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STRATA IN THE DEFORMATION OF REAL ISOLATED SINGULARITIES ARE IN GENERAL NON CONTRACTIBLE

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Introduction

In [4] Looijenga answered positively to a conjecture of Thom on isolated singularities (see [5] page 58) in the case of real simple singularities. The complex form of this conjecture in the case of a simple singularity had been earlier verified (see for example [1] and [2]). There exist some reasons to believe that the conjecture over $\mathbb{C}$ is not true in general and in fact that for elliptic singularities the complement of the (complex) discriminant has a non vanishing second homotopy group (see [3] for an attempt to understand the meaning of this conjectured non-vanishing).

Here we give an example which shows that the conjecture over $\mathbb{R}$ is not true and in fact that a connected component of the complement of the real discriminant[1] for the singularity $x^4 + y^4$ has an infinite fundamental group. This supports, we believe, the idea that the second homotopy group of the complement of the complex discriminant of $x^4 + y^4$ is infinite too.

Consider the semiuniversal deformation of the elliptic singularity $x^4 + y^4$. We will construct a continuous family $\{F_\theta\}_{0 \le \theta \le \pi}$ of non singular real curves of the deformation with the following properties:

i) $F_0 = F_\pi$

ii) the automorphism of F_0, induced by the homotopy class of the loop $\theta \to F_\theta$, is non trivial.

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[1] We follow [4]: the "real discriminant" means the image of the real critical points of the deformation which is contained (generally strictly) in the real part of the complex discriminant.

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This will show that there exists a non contractible connected component in the complement of the real discriminant.

Let F_0 be the curve

$$x^4 + y^4 - 2\eta x^2 + \epsilon = 0$$

with $\eta, \epsilon > 0$. One can easily see that for $0 < \epsilon < \eta^2$ this is a compact irreducible curve with two connected components which are separated by the line $x = 0$.

Let F_θ be the curve

$$x^4 + y^4 - 2\eta\varphi^2 + \epsilon = 0$$

where $\varphi = x\cos\theta + y\sin\theta$. Obviously $F_\pi = F_0$. Moreover for each θ, F_θ has the same properties as F_0 with respect to the line $\varphi = 0$, namely it consists of two connected components separated by the line $\varphi = 0$.

This will be shown, through an isotopy principle, by proving that each F_θ is non singular plus the remark that F_θ does not intersect $\varphi = 0$.

Since, as θ varies in $[0, \pi]$, clearly the halfplanes $\varphi > 0$ and $\varphi < 0$ interchange, the components of F_0 will do the same. It will follow that the connected component, in the complement of the (real) discriminant, containing F_0 has a non trivial fundamental group.

Proof that F_θ is non singular

A singular point $P = (x_0, y_0)$ for F_θ is a solution of

$$(+) \qquad \begin{cases} x^4 + y^4 - 2\eta\varphi^2 + \epsilon = 0 \\ x^3 - \eta\varphi\cos\theta = 0 \\ y^3 - \eta\varphi\sin\theta = 0 \end{cases}$$

from where

$$x^4 + y^4 = \eta\varphi^2 = \epsilon.$$

One has to show that the system (+) has no solutions for η, ϵ sufficiently small.

Choose $\epsilon < \frac{1}{8}\eta^2$.

On one hand: $(x^2 + y^2)^3 \geq x^6 + y^6 = (\eta\varphi\cos\theta)^2 + (\eta\varphi\sin\theta)^2 = \eta^2\varphi^2 = \eta\epsilon$

On the other: $(x^2 + y^2)^2 \leq 2(x^4 + y^4) \leq 2\epsilon$

Hence $(2\epsilon)^3 \geq (\eta\epsilon)^2$; implying $\epsilon \geq \frac{1}{8}\eta^2$.

Contradiction.

Remark 1

Let Γ be the connected component of $S_{\mathbb{R}} - D_{\mathbb{R}}$ (same notations as in [4]) which contains the curves F_θ.

The loop γ we described before generates an infinite subgroup of $\pi_1(\Gamma, F_0)$. In fact, let G be this subgroup: one has a map $b: G \to B(2)$ (the braid group in two strings) induced by the map which to each curve F_θ associates the couple of its baricenters, and clearly γ goes to the generator of $B(2)$.

Remark 2

The two components of F_0 induce linearly independent cycles in $H_1(\tilde{F}_0, \mathbb{Z})$, where $\tilde{F}_0$ denotes the complex fibre, as one easily verifies, for instance, by inspection of the induced ramified covering. It follows that the loop described before also induces a non trivial element in the monodromy of the complex deformations.

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