COMPOSITIO MATHEMATICA

JAMES ARTHUR

A trace formula for reductive groups. II : applications of a truncation operator

Compositio Mathematica, tome 40, nº 1 (1980), p. 87-121

<http://www.numdam.org/item?id=CM_1980__40_1_87_0>

COMPOSITIO MATHEMATICA, Vol. 40, Fasc. 1, 1980, pag. 87–121

Printed in the Netherlands

A TRACE FORMULA FOR REDUCTIVE GROUPS II: APPLICATIONS OF A TRUNCATION OPERATOR

James Arthur*

Introduction

This paper, as promised in the introduction to [1(c)], contains an identity which is valid for any reductive group G over $\mathbb{Q}$, and which generalizes the Selberg trace formula for anisotropic G. We have already shown that a certain sum of distributions on $G(\mathbb{A})^1$, indexed by equivalence classes in $G(\mathbb{Q})$, equals the integral of the function

$$\sum_{\chi\in\mathfrak{X}} k_\chi^T(x,f), \qquad x\in G(\mathbb{Q})\backslash G(\mathbb{A})^1.$$

The main task of this paper is to show that the integral may be taken inside the sum over χ. There does not seem to be any easy way to do this. We are forced to proceed indirectly by first defining and studying a truncation operator Λ^T on functions on $G(\mathbb{Q})\backslash G(\mathbb{A})^1$.

Recall that $k_\chi^T(x,f)$ was obtained by modifying the function $K_\chi(x,x)$. We shall apply the results of §1 to the function

$$\Lambda_1^T\Lambda_2^T K_\chi(x,x), \qquad x\in G(\mathbb{Q})\backslash G(\mathbb{A})^1,$$

*Partially supported by NSF Grant MCS77-0918.

0010-437X/80/01/0087-36 $00.20/0

obtained from $K_\chi(x, y)$ by truncating in each variable separately, and setting $x = y$. It will turn out that the function

$$\sum_{\chi \in \mathscr{X}} |\Lambda_1^T \Lambda_2^T K_\chi(x, x)|$$

is integrable. Then in §2, our main chapter, we shall show that for T sufficiently regular,

$$\sum_{\chi \in \mathscr{X}} \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} (\Lambda_1^T \Lambda_2^T K_\chi(x, x) - k_\chi^T(x, f))\, dx$$

converges absolutely. We shall also show that for each χ, the integral over $G(\mathbb{Q})\backslash G(\mathbb{A})^1$ equals 0. If we set $J_\chi^T(f)$ equal to

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \Lambda_1^T \Lambda_2^T K_\chi(x, x)\, dx = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} k_\chi^T(x, f)\, dx,$$

the identity associated to G is then

$$\sum_{\mathfrak{o} \in \mathcal{O}} J_{\mathfrak{o}}^T(f) = \sum_{\chi \in \mathscr{X}} J_\chi^T(f).$$

We should note that the distributions $J_{\mathfrak{o}}^T$ and J_χ^T are not in general invariant. Moreover, they depend on a choice of maximal compact subgroup and minimal parabolic subgroup. However, it should be possible to modify each of the distributions so that they are invariant and independent of these choices, and so that the identity still holds. We hope to do this in a future paper.

Both formulas for $J_\chi^T(f)$ are likely to be useful. The integral on the right is particularly suited to evaluating J_χ^T on the function obtained by subtracting f from a conjugate of itself by a given element in $G(\mathbb{A})^1$. It can also be used to show that $J_\chi^T(f)$ is a polynomial function in T. We shall not discuss these questions here. On the other hand, the integral on the left can be calculated explicitly if the class χ is unramified. We do this in §4. The result follows from a formula, announced by Langlands in [4(a)], for the inner product of two truncated Eisenstein series. It was by examining Langlands' method for truncating Eisenstein series that I was led to the definition of the operator Λ^T.

1. A truncation operator

Let G be a reductive algebraic group defined over $\mathbb{Q}$. We adopt the definitions and notation of [1(c)]. In particular, K is a maximal compact subgroup of $G(\mathbb{A})$ and P_0 is a fixed minimal parabolic subgroup of G defined over $\mathbb{Q}$. Again we shall use the term 'parabolic subgroup' for a parabolic subgroup P of G, defined over $\mathbb{Q}$, which contains P_0. We would like to prove that the terms on the right hand side of the identity given in Proposition 5.3 of [1(c)] are integrable functions of x. To this end, we shall introduce a truncation operator for functions on $G(\mathbb{Q})\backslash G(\mathbb{A})^1$.

Recall that T is a fixed, suitably regular point in $\mathfrak{a}_0^+$. If ϕ is a continuous function on $G(\mathbb{Q})\backslash G(\mathbb{A})^1$, define $(\Lambda^T\phi)(x)$ to be the function

$$\sum_P (-1)^{\dim(A/Z)} \sum_{\delta \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \phi(n\delta x) \cdot \hat{\tau}_P(H(\delta x) - T).$$

(the sum over P is of course over all parabolic subgroups.) Note the similarity with our definitions of the functions $k_{\mathfrak{o}}^T(x,f)$ and $k_\chi^T(x,f)$ in [1(c)]. If ϕ is a cusp form, $\Lambda^T\phi = \phi$. It is a consequence of [1(c), Corollary 5.2] that if $\phi(x)$ is slowly increasing, in the sense that

$$|\phi(x)| \le C\|x\|^N,$$

for some C and N, then so is $\Lambda^T\phi(x)$.

LEMMA 1.1: *Fix P_1. Then for $\phi \in C(G(\mathbb{Q})\backslash G(\mathbb{A})^1)$,*

$$\int_{N_1(\mathbb{Q})\backslash N_1(\mathbb{A})} \Lambda^T\phi(n_1x)\, \mathrm{d}n_1 = 0$$

unless $\varpi(H_0(x) - T) < 0$ for each $\varpi \in \hat{\Delta}_1$.

PROOF: For any P, let $\Omega(\mathfrak{a}_0; P)$ be the set of $s \in \Omega$ such that $s^{-1}\alpha > 0$ for each $\alpha \in \Delta_0^P$. Applying the Bruhat decomposition to $P(\mathbb{Q})\backslash G(\mathbb{Q})$, we find that $\int_{N_1(\mathbb{Q})\backslash N_1(\mathbb{A})} \Lambda^T\phi(n_1x)\,\mathrm{d}n_1$ equals the sum over P and $s \in \Omega(\alpha_0; P)$ of the integral over n in $N(\mathbb{Q})\backslash N(\mathbb{A})$ of the product of $(-1)^{\dim(A/Z)}$ with

$$\int_{N_1(\mathbb{Q})\backslash N_1(\mathbb{A})} \sum_{\nu \in w_s^{-1}N_0(\mathbb{Q})w_s \cap N_0(\mathbb{Q})\backslash N_0(\mathbb{Q})} \phi(nw_s\nu n_1x) \cdot \hat{\tau}_P(H(w_s\nu n_1x) - T)\, \mathrm{d}n_1.$$

Since $N_1(\mathbb{Q})\backslash N_1(A) = N_0(\mathbb{Q})\backslash N_0^1(\mathbb{Q})N_1(\mathbb{A})$, this last expression equals

$$\int_{w_s^{-1}N_0(\mathbb{Q})w_s \cap N_0(\mathbb{Q})\backslash N_0^1(\mathbb{Q})N_1(\mathbb{A})} \phi(nw_sn_1x)\hat{\tau}_P(H(w_sn_1x)-T)\,\mathrm{d}n_1.$$

Decompose $w_s^{-1}N_0(\mathbb{Q})w_s \cap N_0(\mathbb{Q})\backslash N_0^1(\mathbb{Q})N_1(\mathbb{A})$ as

$$\begin{aligned}
&(w_s^{-1}N_0(\mathbb{Q})w_s \cap N_0(\mathbb{Q})\backslash w_s^{-1}N_0(\mathbb{A})w_s \cap N_0^1(\mathbb{Q})N_1(\mathbb{A}))\\
&\quad\times (w_s^{-1}N_0(\mathbb{A})w_s \cap N_0^1(\mathbb{Q})N_1(\mathbb{A})\backslash N_0^1(\mathbb{Q})N_1(\mathbb{A}))\\
&= (w_s^{-1}N_0(\mathbb{Q})w_s \cap N_1(\mathbb{Q})\backslash w_s^{-1}N_0(\mathbb{A})w_s \cap N_1(\mathbb{A}))\\
&\quad\times (w_s^{-1}N_0(\mathbb{A})w_s \cap N_0^1(\mathbb{Q})N_1(\mathbb{A})\backslash N_0^1(\mathbb{Q})N_1(\mathbb{A})).
\end{aligned}$$

This induces a decomposition of the measure $\mathrm{d}n_1$ as $\mathrm{d}n_*\,\mathrm{d}n^*$. Then write

$$w_sn_*n^* = w_sn_*w_s^{-1}w_sn^* = \tilde{n}_*w_sn^*,$$

and finally, combine the integral over $\tilde{n}_*$ with the integral over n in $N(\mathbb{Q})\backslash N(\mathbb{A})$. Because s lies in $\Omega(\mathfrak{a}_0; P)$, $N_0 \cap w_sN_1w_s^{-1} \cap M$ is the unipotent radical of a standard parabolic subgroup of M. It follows that

$$(N_0 \cap w_sN_1w_s^{-1} \cap M)N = N_s$$

is the unipotent radical of a uniquely determined parabolic subgroup P_s of G, which is contained in P. We have shown that $\int_{N_1(\mathbb{Q})\backslash N_1(\mathbb{A})} \phi(n_1x)\,\mathrm{d}n_1$ equals

$$\sum_P (-1)^{\dim(A/Z)} \sum_{s\in\Omega(\mathfrak{a}_0;\, P)} \int_{w_s^{-1}N_0(\mathbb{A})w_s\cap N_0^1(\mathbb{Q})N_1(\mathbb{A})\backslash N_0^1(\mathbb{Q})N_1(\mathbb{A})} \mathrm{d}n^* \cdot \int_{N_s(\mathbb{Q})\backslash N_s(\mathbb{A})} \phi(nw_sn^*x)\hat{\tau}_P(H(w_sn^*x)-T)\mathrm{d}n.$$

We shall change the order of summation, and consider the set of P which give rise to a fixed P_s. Fix $s \in \Omega$. Define S^1 (resp. S_1) to be the set of $\alpha \in \Delta_0$ such that $s^{-1}\alpha$ is a positive root which is orthogonal (resp. not orthogonal) to $\mathfrak{a}_1$. If P_s is one of the groups that appear in the above formula, Δ_0^s will be a subset of S^1. Those P which give rise to a fixed P_s are exactly the groups for which Δ_0^P is the union of Δ_0^s and a subset S of S_1. Thus, for fixed s with $\Delta_0^s \subset S^1$, we will obtain an alternating sum over $S \subset S_1$ of the corresponding functions $\hat{\tau}_P$. We apply Proposition 1.1 of [1(c)]. Let χ_s be the characteristic function of the set of $H \in \mathfrak{a}_0$ such that for $\alpha \in \Delta_0 - \Delta_0^s \cup S_1$, $\varpi_\alpha(H) > 0$, while $\varpi_\alpha(H) \leq 0$ for α in S_1. Here ϖ_α is the element in $\hat{\Delta}_0$ corresponding to α. Then $\int_{N_1(\mathbb{Q})\backslash N_1(\mathbb{A})} \phi(n_1x)\, dn_1$ is sum over $s \in \Omega$ and over all subsets Δ_0^s of S^1, of the integral over n^* in $w_s^{-1}N_0(\mathbb{A})w_s \cap N_0^1(\mathbb{Q})N_1(\mathbb{A})\backslash N_0^1(\mathbb{Q})N_1(\mathbb{A})$ and n in $N_s(\mathbb{Q})\backslash N_s(\mathbb{A})$ of the product of

$$\phi(nw_sn^*x)\chi_s(H_0(w_sn^*x) - T)$$

with -1 raised to a power equal to the number of roots in $\Delta_0 - S^1 \cup S_1$.

Suppose that for some s, $\chi_s(H_0(w_sn^*\nu) - T)$ does not vanish. Then if

$$H_0(w_sn^*x) - T) = \sum_{\alpha \in \Delta_0} t_\alpha \alpha^\vee, \quad t_\alpha \in \mathbf{R},$$

t_α is positive for α in $\Delta_0 - \Delta_0^s \cup S_1$, and is not positive for $\alpha \in S_1$. If $\varpi \in \hat{\Delta}_1$,

$$\begin{aligned} &\varpi(s^{-1}(H_0(w_sn^*x) - T)) \\ &= \sum_{\alpha \in \Delta_0} t_\alpha \varpi(s^{-1}\alpha^\vee) \\ &= \sum_{\alpha \in \Delta_0 \backslash S^1} t_\alpha \varpi(s^{-1}\alpha^\vee), \end{aligned}$$

where $s^{-1}\alpha$ is orthogonal to $\mathfrak{a}^1$ if $\alpha \in S^1$. This last number is clearly less than or equal to 0. Now

$$s^{-1}(H_0(w_sn^*x) - T) = H_0(x) - T + s^{-1}H_0(w_svw_s^{-1}) + (T - s^{-1}T),$$

for some element $v \in N_0(\mathbb{A})$. If $\varpi \in \hat{\Delta}_0$, it is well known that $\varpi(s^{-1}H_0(w_svw_s^{-1})$ is nonnegative and $\varpi(T - s^{-1}T)$ is strictly positive. Therefore $\varpi(H_0(x) - T)$ is negative for any $\varpi \in \hat{\Delta}_1$. □

From the definition of Λ^T we obtain

COROLLARY 1.2: $\Lambda^T\Lambda^T = \Lambda^T$. □

LEMMA 1.3: *Suppose that ϕ_1 and ϕ_2 are continuous functions on $G(\mathbb{Q})\backslash G(\mathbb{A})^1$. Assume that ϕ_1 is slowly increasing, and that ϕ_2 is rapidly decreasing, in the sense that for any N, the function $\|x\|^N \cdot |\phi_2(x)|$ is bounded on any Siegel set. Then*

$$(\Lambda^T\phi_1, \phi_2) = (\phi_1, \Lambda^T\phi_2).$$

PROOF: The inner product $(\Lambda^T\phi_1, \phi_2)$ is defined by an absolutely convergent integral. It equals

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \sum_P (-1)^{\dim(A/Z)} \sum_{\delta\in P(\mathbb{Q})\backslash G(\mathbb{Q})}$$

$$\times \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \phi_1(n\delta x)\hat{\tau}_P(H(\delta x)-T)\overline{\phi_2(x)}\, dn\, dx$$

$$= \sum_P (-1)^{\dim(A/Z)} \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \int_{P(\mathbb{Q})\backslash G(\mathbb{A})^1} \phi_1(nx)\overline{\phi_2(x)}\hat{\tau}_P(H(x)-T)\, dx\, dn$$

$$= \sum_P (-1)^{\dim(A/Z)} \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \int_{P(\mathbb{Q})\backslash G(\mathbb{A})^1} \phi_1(x)\overline{\phi_2(nx)}\hat{\tau}_P(H(x)-T)\, dx\, dn.$$

This last expression reduces to $(\phi_1, \Lambda^T\phi_2)$. □

REMARK: It can be shown that Λ^T extends to an orthogonal projection on $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})^1)$.

We would like to show that under suitable conditions, $\Lambda^T\phi(x)$ is rapidly decreasing at infinity. The argument begins the same way as the proofs of Theorems 7.1 and 8.1 of [1(c)]. Suppose ϕ is a continuous function on $G(\mathbb{Q})\backslash G(\mathbb{A})^1$. Apply Lemma 6.4 as in the beginning of the proof of Theorem 7.1 of [1(c)]. We find that $\Lambda^T\phi(x)$ is the sum over $\{P_1, P_2\colon P_0 \subset P_1 \subset P_2\}$ and $\delta \in P_1(\mathbb{Q})\backslash G(\mathbb{Q})$, of

$$F^1(\delta x, T)\sigma_1^2(H_0(\delta x)-T)\phi_{P_1,P_2}(\delta x),$$

where

$$\phi_{P_1P_2}(y) = \sum_{\{P:\, P_1\subset P\subset P_2\}} (-1)^{\dim(A/Z)} \int_{N(\mathbb{Q})\backslash N(1)} \phi(ny)\, dn.$$

For the moment, fix δ and x. We regard δ as an element in $G(\mathbb{Q})$ which we are free to left multiply by an element in $P_1(\mathbb{Q})$. We can therefore assume, as in [1(c), §7] that

$$\delta x = n^* n_* m a k,$$

where $k \in K$, n^*, n_*, and m belong to fixed compact subsets of $N_2(\mathbb{A})$, $N^2(\mathbb{A})$ and $M_1(\mathbb{A})^1$ respectively, and a is an element in $A_1(\mathbf{R})^0$

with $\sigma_1^2(H_0(a)-T)\neq 0$. Therefore

$$\begin{aligned}\phi_{P_1,P_2}(\delta x) &= \phi_{P_1,P_2}(n^* n_* mak)\\ &= \phi_{P_1,P_2}(n_* mak)\\ &= \phi_{P_1,P_2}(aa^{-1}n_* mak)\\ &= \phi_{P_1,P_2}(ac),\end{aligned}$$

where c belongs to a fixed compact subset of $G(\mathbb{A})^1$ which depends only on G.

The function ϕ_{P_1,P_2} resembles the function estimated in the corollary of [3, Lemma 10]. We want a slightly different statement of the estimate, however, so we had best re-examine the proof. If $\alpha \in \Delta_1^2$, let P_α, $P_1 \subset P_\alpha \subset P_2$, be the parabolic subgroup such that $\Delta_1^\alpha = \Delta_{P_1}^{P_\alpha}$ is the complement of α in Δ_1^2. For each α, let $\{Y_{\alpha,1},\ldots, Y_{\alpha,n_\alpha}\}$ be a basis of $\mathfrak{n}_\alpha^2(\mathbb{Q})$, the Lie algebra of $N_\alpha^2(\mathbb{Q})$. We shall assume that the basis is compatible with the action of A_1, so that each $Y_{\alpha,i}$ is a root vector corresponding to the root $\beta_{\alpha,i}$ of $(M_2\cap P_1, A_1)$. We shall also assume that if $i\leq j$, the height of $\beta_{\alpha,i}$ is not less than the height of $\beta_{\alpha,j}$. Define $\mathfrak{n}_{\alpha,j}$, $0\leq j\leq n_\alpha$, to be the direct sum of $\{Y_{\alpha,1},\ldots, Y_{\alpha,j}\}$ with the Lie algebra of N_2, and let $N_{\alpha,j}=\exp \mathfrak{n}_{\alpha,j}$. Then $N_{\alpha,j}$ is a normal subgroup of N_1 which is defined over $\mathbb{Q}$. If V is any subgroup of N_1, defined over $\mathbb{Q}$, let $\pi(V)$ be the operator which sends ϕ to

$$\int_{V(\mathbb{Q})\backslash V(\mathbb{A})} \phi(ny)\,\mathrm{d}n, \quad y\in G(\mathbb{A}).$$

Then ϕ_{P_1,P_2} is the transform of ϕ by the product over $\alpha\in\Delta_1^2$ of the operators

$$\pi(N_2)-\pi(N_\alpha)=\sum_{i=1}^{n_\alpha}\pi(N_{\alpha,i-1})-\pi(N_{\alpha,i}).$$

If K_0 is an open compact subgroup of $G(\mathbb{A}_f)$, $G(\mathbb{Q})\backslash G(\mathbb{A})^1/K_0$ is differentiable manifold. We assume from now on that ϕ is a function on this space which is differentiable of sufficiently high order. Suppose that I is a collection of indices

$$\{i_\alpha\colon \alpha\in\Delta_1^2,\ 1\leq i_\alpha\leq n_\alpha\}.$$

Then

$$N_I^- = \prod_\alpha N_{\alpha,i_\alpha-1}$$

and

$$N_I = \prod_\alpha N_{\alpha,i_\alpha}$$

are normal subgroups of N_1. Let $\mathfrak{n}^I$ be the span of $\{Y_{\alpha,i_\alpha}\}$ and let $\mathfrak{n}^I(\mathbb{Q})'$ be the set of elements

$$\xi = \sum_\alpha r_\alpha Y_{\alpha,i_\alpha}, \quad r_\alpha \in \mathbb{Q}^*.$$

Then if n is any positive integer,

$$\xi^n = \prod_\alpha (r_\alpha^n)$$

is a nonzero real number. By the Fourier inversion formula for the group $\mathbb{A}/\mathbb{Q}$, $\phi_{P_1,P_2}(y)$ is the sum over all I of

$$\sum_{\xi\in\mathfrak{n}^I(\mathbb{Q})'} \int_{\mathfrak{n}^I(\mathbb{Q})\backslash\mathfrak{n}^I(\mathbb{A})} \mathrm{d}x \cdot \int_{N_{\bar{I}}(\mathbb{Q})\backslash N_{\bar{I}}(\mathbb{A})} \mathrm{d}u \cdot \phi(ue(X)y)\psi(\langle X, \xi\rangle).$$

Here e and ψ are as in [1(c), §7] and $\langle\, ,\rangle$ is the inner product defined by our basis on $\mathfrak{n}^I$. If n is a positive integer,

$$Y_I^n = \prod_\alpha (-\sqrt{-1}\, Y_{\alpha,i_\alpha})^n$$

can be regarded as an element in $\mathcal{U}(\mathfrak{g}(\mathbf{R})^1 \otimes \mathbf{C})$. Then $\phi_{P_1,P_2}(y)$ equals the sum over I and over $\xi \in \mathfrak{n}^I(\mathbb{Q})'$ of

$$(1.1)\qquad (\xi^n)^{-1} \int_{\mathfrak{n}^I(\mathbb{Q})\backslash\mathfrak{n}^I(\mathbb{A})} \mathrm{d}X \cdot \int_{N_{\bar{I}}(\mathbb{Q})\backslash N_{\bar{I}}(\mathbb{A})} \mathrm{d}u \cdot R_y(\mathrm{Ad}(y^{-1})Y_I^n)\phi(ue(X)y)\psi(\langle X, \xi\rangle).$$

Now, we set

$$y = \delta x = ac,$$

as above. Since $\sigma_1^2(H_0(a) - T) \neq 0$, a belongs to a fixed Siegel set in $M_2(\mathbb{A})$. It follows that the integrand in (1.1), as a function of X, is invariant by an open compact subgroup of $\mathfrak{n}^I(\mathbb{A}_f)$ which is independent of a and c. Consequently, (1.1) vanishes unless ξ belongs to a fixed lattice, $L^I(K_0)$, in $\mathfrak{n}^I(\mathbf{R})$. But for n sufficiently large

$$\sum_{\xi \in \mathfrak{n}^I(\mathbf{Q})' \cap L^I(K_0)} |\xi^n|^{-1}$$

is finite for all I. Let $c_n(K_0)$ be the supremum over all I of these numbers. Then $|\phi_{P_1,P_2}(ac)|$ is bounded by

$$c_n(K_0) \sum_I \int_{N_I(\mathbf{Q})\backslash N_I(\mathbb{A})} |(R(\mathrm{Ad}(c)^{-1}\,\mathrm{Ad}(a)^{-1} Y_I^n)\phi)(uac)|\,\mathrm{d}u.$$

Let $\beta_I = \Sigma_\alpha \beta_{\alpha,i_\alpha}$. Then β_I is a positive sum of roots in Δ_1^2. For any n,

$$\mathrm{Ad}(a^{-1})Y_I^n = \mathrm{e}^{-n\beta_I(H_0(a))}Y_I^n = \mathrm{e}^{-n\beta_I(H_0(\delta x))}Y_I^n.$$

We can choose a finite set of elements $\{X_i\}$ in $\mathcal{U}(\mathfrak{g}(\mathbf{R})^1 \otimes \mathbf{C})$, depending only on n and K_0, such that for any P_1, P_2, I and c,

$$c_n(K_0)\,\mathrm{Ad}(c)^{-1}Y_I^n$$

is a linear combination of $\{X_i\}$. Since c lies in a compact set, we may assume that each of the coefficients has absolute value less than 1. We have thus far shown that $|\Lambda^T\phi(x)|$ is bounded by the sum over all P_1, P_2 and $\delta \in P_1(\mathbf{Q})\backslash G(\mathbf{Q})$ of the product of

$$F^1(\delta x, T)\sigma_1^2(H_0(\delta x) - T)$$

with

$$\sum_I \sum_i \int_{N_I(\mathbf{Q})\backslash N_I(\mathbb{A})} |R(X_i)\phi(u\delta x)|\,\mathrm{d}u \cdot \mathrm{e}^{-n\beta_I(H_0(\delta x))}. \tag{1.2}$$

LEMMA 1.4: *Let $\mathfrak{S}$ be a Siegel set in $G(\mathbb{A})^1$. For any pair of positive integers N' and N, and any open compact subgroup K_0 of $G(\mathbb{A}_f)$, we can choose a finite subset $\{X_i\}$ of $\mathcal{U}(\mathfrak{g}(\mathbf{R})^1 \otimes \mathbf{C})$ and a positive integer r which satisfy the following property: Suppose that $(S, \mathrm{d}\sigma)$ is a measure space and that $\phi(\sigma, x)$ is a measurable function from S to $C^r(G(\mathbf{Q})\backslash G(\mathbb{A})^1/K_0)$. Then for any $x \in \mathfrak{S}$,*

$$\int_S |\Lambda^T\phi(\sigma, x)|\,\mathrm{d}\sigma$$

is bounded by

$$\sum_i \sup_{y\in G(\mathbb{A})^1}\left(\int_S |R(X_i)\phi(\sigma, y)|\,\mathrm{d}\sigma\cdot\|y\|^{-N}\right)\cdot\|x\|^{-N'}.$$

PROOF: Substitute $\phi(\sigma)$ for ϕ in (1.2) and integrate over σ. The result is

$$\sum_I\sum_i\int_{N_I(\mathbb{Q})\backslash N_I(\mathbb{A})}\int_S |R(X_i)\phi(\sigma, u\delta x)|\,\mathrm{d}\sigma\,\mathrm{d}u\cdot \mathrm{e}^{-n\beta_I(H_0(\delta x))}. \tag{1.3}$$

If $\delta x = ac$, with a and c as above,

$$\|\delta x\|\le\|a\|\cdot\|c\|.$$

We are assuming that $\sigma_1^2(H_0(a)-T)\neq 0$. Since β_I is a positive sum of roots in Δ_1^2 we conclude from [1(c), Corollary 6.2] that $\|a\|$ is bounded by a fixed power of

$$\mathrm{e}^{\beta_I(H_0(a))}=\mathrm{e}^{\beta_I(H_0(\delta x))}.$$

It follows that for any positive integers N and N_1 we may choose n so that (1.3) is bounded by a constant multiple of

$$\sum_i \sup_{y\in G(\mathbb{A})^1}\left(\int_S |R(X_i)\phi(\sigma, y)|\,\mathrm{d}\sigma\cdot\|x\|^{-N'}\right)\|\delta x\|^{-N_1}.$$

It is well known (see [2]) that there is a constant c_1 such that for any $\gamma\in G(\mathbb{Q})$ and $x\in\mathcal{S}$,

$$\|\gamma x\|^{-N_1}\le c_1\|x\|^{-N_1}.$$

The only thing left to estimate is

$$\sum_{\delta\in P_1(\mathbb{Q})\backslash G(\mathbb{Q})} F^1(\delta x, T)\sigma_1^2(H_0(\delta x)-T).$$

The summand is the characteristic function, evaluated at δx, of a certain subset of

$$\{y\in G(\mathbb{A})^1 : \varpi(H_0(y)-T)>0, \varpi\in\hat{\Delta}_1\}.$$

The sum is bounded by

$$\sum_{\delta\in P_1(\mathbb{Q})\backslash G(\mathbb{Q})}\hat{\tau}_1(H_0(\delta x)-T).$$

It follows from [1(c), Lemma 5.1] that we can find constants C_2 and N_2 such that for all P_1 this last expression is bounded by $C_2\|x\|^{N_2}$. Set $N_1 = N' + N_2$. N_1 dictates our choice of n, from which we obtain the differential operators $\{X_i\}$. The theorem follows with any r greater than all the degrees of the operators X_i. □

In the next section we will need to have analogues of the operators Λ^T for different parabolic subgroups of G. If P_1 is a parabolic subgroup, and ϕ is a continuous function on $P_1(\mathbb{Q})\backslash G(\mathbb{A})^1$, define

$$\Lambda^{T,P_1}\phi(x) = \sum_{\{R: P_0 \subset R \subset P_1\}} (-1)^{\dim(A_R/A_1)} \sum_{\delta \in R(\mathbb{Q})\backslash P_1(\mathbb{Q})} \times \int_{N_R(\mathbb{Q})\backslash N_R(\mathbb{A})} \phi(n\delta x)\, dn \cdot \hat{\tau}_R^1(H_0(\delta x) - T).$$

LEMMA 1.5: *Suppose that P is a parabolic subgroup and ϕ is a continuous function on $P(\mathbb{Q})\backslash G(\mathbb{A})^1$.* Then

$$\sum_{\{P_1: P_0 \subset P_1 \subset P\}} \sum_{\delta \in P_1(\mathbb{Q})\backslash P(\mathbb{Q})} \Lambda^{T,P_1}\phi(\delta x)\tau_1^P(H_0(\delta x) - T)$$

equals

$$\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \phi(nx)\, dx. \tag{1.4}$$

PROOF: We need to prove that (1.4) is the sum over $\{R: P_0 \subset R \subset P\}$ and $\delta \in R(\mathbb{Q})\backslash P(\mathbb{Q})$ of the product of

$$\int_{N_R(\mathbb{Q})\backslash N_R(\mathbb{A})} \phi(n\delta x)\, dn$$

with

$$\sum_{\{P_1: R \subset P_1 \subset P\}} (-1)^{\dim(A_R/A_1)} \hat{\tau}_R^1(H_0(\delta x) - T) \cdot \tau_1^P(H_0(\delta x) - T). \tag{1.5}$$

Consider Lemma 6.3 of [1(c)], with Λ a point in $-(\mathfrak{a}_0^*)^+$. The sum given in that lemma then reduces to (1.5). It follows from [1(c), Prop. 1.1] that (1.5) vanishes if $R \neq P$ and equals 1 if $R = P$. This establishes Lemma 1.5. □

2. Integrability of $k_\chi^T(x, f)$

We take r to be a sufficiently large integer, and continue to let T be a suitably regular point in $\mathfrak{a}_0^+$. In [1(c)] we associated to every $f \in C_c^r(G(\mathbb{A})^1)$ a function, $k_\chi^T(x, f)$, on $G(\mathbb{Q})\backslash G(\mathbb{A})^1$.

THEOREM 2.1: *For sufficiently regular T,*

$$\sum_{\chi \in \mathscr{X}} \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} |k_\chi^T(x, f)| \, dx$$

is finite.

We will not prove the theorem directly. Rather, we shall relate $k_\chi^T(x, f)$ to the truncation operators whose asymptotic properties we have just studied. We shall operate on $K_{P,\chi}(x, y)$, which of course is a function of two variables. If $P_1 \subset P_2$, we shall write Λ_1^{T,P_1} (resp. Λ_2^{T,P_1}) for the operator Λ^{T,P_1}, acting on the first (resp. second) variable.

LEMMA 2.2: *For any $\chi \in \mathscr{X}$, $k_\chi^T(x, f)$ equals*

$$\sum_{\{P_1, P_2 :\, P_0 \subset P_1 \subset P_2\}} \sum_{\delta \in P_1(\mathbb{Q})\backslash G(\mathbb{Q})} \sigma_1^2(H_0(\delta x) - T)$$

$$\times \left\{ \sum_{\{P :\, P_1 \subset P \subset P_2\}} (-1)^{\dim(A/Z)} \Lambda_2^{T,P_1} K_{P,\chi}(\delta x, \delta x) \right\}.$$

PROOF: The given expression is the sum over all chains $P_1 \subset P \subset P_2 \subset P_3$ and over $\delta \in P_1(\mathbb{Q})\backslash G(\mathbb{Q})$, of

$$(-1)^{\dim(A_3/A_2)} \hat{\tau}_3(H_0(\delta x) - T) \cdot \tau_1^3(H_0(\delta x) - T)(-1)^{\dim(A/Z)} \Lambda_2^{T,P_1} K_{P,\chi}(\delta x, \delta x).$$

As we have done many times, we appeal to [1(c), Prop. 1.1]. We see that the sum over P_2 equals 0 unless $P = P_3$. Therefore the given expression equals

$$\sum_{\{P_1, P :\, P_0 \subset P_1 \subset P\}} (-1)^{\dim(A/Z)} \sum_{\delta \in P_1(\mathbb{Q})\backslash G(\mathbb{Q})} \hat{\tau}_P(H_0(\delta x) - T)$$

$$\cdot \, \tau_{P_1}^P(H_0(\delta x) - T) \cdot \Lambda_2^{T,P_1} K_{P,\chi}(\delta x, \delta x).$$

Apply Lemma 1.5 to the sum over P_1. We obtain

$$\sum_{\{P:\,P_0\subset P\}}(-1)^{\dim(A/Z)}\sum_{\delta\in P(\mathbb{Q})\backslash G(\mathbb{Q})}\hat{\tau}_P(H_0(\delta x)-T)\cdot\int_{N(\mathbb{Q})\backslash N(\mathbb{A})}K_{P,\chi}(\delta x, n\delta x)\,dn.$$

Since

$$K_{P,\chi}(\delta x, n\delta x)=K_{P,\chi}(\delta x,\delta x),\quad n\in N(\mathbb{A}),$$

this last expression equals $k_\chi^T(x,f)$, as required. □

Fix $P_1\subset P_2$. Motivated by the last lemma, we shall examine the expression

$$\sum_{\{P:\,P_1\subset P\subset P_2\}}(-1)^{\dim(A/A_2)}\int_{N_1(\mathbb{Q})\backslash N_1(\mathbb{A})}K_P(x,ny)\,dn.$$

It equals

$$\sum_{\{P:\,P_1\subset P\subset P_2\}}(-1)^{\dim(A/A_2)}\sum_{\gamma\in P_1(\mathbb{Q})\backslash P(\mathbb{Q})}\int_{N_1(\mathbb{A})}\sum_{\eta\in M_1(\mathbb{Q})}f(x^{-1}\gamma^{-1}n\eta y)\,dn$$
$$=\sum_P(-1)^{\dim(A/A_2)}\sum_{\gamma\in P_1(\mathbb{Q})\backslash P(\mathbb{Q})}K_{P_1}(\gamma x,y).$$

Let $F(P_1,P_2)$ be the set of elements in $P_1(\mathbb{Q})\backslash P_2(\mathbb{Q})$ which do not belong to $P_1(\mathbb{Q})\backslash P(\mathbb{Q})$ for any P, with $P_1\subset P\subsetneq P_2$. By [1(c), Prop. 1.1] the above expression equals

$$\sum_{\gamma\in F(P_1,P_2)}K_{P_1}(\gamma x,y).$$

In this last formula we have affected the cancellation implicit in the alternating sum over P. In order to exploit the equation we have just derived, we interrupt with a lemma.

LEMMA 2.3: *Suppose that for each i, $1\le i\le n$, we are given a parabolic subgroup $Q_i\supset P_1$, points x_i, $y_i\in G(\mathbb{A})$ and a number c_i such that*

$$\sum_{i=1}^n c_i\int_{N_1(\mathbb{Q})\backslash N_1(\mathbb{A})}K_{Q_i}(x_i,nmy_i)\,dn$$

vanishes for all $m \in M_1(\mathbb{Q})\backslash M_1(\mathbb{A})^1$. Then for any $\chi \in \mathscr{X}$,

$$h_\chi(m) = \sum_{i=1}^{n} c_i \int_{N_1(\mathbb{Q})\backslash N_1(\mathbb{A})} K_{Q_i,\chi}(x_i, nmy_i)\,\mathrm{d}n$$

also vanishes for all $m \in M_1(\mathbb{Q})\backslash M_1(\mathbb{A})^1$.

PROOF: Suppose that for a given $\chi' \in \mathscr{X}$, there is a group R in $P_{\chi'}$ which is contained in P_1. We would like to prove that for any function $\phi_{\chi'} \in L^2(M_R(\mathbb{Q})\backslash M_R(\mathbb{A})^1)_{\chi'}$, the integral

$$\int_{M_R(\mathbb{Q})\backslash M_R(\mathbb{A})^1} \int_{N_R^1(\mathbb{Q})\backslash N} h_\chi(nm)\phi_{\chi'}(m)\,\mathrm{d}n\,\mathrm{d}m \tag{2.1}$$

vanishes for $\chi \neq \chi'$. Suppose that $\chi \neq \chi'$, and that $\phi \in \mathscr{H}_Q(\pi)_\chi$ for some $Q \subset Q_i$, and some $\pi \in \Pi(M_Q)$. The construction of Eisenstein series is such that if the function

$$m \to \int_{N_1(\mathbb{Q})\backslash N_1(\mathbb{A})} E_{Q_i}(nmy, \phi)\,\mathrm{d}n, \quad m \in M_1(\mathbb{Q})\backslash M_1(\mathbb{A})^1,$$

is substituted for h_χ in (2.1), the result is 0. It follows from the estimates of [1(c), §4] that (2.1) itself is 0. The same estimates yield constants c and N such that

$$\sum_{\chi \in \mathscr{X}} |h_\chi(m)| \leq c\|m\|^N, \qquad m \in M_1(\mathbb{Q})\backslash M_1(\mathbb{A})^1.$$

By assumption, $\Sigma_\chi h_\chi(m)$ equals 0. Consequently (2.1) is zero even when $\chi = \chi'$. The function h_χ is continuous. Because (2.1) vanishes for all $\phi_{\chi'}$, h_χ satisfies the hypotheses of [4(b), Lemma 3.7]. h_χ is therefore zero. □

To return to the proof of the theorem, we look for conditions imposed on x, y and γ by the nonvanishing of

$$K_{P_1}(\gamma x, my), \qquad m \in M_1(\mathbb{Q})\backslash M_1(\mathbb{A})^1. \tag{2.2}$$

Set

$$y = y_1 k, \qquad y_1 \in P_1(\mathbb{A}) \cap G(\mathbb{A})^1, \quad k \in K.$$

There is a compact subset of $G(\mathbb{A})^1$, depending only on the support of

f, which contains some point

$$x^{-1}\gamma^{-1}n\eta m y_1, \quad n\in N_1(\mathbb{A}),\ \eta\in M_1(\mathbb{Q}),$$

whenever (2.2) does not vanish. Fix $\varpi\in\hat{\Delta}_1$ and let Λ be a rational representation of G with highest weight $d\varpi$, $d>0$. Choose a height function $\|\ \|$ as in [1(c), §1]. If v is a highest weight vector, we can choose a constant c_1 such that

$$\|\Lambda(x^{-1}\gamma n\eta m y_1)v\|\le c_1$$

whenever $x^{-1}\gamma n\eta m y_1$ lies in the given compact subset of $G(\mathbb{A})^1$. The left side of this inequality equals

$$e^{d\varpi(H_0(y_1))}\|\Lambda(x^{-1}\gamma^{-1})v\| = e^{d\varpi(H_0(y))}\|\Lambda(x^{-1}\gamma^{-1})v\|,$$

which is no less than a constant multiple of

$$e^{d\varpi(H_0(y))}\,e^{-d\varpi(H_0(\gamma x))}.$$

In other words, $\varpi(H_0(\gamma x)-H_0(y))$ is no less than a fixed constant. It follows from this observation that we may choose a point $T_0\in\mathfrak{a}_0$, depending only on the support of f, such that

$$\hat{\tau}_1(H_0(\gamma x)-H_0(y)-T_0)=1 \tag{2.3}$$

whenever (2.2) does not vanish identically in m. We conclude from Lemma 2.3 that if (2.3) fails to hold for a given x, y and γ, then

$$K_{P_1,\chi}(\gamma x, my), \qquad m\in M_1(\mathbb{Q})\backslash M_1(\mathbb{A})^1, \tag{2.4}$$

vanishes for all χ and m.

Combining [1(c), Lemma 5.1] with what we have just shown, we conclude that for fixed x and y,

$$K_{P_1,\chi}(\gamma x, y), \qquad \gamma\in F(P_1,P_2),$$

vanishes unless γ belongs to a finite subset of $F(P_1,P_2)$, independent of χ. Therefore the sums in

$$\sum_{\{P:\,P_1\subset P\subset P_2\}}(-1)^{\dim(A/A_2)}\int_{N_1(\mathbb{Q})\backslash N_1(\mathbb{A})}K_P(x,nmy)\,\mathrm{d}n - \sum_{\gamma\in F(P_1,P_2)}K_{P_1}(\gamma x, my)$$

are finite. Since the expression vanishes for all m in $M_1(\mathbb{Q})\backslash M_1(\mathbb{A})^1$, we can apply Lemma 2.3. We obtain an equality of functions of y for each χ. We are certainly at liberty to apply our truncation operator to those functions. It follows that for any $\chi \in \mathscr{X}$,

$$\sum_{\{P:\, P_1 \subset P \subset P_2\}} (-1)^{\dim(A/Z)} \Lambda_2^{T,P_1} K_{P,\chi}(x, y)$$

equals

$$(-1)^{\dim(A_2/Z)} \sum_{\gamma \in F(P_1,P_2)} \Lambda_2^{T,P_1} K_{P_1,\chi}(\gamma x, y).$$

We have thus far shown that

$$\sum_\chi \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} |k_\chi^T(x, f)| \, dx$$

is bounded by the sum over $\{P_1, P_2: P_0 \subset P_1 \subset P_2\}$ of

$$\int_{P_1(\mathbb{Q})\backslash G(\mathbb{A})^1} \sum_\chi \sum_{\gamma \in F(P_1,P_2)} \sigma_1^2(H_0(x) - T) |\Lambda_2^{T,P_1} K_{P_1,\chi}(\gamma x, x)| \, dx.$$

Let $\mathfrak{S}$ be a fixed Siegel set in $M_1(\mathbb{A})^1$ with $M_1(\mathbb{Q})\mathfrak{S} = M_1(\mathbb{A})^1$, and let Γ be a compact subset of $N_1(\mathbb{A})$ with $N_1(\mathbb{Q})\Gamma = N_1(\mathbb{A})$. Then the last integral is bounded by the integral over $n \in \Gamma$, $m \in \mathfrak{S} \cap P_0(\mathbb{A})$, $a \in A_1(\mathbb{R})^0 \cap G(\mathbb{A})^1$, and $k \in K$, of

$$e^{-2\rho_{P_1}(H_0(a))} \sigma_1^2(H_0(a) - T) \sum_\chi \sum_\gamma |\Lambda_2^{T,P_1} K_{P_1,\chi}(\gamma nmak, mak)|.$$

Suppose that for n, m, a and k as above, and for some $\tilde{m} \in M_1(\mathbb{A})^1$, $\gamma \in F(P_1, P_2)$ and $\chi \in \mathscr{X}$,

$$K_{P_1,\chi}(\gamma nmak, \tilde{m}ak) \neq 0.$$

Write $\gamma = \nu w_s \pi$, for $\nu \in N_0^2(\mathbb{Q})$, $\pi \in P_0(\mathbb{Q})$ and $s \in \Omega^{M_2}$, the Weyl group of (M_2, A_0). It follows from Lemma 2.3 that there is a fixed compact subset of $G(\mathbb{A})^1$ which contains

$$a^{-1}m^{-1}n_1 w_s p_1 a,$$

for points $n_1 \in N_0(\mathbb{A})$ and $p_1 \in M_1(\mathbb{A})^1 N_1(\mathbb{A})$. Fix $\varpi \in \hat{\Delta}$, and let Λ and

v be as above. $\Lambda(w_s)v$ is a weight vector, with weight $s\varpi$. The vector

$$\Lambda(a^{-1}m^{-1}n_1w_sp_1a)v - e^{d(\varpi - s\varpi)(H_0(a))}\, e^{-ds\varpi(H_0(m))}v$$

can be written as a sum of weight vectors, with weights higher than $s\varpi$. By the construction of our height function,

$$e^{d(\varpi - s\varpi)(H_0(a))}\, e^{-ds\varpi(H_0(m))}\, \|v\| \leq \|\Lambda(a^{-1}m^{-1}n_1w_sp_1a)v\|.$$

It follows that there are constants c' and c, depending only on the support of f, such that

$$|(\varpi - s\varpi)H_0(a))| \leq c'|s\varpi(H_0(m))| \leq c(1+\log\|m\|).$$

Since s fixes $\mathfrak{a}_2$ pointwise, the inequality

$$|(\varpi - s\varpi)(H_0(a))| \leq c(1+\log\|m\|)$$

holds for the projection of ϖ onto $\mathfrak{a}_1^2$. In other words, we may take ϖ to be an element in $\hat{\Delta}_1^2$. For each such ϖ, $\varpi - s\varpi$ is a nonnegative integral sum of roots in Δ_1^2. We claim that the coefficient of the element α in Δ_1^2, such that $\varpi = \varpi_\alpha$, is not zero. Otherwise we would have $(\varpi - s\varpi)(\varpi^\vee) = 0$, or equivalently, $s\varpi = \varpi$. This would force s to belong to Ω^M, for some parabolic subgroup P, $P_1 \subset P \subsetneq P_2$. This contradicts the assumption that $\gamma = \nu w_s \pi$ belongs to $F(P_1, P_2)$, so the coefficient of α is indeed positive. We can assume that a has the additional property that

$$\sigma_1^2(H_0(a) - T) \neq 0.$$

It follows from Corollary 6.2 of [1(c)] that for any Euclidean norm $\|\ \|$ on $\mathfrak{a}_0$ there is a constant c such that

$$\|(H_0(a)\| \leq c(1+\log\|m\|). \tag{2.5}$$

We have shown that if $a \in A_1(\mathbb{R})^0 \cap G(\mathbb{A})^1$ is such that for some χ, γ, n, m, $\tilde{m}$ and k,

$$|\sigma_1^2(H_0(a) - T)K_{P_1,\chi}(\gamma nma, \tilde{m}ak)| \tag{2.6}$$

does not vanish, then the inequality (2.5) holds.

Suppose that f is right invariant under an open compact subgroup

K_0 of $G(\mathbb{A}_f)$. Then if $I_{P_1}(\pi, f)\phi \neq 0$ for some π and $\phi \in \mathscr{B}_{P_1}(\pi)_\chi$, the function $E(y, \phi)$ is right K_0-invariant in y. Therefore for any x, γ and χ, $K_{P_1,\chi}(\gamma x, y)$ is right K_0-invariant in y. It follows that (2.6) is right invariant in $\tilde{m}$ under the open compact subgroup

$$\bigcap_{k_1 \in K} (k_1 k_0 k_1^{-1}) \cap M_1(\mathbb{A}_f)^1$$

of $M_1(\mathbb{A}_f)^1$. We apply Lemma 1.4 with the group G replaced by M_1. For any positive integers N_1 and N_1' we can choose a finite set $\{X_i\}$ of elements in $\mathscr{U}(\mathfrak{m}_1(\mathbb{R})^1 \otimes \mathbb{C})$, the universal enveloping algebra of the complexification of the Lie algebra of $M_1(\mathbb{R})^1$, such that for all $n \in \Gamma$, $m \in \mathfrak{s} \cap P_0(\mathbb{A})$, $\tilde{m} \in \mathfrak{s}$, $a \in A_1(\mathbb{R})^0 \cap G(\mathbb{A})^1$ and $k \in K$,

$$\sum_{\gamma \in F(P_1,P_2)} \sum_{\chi} |\Lambda_2^{T,P_1} K_{P_1,\chi}(\gamma nmak, \tilde{m}ak)| \tag{2.7}$$

is bounded by

$$\sum_i \sup_{u \in M_1(\mathbb{A})^1} \Big(\sum_\gamma \sum_\chi |R_u(X_i) K_{P_1,\chi}(\gamma nmak, uak)| \cdot \|u\|^{-N_1} \Big) \cdot \|\tilde{m}\|^{-N_1'}. \tag{2.8}$$

We can choose elements $\{Y_j\}$ in $\mathscr{U}(\mathfrak{g}(\mathbb{R})^1 \otimes \mathbb{C})$ such that

$$\mathrm{Ad}(ak)^{-1} X_i = \mathrm{Ad}(k)^{-1} X_i = \sum_j c_{ij}(k) Y_j,$$

where $c_{ij}(k)$ are continuous functions on K. Recall that $K_{P_1,\chi}(x, y)$ is ultimately defined in terms of f. The function $R_y(Y_j)K_{p_1,\chi}(x, y)$ is defined the same way, but with f replaced by $f * \bar{Y}_j^*$. The support of $f * \bar{Y}_j^*$ is contained in the support of f, so we can assume that (2.3) is valid whenever $R_y(Y_i)K_{P_1,\chi}(\gamma x, y)$ does not vanish. By Corollary 4.6 of [1(c)],

$$\sum_\chi |R_y(Y_j) K_{P_1,\chi}(x, y)|$$

is bounded by a constant multiple of a power of $\|x\| \cdot \|y\|$. It follows from Corollary 5.2 of [1(c)] that the expression

$$\sum_{\gamma \in F(P_1,P_2)} \sum_\chi |R_y(Y_j) K_{P_1,\chi}(\gamma x, y)|$$
$$= \sum_{\gamma \in F(P_1,P_2)} \sum_\chi |R_y(Y_j) K_{P_1,\chi}(\gamma x, y)| \cdot \hat{\tau}_1(H_0(\gamma x) - H_0(y) - T_0)$$

is also bounded by a constant multiple of a power of $\|x\| \cdot \|y\|$. By taking N_1 to be large enough we obtain constants C_2 and N_2 such (2.8), and therefore (2.7), is bounded by

$$C_2\|m\|^{N_2}\|a\|^{N_2}\|\tilde{m}\|^{-N_1'}.$$

Set $\tilde{m} = m$ in (2.7). Integrate the resulting expression over $n \in \Gamma$, $m \in \mathfrak{s} \cap P_0(\mathbb{A})$, $k \in K$ and a in the subset of elements in $A_1(\mathbb{R})^0 \cap G(\mathbb{A})^1$ which satisfy (2.5). There are constants C_3 and N_3 such that the result is bounded by

$$C_3 \int_{\mathfrak{s}} \|m\|^{N_3 - N_1'} \, dm.$$

If we set $N_1' = N_3$, this is finite. The proof of Theorem 2.1 is complete. □

LEMMA 2.4: *For T sufficiently regular, and r sufficiently large,*

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} k_\chi^T(x, f)\, dx = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \Lambda_2^T K_\chi(x, x)\, dx,$$

for all $f \in C_c^r(G(\mathbb{A})^1)$ and $\chi \in \mathfrak{X}$.

PROOF: It follows from the proof of Theorem 2.1 that the integral of $k_\chi^T(x, f)$ is the sum over all $P_1 \subset P_2$ of the product of $(-1)^{\dim(A_2/Z)}$ with

$$\int_{P_1(\mathbb{Q})\backslash G(\mathbb{A})^1} \sum_{\gamma \in F(P_1, P_2)} \sigma_1^2(H_0(x) - T) \cdot \Lambda_2^{T,P_1} K_{P_1,\chi}(\gamma x, x)\, dx.$$

As a double integral over x and γ this converges absolutely. If $P_1 = P_2 \neq G$, the integrand is zero. If $P_1 = P_2 = G$, the result is the integral of $\Lambda_2^T K_\chi(x, x)$. We have only to show that if $P_1 \subsetneq P_2$, the result is zero. Let $\Omega(P_1, P_2)$ be the set of elements s in Ω^{M_2} such that $s\alpha$ and $s^{-1}\alpha$ are positive roots for each $\alpha \in \Delta_0^1$ and such that s does not belong to any Ω^M, with $P_1 \subset P \subsetneq P_2$. Then the above integral equals the sum over all $s \in \Omega(P_1, P_2)$ of

$$\int_{P_1(\mathbb{Q})\backslash G(\mathbb{A})^1} \sum_{\gamma \in P_1(\mathbb{Q}) \cap w_s^{-1} P_1(\mathbb{Q}) w_s \backslash P_1(\mathbb{Q})} \sigma_1^2(H_0(x) - T) \cdot \Lambda_2^{T,P_1} K_{P_1,\chi}(w_s \gamma x, x)\, dx.$$

Since

$$\sigma_1^2(H_0(x) - T) \cdot \Lambda_2^{T,P_1} K_{P_1,\chi}(w_s\gamma x, x) = \sigma_1^2(H_0(\gamma x) - T) \cdot \Lambda_2^{T,P_1} K_{P_1,\chi}(w_s\gamma x, x)$$

for any $\gamma \in P_1(\mathbb{Q})$, this equals

$$(2.9) \qquad \int_{(P_1(\mathbb{Q})\cap w_s^{-1}P_1(\mathbb{Q})w_s)\backslash G(\mathbb{A})^1} \sigma_1^2(H_0(x)-T)\cdot \Lambda_2^{T,P_1}K_{P_1,\chi}(w_s x, x)\,\mathrm{d}x.$$

If $s\in\Omega(P_1,P_2)$, $w_s^{-1}P_0w_s\cap M_1$ is the standard minimal parabolic subgroup of M_1, since $s^{-1}\alpha>0$ for $\alpha\in\Delta_0^1$. Therefore $M_1\cap w_s^{-1}P_1w_s$ equals $M_1\cap P_s$, for a unique parabolic subgroup P_s of G, with $P_0\subset P_s\subset P_1$. Write the integral in (2.9) as a double integral over $M_s(\mathbb{Q})N_s(\mathbb{A})\backslash G(\mathbb{A})^1\times(P_1(\mathbb{Q})\cap w_s^{-1}P_1(\mathbb{Q})w_s)\backslash M_s(\mathbb{Q})N_s(\mathbb{A})$. $P_1\cap w_s^{-1}P_1w_s$ is the semi-direct product of $M_1\cap w_s^{-1}P_1w_s$ and $N_1\cap w_s^{-1}P_1w_s$, and $M_1\cap w_s^{-1}P_1w_s$ decomposes further as the semidirect product of $M_s(\mathbb{Q})$ and $N_s^1(\mathbb{Q})$. Therefore, (2.9) equals the integral over x in $M_s(\mathbb{Q})N_s(\mathbb{A})\backslash G(\mathbb{A})^1$ of the product of $\sigma_1^2(H_0(x)-T)$ and

$$(2.10) \qquad \int_{N_s^1(\mathbb{Q})\backslash N_s^1(\mathbb{A})}\mathrm{d}n\int_{N_1(\mathbb{A})\cap w_s^{-1}P_1(\mathbb{A})w_s\backslash N_1(\mathbb{A})}\mathrm{d}n_1\cdot \Lambda_2^{T,P_1}K_{P_1,\chi}(w_sn_1nx, n_1nx).$$

This last expression equals

$$\int\int \Lambda_2^{T,P_1}K_{P_1,\chi}(w_sn_1nx, nx)\,\mathrm{d}n_1\,\mathrm{d}n$$
$$=\int\int \Lambda_2^{T,P_1}K_{P_1,\chi}(w_sn_1x, nx)\,\mathrm{d}n_1\,\mathrm{d}n.$$

We apply Lemma 1.1 to the parabolic subgroup $M_1\cap P_s$ of M_1. Then this expression vanishes unless $\varpi(H_0(x)-T)$ is negative for each $\varpi\in\hat{\Delta}_s^1$. On the other hand, we can assume that (2.3) holds, with γ, x, and y replaced by w_s, n_1x, and nx respectively. In other words,

$$\varpi(H_0(w_sn_1x))\geq\varpi(H_0(x))+\varpi(T_0)$$

for each $\varpi\in\hat{\Delta}_1$. But it is well known that

$$\varpi(H_0(w_sn_1x))\leq\varpi(sH_0(x)),$$

so there is a constant C, depending only on the support of f, such that

$$\varpi(H_0(x)-sH_0(x))\leq C$$

for every ϖ in $\hat{\Delta}_1$. These two conditions on $H_0(x)$, we repeat, are based on the assumption that (2.10) does not vanish. We obtain a third condition by demanding that $\sigma_1^2(H_0(x)-T)$ not vanish. We shall show that these three conditions are incompatible if T is sufficiently regular.

We write the projection of $H_0(x)-T$ on $\mathfrak{a}_s^2$ as

$$-\sum_{\alpha\in\Delta_s^1} c_\alpha\alpha^\vee + \sum_{\varpi\in\hat{\Delta}_1^2} c_\varpi\varpi^\vee.$$

The first and third conditions on $H_0(x)$ translate to the positivity of each c_α and c_ϖ. Now the Levi component of P_s equals $M_1\cap w_s^{-1}M_1w_s$. Therefore $s\mathfrak{a}_0^s$ is orthogonal to $\mathfrak{a}_1$. Then for $\varpi_0\in\hat{\Delta}_1$,

$$\varpi_0(H_0(x)-sH_0(x))$$

equals

$$\varpi_0(T-sT)+\sum_{\alpha\in\Delta_s^1} c_\alpha\varpi_0(s\alpha^\vee)+\sum_{\varpi\in\hat{\Delta}_1^2} c_\varpi\varpi_0(\varpi^\vee-s\varpi^\vee).$$

Now $\varpi^\vee - s\varpi^\vee$ is a nonnegative sum of co-roots, so the sum over ϖ is nonnegative. Moreover we can replace each α in the sum over Δ_s^1 by the corresponding root in $\Delta_0^1\backslash\Delta_0^s$. Since s maps the roots in this latter set to positive roots, the sum over α is also nonnegative. Finally, for any ϖ_0, $\varpi_0(T-sT)$ can be made arbitrarily large for T sufficiently regular. We thus contradict the second condition on $H_0(x)$. Therefore (2.10) is always zero so the integral of $k_\chi^T(x,f)$ equals that of $\Lambda_2^T K_\chi(x,x)$. □

3. The operator $M_P^T(\pi)$

For any $\chi\in\mathscr{X}$, set

$$J_\chi^T(f)=\int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} k_\chi^T(x,f)\,dx.$$

In this section we shall give another formula, which reveals a different set of properties of the distributions J_χ^T. We shall build on Lemma 2.4, which is a partial step in this direction.

Fix P, $\pi\in\Pi(M)$ and $\chi\in\mathscr{X}$. Suppose that A is a linear operator on $\mathscr{H}_P(\pi)$ under which one of the spaces $\mathscr{H}_P(\pi)_\chi$, $\mathscr{H}_P(\pi)_{\chi,K_0}$ or $\mathscr{H}_P(\pi)_{\chi,K_0,W}$ is invariant. Here K_0 is an open compact subgroup of

$G(\mathbb{A}_f)$ and W is an equivalence class of irreducible representations of $K_{\mathbf{R}}$. We shall write A_χ, A_{χ,K_0} or $A_{\chi,K_0,W}$ for the restriction of A to the subspace in question.

Suppose that $\mathfrak{S}$ is a Siegel set in $G(\mathbb{A})^1$. It is a consequence of Lemma 1.4 and [1(c), (3.1)] that given any integer N' and a vector $\phi \in \mathcal{H}_P^0(\pi)_\chi$, we can choose a locally bounded function $c(\zeta)$ on the set of $\zeta \in \mathfrak{a}^*_{P,\mathbf{C}}$ at which $E(x, \phi_\zeta)$ is regular, such that

$$|\Lambda^T E(x, \phi_\zeta)| \leq c(\zeta) \cdot \|x\|^{-N'},$$

for all $x \in \mathfrak{S}$. It follows that for $\phi, \psi \in \mathcal{H}_P^0(\pi)_\chi$, the integrals

$$\int_{G(\mathbf{Q})\backslash G(\mathbb{A})^1} \Lambda^T E(x, \phi_\zeta)\overline{\Lambda^T E(x, \psi_\eta)}\,dx$$

and

$$\int_{G(\mathbf{Q})\backslash G(\mathbb{A})^1} E(x, \phi_\zeta)\overline{\Lambda^T E(x, \psi_\eta)}\,dx$$

converge absolutely, and define meromorphic functions in $(\zeta, \bar{\eta})$ which are regular whenever the integrands are. By Corollary 1.2 and Lemma 1.3 these meromorphic functions are equal. Thus we obtain a linear operator $M_P^T(\pi)$ on $\mathcal{H}_P^0(\pi)$ by defining

$$\begin{aligned}(M_P^T(\pi)_\chi \phi_1, \phi_2) &= \int_{G(\mathbf{Q})\backslash G(\mathbb{A})^1} \Lambda^T E(x, \phi_1) \cdot \overline{\Lambda^T E(x, \phi_2)}\,dx \\ &= \int_{G(\mathbf{Q})\backslash G(\mathbb{A})^1} E(x, \phi_1) \cdot \overline{\Lambda^T E(x, \phi_2)}\,dx,\end{aligned}$$

for every pair ϕ_1 and ϕ_2 in $\mathcal{H}_P^0(\pi)$. $M_P^T(\pi)$ depends only on the orbit of π in $\Pi^G(M)$. It is clear that $M_P^T(\pi)_\chi$ is self-adjoint and positive definite. Notice also that

$$I_P(\pi, k) \cdot M_P^T(\pi)_\chi = M_P^T(\pi)_\chi \cdot I_P(\pi, k)$$

for all $k \in K$. It follows that for any K_0 and W, $M_P^T(\pi)_\chi$ leaves the finite dimensional space $\mathcal{H}_P(\pi)_{\chi,K_0,W}$ invariant.

Recall that in the proof of Lemma 4.1 of [1(c)], we fixed an elliptic element Δ in $\mathcal{U}(\mathfrak{g}(\mathbf{R})^1 \otimes \mathbf{C})^{K_\mathbf{R}}$. For any K_0 and W, $\mathcal{H}_P(\pi)_{\chi,K_0,W}$ is an invariant subspace for the operator $I_P(\pi, \Delta)$. Choose Δ so that for any χ, π, W and K_0, such that $\mathcal{H}_P(\pi)_{\chi,K_0,W} \neq \{0\}$, $I_P(\pi, \Delta)_{\chi,K_0,W}$ is the product of the identity operator with a real number which is larger than 1. For example, we could take Δ to equal $1+\Delta_1^*\Delta_1$, where Δ_1 is a suitable linear combination of the Casimir elements for $G(\mathbf{R})^1$ and $K_{\mathbf{R}}$.

If A is any operator on a Hilbert space, $\|A\|_1$ denotes the trace class norm of A.

THEOREM 3.1: *There is a positive integer n such that for any open compact subgroup K_0 of $G(\mathbb{A}_f)$,*

$$\sum_{\chi}\sum_{P} n(A)^{-1}\int_{\Pi^G(M)} \|M_P^T(\pi)_{\chi,K_0}\cdot I_P(\pi,\Delta^n)_{\chi,K_0}^{-1}\|_1 \,\mathrm{d}\pi$$

is finite.

Assume the proof of the theorem for the moment and take $r_1 = \deg \Delta^n$. Suppose that f is a function in $C_c^{r_1}(G(\mathbb{A})^1)$, which is bi-invariant under K_0. Then

$$\begin{aligned}
&\|M_P^T(\pi)\cdot I_P(\pi,f)\chi\|_1\\
&=\|M_P^T(\pi)_{\chi,K_0}\cdot I_P(\pi,f)\|_1\\
&=\|M_P^T(\pi)_{\chi,K_0}\cdot I_P(\pi,\Delta^n)^{-1}I_P(\pi,\Delta^n * f)\|_1\\
&\le\|M_P^T(\pi)_{\chi,K_0}\cdot I_P(\pi,\Delta^n)_{\chi,K_0}^{-1}\|_1\cdot\|I_P(\pi,\Delta^n * f)\|.
\end{aligned}$$

For any π the norm of the operator $I_P(\pi,\Delta^n * f)$ is bounded by

$$\int_{G(\mathbb{A})^1} |(\Delta^n * f)(x)| \,\mathrm{d}x.$$

Thus, Theorem 3.1 implies that for every $f\in C_c^{r_1}(G(\mathbb{A})^1)$,

$$\sum_{\chi}\sum_{P} n(A)^{-1}\int_{\Pi^G(M)} \|M_P^T(\pi)_{\chi}\cdot I_P(\pi,f)_{\chi}\|_1 \,\mathrm{d}\pi \tag{3.1}$$

is finite, and in fact defines a continuous seminorm on $C_c^{r_1}(G(\mathbb{A})^1)$. In particular, the operator $M_P^T(\pi)_\chi\cdot I_P(\pi,f)_\chi$ is of trace class for almost all π.

THEOREM 3.2: *There is an $r\ge r_1$ such that for any χ and any $f\in C_c^r(G(\mathbb{A})^1)$,*

$$J_\chi^T(f)=\sum_{P} n(A)^{-1}\int_{\Pi^G(M)} \operatorname{tr}(M_P^T(\pi)_\chi\cdot I_P(\pi,f)_\chi)\,\mathrm{d}\pi.$$

We shall prove the two theorems together, Let N and r_0 be the positive integers of Lemma 4.4 in [1(c)]. Choose an open compact subgroup, K_0, of $G(\mathbb{A}_f)$ and a Siegel set $\mathfrak{s}$ in $G(\mathbb{A})^1$. According to Lemma 1.4 and the lemma just quoted from [1(c)], we may choose a finite set $\{Y_i\}$ of elements in $\mathcal{U}(\mathfrak{g}(\mathbf{R})^1 \otimes \mathbf{C})$ such that for $x \in G(\mathbb{A})^1$, $y \in \mathfrak{s}$,

$$r \geq r_0 + \sum_i \deg Y_i,$$

and f a K-finite function in $C_c^r(G(\mathbb{A})^1/K_0)$,

$$\sum_P n(A)^{-1} \int_{\Pi^G(M)} \Big| \sum_{\phi \in \mathcal{B}_P(\pi)_\chi} E(x, I_P(\pi, \phi)f) \cdot \overline{\Lambda^T E(y, \phi)} \Big| \, d\pi$$

is bounded by

$$\sum_i \|f * Y_i\|_{r_0} \cdot \|x\|^N \cdot \|y\|^{-N}.$$

When we set $x = y$ and integrate the above expression over $G\mathbb{Q})\backslash G(\mathbb{A})^1$, the result is bounded by

$$\mathrm{vol}(G(\mathbb{Q})\backslash G(\mathbb{A})^1) \cdot \sum_i \|f * Y_i\|_{r_0}.$$

Suppose $\omega = (W_1, W_2)$ is a pair of equivalence classes of irreducible representations of $K_{\mathbf{R}}$. We defined the function

$$f_\omega(x) = \deg W_1 \cdot \deg W_2 \cdot \int_{K_R \times K_R} ch_{W_1}(k_1)\, f(k_1^{-1} x k_2^{-1}) ch_{W_2}(k_2) \, dk_1 \, dk_2$$

in §4 of [1(c)]. We also defined the positive integer ℓ_0. Let

$$r_2 = r_0 + \ell_0 + \sum_i \deg Y_i.$$

As we saw in §4 of [1(c)],

$$\|f\|_{r_2} = \mathrm{vol}(G(\mathbb{Q})\backslash G(\mathbb{A})^1) \sum_\omega \sum_i \|f_\omega * Y_i\|_{r_0}, \quad f \in C_c^{r_2}(G(\mathbb{A})^1),$$

is a continuous seminorm on $C_c^{r_2}(G(\mathbb{A})^1)$. We have shown that

$$(3.2)\qquad \sum_{\omega}\int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1}\sum_{\chi}\sum_{P} n(A)^{-1}\int_{\Pi^G(M)}\Big|\sum_{\phi\in\mathscr{B}_P(\pi)} E(x, I_P(\pi, f_\omega)\phi) \cdot \overline{\Lambda^T E(x,\phi)}\Big|\,\mathrm{d}\pi\,\mathrm{d}x$$

is bounded by $\|f\|_{r_2}$, for every $f \in C_c^{r_2}(G(\mathbb{A})^1)$.

Let r by any integer larger than r_2 for which Lemma 2.4 is valid. Then if $f \in C_c^r(G(\mathbb{A})^1)$, and χ is fixed,

$$\sum_P n(A)^{-1}\int_{\Pi^G(M)}\sum_{\omega}\operatorname{tr}(M_P^T(\pi)_\chi \cdot I_P(\pi, f_\omega)_\chi)\,\mathrm{d}\pi$$

$$= \sum_P n(A)^{-1}\int_{\Pi^G(M)}\sum_{\omega}\int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1}\Big(\sum_{\phi\in\mathscr{B}_P(\pi)_\chi} E(x, I_P(\pi, f_\omega)\phi) \cdot \overline{\Lambda^T E(x,\phi)}\Big)\,\mathrm{d}x\cdot\mathrm{d}\pi$$

$$= \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1}\sum_{\omega}\sum_{P} n(A)^{-1}\int_{\Pi^G(M)}\Big(\sum_{\phi} E(x, I_P(\pi, f_\omega)\phi) \cdot \overline{\Lambda^T E(x,\phi)}\Big)\,\mathrm{d}\pi\cdot\mathrm{d}x,$$

by Tonelli's theorem. The operator Λ^T is defined in terms of sums and integrals over compact sets. If we combine Tonelli's theorem with the estimates of [1(c), §4] we find that we can take Λ^T outside the sums over ϕ, P and ω, and the integral over π. The result is

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1}\Lambda_2^T K_\chi(x,x)\,\mathrm{d}x,$$

which by Lemma 2.4 equals $J_\chi^T(f)$. The proof of Theorem 3.2 will now follow from Theorem 3.1 if we take r to be larger than r_1.

The only remaining thing to prove is Theorem 3.1. We shall use Lemma 4.1 of [1(c)]. We can choose n, and functions $g_{\mathbb{R}}^1 \in C_c^{r_2}(G(\mathbb{A})^1)^{K_{\mathbb{R}}}$ and $g_{\mathbb{R}}^2 \in C_c^\infty(G(\mathbb{A})^1)^{K_{\mathbb{R}}}$ such that $\Delta^n * g_{\mathbb{R}}^1 + g_{\mathbb{R}}^2$ is the Dirac distribution at 1 in $G(\mathbb{R})^1$. If $i = 1, 2$, set

$$g_i(x_{\mathbb{R}}\cdot x_f) = \operatorname{vol}(K_0)^{-1}\cdot g_{\mathbb{R}}^i(x_{\mathbb{R}})\cdot ch_{K_0}(x_f),$$

$$x_{\mathbb{R}} \in G(\mathbb{R})^1,\ x_f \in G(\mathbb{A})^1,$$

where ch_{K_0} is the characteristic function of K_0. Then

$$I_P(\pi, \Delta^n)^{-1}_{\chi,K_0} = I_P(\pi, g_1)_\chi + I_P(\pi, \Delta^n)^{-1} I_P(\pi, g_2)_\chi.$$

Suppose that W is an irreducible $K_{\mathbf{R}}$-type and that $\omega = (W, W)$. Then the trace of the restriction of $M_P^T(\pi)_\chi \cdot I_P(\pi, \Delta^n)^{-1}$ to $\mathscr{H}_P(\pi)_{\chi,K_0,W}$ is

$$\mathrm{tr}(M_P^T(\pi)_\chi \cdot I_P(\pi, g_{1,\omega})_\chi + M_P^T(\pi)_\chi \cdot I_P(\pi, \Delta^n)^{-1} I_P(\pi, g_{2,\omega})_\chi).$$

Since the eigenvalues of $I_P(\pi, \Delta^n)$ are all larger than 1, this last expression is bounded by

$$\sum_{i=1}^{2} |\mathrm{tr}(M_P^T(\pi)_\chi \cdot I_P(\pi, g_{i,\omega})_\chi)|.$$

Now the trace class norm of the operator

$$M_P^T(\pi)_{\chi,K_0} \cdot I_P(\pi, \Delta^n)^{-1}_{\chi,K_0}$$

is the sum of the traces of its restriction to each of the subspaces $\mathscr{H}_P(\pi)_{\chi,K_0,W}$. Therefore

$$\sum_\chi \sum_P n(A)^{-1} \int_{\Pi^G(M)} \|M_P^T(\pi)_{\chi,K_0} \cdot I_P(\pi, \Delta^n)^{-1}_{\chi,K_0}\|_1 \, \mathrm{d}\pi$$

is bounded by the sum over $i = 1, 2$ of

$$\sum_\chi \sum_P n(A)^{-1} \int_{\Pi^G(M)} \sum_\omega \Big| \sum_{\phi \in \mathscr{B}_P(\pi)_\chi} \int_{G(\mathbf{Q})\backslash G(\mathbf{A})^1} E(x, I_P(\pi, g_{i,\omega})\phi) \cdot \overline{\Lambda^T E(x, \phi)} \, \mathrm{d}x \Big| \, \mathrm{d}\pi.$$

This in turn is bounded by

$$\sum_\omega \int_{G(\mathbf{Q})\backslash G(\mathbf{A})^1} \sum_\chi \sum_P n(A)^{-1} \int_{\Pi^G(M)} \Big| \sum_{\phi \in \mathscr{B}_P(\pi)_\chi} E(x, I_P(\pi, g_{i,\omega})\phi) \cdot \overline{\Lambda^T E(x, \phi)} \, \mathrm{d}x \Big| \, \mathrm{d}\pi,$$

which is just (3.2) with f replaced by g_i. Theorem 3.1, as well as Theorem 3.2, is now proved. □

4. Evaluation in a special case

In this section we shall give an explicit formula for $J_\chi^T(f)$ for a particular kind of class $\chi \in \mathscr{X}$. These special χ we will call unramified; they are analogues of the unramified classes $\mathfrak{o} \in \mathcal{O}$ for which we were able to calculate $J_\mathfrak{o}^T(f)$ in [1(c), §8]. The formula for $J_\chi^T(f)$ is a consequence of an inner product formula of Langlands which was announced in [4(a), §9]. Most of this section will be taken up with the proof, essentially that of Langlands, for the formula. First, however, we must demonstrate a connection between the truncation operator Λ^T and the modified Eisenstein series defined by Langlands in [4(a)].

Fix a parabolic subgroup P_1 and a representation $\pi \in \Pi(M_1)$. If $\phi \in \mathscr{H}_{P_1}^0(\pi)$ and $\zeta \in \mathfrak{a}_{1,\mathbb{C}}^*$, write

$$E_P(x, \phi, \zeta) = E_P(x, \phi_\zeta), \quad P \supset P_1.$$

If $s \in \Omega(\mathfrak{a}_1, \mathfrak{a}_2)$, define $M(s, \pi, \zeta) = M(s, \zeta)$ by

$$M(s, \zeta)\phi = M(s, \pi_\zeta)\phi_\zeta)_{-s\zeta}.$$

$M(s, \zeta)$ maps $\mathscr{H}_{P_1}^0(\pi)$ to $\mathscr{H}_{P_2}^0(s\pi)$. Suppose that $\chi \in \mathscr{X}$ is such that $P_1 \in P_\chi$. Then for all $\chi \in G(\mathbb{A})^1$,

$$\phi(mx), \quad m \in M_1(\mathbb{Q})\backslash M_1(\mathbb{A})^1,$$

is a cusp form in m. If P_2 is a second group in $\mathscr{P}_\chi$, we have the following basic formula from the theory of Eisenstein series:

$$\int_{N_2(\mathbb{Q})\backslash N_2(\mathbb{A})} E(nx, \phi, \zeta)\, \mathrm{d}n = \sum_{s \in \Omega(\mathfrak{a}_1, \mathfrak{a}_2)} (M(s, \zeta)\phi)(x) \cdot \mathrm{e}^{(s\zeta + \rho_{P_2})(H(x))}.$$

A formula like this exists if P_2 is replaced by an arbitrary (standard) parabolic subgroup, P. Recall that $\Omega(\mathfrak{a}_1; P)$ is defined to be the union over all $\mathfrak{a}_2$ of those elements $s \in \Omega(\mathfrak{a}_1, \mathfrak{a}_2)$ such that $s\mathfrak{a}_1 = \mathfrak{a}_2$ contains $\mathfrak{a}$, and $s^{-1}\alpha$ is positive for each $\alpha \in \Delta_2^P$. Then we have

$$(4.1) \qquad \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} E(nx, \phi, \zeta)\, \mathrm{d}n = \sum_{s \in \Omega(\mathfrak{a}_1; P)} E_P(x, M(s, \zeta)\phi, s\zeta).$$

The verification of this formula is a simple exercise which we can leave to the reader. It can be proved directly from the series definition of $E(x, \phi, \zeta)$. Alternatively, one can prove it by induction on $\dim A$, applying [4(b), Lemma 3.7] to the group M.

LEMMA 4.1: *Suppose that $P_1 \in \mathcal{P}_\chi$ as above, that $\phi \in \mathcal{H}^0_{P_1}(\pi)_\chi$ and that ζ is a point in $\mathfrak{a}^*_{1,\mathbf{C}}$ whose real part $\zeta_{\mathbf{R}}$ lies in $\rho_1 + (\mathfrak{a}^*_1)^+$. Then* $\Lambda^T E(x, \phi, \zeta)$ equals

$$\sum_{P_2} \sum_{\delta \in P_2(\mathbf{Q})\backslash G(\mathbf{Q})} \sum_{s \in \Omega(\mathfrak{a}_1, \mathfrak{a}_2)} \epsilon_2(s\zeta_{\mathbf{R}})\phi_2(s\zeta_{\mathbf{R}}, H_0(\delta x) - T)) \cdot e^{(s\zeta+\rho_2)(H_0(\delta x))}(M(s, \zeta)\phi)(\delta x), \tag{4.2}$$

with the sum over δ converging absolutely. (The functions ϵ_2 and ϕ_2 are as [1(*c*), §8].)

PROOF: Suppose that P_2 and $s \in \Omega(\mathfrak{a}_1, \mathfrak{a}_2)$ are given. In the process of verifying the equality of (8.5) and (8.6) in [1(c)], we ended up proving that for all $H \in \mathfrak{a}_0$,

$$\epsilon_2(s\zeta_{\mathbf{R}})\phi_2(s\zeta_{\mathbf{R}}, H)$$

was equal to

$$\sum_{\{P:\, P \supset P_2, s \in \Omega(\mathfrak{a}_1;\, P)\}} (-1)^{\dim(A_P/Z)} \hat{\tau}_P(H).$$

Apply this to (4.2). Then decompose the sum over $P_2(\mathbf{Q})\backslash G(\mathbf{Q})$ into a sum over $P_2(\mathbf{Q})\backslash P(\mathbf{Q})$ and $P(\mathbf{Q})\backslash G(\mathbf{Q})$. The sum over $P(\mathbf{Q})\backslash G(\mathbf{Q})$ will be finite by [1(c), Lemma 5.1]. If $\alpha \in \Delta_2^P$, $s^{-1}\alpha^\vee$ is a nonnegative sum of elements of the form $\beta^\vee$, for $\beta \in \Delta_1$. It follows that

$$(s\zeta_{\mathbf{R}} - \rho_2)(\alpha^\vee) = (\zeta_{\mathbf{R}} - \rho_1)(s^{-1}\alpha^\vee) + \rho_1(s^{-1}\alpha^\vee) - \rho_2(\alpha^\vee)$$

is positive. Therefore the sum

$$\sum_{\xi \in P_2(\mathbf{Q})\backslash P(\mathbf{Q})} e^{(s\zeta+\rho_2)(H_0(\xi\delta x))} \cdot (M(s, \zeta)\phi)(\xi\delta x)$$

is absolutely convergent, and in fact equal to $E_P(\delta x, M(s, \zeta)\phi, s\zeta)$. In particular, the original sum over δ in (4.2) is absolutely convergent.

We find that (4.2) equals

$$\sum_P (-1)^{\dim(A/Z)} \sum_{\delta \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \left\{ \sum_{s \in \Omega(\mathfrak{a}_1; P)} E_P(\delta x, M(s,\zeta)\phi, s\zeta) \right\} \hat{\tau}_P(H(\delta x) - T).$$

If the left hand side of (4.1) is substituted into the brackets, the result is $\Lambda^T E(x, \phi, \zeta)$. □

To simplify the notation, we shall assume that $\pi(a)$ is the identity operator for all $a \in A_1(\mathbf{R})^0$. This entails no loss of generality, since any $\pi_1 \in \Pi(M_1)$ equals π_η, for some such π and some $\eta \in i\mathfrak{a}_1^*$. Given P_2, define

$$\psi_2(x) = \sum_{s \in \Omega(\mathfrak{a}_1, \mathfrak{a}_2)} \epsilon_2(s\zeta_{\mathbf{R}}) \phi_2(s\zeta_{\mathbf{R}}, H_0(x) - T) \cdot \mathrm{e}^{(s\zeta + \rho_2)(H_0(x))} (M(s,\zeta)\phi)(x).$$

If $\Lambda \in i\mathfrak{a}_2^*$, define

$$\Psi_2(\Lambda, x) = \int_{A_2(\mathbf{R})^0 \cap G(\mathbb{A})^1} (\mathrm{e}^{-(\Lambda + \rho_2)(H_0(ax))} \psi_2(ax))\, \mathrm{d}a,$$

for $x \in G(\mathbb{A})^1$. This function is not hard to compute. We have only to evaluate

$$\int_{A_2(\mathbf{R})^0 \cap G(\mathbb{A})^1} \mathrm{e}^{(s\zeta - \Lambda)(H_0(ax))} \epsilon_2(s\zeta_{\mathbf{R}}) \phi_2(s\zeta_{\mathbf{R}}, H_0(ax) - T)\, \mathrm{d}a$$

Since $a \to H_2(ax)$ is a measure preserving diffeomorphism from $A_2(\mathbf{R})^0 \cap G(\mathbb{A})^1$ onto $\mathfrak{a}_2^G$, this last expression equals

$$\int_{\mathfrak{a}_2^G} \mathrm{e}^{(s\zeta - \Lambda)(H)} \epsilon_2(s\zeta_{\mathbf{R}}) \phi_2(s\zeta_{\mathbf{R}}, H - T)\, \mathrm{d}H.$$

Make a further change of variables

$$H = \sum_{\alpha \in \Delta_2} t_\alpha \alpha^\vee, \quad t_\alpha \in \mathbf{R}.$$

Of course, we will have to multiply by the Jacobian of this change of measure. It is the volume of $\mathfrak{a}_2^G$ modulo the lattice, L_2, spanned by $\{\alpha^\vee : \alpha \in \Delta_2\}$. The integral becomes a product of integrals of decreasing functions over half lines; it is easy to evaluate (see [1(b), Lemma

3.4]). We find that $\Psi_2(\Lambda, x)$ equals

$$\mathrm{vol}(\mathfrak{a}_2^G/L_2)\cdot\sum_{s\in\Omega(\mathfrak{a}_1,\mathfrak{a}_2)}\frac{e^{(s\zeta-\Lambda)(T)}}{\prod_{\alpha\in\Delta_2}(s\zeta-\Lambda)(\alpha^\vee)}\cdot(M(s,\zeta)\phi)(x).$$

We have been assuming that $\zeta_{\mathbf{R}}$ is a point in $\rho_1+(\mathfrak{a}_1^*)^+$. Let us suppose from now on that it is suitably regular. Then $\Psi_2(\Lambda, x)$ can be analytically continued as a holomorphic function, for Λ in a tube in $\mathfrak{a}_{2,\mathbf{C}}^*$ over a ball B_{P_2} in $\mathfrak{a}_2^*$, centered at the origin, of arbitrarily large radius. The functions

$$\Psi_2(\Lambda): x\to\Psi_2(\Lambda, x),$$

indexed by Λ, span a finite dimensional subspace of $L^2(M_2(\mathbf{Q})\backslash M_2(\mathbb{A})^1\times K)$. For fixed Λ_0 in B_{P_2}, $\Psi_2(\Lambda)$ is a square integrable function from $\Lambda_0+i(\mathfrak{a}_2^G)^*$ to this finite dimensional space.

Suppose that P_1' is another group in $\mathscr{P}_\chi$. Pick a class $\pi'\in\Pi(M')$, a vector $\phi'\in\mathscr{H}_{P_1'}^0(\pi')$ and a point $\zeta'\in\mathfrak{a}_{P_1',\mathbf{C}}^*$ to satisfy the same conditions as above, and define the functions ψ_2' and Ψ_2' associated to any other group P_2' in $\mathscr{P}_\chi$. Then

$$\int_{G(\mathbf{Q})\backslash G(\mathbb{A})^1}\Lambda^T E(x,\phi,\zeta)\cdot\overline{\Lambda^T E(x,\phi',\zeta')}\,dx \tag{4.3}$$

is the sum over P_2 and P_2' in $\mathscr{P}_\chi$ of

$$\int_{G(\mathbf{Q})\backslash G(\mathbb{A})^1}\Big(\sum_{\delta\in P_2(\mathbf{Q})\backslash G(\mathbf{Q})}\psi_2(\delta x)\Big)\overline{\Big(\sum_{\delta\in P_2'(\mathbf{Q})\backslash G(\mathbf{Q})}\psi_2'(\delta x)\Big)}\,dx.$$

This last inner product is given by a basic formula in the theory of Eisenstein series ([4(a), Lemma 4.6]). It equals

$$\int_{\Lambda_0+i(\mathfrak{a}_2^G)^*}\sum_{t\in\Omega(\mathfrak{a}_2,\mathfrak{a}_2')}(M(t,\Lambda)\Psi_2(\Lambda),\Psi_2'(-t\bar{\Lambda}))\,d\Lambda,$$

where Λ_0 is any point in $B_{P_2}\cap(\rho_2+(\mathfrak{a}_2^*)^+)$, and $d\Lambda$ is the Haar measure on $i(\mathfrak{a}_2^G)^*$ which is dual to our Haar measure on $\mathfrak{a}_2^G$. Therefore, (4.3) equals the sum over P_2 and $s\in\Omega(\mathfrak{a}_1,\mathfrak{a}_2)$, of the integral over Λ, of the product of

$$\mathrm{vol}(\mathfrak{a}_2^G/L_2)^2\frac{e^{(s\zeta-\Lambda)(T)}}{\prod_{\alpha\in\Delta_2}(s\zeta-\Lambda)(\alpha^\vee)} \tag{4.4}$$

and

$$\sum_{P_2'} \sum_{t \in \Omega(\mathfrak{a}_2, \mathfrak{a}_2')} \sum_{s' \in \Omega(\mathfrak{a}_1', \mathfrak{a}_2')} \frac{e^{(s'\bar{\zeta}' + t\Lambda)(T)}}{\prod_{\alpha \in \Delta_2'} (s'\bar{\zeta}' + t\Lambda)(\alpha^\vee)} \times (M(t, \Lambda)M(s, \zeta)\phi, M(s', \zeta')\phi'). \tag{4.5}$$

We shall show that (4.5) is a regular function of Λ on the tube over $\rho_2 + (\mathfrak{a}_2^*)^+$. The functions $M(t, \Lambda)$ are regular on this tube, so the only signularities are along hyperplanes

$$\{\Lambda: (s'\bar{\zeta}' + t\Lambda)(\alpha^\vee) = 0\},$$

for fixed s', t, ζ' and $\alpha \in \Delta_2'$. Let $s_\alpha \in \Omega(\mathfrak{a}_2', \mathfrak{a}_2'')$ be the simple reflection belonging to α (see [4(b), Pg. 35]). Then $\beta = -s_\alpha \alpha$ is a root in $\Delta_{P_2''}$, and

$$\{\Lambda: (s_\alpha s'\bar{\zeta}' + st\Lambda)(\beta^\vee) = 0\}$$

is the same hyperplane as above. Thus, the summands in (4.5) which are singular along a given hyperplane occur naturally in pairs. We shall show that the two residues around the hyperplane add up to 0. Assume that $(s'\bar{\zeta}' + t\Lambda)(\alpha^\vee) = 0$. Then $(s_\alpha s'\bar{\zeta}' + s_\alpha t\Lambda)(\beta^\vee) = 0$. The inner product from the summand of (4.5) corresponding to P_2'', $s_\alpha s'$, $s_\alpha t$ equals

$$\begin{aligned} &(M(s_\alpha t, \Lambda)M(s, \zeta)\phi, M(s_\alpha s', \zeta')\phi') \\ &= (M(s_\alpha, s'\zeta')^* M(s_\alpha, t\Lambda) \cdot M(t, \Lambda)M(s, \zeta)\phi, M(s', \zeta')\phi') \end{aligned} \tag{4.6}$$

by the functional equations. But

$$M(s_\alpha, s'\zeta')^* = M(s_\alpha, -s'\bar{\zeta}')^{-1} = M(s_\alpha, t\Lambda)^{-1},$$

since $M(s_\alpha, t\Lambda)$ depends only on the projection of $t\Lambda$ onto α. Therefore (4.6) equals

$$(M(t, \Lambda)M(s, \zeta)\phi, M(s', \zeta')\phi'),$$

which is the inner product from the summand of (4.5) corresponding to P_2', s', t. It follows that the residues of the two summands do add up to zero. Therefore (4.5) is regular at the hyperplane under consideration, and so is regular on the tube over $\rho_2 + (\mathfrak{a}_2^*)^+$.

Next we shall show that if $s \neq 1$, the integral in Λ of the product of (4.4) and (4.5) equals 0. Given such an s, choose a root $\alpha \in \Delta_2$ such that $(s\zeta_{\mathbf{R}})(\alpha^\vee) < 0$. Change the path of integration from $\operatorname{Re} \Lambda = \Lambda_0$ to $\operatorname{Re} \Lambda = \Lambda_0 + N\varpi_\alpha$, where N is a positive integer which we let approach ∞. We can do this by virtue of the regularity of (4.5) and the fact that the numbers

$$\{\|M(t, \Lambda)\| \colon \operatorname{Re} \Lambda = \Lambda + N\varpi_\alpha\}$$

are bounded independently of N. Notice that

$$|e^{-\Lambda(T)} e^{(t\Lambda)(T)}| = e^{(t\Lambda_0 - \Lambda_0 + N(t\varpi_\alpha - \varpi_\alpha))(T)}$$

is no greater than 1. Therefore, the integral over $\operatorname{Re} \Lambda = \Lambda_0 + N\varpi_\alpha$ approaches 0 as N approaches ∞. It follows that the original integral equals zero.

We have only to set $s = 1$ in (4.4), multiply the result by (4.5), and then integrate over Λ. Make a change of variables in the integral over Λ, setting

$$\Lambda = \sum_{\alpha \in \Delta_2} z_\alpha \varpi_\alpha, \quad z_\alpha \in \mathbf{C}.$$

With this change of measures, we must multiply the result by the volume of $i(\mathfrak{a}_2^G)^*$ modulo the lattice spanned by $\{\varpi_\alpha \colon \alpha \in \Delta_2\}$. Since $d\Lambda$ represents the measure on $i(\mathfrak{a}_2^G)^*$ dual to that on $\mathfrak{a}_2^G$, and since $\{\varpi_\alpha\}$ and $\{\alpha^\vee\}$ are dual bases, this factor equals

$$\left(\frac{1}{2\pi i}\right)^{\dim(A_2/Z)} \operatorname{vol}(\mathfrak{a}_2^G/L_2)^{-1}.$$

The product of this factor with (4.4) then equals

$$\left(\frac{1}{2\pi_i}\right)^{\dim(A_2/Z)} \operatorname{vol}(\mathfrak{a}_2^G/L_2)\, e^{\zeta(T)} \prod_{\alpha \in \Delta_2} \frac{e^{z_\alpha \cdot \varpi_\alpha(T)}}{\zeta(\alpha^\vee) - z_\alpha}.$$

Each z_α is to be integrated over the line $\Lambda_0(\alpha^\vee) + i\mathbf{R}$). We replace this contour with the line $\Lambda_0(\alpha^\vee) + N + i\mathbf{R}$, and let N approach ∞. According to our assumptions on ζ, $\zeta_{\mathbf{R}}(\alpha^\vee) > \Lambda_0(\alpha^\vee)$, so we will pick up a residue at $z_\alpha = \zeta(\alpha^\vee)$. By the arguments of the previous paragraph, the integral of z_α over the line $\Lambda_0(\alpha^\vee)+N+i\mathbf{R}$ approaches 0 as N approaches ∞. Therefore the integral of z_α over $\Lambda_0(\alpha^\vee)+i\mathbf{R}$ equals the residue of the integrand at $z_\alpha=\zeta(\alpha^\vee)$. It follows that (4.3) is the product of $\mathrm{vol}(\mathfrak{a}_2^G/L_2)$ with the value of (4.5) at $s=1$ and $\Lambda=\zeta$. We have proved

LEMMA 4.2: (*Langlands*) *Suppose that $P_1, P_1'\in P_\chi$, that $\phi\in\mathcal{H}^0_{P_1}(\pi)_\chi$, $\phi'\in\mathcal{H}^0_{P_1'}(\pi')_\chi$ and that ζ and ζ' are vectors in $\mathfrak{a}^*_{\mathfrak{p}_1,\mathbf{C}}$ and $\mathfrak{a}^*_{\mathfrak{p}_1',\mathbf{C}}$ whose real parts are suitably regular points in $(\mathfrak{a}^*_{\mathfrak{p}_1})^+$ and $(\mathfrak{a}^*_{\mathfrak{p}_1'})^+$ respectively. Then*

$$\int_{G(\mathbf{Q})\backslash G(\mathbf{A})^1} \Lambda^T E(x,\phi,\zeta)\overline{\Lambda^T E(x,\phi',\zeta')}\,\mathrm{d}x$$

equals the sum over $P_2\in\mathcal{P}_\chi$, $s\in\Omega(\mathfrak{a}_1,\mathfrak{a}_2)$, and $s'\in\Omega(\mathfrak{a}_1',\mathfrak{a}_2)$ of

$$\mathrm{vol}(\mathfrak{a}_2^G/L_2)\cdot\frac{\mathrm{e}^{(s\zeta+s'\bar{\zeta}')(T)}}{\prod_{\alpha\in\Delta_2}(s\zeta+s'\bar{\zeta}')(\alpha^\vee)}(M(s,\zeta)\phi, M(s',\zeta')\phi'). \qquad \square$$

Both sides of the identity of the lemma are meromorphic functions in $(\zeta,\bar{\zeta}')$. Therefore the identity is valid for all regular points ζ and ζ'.

Recall that the elements of $\mathcal{X}$ are equivalence classes of pairs (M_1,ρ_1). We shall say that χ is *unramified* if for any pair (M_1,ρ) in χ, the only element $s\in\Omega(\mathfrak{a}_1,\mathfrak{a}_1)$ for which $s\rho=\rho$ is the identity. For the remainder of this section, assume that χ is unramified. Suppose that $P_1=P_1'=P$ and that $\pi=\pi'$. Then if ϕ, ϕ', s and s' are as in the lemma,

$$(M(s,\zeta)\phi, M(s',\zeta')\phi')=0$$

unless $s=s'$. It follows that for $\eta\in i\mathfrak{a}^*$, $(M_P^T(\pi_\eta)_\chi\phi_\eta,\phi'_\eta)$ equals

$$\lim_{\zeta\to 0}\sum_{P_2\in\mathcal{P}_\chi}\sum_{s\in\Omega(\mathfrak{a},\mathfrak{a}_2)}\mathrm{vol}(\mathfrak{a}_2^G/L_2)\cdot\frac{\mathrm{e}^{(s\zeta)(T)}}{\prod_{\alpha\in\Delta_2}(s\zeta)(\alpha^\vee)}(M(s,\eta+\zeta)\phi, M(s',\eta)\phi').$$

We can now take π to be any class in $\Pi(M)$. We have shown that for $P\in P_\chi$ and $\pi\in\Pi(M)$,

$$M_P^T(\pi)_\chi=\mathrm{vol}(\mathfrak{a}_P^G/L_P)\cdot\lim_{\zeta\to 0}\sum_{P_2\in\mathcal{P}_\chi}\sum_{s\in\Omega(\mathfrak{a},\mathfrak{a}_2)}\times\frac{\mathrm{e}^{(s\zeta)(T)}M(s,\pi)^{-1}M(s,\pi,\zeta)}{\prod_{\alpha\in\Delta_2}(s\zeta)(\alpha^\vee)}.$$

On the other hand, if P does not belong to $\mathcal{P}_\chi$ and $\pi\in\Pi(M)$, then $\mathcal{H}_P(\pi)_\chi=\{0\}$. This fact can be extracted from the results of [4(b), §7].

We can therefore write

$$J_\chi^T(f)=\sum_{P\in\mathcal{P}_\chi} n(A)^{-1}\int_{\Pi^G(M)} \operatorname{tr}(M_P^T(\pi)_\chi\cdot I_P(\pi,f)_\chi)\,\mathrm{d}\pi,$$

with $M_P^T(\pi)_\chi$ given explicitly above in terms of the global intertwining operators. If we wanted to pursue the analogy with §8 of [1(c)], we might regard this formula as a linear combination of 'weighted characters' of f.

5. Conclusion

The results of this paper, and of [1(c)] can be summarized as an identity for the reductive group G. Namely, there is an integer $r>0$ such that for any $f\in C_c^r(G(\mathbb{A})^1)$ and any suitably regular point $T\in\mathfrak{a}_0^+$,

$$\sum_{\mathfrak{o}\in\mathcal{O}} J_{\mathfrak{o}}^T(f)=\sum_{\chi\in\mathcal{X}} J_\chi^T(f),$$

where

$$J_{\mathfrak{o}}^T(f)=\int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} k_{\mathfrak{o}}^T(x,f)\,\mathrm{d}x$$
$$=\int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} j_{\mathfrak{o}}^T(x,f)\,\mathrm{d}x,$$

and

$$J_\chi^T(f)=\int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} k_\chi^T(x,f)\,\mathrm{d}x$$
$$=\sum_P n(A)^{-1}\int_{\Pi^G(M)} \operatorname{tr}(M_P^T(\pi)_\chi\cdot I_P(\pi,f)_\chi)\,\mathrm{d}\pi.$$

Let R_{cusp} be the restriction of the representation R to $L^2_{\text{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A})^1)$. Let $\mathcal{X}(G)$ be the set of classes $\chi\in\mathcal{X}$ such that $\mathcal{P}_\chi=\{G\}$. Then R_{cusp} is the direct sum over all χ in $\mathcal{X}(G)$ of the representations R_χ. If $\chi\in\mathcal{X}(G)$ and $\pi\in\Pi(G)$, $M_G^T(\pi)_\chi$ is the identity operator. It follows from the finiteness of (3.1) that if f is in $C_c^{r_1}(G(\mathbb{A})^1)$, $R_{\text{cusp}}(f)$ is of trace class. (This fact also follows from [3, Pg. 14] and [1(c), Corollary 4.2].) Moreover if $f\in C_c^r(G(\mathbb{A})^1)$, for r as in Theorem 3.2,

$$\begin{aligned}
&\operatorname{tr} R_{\text{cusp}}(f)\\
&= \sum_{\chi\in\mathscr{X}(G)} \operatorname{tr} R_\chi(f)\\
&= \sum_{\chi\in\mathscr{X}(G)} \int_{\Pi^G(G)} \operatorname{tr}(I_G(\pi, f))\,\mathrm{d}\pi\\
&= \sum_{\chi\in\mathscr{X}(G)} J_\chi^T(f).
\end{aligned}$$

Thus

$$\operatorname{tr} R_{\text{cusp}}(f) = \sum_{\mathfrak{o}\in\mathcal{O}} J_{\mathfrak{o}}^T(f) - \sum_{\chi\in\mathscr{X}\setminus\mathscr{X}(G)} J_\chi^T(f).$$

REFERENCES

[1] J. Arthur: (a) The characters of discrete series as orbital integrals. *Inv. Math. 32* (1976) 205–261.

(b) Eisenstein series and the trace formula, in *Automorphic Forms, Representations and L-functions*, A. M. S., 1979.

(c) A trace formula for reductive groups I. *Duke Math. J. 45* (1978) 911–952.

[2] A. Borel: *Ensembles fundamentaux pour les groups arithmétiques et forms automorphes*, E. N. S., 1967.

[3] Harish-Chandra: *Automorphic forms on semisimple Lie groups.* Springer-Verlag, 1968.

[4] R. Langlands: (a) Eisenstein series, in *Algebraic groups and discontinuous subgroups.* A. M. S. 1966.

(b) *On the functional equations satisfied by Eisenstein*, series, Springer-Verlag, 1976.

(Oblatum 18-IV-1978 & 3-XI-1978)

Department of Mathematics
University of Toronto
Toronto, Canada, M551A1