme. Is this true more in general?

(iv) See [40], Prop. 2.6 and [41] for interesting results on blowing up conductors, to be compared with ours.

8. Seminormality of local Gorenstein rings of dimension 1

Our purpose is now to study SN Gorenstein schemes (in particular: SN complete intersections). By 2.7 the problem is reduced to one-dimensional local rings which are Gorenstein. Such rings are studied in this section (following [12]), while geometrical interpretations of the results will be discussed in the next one. For informations on Gorenstein rings see [3] and [19].

8.1. THEOREM: *Let A be a Mori local ring of dimension 1 which is not normal. Let $\mathfrak{m}$ be the maximal ideal of A and put $k = A/\mathfrak{m}$. The following conditions are equivalent*:

(i) *A is* SN *and $\mathfrak{m}$ is generated by two elements*;

(ii) *A is* SN *and Gorenstein*;

(iii) $\mathfrak{m} = \operatorname{rad} \bar{A}$ *and* $\dim_k \bar{A}/\mathfrak{m} = 2$;

(iv) $\mathfrak{m} = \operatorname{rad} \bar{A}$ *and either*:

(iv)′ *$\bar{A}$ is local and its residue field is a quadratic extension of k, or*

(iv)″ *$\bar{A}$ has exactly two maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2$ and the residue fields $A/\mathfrak{m}_i$ coincide with k*;

(v) $\mathfrak{m} = \operatorname{rad} \bar{A}$ *and the Hilbert polynomial of A is $p(n) = 2n - 1$*;

(vi) *the multiplicity $e(A)$ of A is equal to 2, $\bar{A}/\mathfrak{m}\bar{A}$ is reduced, and* $\operatorname{Spec}(\bar{A})$ *is the scheme obtained by blowing up* $\operatorname{Spec}(A)$ *along $\mathfrak{m}$*;

(vii) $e(A) = 2$ *and* $\operatorname{Proj}(\operatorname{gr}(A))$ *is a reduced scheme*;

(viii) *The completion $\hat{A}$ of A is of the form $R/(f)$ where R is a local regular ring of dimension 2 and the leading form of f in* $\operatorname{gr}(R)$ *has degree 2 and is not a square*;

(ix) $\operatorname{gr}(A) = k[X, Y]/(\phi)$ *where ϕ is a form of degree 2 which is not a square*;

(x) *$e(A)=2$ and* $\mathrm{gr}(A)$ *is reduced*;

(xi) *$e(A)=2$ and $B/\mathfrak{m}B$ is reduced, where B is the first neighborhood of A (as defined by Northcott, see* [25], *ch.* 12)

If moreover A contains a field the above are equivalent to:

(xii) *either* $\hat{A}=k[[X,Y]]/(XY)$, *or* $\hat{A}=k[[T,uT]]\subset k(u)[[T]]$, *where $k(u)$ is a field extension of degree* 2 *of k*;

(xiii) $\hat{A}=k[[X,Y]]/(X^2+bXY+cY^2)$ *where $b,c\in k$ and either $b^2-4c\neq 0$, or k has characteristic* 2, *$b=0$ and c is not a square in k.*

PROOF:

(i)$\to$(ii). This is immediate by [25], 13.2.

(ii)$\to$(iii). Clearly $\mathfrak{m}$ is the conductor of A. Hence $l(\bar{A}/\mathfrak{m})=2l(A/\mathfrak{m})$, where $l(\,)$ denotes the length of an A-module (see [19], 3.5). The conclusion follows easily.

(iii)$\leftrightarrow$(iv). This is easy.

(iii)$\to$(v). It is easy to see that $\mathfrak{m}$ is a rank 1 free ideal of $\bar{A}$. Hence for all $n\geq 1$ we have an isomorphism of $\bar{A}$-modules (and a fortiori of A-modules): $\mathfrak{m}^n/\mathfrak{m}^{n+1}=\bar{A}/\mathfrak{m}$. Thus for all $n\geq 1$ we have: $l_A(\mathfrak{m}^n/\mathfrak{m}^{n+1})=2$. The conclusion follows by a straightforward computation.

(v)$\to$(vi). By the definition of multiplicity we have $e(A)=2$. Moreover $\mathfrak{m}=\mathfrak{m}\bar{A}$, whence $\bar{A}/\mathfrak{m}\bar{A}$ is reduced. The conclusion follows by 7.5, since $\mathfrak{m}$ is clearly the conductor and $\bar{A}$ is regular.

(vi)$\to$(vii). We have $\mathrm{Proj}(\mathrm{gr}(A))=\mathrm{Spec}(A/\mathfrak{m}A)$ ([18], IV_4.19.8.2).

In order to continue our proof we need an easy Lemma.

8.2. LEMMA: *Let A be a complete local ring with maximal ideal $\mathfrak{m}$. If $\mathfrak{m}=(x_1,\dots,x_n)$, there is a regular local ring R of dimension n such that $A=R/I$. If moreover A is equidimensional of dimension $n-1$, then I is principal.*

PROOF: It is well known that there is a surjective homomorphism $f: W[[X_1,\dots,X_n]]=S\to A$ such that $f(X_i)=x_i$ ([18], proof of 0.19.8.8), where W is either a field or a DVR. In the former case take $R=S$. Otherwise let X be a regular parameter of W. Then there are $a_1,\dots,a_n\in S$ such that $X_0=X-\Sigma\, a_iX_i$ belongs to Ker(f). Clearly $X_0,\dots,X_n$ form a regular system of parameters of S. Thus we may take $R=S/X_0S$. This proves the first statement. The second one is an easy consequence of the unique factorization in R.

Now we can continue the proof of 8.1.

(vii)$\to$(viii). Since (vii) is clearly preserved by completions, we may assume A is complete. Since $e(A)=2$, $\mathfrak{m}$ is generated by two elements ([25], 12.17). Hence by 8.2 we have $A=R/(f)$ where R is a two-dimensional regular local ring. Thus $\mathrm{gr}(A) = \mathrm{gr}(R)/(\phi)$ where ϕ is the leading form f ([20], p. 190, Lemma 5). Since R is regular we have then $\mathrm{gr}(A) = k[X, Y]/(\phi)$. From the definitions of $\mathrm{gr}(A)$ and of multiplicity it follows easily that ϕ has degree 2. Finally ϕ is not a square, otherwise $\mathrm{Proj}(\mathrm{gr}(A))$ would be non reduced.

(viii) $\to$ (ix) $\to$ (x). These implications are obvious.

(x) $\to$ (xi). We have: $A \subset B \subset \bar{A}$ ([25], 12.1) and $\mathfrak{m}B$ is a free ideal of rank 1 ([25], 12.2). By 7.2 $\mathrm{Spec}(\bar{A})$ is obtained by blowing up $\mathrm{Spec}(A)$ along $\mathfrak{m}$, whence $\bar{A} = B$ by 7.1. Moreover $\mathrm{Spec}(\bar{A}/m\bar{A}) = \mathrm{Proj}(\mathrm{gr}(A))$ by 7.2, and the conclusion follows.

(xi) $\to$ (i). Since $e(A) = 2$ we have $\mathfrak{m}B \subset A$ ([25], 12.17 and 12.15). Moreover since $\mathfrak{m}B$ is principal ([25], 12.2) and is contained in $\mathrm{rad}(B)$ ([25], 12.1], it is easy to see that B is normal. Hence $B = \bar{A}$ (l.cit.), $\mathfrak{m}$ is the conductor of A, and $\bar{A}/\mathfrak{m}$ is reduced. The conclusion follows by 2.7(vii).

Assume now that A contains a field.

(iv) $\to$ (xii). Since A is Mori of dimension 1 it is easy to see that $\hat{\bar{A}} = \bar{\hat{A}}$, so that we may assume A is complete. If $\bar{A}$ is not local we have $\bar{A} = k[[T]]xk[[U]]$, where k is identified to a coefficient field of A. Put $C = k[[X, Y]]/(XY)$ and identify C to a subring of $\bar{A}$ by mapping $f(X, Y)$ to the pair $(f(T, 0), f(0, U))$. Then $\bar{C} = \bar{A}$ and the conductor of C is $(x, y)C = \mathrm{rad}(\bar{A})$, where x, y are the images of X, Y in C. Since $\mathrm{rad}(\bar{A}) \subset A$ it is easy to see that $C \subset A$. But C is Gorenstein and hence $l(\bar{A}/C) = l(C/(x, y)) = 1$ ([19], proof of 3.5) which implies $A = C$.

Assume now $\bar{A}$ is local. Let $K = k(u)$ be the residue field of $\bar{A}$ and let $k' = K \cap A$ (identify K to a fixed coefficient field of $\bar{A}$). Then k' is a subfield of A and since $\mathfrak{m} = \mathrm{rad}(\bar{A})$ it is easy to see that k' is a coefficient field of A. After identifying k and k', we see that $\bar{A}$ has a coefficient field of the form $k(u)$ where k is a coefficient field of A and u is algebraic of degree 2 over k. We have then $\bar{A} = k(u)[[T]]$. Put $C = k[[T, uT]]$. Then the argument used in the previous case can be used to show that $A = C$.

(xiii) $\leftrightarrow$ (xii) $\to$ (iv). These follow by straightforward computations. The proof of 8.1 is then complete.

8.3. Remarks:

(i) By 8.1 a SN one-dimensional local ring of multiplicity strictly greater than 2 is not Gorenstein. In [5] it is shown that SN one-dimensional rings of multiplicity n can occur as local rings at singular points of curves in n-space. This provides a large class of SN non-Gorenstein rings.

(ii) A one-dimensional local Gorenstein ring A with $\mathrm{gr}(A)$ reduced is not necessarily SN: take for example the local ring of a plane curve at a point of multiplicity strictly greater than 2, and with distinct tangent lines.

Now we deduce from 8.1 a slight generalization of a result of P. Salmon [29].

8.4. PROPOSITION: *Let $C = \mathrm{Spec}(k[X, Y]/(f))$ be a plane reduced curve over the field k. Let z be a point of C rational over k and let $A = \mathcal{O}_{C,z}$. Then A is* SN *if and only if either z is an ordinary node, or* $\mathrm{gr}(A) = k[U, V]/(U^2 + tV^2)$ *where $t \in k$ is not a square.*

PROOF: We may assume that z is the origin, so that $\mathrm{gr}(A) = k[X, Y]/(\varphi)$, where φ is the leading form of f. Since A is Gorenstein the conclusion follows from 8.1.

8.5. PROPOSITION: *Let k be a perfect field and let C be as in* 8.4. *Let k' be an algebraic closure of k and put $C' = C \otimes k'$. Then the following are equivalent*:

(a) *C is* SN;
(b) *C' is* SN;
(c) *C' has at most ordinary double points.*

PROOF: The equivalence of (a) and (b) follows by 5.7. The equivalence of (b) and (c) follows by 8.4 and 2.7.

8.6. REMARK: The implication (a)$\rightarrow$(b) of 8.5 is false if k is not assumed to be perfect. Indeed C' might be non-reduced. Here is a counterexample where C' is integral. Let k be a (non-perfect) field of characteristic 2, and let $k(t)$ be a purely inseparable extension of k having degree 2. Let $A = k[X, Y]/(X^2 + t^2Y^2 + Y^3)$. Then A is SN by 8.4, and it is easy to check that $A \otimes k'$ is a domain (k' an algebraic closure of k). However $A \otimes k'$ is not SN by 8.4.

8.7. COROLLARY: *Let C be a* SN *plane curve over an algebraically closed field, and let $p : \bar{C} \rightarrow C$ be its normalization. Then if x is a singular point of C, $p^{-1}(x)$ consists of two distinct reduced points.*

8.8. REMARK: The converse of the above corollary is false. Indeed let C be the complex plane curve having equation $f = 0$, where $f(X, Y) = X^2 + 2XY^2 + Y^5$. This curve is not SN by 8.5. Let A be the ring of C at the origin. Then $\hat{A} = \mathbf{C}[[X, Y]]/(f)$. The discriminant of f

as a polynomial in X is $Y^4(1-Y)$, which is clearly a square in $\mathbf{C}[[Y]]$. Hence f splits into the product of two distinct factors in $\mathbf{C}[[X, Y]]$, each having order 1. Thus $\bar{A}$ has exactly two maximal ideals, and if $p: \bar{C} \to C$ is the normalization it is not difficult to see that the preimage of the origin consists of two distinct reduced points. Since C has no other singular points, the conclusion of 8.7 holds.

8.9. Remark: As the referee pointed out the results of the present section are strictly related to (and some contained in) Davis [39]. However we prefer not to change the present setting, because our approach is somewhat different from Davis', and was announced in a previous paper (see [12]).

9. Seminormal Gorenstein schemes

We characterize SN schemes which are S_2 and Gorenstein in codimension 1 (in particular Gorenstein schemes and complete intersections) by describing their singularities in codimension 1, and by giving explicitly their local equations and the equation of the tangent cone at each point of an open subscheme whose complement has codimension not less than 2. Most of our results hold for excellent schemes, but the more complete description is obtained for algebraic schemes over an algebraically closed field of characteristic different from 2. Some of the results of this section where announced without proof in [12].

In this section we deal with a scheme X. We always assume that X is reduced and we denote by $p: X' \to X$ the normalization of X. Moreover we call I the set of all closed singular reduced and irreducible subschemes of X having codimension one.

9.1. Lemma: *If X is a Mori scheme (see 3.1) I is locally finite.*

Proof: If $U = \operatorname{Spec}(A)$ is an open affine subscheme with A noetherian the elements of I which intersect U correspond to the prime ideals of height one of A which contain the conductor of A. Since A is Mori the conductor cannot have height zero and the conclusion follows.

9.2. Definition: The scheme X is said to be G1 if it is S_2 and "Gorenstein in codimensional 1", that is $\mathcal{O}_{X,x}$ is Gorenstein whenever it has dimension ≤ 1. This is clearly a local property. A Gorenstein scheme (i.e. a scheme whose local rings are Gorenstein) is G1. In particular a scheme which is locally a complete intersection is G1.

(Property G1 was introduced in [32] with a different name. Generalizations and variations are due to many authors; we mention only [21] and [24]. See also [23] for more information and bibliography).

9.3. PROPOSITION: *If X is a Mori scheme the following conditions are equivalent*:

(i) *X is $G1$ and* SN;

(ii) *X is S_2 and for any $x \in X$ such that $\mathcal{O}_{X,x}$ is non-normal of dimension* 1, *$\mathcal{O}_{X,x}$ verifies the equivalent conditions of* 8.1;

(iii) *X is S_2 and there is an open subscheme U of X which is* SN *and $G1$ and such that* $\operatorname{codim}(X - U) \geq 2$.

PROOF: It follows by 8.1 and 3.3.

9.4. COROLLARY: *Assume X is* SN *and $G1$. Then for any $Y \in I$, $p^{-1}(Y) = Y'$ is a reduced subscheme of X'. Moreover either Y' is irreducible and the induced morphism $p: Y' \to Y$ has degree* 2, *or Y' has exactly two irreducible components which are birationally equivalent to Y (via p).*

PROOF: It follows easily by 9.3 and 8.1(iv).

9.5. REMARK: The converse of 9.4 is false, as shown in 8.8. The next proposition describes the behavior of a SN $G1$ excellent scheme in an open subset whose complement has codimension at least 2. Recall that a scheme is *excellent* if it can be covered by affine open subschemes whose rings are excellent (see [18], IV_2.7.8.5). In particular an algebraic scheme over a field is excellent. We note also that a reduced excellent scheme is Mori (see 3.1).

9.6. PROPOSITION: *Assume the scheme X is S_2 and excellent. Then the following conditions are equivalent*:

(i) *X is* SN *and $G1$*;

(iv) *there is an open subscheme U of X such that*:

- (a) $\operatorname{codim}(X - U) \geq 2$,
- (b) *the normalization of U is regular,*
- (c) *any singular point of U belongs to one and only one $Y \in I$,*
- (d) *for any singular $x \in U$ there is a complete regular local ring R, a regular system of parameters $X_1, \ldots, X_n$ of R and an $f \in (X_1, X_2) = \mathfrak{P}$ such that*:

(d1) $\hat{\mathcal{O}}_x = R/(f)$,

(d2) $\mathfrak{P}/(f)$ *is the conductor of* $\hat{\mathcal{O}}_x$,

(d3) *the leading form of* f *in* $\mathrm{gr}(R_{\mathfrak{P}})$ *has degree* 2 *and is not a square*;

(v) *there is an open subscheme* U *of* X *which is* SN *and Gorenstein and such that* $\mathrm{codim}(X - U) \geq 2$.

PROOF: (i)$\rightarrow$(iv). Let $Y \in I$ and let $\mathcal{J}$ be the sheaf of ideals which defines Y. Let y be the generic point of Y. Since $\mathcal{O}_y$ is SN and Gorenstein of dimension 1 the stalk $\mathcal{J}_y$ is generated by two elements over $\mathcal{O}_y$ (by 8.1), whence there is an open neighborhood V of y over which $\mathcal{J}$ is generated by two sections. Since X is excellent we may assume that every point of $V \cap Y$ is regular in Y, and since I is locally finite by 9.1 we may assume that $V \cap Y \cap Y' = \emptyset$ if $Y' \in I$ and $Y \neq Y'$.

Put $Z_Y = Y - V$ and let Z' be the union of the Z_Y's as Y ranges in I. By 9.1 Z' is closed of codimension ≥ 2. Let Z'' be the image (under p) of the singular locus of X'. Since X' is normal and p is closed, Z'' is closed of codimension ≥ 2. Put $U = X - (Z' \cup Z'')$. Then U verifies (a) and (b). Moreover if $x \in U$, $p^{-1}(x)$ consists of regular points; hence if x is singular it cannot be normal, so that it belongs to some Y of I. This Y is clearly unique and we have (c).

Let now x be a singular point of U and let Y be the unique element of I containing it. Put $B = \mathcal{O}_{X,x}$ and let $\mathfrak{p}$ be the prime ideal of B corresponding to Y. Then $B/\mathfrak{p}$ is regular and $\mathfrak{p} = (x_1, x_2)$ by our choice of U. Put $A = \hat{B}$ and $\mathfrak{P} = \hat{\mathfrak{p}}$. Since $A/\mathfrak{P} = (B/\mathfrak{p})\hat{}$ it follows that $A/\mathfrak{P}$ is regular (e.g. [14], 8.3) and $\mathfrak{P}$ is prime. Moreover $\mathfrak{P} = (x_1, x_2)A$. Since B is catenary and S_2 it is equidimensional ([18], IV_2.5.10.9), whence dim $A/\mathfrak{P} = \dim B/\mathfrak{p} = \dim B - 1 = \dim A - 1$ (see [14], 7.3). Then the maximal ideal of A is generated by $x_1, \ldots, x_n$ where $n = \dim A + 1$ and $x_3, \ldots, x_n$ map to a regular system of parameters of $A/\mathfrak{P}$. Since B is excellent and equidimensional, also A is equidimensional ([18], IV_2.7.8.3(x)) and thus, by 8.2, $A = R/(f)$ where R is a regular local ring of dimension n. For each i let X_i be a preimage of x_i. Then the X_i's form a regular system of parameters of R and $\mathfrak{P} = P/(f)$ where $P = (X_1, X_2)$. This proves (d1).

By our choice of U it is clear that $\mathfrak{p}$ is the radical of the conductor of B, hence it is the conductor by 1.4. Thus by flatness we have (d2).

By 5.3 A is SN, and since $\mathfrak{P}$ is generated by 2 elements it is easy to see that (d3) follows by 8.1.

(iv)$\rightarrow$(v). Let U be as in (iv). Then U is Gorenstein by (d1) and

[14], 9.7. Moreover by applying (iv) to the singular points y of U such that $\dim \mathcal{O}_y = 1$ we see that U is SN by 9.3 and 8.1.

(v) $\to$ (i). This is an easy consequence of 9.3 and 8.1. This concludes the proof of 9.6.

In the above proposition we have seen that a SN $G1$ scheme can be defined by a single equation (analytically) at each point of a suitable "large" open subscheme. Now we wish to write explicitly these equations. This is possible under an extra condition, which is always verified by a scheme over a field of characteristic $\neq 2$. We begin by discussing this condition.

9.7. LEMMA: *Let X be a Mori scheme. Then the following conditions are equivalent*:

(a) *for any $y \in X$ such that $\dim \mathcal{O}_y = 1$ the fiber $p^{-1}(y)$ is geometrically reduced* ([18], IV_2.6.7.6);

(b) *for any y as above $k(y) \otimes_{\mathcal{O}_y} \bar{\mathcal{O}}_y$ is an étale $k(y)$-algebra* (*see* [18], IV_4.17.6).

(c) *for any $Y \in I$, if $Y' = p^{-1}(Y)$, the morphism $p: Y' \to Y$ is étale at some point of Y*;

(d) *for any $Y \in I$ there is a non-empty open $V \subset Y$ such that $p: Y' \to Y$ is étale at each point of V.*

Moreover if X is a SN *$G1$ k-scheme where k is a field of characteristic $\neq 2$, the above conditions are verified.*

PROOF: The equivalence of (a) and (b) is clear by definition, and since p is finite the remaining equivalences follow easily by [18], IV_3.12.1.6(iii).

Finally if X is SN and $G1$ the fiber $p^{-1}(y)$ consist of at most two geometrical points (by 9.3 and 8.1), and the final statement follows easily.

9.8. PROPOSITION: *Assume the scheme X is excellent and S_2, and verifies the equivalent conditions of* 9.7. *Then the following conditions are equivalent*:

(i) *X is* SN *and $G1$*;

(vi) *there is an open subscheme U of X such that*:

(a) $\operatorname{codim}(X - U) \geq 2$,

(b) *for each singular $x \in U$ there is a complete regular local ring R and a regular system of parameters $X_1, \ldots, X_n$ of R such that*:

(b1) $\hat{\mathcal{O}}_x = R/(f)$, *where* $f = X_1^2 + bX_1X_2 + cX_2^2$, *and* (b2) *$b^2-4c$ is a unit and is not a square in R.*[1]

Moreover if K is a fixed coefficient field of $\mathcal{O}_x$, in (b) *we may choose b, c in K.*

PROOF: (i)$\to$(vi). We show that (iv) of 9.6 implies (vi). Let U be as in (iv). By 9.7 we may assume that for all $Y \in I$ the morphism $p: p^{-1}(Y) \to Y$ is étale at each point of $U \cap Y$. Let $x \in U$ be singular and put $A = \hat{\mathcal{O}}_x$. Write $A = R/(f)$ as in (iv) and put $\mathfrak{p} = P/(f)$, where $P = (X_1, X_2)$ and $X_1, \ldots, X_n$ is a regular system of parameters of R. By assumption $\mathfrak{p}$ is the conductor of A, and by our choice of U the canonical homomorphism $A/\mathfrak{p} \to \bar{A}/\mathfrak{p}$ is étale, and since $A/\mathfrak{p}$ is regular (see proof of 9.6) it follows that $\bar{A}/\mathfrak{p}$ is regular ([18], IV_4.17.6.1 and IV_2.6.5.2). Moreover since $\mathcal{O}_x$ is excellent the ring $\bar{A}$ coincides with the completion of the normalization of $\mathcal{O}_x$ ([18], IV_2.7.8.3(vii), and hence it is regular ([14], 8.3). Since A is SN by 5.3 it follows by 7.3 that:

$$\operatorname{Spec}(\bar{A} \otimes_A k) = \operatorname{Proj}(k \otimes_A \operatorname{gr}_{\mathfrak{p}}(A)) = \operatorname{Proj}(k[X_1, X_2]/(\phi)) \qquad (\circ)$$

where $k = k(x)$ and ϕ is the form obtained by reducing modulo $\mathfrak{m}$ the coefficients of the leading form of f in $\operatorname{gr}_P(R) = (R/P)[X_1, X_2]$. Since $\bar{A}/\mathfrak{p}$ is étale over $A/\mathfrak{p}$ it follows by [18], IV_4.17.3.3 that $\bar{A} \otimes_A k$ is an étale k-algebra and, in particular it is reduced. We cannot have $\bar{A} \otimes_A k = k$ otherwise $A = \bar{A}$ by Nakayama, whence A is regular, contrary to our choice of x. Thus $\dim_k(\bar{A} \otimes_A k) \geq 2$. On the other hand ϕ has degree at most 2, whence $\dim_k(\bar{A} \otimes_A k) = 2$. From this and ($\circ$) it follows that ϕ has degree 2. Write $\phi = X_1^2 + \beta X_1 X_2 + \gamma X_2^2$, $\beta, \gamma \in k$. By ($\circ$) we have $\bar{A} \otimes_A k = k[T]/(T^2 + \beta T + \gamma)$, and since $\bar{A} \otimes_A k$ is étale over k we have $\beta^2 - 4\gamma \neq 0$ ([18], IV_4.18.4.3). Put $q = T^2 + \beta T + \gamma$. If q is reducible then $\bar{A}$ has two maximal ideals, and since A is complete this means that A has two minimal primes. Thus $f = gh$ where g and h have order 1 in R. By looking at the leading forms of f, g, h in $\operatorname{gr}_P(R)$ it is easy to see that the leading forms of g, h are linearly independent over R/P. Thus $P = (g, h)$ and the conclusion is clear in this case. If q is irreducible, $\bar{A}$ is a local domain and its residue field is $K = k[T]/(q)$. Let $b, c \in R$ be preimages of β, γ and put $Q = T^2 + bT + c$.

[1] This means that we have essentially two cases: either $f = X_1X_2$ or b^2-4c is not a square modulo the maximal ideal of R. In particular if the residue field of R (that is $k(x)$) is algebraically closed only the former case occurs (see also the proof of (i)$\to$(vi)).

Let $R' = R[T]/(Q)$. Then R' is a local domain and is an étale R-algebra; thus R' is regular. Moreover if $P' = PR'$, R'/P' is étale over R/P (l.cit.) and P' is generated by two regular parameters of R'. Put $A' = R'/fR'$ and $\mathfrak{P}' = \mathfrak{P}A' = P'/fR'$. Then $\mathfrak{P}'$ is the conductor of A' by flatness. Finally since étale base change commutes with normalization ([18], IV_2.6.14) we have that $\bar{A}'/\mathfrak{P}'$ is étale over $A'/\mathfrak{P}'$. Thus we may repeat the above argument, with A' in place of A, to show that $f = Y_1 Y_2$ where $Y_1, \ldots, Y_n$ form a regular system of parameters of R'.

Let t, u be the roots of Q in R', and let σ be the unique R-automorphism of R' which maps t into u. Since $t - u$ is a unit in R' it follows that the fixed ring of σ is R. Thus by the unique factorization in R, and since $f \in R$, we have $\sigma(Y_1) = Y_2$. Since $1, t$ is a basis of R' over R we can write $Y_1 = X_1 - X_2 t$, with $X_1, X_2 \in R$. Then $Y_2 = X_1 - X_2 u$ and $Y_1 Y_2 = X_1^2 + bX_1X_2 + cX_2^2$. Moreover since $t - u$ is a unit $X_1, X_2, Y_3, \ldots, Y_n$ is a regular system of parameters of R', whence X_1, X_2 are regular parameters of R. Thus (i)$\rightarrow$(vi). If K is a coefficient field of A it is easy to see that in the above argument we may take b, c in K. This proves the last statement. Finally by restricting U as in the proof of 9.7 we see that (vi)$\rightarrow$(iv) and the proof is complete.

The next proposition gives relations between seminormality and the tangent cones.

9.9. PROPOSITION: *Assume the scheme X is excellent and S_2 and consider the following condition*:

(vii) *there is an open subscheme U of X such that* $\operatorname{codim}(X - U) \geq 2$ *and such that for any singular $x \in U$*
$\operatorname{gr}(\mathcal{O}_x) = (K[X_1, X_2)/(\phi))[X_3, \ldots, X_n]$
where ϕ is a form of degree 2 which is not a square ($K = k(x)$).
Then we have:

(a) *If X verifies* (vii), *X is* SN *and $G1$*;

(b) *if X is* SN *and $G1$ and verifies the equivalent conditions of* 9.7, *then X verifies* (vii).

PROOF: To prove (a) use (vii) when $\dim \mathcal{O}_x = 1$ and apply 9.3. To prove (b) observe that if X is SN and $G1$ and verifies the equivalent conditions of 9.7 then X verifies 9.8(vi), which in turn implies (vii).

Now we give the main theorem of this section.

9.10. THEOREM: *Let X be an algebraic scheme over the algebraic-*

ally closed field k, and assume X is reduced and S_2*. Consider the following conditions*:

(i) X *is* SN *and* $G1$;

(viii) *there is an open subset* $U \subset X$ *such that* $\operatorname{codim}(X - U) \geq 2$ *and for any closed singular point* $x \in U$ *one has* $\hat{\mathcal{O}}_x = k[[X_1, \ldots, X_n]]/(X_1X_2)$;

(ix) *there is* U *as above such that for any singular closed* x *in* U *one has* $\operatorname{gr}(\mathcal{O}_x) = k[X_1, \ldots, X_n]/(X_1X_2)$;

(x) *any singular reduced irreducible closed subscheme* Y *of codimension* 1 *is bi-hyperplanar* (*see* 3.8).

Then we have the following implications: (x)$\leftrightarrow$(ix)$\leftrightarrow$(viii)$\rightarrow$(i). *Moreover if the characteristic of k is not* 2 *then* (i)$\rightarrow$(viii).

PROOF: (viii)$\rightarrow$(ix)$\leftrightarrow$(x) are obvious.

(ix)$\rightarrow$(viii). After restricting U if necessary we may assume that for any $Y \in I$, X is normally flat along Y at each point of $U \cap Y$ ([20], Cor. on p. 189). We may also assume, as in the proof of 9.8, that the normalization of U is regular, that any singular point of U belongs to a unique $Y \in I$, and that all $Y \in I$ are regular at the points of U. Finally we may assume that if $p : U' \to U$ is the normalization of U then $p^{-1}(U \cap Y)$ is unmixed as a subscheme of U' (eventual embedded components have codimension at least 2 and can be avoided).

Let now x be a singular closed point of U and put $B = \mathcal{O}_x$. We shall see that $\bar{B}$ is not local. Assume this for a moment. Then $A = \hat{B}$ is not a domain (see e.g. [18], IV_2.7.6.2). Moreover by (ix) it follows easily that $\dim A = n - 1$ and that the maximal ideal of A is generated by n elements. Then by the Cohen Structure Theorem and by [20], Lemma 6, p. 190, we have $A = k[[X_1, \ldots, X_n]]/(f)$ where f is reducible and has leading form X_1X_2. Then $f = f_1f_2$ where the leading form of f_i is X_i; thus after replacing X_i by f_i we have the conclusion. Now we prove that $\bar{B}$ is not local. Let Y be the unique element of I containing x and let $\mathfrak{p}$ be the prime ideal of B corresponding to Y. By our choice of U, $B/\mathfrak{p}$ is regular and $\operatorname{gr}_{\mathfrak{p}}(B)$ is flat over $B/\mathfrak{p}$. Hence we have: $e(B_{\mathfrak{p}}) = e(B)$ ([20], Cor. 2 on p. 186), and since clearly $e(B) = 2$ it follows by [25] (Th. 12.10) that

$$\dim_{k(\mathfrak{p})}(\mathfrak{p}^n/\mathfrak{p}^{n+1}) = 2 \quad \text{for } n \geq 1.$$

Thus $\mathfrak{p} = (x_1, x_2)$ whence $\operatorname{gr}_{\mathfrak{p}}(B) = (B/\mathfrak{p})[X_1, X_2]/\mathfrak{a}$. By looking at the ranks of the homogeneous components we see that $\mathfrak{a}$ is generated by a form ϕ of degree 2.

Let $\mathfrak{m}$ be the maximal ideal of B. Then $\mathfrak{m} = (x_1, \ldots, x_n)$ where $x_3, \ldots, x_n$ lift a regular system of parameters of $B/\mathfrak{p}$. By [20], Prop. 1 on p. 184 we have an isomorphism of graded k-algebras (depending on the choice of the above generators of $\mathfrak{m}$):

$$\psi : (\mathrm{gr}_{\mathfrak{p}}(B) \otimes k) \otimes_k \mathrm{gr}(B/\mathfrak{p}) \xrightarrow{\sim} \mathrm{gr}(B) = k[X_1, \ldots, X_n]/(X_1X_2)$$

By looking at the definition of ψ we see that ψ induces the isomorphism

$$((B/\mathfrak{m})[X_1, X_2]/(\bar{\phi}))[X_3, \ldots, X_n] \to k[X_1, \ldots, X_n]/(X_1X_2).$$

where $\bar{\phi}$ is the image of ϕ. It follows that $\bar{\phi}$ is associated to X_1X_2 whence ϕ has no multiple factors, and since $B/\mathfrak{p}$ is UFD it follows that $\mathrm{gr}_{\mathfrak{p}}(B)$ is reduced. Moreover by our choice of U it is easy to see that $\mathfrak{p}$ is the radical of the conductor of B. Finally since $\bar{B}$ is locally UFD and $\mathfrak{p}\bar{B}$ is unmixed of height 1 (again by the choice of U) it is easy to see that $\mathfrak{p}\bar{B}$ is invertible. Hence we can apply 7.2 to get an isomorphism of schemes:

$$\mathrm{Spec}(\bar{B}/\mathfrak{p}\bar{B}) = \mathrm{Proj}(\mathrm{gr}_{\mathfrak{p}}(B))$$

and tensoring by $k = B/\mathfrak{m}$:

$$\mathrm{Spec}(\bar{B}/\mathfrak{m}\bar{B}) = \mathrm{Proj}(k[X_1, X_2]/(X_1X_2))$$

It follows that $\mathrm{Spec}(\bar{B}/\mathfrak{m}\bar{B})$ consists of two points, whence $\bar{B}$ is not local. This completes the proof of (ix)$\to$(viii).

(viii)$\to$(i). A straightforward computation shows that $\hat{\mathcal{O}}_x$ is Gorenstein and SN for all closed points of X contained in U. Hence all closed points of X contained in U are SN by 1.7. Moreover since X is a Jacobson scheme it follows that any point of U is a generalization of a closed point of X contained in U ([17], I.6.4); the conclusion follows by 2.7, and 9.6(v).

(i)$\to$(viii) (when char $k \neq 2$). This is an immediate consequence of 9.8.

9.11. REMARKS:

(i) If in 9.10 the characteristic of k is assumed to be $\neq 2$, the proof of (ix)$\to$(viii) can be given by using 3.10 in order to have (ix)$\to$(i) and then applying 9.9.

(ii) Theorem 9.10 remains valid for schemes which are locally of the form $\mathrm{Spec}(k\{X_1, \ldots, X_n\}/\mathfrak{a})$ where k is an algebraically closed and complete valued field and $k\{X_1, \ldots, X_n\}$ is the ring of restricted power series over k. Indeed the main facts used in the proof (such as: excellence, Hilbert Nullstellensatz etc.) are verified in this case (see e.g. [16] and its bibliography).

(iii) The trouble given by the characteristic 2 in the present paragraph should probably disappear by considering the definition of seminormalization given by Andreotti-Bombieri [1], since it allows purely inseparable residue field extensions. We believe that the above results (with no restrictions on the characteristic) should be valid in this setting also.

(iv) By using 9.10 one can easily give examples of seminormal hypersurfaces which are not generic projections. For example it is known that a complex surface in $\mathbf{P}^3_{\mathbf{C}}$ which is a generic projection cannot have points of multiplicity >3. Hence a cone on a plane seminormal quartic provides an example of a seminormal surface which is not a generic projection.

More information on seminormal surfaces of $\mathbf{P}^3_C$ (especially of order ≤ 4) can be found in [9].

(v) The conditions (i) and (ix) of 9.10 are not equivalent in characteristic 2. Indeed let X be the surface $\mathrm{Spec}(k[T, U, V)/(U^2 + TV^2))$ where k is an algebraically closed field of characteristic 2. The singular locus of X is the line $U = V = 0$, and if y is its generic point, it is easy to see that $\mathcal{O}_{X,y} \cong k(T)[U, V]/(U^2 + TV^2)$, which is SN by 8.1. Thus X is SN (e.g. by 9.3). However the tangent cone at the point $(t, 0, 0)$ is $\mathrm{Spec}(k[T, U, V]/(U^2 + tV^2))$, and it is not reduced since k is algebraically closed of characteristic 2. Thus, with the notations as in 9.10 we have that (i) does not imply (ix).

(vi) The above example shows also that statement (b) of 9.9 is false in characteristic 2, and that conditions (i) and (vi) of 9.8 are not equivalent in general. We leave the details to the reader.

(vii) We observe that the surface X considered in (v) *is not* SN according to Andreotti-Bombieri [1] (its seminormalization coincides with its normalization and is isomorphic to an affine plane). Compare with (iii) above.

(viii) Let X be an algebraic variety over k algebraically closed, and let x be a closed point of X. Then it is not true, in general, that $\mathrm{gr}(\mathcal{O}_{X,x}) = k[X_1, \ldots, X_n]/(X_1X_2)$ implies $\hat{\mathcal{O}}_{X,x} \cong k[[X_1, \ldots, X_n]]/(X_1X_2)$ (compare with 9.10(viii) and (ix)). Indeed the surface $X = \mathrm{Spec}(k[X, Y, Z]/(XY + Z^3))$ is normal and hence, if x is the origin, $\hat{\mathcal{O}}_{X,x}$ is a domain, while $\mathrm{gr}(\mathcal{O}_{X,x}) = k[X, Y, Z]/(XY)$.

REFERENCES

[1] A. ANDREOTTI and E. BOMBIERI: Sugli omeomorfismi delle varietà algebriche. *Ann. Sc. Norm. Sup. Pisa 23* (1969) 430–450.

[2] A. ANDREOTTI and F. NORGUET: La convexité holomorphe dans l'espace analytique des cycles d'une variété algébrique. *Ann. Sc. Norm. Sup. Pisa 21* (1967) 31–82.

[3] H. BASS: On the ubiquity of Gorenstein rings. *Math. Z. 82* (1963) 8–28.

[4] H. BASS and P. MURTHY: Grothendieck groups and Picard groups of abelian group rings. *Ann. of Math. 86* (1967) 16–73.

[5] E. BOMBIERI: Seminormalità e singolarità ordinarie. *Symp. Math. XI* (1973) 205–210.

[6] N. BOURBAKI: *Algèbre Commutative. ch. I, II*, Hermann, Paris 1961.

[7] N. BOURBAKI: *Algèbre Commutative. ch. III, IV*, Hermann, Paris 1961.

[8] N. BOURBAKI: *Algèbre Commutative. ch. V, VI*, Hermann, Paris 1964.

[9] N. CHIARLI: Sulla seminormalità di certe superfici algebriche. *Atti Acc. Sc. Torino 108* (1973–74) 605–611.

[10] A. FRANCHETTA: Sulla varietà doppia della proiezione generica di una varietà algebrica non singolare. *Acc. Scienze Lettere ed Arti di Palermo 14* (1953–54) 1–12.

[11] S. GRECO: Anelli Henseliani in *Categories and Commutative Algebra.* Cremonese, Roma, 1974.

[12] S. GRECO: Seminormalità delle varietà di Gorenstein. *Atti Acc. Naz. Lincei 58* (1975) 556–558.

[13] S. GRECO: Two theorems on excellent rings. *Nagoya Math. J. 60* (1976) 139–149.

[14] S. GRECO and P. SALMON: *Topics in m-adic topologies. Erg. Der Math. b.* 58, Springer-Verlag 1971.

[15] S. GRECO and N. SANKARAN: On the separable and algebraic closedness of a Hensel couple in its completion. *J. of Algebra 39* (1976) 335–348.

[16] S. GRECO and P. VALABREGA: On the excellent property for strictly convergent power series rings over a complete non Archimedean valued field. *Atti Acc. Sc. Torino 108* (1973–74) 529–539.

[17] A. GROTHENDIECK and J. DIEUDONNE: *Eléments de Géométrie Algébrique I. Grund. der Math. 166*, Springer-Verlag 1971.

[18] A. GROTHENDIECK and J. DIEUDONNE: Eléments de Géométrie Algébrique II, III, IV, *Publ. Math. 8*,..., Paris 1961.

[19] J. HERZOG and E. KUNZ: *Der kanonische Modul eines Cohen-Macaulay Rings. Lect. Notes Math. 238*, Springer-Verlag 1971.

[20] H. HIRONAKA: Resolution of singularities of an algebraic variety over a field of characteristic zero. *Ann. of Math. 79* (1964) 109–326.

[21] F. ISCHEBECK: Eine dualitat zwischen Ext und Tor. *J. of Algebra 11* (1969) 510–531.

[22] F. ISCHEBECK: Zur Picard-Gruppe eines Products. *Math. Z. 139* (1974) 141–157.

[23] M. MARINARI: Su alcune classi di anelli noetheriani che comprendono gli anelli di Gorenstein. *Ann. Univ. Ferrara, Sez. VII, 19* (1974) 135–143.

[24] M. MARINARI: Gorenstein sequences and G_n condition. *J. of Algebra 39* (1976) 349–359.

[25] E. MATLIS: *1-dimensional Cohen-Macaulay rings, Lect. Notes Math. 327*, Springer-Verlag 1973.

[26] H. MATSUMURA: *Commutative Algebra*, Benjamin, New York, 1970.

[27] J. ROBERTS: Generic projections of algebraic varieties. *Am. J. Math. 93* (1971) 191–214.

[28] P. SALMON: Sur les séries formelles restreintes. *Bull. Soc. Math. France 92* (1964) 385–410.

[29] P. SALMON: Singolarità e gruppo di Picard. *Symp. Math. II* (1968) 341–345.

[30] J.P. SERRE: Géométrie algébrique et Géométrie Analytique. *Ann. Inst. Fourier 6* (1955) 1–42.
[31] C. TRAVERSO: Seminormality and Picard Group. *Ann. Sc. Norm. Sup. Pisa 24* (1970) 585–595.
[32] W. VASCONCELOS: Quasi-normal rings, Ill. *J. Math 14* (1970) 268–273.
[33] G. FISCHER: *Complex Analytic Geometry, Lect. Notes Math. 538*, Springer-Verlag 1976.
[34] J. FRISCH: Points de platitude d'un morphisme d'espaces analytiques complexes. *Inv. Math. 4* (1967) 118–138.
[35] R.C. GUNNING and H. ROSSI: *Analytic functions of several complex variables*, Prentice Hall Inc., 1965.
[36] H. MATSUMURA: Formal power series rings over polynomial rings I, *Number Theory, Algebraic Geometry and Commutative Algebra in honor of Y. Akizuki*, Kinokuniya Tokyo 1973, 511–520.
[37] R. NARASIMHAN: *Introduction to the theory of Analytic Spaces, Lect. Notes Math. 25*, Springer-Verlag 1966.
[38] R. NARASIMHAN: Imbedding of holomorphically complete complex spaces. *Am. J. Math. 82* (1960) 917–934.
[39] E.D. DAVIS: On the geometric interpretation of seminormality, *Proc. AMS 68* (1978) 1–5.
[40] R. PIENE: Polar classes of singular varieties. *Ann. Ec. Norm. Sup. 4 ser., Vol 11* (1978) 247–276.
[41] P.M.H. WILSON: On blowing up conductor ideals, *Math. Proc. Camb. Phil. Soc. 83* (1978) 445–450.

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