IZU VAISMAN

Remarkable operators and commutation formulas on locally conformal Kähler manifolds

Compositio Mathematica, tome 40, nº 3 (1980), p. 287-299

<http://www.numdam.org/item?id=CM_1980__40_3_287_0>

COMPOSITIO MATHEMATICA, Vol. 40, Fasc, 3, 1980, pag. 287–299

Printed in the Netherlands

REMARKABLE OPERATORS AND COMMUTATION FORMULAS ON LOCALLY CONFORMAL KÄHLER MANIFOLDS*

Izu Vaisman

Summary

Generalization of the remarkable commutation formulas on Kähler manifolds [9]. Definition of an adapted cohomology. Applications: the expression of the fundamental form of a particular class of compact locally conformal Kähler manifolds; generalized semi-Kähler locally conformal Kähler manifolds are Kähler; etc.

It is known that the geometry and the topology of Kähler manifolds is strongly influenced by the existence of some remarkable operators and commutation formulas on such a manifold. E.g., see [9,1,5] for a discussion of such operators and formulas.

In this Note, we prove that the commutation formulas mentioned above admit interesting generalizations to the case of *locally conformal Kähler* (shortly, l.c.K.) manifolds [4,7,8].

On another hand, we define and discuss the cohomology with coefficients in the sheaf of germs of C^∞-functions f such that, if $f > 0$, $f^{-1}g$ is a Kähler metric. (Here, g is the given l.c.K. metric of the manifold.)

As a consequence of the above mentioned development, some information about the l.c.K. manifolds will be derived. In particular, we determine the expression of the fundamental form of a special class of compact l.c.K. manifolds, and we prove that every l.c.K. manifold which is generalized semi-Kähler is Kähler.

* AMS (MOS) Classification Subjects (1970): 53 C 55.

0010–437X/80/03/0287–13$00.20/0

1. Definitions and known results

We recall [4,7] that an l.c.K. manifold is a Hermitian manifold $M^{2n}(n > 1)$ with complex structure J and metric g, which has an open covering $\{U_\alpha\}$ endowed with differentiable functions $\sigma_\alpha : U_\alpha \to R$ such that the local metrics $e^{-\sigma_\alpha} g$ are Kähler metrics. (In the whole paper, differentiable means C^∞.)

In [4,7], it is shown that the l.c.K. manifolds are characterized by

$$\mathrm{d}\Omega = \omega \wedge \Omega, \tag{1.1}$$

where Ω is the fundamental form of M and ω is a closed Pfaff form called the *Lee form* of M. Locally, the Lee form is given by $\omega = \mathrm{d}\sigma_\alpha$. Iff ω is exact, we can take $U_\alpha = M$, and then g is a globally conformal Kähler metric, i.e. M is Kähler with respect to a metric conformal to g.

The simplest interesting example of an l.c.K. manifold is offered by the Hopf manifolds [7].

Now, let M be an l.c.K. manifold. Its Hermitian structure allows us to define differential forms and operators of the type (p,q) and to consider the following classical operators on forms:

$$*, \mathrm{d}, \delta, \mathrm{d}', \mathrm{d}'', \delta', \delta'', \Delta, \Box, C, \mathrm{e}(\varphi), i(\varphi), L, \Lambda. \tag{1.2}$$

Here, $\mathrm{d} = \mathrm{d}' + \mathrm{d}''$ is the exterior differential with its decomposition into terms of the type $(1,0)$ and $(0,1)$, δ, δ', δ'', are the corresponding co-differentials, Δ is the Laplacian of d, $\Box$ is the Laplacian of d'', $\mathrm{e}(\varphi)$, $i(\varphi)$ are, respectively, exterior and interior multiplication by the form φ, and $L = \mathrm{e}(\Omega)$, $\Lambda = i(\Omega)$. We assume that the definition and the properties of all of these operators (including $*$ and C) are known, and refer to [9,1,5] for the development of this subject.

Furthermore, let us recall that a p-form η on M is called a *primitive form* if $\Lambda\eta = 0$. The primitive p-forms satisfy a lot of important identities [9] and among them:

$$\begin{aligned}\Lambda^s L^r = {} & r(r-1)\cdots(r-s+1)(n-p-r+1)(n-p-r+2) \\ & \times\cdots(n-p-r+s)L^{r-s}\eta \quad (r \geq s \geq 1),\end{aligned} \tag{1.3}$$

$$L^q\eta = 0 \quad (q \geq \max(0, n-p+1)), \tag{1.4}$$

$$* L^r\eta = (-1)^{p(p+1)/2}\frac{r!}{(n-p-r)!}L^{n-p-r}C\eta, \quad \text{etc.} \tag{1.5}$$

One can also prove the following very important result: every p-form φ on M has a unique decomposition of the form

$$\varphi = \sum_{r \geq \max(0,\, p-n)} L^r \eta_r, \tag{1.6}$$

where all of the η_r are primitive forms of a corresponding degree.

For instance, since ω is a 1-form, it is primitive and, by (1.3), we have

$$\Lambda L\omega = (n-1)\omega. \tag{1.7}$$

By (1.1) and the definition of L, this yields

$$\omega = \frac{1}{n-1} \Lambda \mathrm{d}\Omega. \tag{1.8}$$

Formula (1.8) can be used as the definition of a *Lee form* on an arbitrary almost Hermitian manifold.

2. Commutation formulas on l.c.K. manifolds

In the case of a Kähler manifold, a host of commutation formulas of the operators (1.3) are available, and they have highly important geometric consequences. It is our aim here to produce corresponding commutation formulas in the l.c.K. case.

To do this, first, we introduce the operators

$$\mathrm{e} = \mathrm{e}(\omega),\ \epsilon = i(\omega) = *\mathrm{e}*. \tag{2.1}$$

Next, we introduce the auxiliary operators

$$\tilde{\mathrm{d}} = \mathrm{d} - \frac{p}{2}\mathrm{e},\ \ \tilde{\delta} = -*d* = \delta + \left(n - \frac{p}{2}\right)\epsilon, \tag{2.2}$$

where p is the degree of the form acted on. $\tilde{\mathrm{d}}$ is an antiderivation of differential forms and it is easy to see that

$$\tilde{\mathrm{d}}\Omega = 0,\ \ \tilde{\mathrm{d}}^2 = -\tfrac{1}{2}\mathrm{e}\mathrm{d}. \tag{2.3}$$

In the sequel, if A, B are operators of the degrees h, k, respectively, their commutant is defined as

$$[A, B] = AB - (-1)^{hk} BA$$

and, in addition, we put $A^C = C^{-1}AC$.

PROPOSITION 2.1: *The following commutation formulas hold on every l.c.K. manifold*:

$$\text{(2.4)}\qquad \begin{gathered}[\Lambda, L]\alpha = (n - \deg\alpha)\text{id.}, [L, e] = 0, [\Lambda, \epsilon] = 0,\\ [L,\epsilon] = -e^C,\ [\Lambda,e] = \epsilon^C,\ [e,\epsilon] = \|\omega\|^2 \text{ id.}\end{gathered}$$

PROOF: The first formula is known [9]. The second is obvious. The third follows from the second by (2.1) and the fact that $\Lambda = *^{-1}L*$, which is a known relation between L and Λ.

In order to derive the two following formulas, we use componentwise computations [5]. With an obvious notation, we have

$$(\Lambda e\alpha)_{s_1\cdots s_{p-1}} = \Omega^{ij}\omega_i\alpha_{js_1\cdots s_{p-1}} + \tfrac{1}{2}\sum_{\alpha=1}^{p-1}(-1)^{\alpha-1}\omega_{s_\alpha}\Omega^{ij}\alpha_{ijs_1\ldots\hat{s}_\alpha\ldots s_{p-1}},$$

$$\begin{aligned}(e\Lambda\alpha)_{s_1\cdots s_{p-1}} &= \frac{1}{2(p-2)!}\delta^{\lambda u_1\cdots u_{p-2}}_{s_1\cdots s_{p-1}}\,\omega_\lambda\Omega^{ij}\alpha_{iju_1\ldots u_{p-2}}\\ &= \tfrac{1}{2}\sum_{\alpha=1}^{p-1}(-1)^{\alpha-1}\omega_{s_\alpha}\Omega^{ij}\alpha_{ijs_1\ldots\hat{s}_\alpha\ldots s_{p-1}},\end{aligned}$$

where the sign $\hat{}$ denotes the absence of the respective index. $\delta^{\because}_{\because}$ is the Kronecker symbol, and we have taken into account the general identity

$$\delta^{hi_1\ldots i_p}_{j_1\ldots j_{p+1}} = \sum_{\alpha=1}^{p+1}(-1)^{\alpha-1}\delta^h{}_{j_\alpha}\delta^{i_1\ldots\ldots\ldots i_p}_{j_1\ldots\hat{j}_\alpha\ldots j_{p+1}},$$

which holds because the Kronecker index is a determinant.

It follows

$$\text{(2.5)}\qquad [\Lambda,e] = i(\theta),$$

where

$$\text{(2.6)}\qquad \theta = \epsilon\Omega.$$

By using $\Lambda = *^{-1}L*$ and $i(\theta) = *e(\theta)*$ [5], (2.5) yields

$$\text{(2.7)}\qquad [L,\epsilon] = -e(\theta),$$

and, in order to get the stated formulas, it remained to prove that $\mathrm{e}(\theta) = \mathrm{e}^C$.

To do this, let α be a (p, q)-form on M and consider the decomposition of ω into parts of the types (1,0) and (0,1), $\omega = \omega' + \omega''(\omega'' = \bar{\omega}')$, which gives corresponding decompositions $\mathrm{e} = \mathrm{e}' + \mathrm{e}''$ and $\epsilon = \epsilon' + \epsilon'' = i(\omega'') + i(\omega')$. Then we get

$$\mathrm{e}^C\alpha = -\sqrt{-1}(\omega' - \omega'') \wedge \alpha,$$

which means

$$\mathrm{e}^C = -\mathrm{e}(\omega \circ J). \tag{2.8}$$

Furthermore, let us denote by B the vector field defined by $g(B,X) = \omega(X)$, and by A the vector field defined by $\Omega(A,X) = \omega(X)$. We shall be using the definition $\Omega(X,Y) = g(JX,Y)$, which differs in sign from the definition used in [7]. One has then $B = JA$, and (2.6) yields

$$\theta(X) = \Omega(B,X) = \Omega(JA,X) = -\omega \circ J(X). \tag{2.9}$$

Together with (2.8), this implies $\mathrm{e}(\theta) = \mathrm{e}^C$, Q.E.D.

Finally, the last formula (2.4) can be established by similar componentwise computations, and this ends the proof of Proposition 2.1.

PROPOSITION 2.2: *The following relations hold on every l.c.K. manifold*:

$$\begin{gathered}[\mathrm{d},\mathrm{e}] = 0,\ [\delta,\epsilon] = 0,\ [L,\tilde{\mathrm{d}}] = 0,\ [\Lambda,\tilde{\delta}] = 0,\\ [L^h,\mathrm{d}] = -h\mathrm{e}L^h,\ [\Lambda^h,\delta] = h\epsilon\Lambda^h \ (h \geq 0).\end{gathered} \tag{2.10}$$

PROOF: All of these formulas are easy consequences of the definitions of the respective operators. The proof of the last two of them also requires induction.

In the Kähler case, the most important commutation formulas are for $[\Lambda,\mathrm{d}]$ and $[L,\delta]$. The generalization of these formulas is given by

PROPOSITION 2.3: *The following commutation formulas, where p is the degree of the form acted on, hold on every l.c.K. manifold*:

$$\begin{aligned}&[\Lambda,\tilde{\mathrm{d}}] = -\tilde{\delta}^C,\ [L,\tilde{\delta}] = \tilde{\mathrm{d}}^C,\\ &[\Lambda,\mathrm{d}] = -\delta^C - (n-p)\epsilon^C + \mathrm{e}\Lambda,\\ &[L,\delta] = \mathrm{d}^C + (n-p)\mathrm{e}^C - \epsilon L.\end{aligned} \tag{2.11}$$

PROOF: By taking into account the various relations exhibited until now between the operators involved, all these relations follow easily from the first one. Furthermore, the proof of the first relation is similar to the proof of the corresponding Kählerian relation [9]. I.e., using (1.6), we see that it suffices to prove the required formula for forms $L^r\eta\,(r\geq 0)$, where η is a primitive p-form. In this case, $L^{n-p+1}\eta=0$ by (1.4), and this implies $L^{n-p+1}\tilde{\mathrm{d}}\eta=0$. It follows that the decomposition (1.6) of $\tilde{\mathrm{d}}\eta$ is $\tilde{\mathrm{d}}\eta=\eta_0+L\eta_1$, where η_0 and η_1 are primitive forms. Next, one computes $[\Lambda,\tilde{\mathrm{d}}](L^r\eta)$ and $-\tilde{\delta}^C(L^r\eta)$ by using (1.3) and (1.5), and one finds equal results. Q.E.D.

REMARK: There are two interesting alternate ways to prove Proposition 2.3. The first consists of proving the second formula (2.11) over a neighbourhood where a conformal Kähler metric is available. This can be done by using the corresponding commutation formula for the local conformal Kähler metric.

The second way consists of generalizing the "geodesic coordinates" [1, p.173]. From the existence of such coordinates in the Kählerian case, we easily derive that a Hermitian manifold M is l.c.K. iff every point $p\epsilon M$ has an open neighbourhood endowed with the local complex analytic coordinates z^j such that $z^j(p)=0$ and the metric is of the form

$$\mathrm{d}s^2=(a+\sum_{h=1}^{n}b_hz^h+\sum_{h=1}^{n}\bar{b}_h\bar{z}^h)\sum_{j=1}^{n}\mathrm{d}z^j\mathrm{d}\bar{z}^j+\sigma\ (2), \tag{2.12}$$

where $a>0$ is a real constant and b_h are complex constant numbers.

Then, since the required identity contains only the first derivatives of the metric tensor, it suffices to prove it for C^n at 0 and with the metric given by (2.12) without the term σ (2).

Now, the generalization of the other commutation formulas of [9] is but a technical matter, and we shall stop here. However, we should like to mention the following consequence (which requires a lengthy computation):

$$\Delta=2\Box-A, \tag{2.13}$$

Here, A is a first order operator, which vanishes if $\omega=0$, namely

$$A=(\mathrm{e}'\delta'-\mathrm{e}''\delta'')-(\epsilon'\mathrm{d}'-\epsilon''\mathrm{d}'')+(n-p)([\mathrm{d}',\epsilon']-[\mathrm{d}'',\epsilon'']+\|\omega\|^2\mathrm{id}.$$
$$-2\mathrm{e}''\epsilon'')+(\mathrm{e}'\delta''+\mathrm{e}''\delta')-(\epsilon'\mathrm{d}''+\epsilon''\mathrm{d}')+(n-p)([\mathrm{d}',\epsilon'']+[\mathrm{d}'',\epsilon']),$$

where we have written A such as to make clear its effect on the type of forms.

Finally it is worth noting two additional commutation formulas:

$$(2.14) \qquad [\mathrm{d},\epsilon] = L_B,$$

where L_B denotes the Lie derivative with respect to the vector field B associated to ω (which is a well known formula), and

$$(2.15) \qquad [\delta,\mathrm{e}] = -*^{-1}L_B*,$$

which follows from (2.14) by using (2.1) and $\delta = -*\mathrm{d}*$.

3. The adapted cohomology of an l.c.K. manifold

In this section, we shall consider another remarkable operator, which leads to some new cohomology spaces of an l.c.K. manifold. We shall also make some comments on the cohomological significance of the operator $\tilde{\mathrm{d}}$.

As a matter of fact, this cohomology can be defined in a more general setting. Namely, in this section, M will be a paracompact m-dimensional differentiable manifold and ω will be a closed 1-form on M. Then, the operator to be considered is defined by

$$(3.1) \qquad \mathrm{d}_\omega = \mathrm{d} - \mathrm{e}(\omega).$$

It is simple to see that $\mathrm{d}_\omega^2 = 0$, which shows that the differentiable forms on M, together with the operator d_ω, define a cochain complex, and we denote by $H^p_\omega M)$ $(p = 0, 1, 2, \ldots)$ the corresponding cohomology spaces. We are calling them the *adapted cohomology spaces of the pair* (M,ω).

The spaces $H^p_\omega(M)$ can also be obtained as the cohomology spaces of M with coefficients in a sheaf. Namely, let us denote by $\mathscr{F}_\omega(M)$ the sheaf of germs of differentiable functions $f: M \to R$ which are such that

$$(3.2) \qquad \mathrm{d}_\omega f = \mathrm{d}f - f\omega = 0.$$

Then, it is easy to prove

PROPOSITION 3.1: (*The adapted de Rham theorem*). *For every pair* (M,ω) *as introduced above, one has the isomorphisms*

$$H^p(M,\mathscr{F}_\omega(M)) \approx H^p_\omega(M).$$

PROOF: First, we shall note that d_ω satisfies a Poincaré lemma. Indeed, let α be a local form such that $\mathrm{d}_\omega\alpha = 0$. Since ω is closed and the lemma has to be local, we may suppose

$$\omega = -\mathrm{d}b/b, \tag{3.3}$$

where b is a nonzero differentiable function. Then, $\mathrm{d}_\omega\alpha = 0$ means $\mathrm{d}(b\alpha) = 0$, whence $\alpha = \mathrm{d}_\omega(\beta/b)$ for some local form β. This is exactly the requested result.

Then, if we denote by $\mathscr{A}^p(M)$ the sheaf of germs of differentiable p-forms on M, we see that

$$0 \to \mathscr{F}_\omega(M) \overset{\subset}{\to} \mathscr{A}^0(M) \overset{\mathrm{d}_\omega}{\to} \mathscr{A}^1(M) \overset{\mathrm{d}_\omega}{\to} \ldots \tag{3.4}$$

is a fine resolution of $\mathscr{F}_\omega(M)$, which proves the Proposition.

Moreover, it is obvious that d_ω and d are differential operators with the same symbol, hence $H^p_\omega(M)$ are the cohomology spaces of an elliptic complex, and, if M is a compact orientable Riemann manifold, a corresponding theory of harmonic forms is available. (See, for instance, [6] for a formulation of the main theorems regarding elliptic complexes.)

In this case, we also get that $H^p_\omega(M)$ are finite dimensional linear spaces over the real field R, and we shall denote

$$b^p_\omega(M) = \dim H^p_\omega(M). \tag{3.5}$$

Let us note the following properties of the sheaf $\mathscr{F}_\omega(M)$. If U is an open connected subset of M such that ω/U is exact, then $\mathscr{F}_\omega(M)/U \approx R/U$ (where R denotes the real constant sheaf), and, if ω/U is not exact, the only section of $\mathscr{F}_\omega(M)$ over U is 0. Indeed, in the first case, (3.3) holds over U, and every section of $\mathscr{F}_\omega(M)/U$ is of the form c/b with $c\epsilon R$. In the second case, if f is a section over U, then either f vanishes at least at one point $u\epsilon U$ or ω/U is exact. But ω is not exact on U, and, on another hand, $f(u) = 0$ implies $f \equiv 0$ on U, as shown by the following reasoning: as above, $f = c/b$ in a neighbourhood of u, whence $c = 0$, i.e. $f = 0$; next, this fact can be propagated to any point of U along a chain of consecutively intersecting similar neighbourhoods.

In the case of an l.c.K. manifold M, the spaces $H^p_\omega(M)$ can be defined for the Lee form ω of M. These will be called the *adapted cohomology spaces of the l.c.K. manifold.* The sheaf $\mathcal{F}_\omega(M)$, which defines the adapted cohomology has now the following interpretation: $f>0$ is a germ in $\mathcal{F}_\omega(M)$ iff $f^{-1}g$ is the germ of a Kähler metric on M.

At this point, we should like to make also some comments about the cohomological significance of the operator $\tilde{\mathrm{d}}$ used in Section 2.

Let us consider again a pair (M,ω), where M is a differentiable manifold and ω is a closed 1-form on M. Then, we can define the operator $\tilde{\mathrm{d}}$ by (2.2). $\tilde{\mathrm{d}}$ defines a *twisted cohomology* of the differential forms of M [3], which is given by

$$H^*_\sim(M)=\frac{\operatorname{Ker}\tilde{\mathrm{d}}}{\operatorname{Im}\tilde{\mathrm{d}}\cap\operatorname{Ker}\tilde{\mathrm{d}}} \tag{3.6}$$

and is isomorphic [3] to the cohomology of the cochain complex $\tilde{\Omega}(M)$ consisting of the differential forms λ on M satisfying

$$\tilde{\mathrm{d}}^2\lambda=-\omega\wedge\mathrm{d}\lambda=0, \tag{3.7}$$

together with the operator $\tilde{\mathrm{d}}$.

The complex $\tilde{\Omega}(M)$ has an interesting subcomplex $\Omega_\omega(M)$, namely, the ideal generated by ω. On this subcomplex, $\tilde{\mathrm{d}}=\mathrm{d}$, which means that it is also a subcomplex of the usual de Rham complex of M. Hence, one has homomorphisms

$$\begin{aligned} a&:H^p(\Omega_\omega(M))\to H^p_\sim(M),\\ b&:H^p(\Omega_\omega(M))\to H^p(M,R). \end{aligned} \tag{3.8}$$

We can easily construct a homomorphism

$$c:\ H^p_\sim(M)\to H^{p+1}(M,R). \tag{3.9}$$

Namely, if $u\in H^p_\sim(M)$ and $u=[\lambda]$, where λ is a $\tilde{\mathrm{d}}$-closed form, then we put $c(u)=[\omega\wedge\lambda]$, and this produces the homomorphism (3.9). (Brackets denote cohomology classes.)

The existence of c gives some relation between $\tilde{\mathrm{d}}$ and the real cohomology of M.

If $\omega\neq 0$ at every point of M, $\omega=0$ defines a foliation $\mathcal{F}$ of codimension 1, and a usual sheaf-theoretic argument proves that

$$H^p(\Omega_\omega(M))\approx H^p(M,\Phi), \tag{3.10}$$

where Φ is the sheaf of germs of the differentiable functions on M, which are constant on the leaves of $\mathcal{F}$. In this case, the homomorphisms a and b become, as well, significant for the topology of M.

4. Some applications of the commutation formulas

In this section, we come back to the l.c.K. manifolds M^{2n}, endowed with the Lee form ω. The Lee form allows us to use the results of Section 3. In addition, we also have the commutation formulas of Section 2, and we should like to produce some applications.

PROPOSITION 4.1: *Let M be a compact l.c.K. manifold with $b^2_\omega(M) = 0$. Then: (a) there are 1-forms φ on M such that the fundamental form Ω can be written as*

$$\Omega = \varphi \wedge \omega + \mathrm{d}\varphi; \tag{4.1}$$

(b) the form φ satisfies the condition

$$\omega \wedge \varphi \wedge (\mathrm{d}\varphi)^{n-1} \not\equiv 0; \tag{4.2}$$

(c) the global scalar product $(\varphi, \omega \circ J)$ is nonzero.

PROOF: (a) We use the d_ω-cohomology on M, with its theory of harmonic forms, which is available in this case. Obviously, Ω is d_ω-closed, therefore we must have

$$\Omega = H_\omega \Omega + \mathrm{d}_\omega \varphi, \tag{4.3}$$

where φ is some 1-form on M and $H_\omega \Omega$ is d_ω-harmonic. But, from Proposition 3.1 and because of the hypothesis $b^2_\omega(M) = 0$, we have $H_\omega \Omega = 0$. This proves (4.1). The form φ is determined up to a d_ω-exact 1-form.

(b) From (4.1) we get easily

$$\Omega^n = (\mathrm{d}\varphi)^n + n(\mathrm{d}\varphi)^{n-1} \wedge \varphi \wedge \omega = \mathrm{d}(\varphi \wedge (\mathrm{d}\varphi)^{n-1}) + n(\mathrm{d}\varphi)^{n-1} \wedge \varphi \wedge \omega,$$

where, of course, exponents denote exterior powers. But, since Ω^n is a volume element and M is compact, Ω^n cannot be exact, and we must have (4.2).

(c) By applying Λ to (4.1) and using the commutation formulas for

$[\Lambda,\mathrm{d}]$ and $[\Lambda,\mathrm{e}]$, (2.11) and (2.4), we obtain $\delta C\varphi + n\epsilon C\varphi + n = 0$, whence, by integrating over M;

$$V = -(\varphi, \omega \circ J), \tag{4.4}$$

where V is the volume of the manifold M. This yields the conclusion stated.

PROPOSITION 4.2: *On every l.c.K. manifold, one has*

$$\delta(\Omega^h) = h(n-h)(C\omega) \wedge \Omega^{h-1}, \tag{4.5}$$

where $h = 0, 1, 2, \cdots$, *and the exponents of* Ω *denote exterior powers.*

PROOF: First, using the operator $\tilde{\delta}$ of Section 2, we prove inductively that

$$\tilde{\delta}(\Omega^h) = 0 \quad (h \geq 0). \tag{4.6}$$

Indeed, this is trivial for $h = 0$. Next, using (2.11), we have

$$\tilde{\delta}(\Omega^h) = \tilde{\delta}L(\Omega^{h-1}) = L\tilde{\delta}(\Omega^{h-1}) - \tilde{\mathrm{d}}^C(\Omega^{h-1}),$$

and, since Ω is of the type (1, 1) and $\tilde{\mathrm{d}}$-closed, this shows that (4.6) for $h-1$ implies (4.6) for h. Q.E.D.

Now, by (2.2), (4.6) yields

$$\delta(\Omega^h) + (n-h)\epsilon(\Omega^h) = 0. \tag{4.7}$$

An easy induction procedure, based on the commutation formula (2.4) for $[\epsilon, L]$ gives

$$\epsilon\Omega^h = hC^{-1}\mathrm{e}\Omega^{h-1}, \tag{4.8}$$

and, if we combine (4.7) and (4.8) we obtain exactly the required formula (4.5).

REMARKS: (1) By taking $h = n$ in (4.5), we get $\delta(\Omega^n) = 0$, i.e., if M is compact, Ω^n is harmonic.

(2) Using (1.1), (4.5) can be put into the following nice form

$$(h-1)\delta(\Omega^h) - h(n-h)C\mathrm{d}(\Omega^{h-1}) = 0. \tag{4.9}$$

(3) Since $\delta^2 = 0$, (4.7) implies $\delta\epsilon(\Omega^h) = 0$ and, for $h = 1, \delta\theta = 0$, where θ is given by (2.6).

The geometric content of Proposition 4.2 is pointed out by

COROLLARY 4.3: *If M is l.c.K. and if for some $h \neq 0, n$ we have $\delta(\Omega^h) = 0$, then M is a Kähler manifold. Particularly, any semi-Kähler l.c.K. manifold is Kähler. (Moreover, if $h \neq 1$, the hypothesis can be replaced by* $\mathrm{d}(\Omega^{h-1}) = 0$.)

PROOF: The alternate use of the two stated hypotheses follows by (4.9). For $h = 1, \delta\Omega = 0$ implies by (4.5) $\omega \circ J = 0$, i.e. $\omega = 0$. This is the semi-Kähler case, and, in this case, the result fòllows also from [2].

For $h > 1, \delta(\Omega^h) = 0$ implies by (4.5) $\omega \wedge \Omega^{h-1} = 0$. But, if for some point $x \epsilon M$, $\omega_x \neq 0$, we have

$$\Omega_x = \omega_x \wedge \lambda_x + \Theta_x,$$

where λ_x and Θ_x are forms which, if expressed by means of a basis whose first element is ω_x, do not contain ω_x, and rank $\Theta_x = 2n - 2$. Then, $\omega_x \wedge \Omega_x^{h-1} \neq 0$ since $h \neq n$, which contradicts $\omega \wedge \Omega^{h-1} = 0$. This contradiction implies $\omega = 0$, Q.E.D.

If $\delta(\Omega^h) = 0, h \geq 1$, we shall say that M is a *generalized semi-Kähler manifold.*

REMARK: As a matter of fact, the hypotheses of Corollary 4.3 can be weakened. Namely, since ω, Ω are real forms, it suffices to ask that one of the following relations holds

$$\text{(4.10)} \qquad \mathrm{d}'(\Omega^{h-1}) = 0, \mathrm{d}''(\Omega^{h-1}) = 0, \delta'(\Omega^h) = 0, \delta''(\Omega^h) = 0.$$

Finally, we should like to note an easy consequence of formula (2.15):

PROPOSITION 4.4: *The Lee form ω of a compact l.c.K. manifold M is harmonic iff its corresponding vector field B is a volume preserving infinitesimal transformation on M.*

PROOF: From (2.15) we get

$$\delta\omega = \delta \mathrm{e}(1) = - *^{-1} L_B *(1),$$

and, since $*1$ is the volume element of M, the Proposition is proven.

REFERENCES

[1] S.I. GOLDBERG: *Curvature and Homology*. Academic Press, New York, 1962.

[2] A. GRAY and L.M. HERVELLA: *The sixteen classes of almost Hermitian manifolds and their linear invariants* (preprint).

[3] S. HALPERIN and D. LEHMANN: Cohomologies et classes caractéristiques des choux de Bruxelles. *Diff. Topology and Geometry, Proc. Colloq. Dijon 1974.* Lecture Notes in Math 484, Springer-Verlag, Berlin, 1975, 79–120.

[4] P. LIBERMANN: Sur les structures presque complexes et autres structures infinitésimales régulières. *Bull Soc. Math. France*, 83 (1955), 195–224.

[5] A. LICHNEROWICZ: *Théorie globale des connexions et des groupes d'holonomie*, Edizione Cremonese, Roma, 1955.

[6] I. VAISMAN: *Cohomology and Differential Forms*, M. Dekker, Inc., New York, 1973.

[7] I. VAISMAN: On locally conformal almost Kähler manifolds, *Israel J. of Math.* 24 (1976), 338–351.

[8] I. VAISMAN: Locally conformal Kähler manifolds with parallel Lee form, *Rendiconti di Mat ematica* Roma (to appear).

[9] A. WEIL: *Introduction à l'étude des variétés Kählériennes*, Hermann, Paris, 1958.

(Oblatum 8–II–1977 & 28–VI–1979)

Department of Mathematics
University of Haifa, Israel.