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UNICITY OF THE LIE PRODUCT

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1. Statement of the result

For a C^∞ manifold M, $\mathscr{X}(M)$ denotes the linear space of C^∞ vectorfields on M. Let χ: $\mathscr{X}(M) \times \mathscr{X}(M) \to \mathscr{X}(M)$ be a bilinear operator, defined for every n dimensional manifold M. This operator is called natural if for every smooth open embedding f: $N \to M$ the following diagram commutes:

$$\begin{array}{ccc} \mathscr{X}(M) \times \mathscr{X}(M) & \xrightarrow{\chi} & \mathscr{X}(M) \\ \Big\downarrow{\scriptstyle f^* \times f^*} & & \Big\downarrow{\scriptstyle f^*} \\ \mathscr{X}(N) \times \mathscr{X}(N) & \xrightarrow{\chi} & \mathscr{X}(n) \end{array}$$

where M, N are C^∞ manifolds and f^* is the composition $\mathscr{X}(M) \xrightarrow{r} \mathscr{X}(f(N)) \xrightarrow{(f^{-1})_*} \mathscr{X}(N)$, r the restriction operator, i.e. $f^*X(x) = \mathrm{d}f(x)^{-1}(X(f(x)))$ for $X \in \mathscr{X}(M)$. In this note I shall prove that the Lie-product ($[X, Y] = X \cdot Y - Y \cdot X$ for $X, Y \in \mathscr{X}(M)$) is characterised by this property:

THEOREM: *Let χ be a bilinear natural operator in the above sense, then there exists a constant $\lambda \in \mathbb{R}$ such that $\chi(X, Y) = \lambda \cdot [X, Y]$, for all $X, Y \in \mathscr{X}(M)$.*

Palais and others [3], [4], [5] prove analogous results for operations on differential forms. Peetre [6] has a similar characterisation of linear (not bilinear) differential operators. The formal techniques are similar to those in [7]. I am indebted to my supervisor Prof. Floris Takens, for suggesting the problem and for his encouragement.

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2. The proof

The naturality of χ implies that it is a local operator, i.e. for U open in M

$$\chi(X, Y) \mid U = \chi(X|U, Y|U).$$

Furthermore if $U, V \subset M$, U, V diffeomorphic and $\chi(X, Y) = \lambda \cdot [X, Y]$ for some constant and all $X, Y \in \mathscr{X}(U)$, then also $\chi(X, Y) = \lambda \cdot [X, Y]$ for all $X, Y \in \mathscr{X}(V)$. Therefore I may assume $M = \mathbb{R}^n$. It is sufficient to prove

$$\chi(X, Y)(0) = \lambda \cdot [X, Y](0), \quad \forall X, Y \in \mathscr{X}(\mathbb{R}^n),$$

because χ commutes with translations. Of course naturality implies

$$f_*\chi(X, Y)(0) = \chi(f_*X, f_*Y)(0) \tag{1}$$

for every diffeomorphism f and every $X, Y \in \mathscr{X}(\mathbb{R}^n)$.

The main step in the proof is $\chi(X, Y)(0) = \chi(j^1X(0), j^1Y(0))(0)$. (Where, for $s \in \mathbb{N}$, $j^sX(p)$ is the polynomial vectorfield of degree s corresponding to the s-jet of X in p, that is, the first s terms of the Taylor expansion of X in p.) In lemma 1 I use naturality to prove this for polynomial vectorfields. In lemmas 2 and 3 this is shown for arbitrary smooth vectorfields, by proving $\chi(X, Y)(0) = 0$ if $X(p)$ or $Y(p)$ has in $p = 0$ a zero of sufficiently high order.

In lemma 4 I show that there exist constants $\gamma_1, \ldots, \gamma_4$ such that:

$$\chi(X, Y)(0) = \gamma_1 \cdot \nabla_X Y(0) + \gamma_2 \cdot \nabla_Y X(0) + \gamma_3 \cdot ((\operatorname{div} Y)(0)) \cdot X(0) + \gamma_4 \cdot ((\operatorname{div} X)(0)) \cdot Y(0).$$

$$\left(\text{Where } \nabla_X Y = \sum X_j \frac{\partial Y_i}{\partial x_j} \frac{\partial}{\partial x_i}, \quad \text{if } X = \sum X_i \frac{\partial}{\partial x_i}, \ Y = \sum Y_i \frac{\partial}{\partial x_i}.\right)$$

In these lemmas I use the naturality property, but only with affine diffeomorphisms f in equation (1).

Finally in the proof of the theorem one needs non-linear diffeomorphisms f in (1) to show that the constants $\gamma_1, \ldots, \gamma_4$ satisfy $\gamma_1 = -\gamma_2$, $\gamma_3 = \gamma_4 = 0$; i.e.: $\chi(X, Y) = \gamma_1[X, Y]$.

LEMMA 1: *For monomial vectorfields*

$$X(x_1, \ldots, x_n) = x_1^{\alpha_1} \ldots x_n^{\alpha_n} \frac{\partial}{\partial x_i},$$

$$Y(x_1, \ldots, x_n) = x_1^{\beta_1} \ldots x_n^{\beta_n} \frac{\partial}{\partial x_j},$$

$\chi(X, Y)(0) = 0$ if $\Sigma\, \alpha_i + \Sigma\, \beta_i \neq 1$.

PROOF: Let

$$\chi(X, Y)(0) = \chi \left(x_1^{\alpha_1} \ldots x_n^{\alpha_n} \frac{\partial}{\partial x_i}, x_1^{\beta_1} \ldots x_n^{\beta_n} \frac{\partial}{\partial x_j} \right)(0)$$
$$= c_1 \frac{\partial}{\partial x_1}\bigg|_0 + \cdots + c_n \frac{\partial}{\partial x_n}\bigg|_0.$$

Define a diffeomorphism by $\Phi(x) = \lambda \cdot x$, $\lambda \neq 0$. Then

$$\Phi_* X = \lambda^{-\Sigma\alpha_i + 1} \cdot X, \quad \Phi_* Y = \lambda^{-\Sigma\beta_i + 1} \cdot Y,$$

hence, using (1),

$$\Phi_*(\chi(X, Y)) = \chi(\Phi_* X, \Phi_* Y) = \lambda^{-\Sigma\alpha_i - \Sigma\beta_i + 2} \cdot \chi(X, Y).$$

However, the left side at 0 is equal to $\lambda \cdot \chi(X, Y)(0)$. This proves the lemma.

LEMMA 2: *For X a C^∞ vectorfield, there exists a C^∞ vectorfield $\bar{X}$ and sequences $p_s \to 0$, $q_s \to 0$ such that*: (1) $\bar{X} \mid U_s = (X - j^s X(p_s)) \mid U_s$, *$U_s$ a neighbourhood of p_s*. (2) $\bar{X} \mid V_s = 0$, *V_s a neighbourhood of q_s*.

PROOF: In fact lemmas 2 and 3 use a classical theorem of E. Borel and a technique of J. Peetre [6]. Take for example $p_s = (1/s, 0, \ldots, 0)$, $q_s = -p_s$. Define a smooth function $\alpha: \mathbf{R}^n \to \mathbf{R}$ such that

$$\alpha(x) = \begin{cases} 1 \text{ for } \|x\| \leq \frac{1}{2} \\ 0 \text{ for } \|x\| \geq 1. \end{cases}$$

and $\tilde{X}_s = X - j^s X(p_s)$, for every $s \in \mathbf{N}$. Choose $\epsilon_s > 0$ so small that

$$\frac{1}{s+1} + \epsilon_{s+1} < \frac{1}{s} - \epsilon_s$$

so that

$$\left\| \alpha \left(\frac{x - p_s}{\epsilon_s} \right) \tilde{X}_s \right\|_i < 2^{-i} \quad \text{for } i = 1, \ldots, s-1;$$

$$\|f\|_i = \sup_{\substack{|\nu|=i \\ x \in \mathbf{R}^n}} |D^\nu f(x)|.$$

Then

$$\bar{X} = \sum_s \alpha \left(\frac{x - p_s}{\epsilon_s} \right) \tilde{X}_s$$

converges and has the desired properties.

LEMMA 3: *For all $X, Y \in \mathscr{X}(\mathbf{R}^n)$*

$$\chi(X, Y)(0) = \chi(j^1X(0), j^1Y(0))(0).$$

PROOF: This lemma can also be immediately deduced from Peetre [6], because $X \to \chi(X, Y)$ is a local linear operator. But for the sake of completeness an elementary proof will be given here. Take $\bar{X}, \bar{Y}$ as in lemma 2.

First I shall prove that for all $p \in \mathbf{R}^n$:

$$(2) \qquad \chi \left(x_1^{\alpha_1} \ldots x_n^{\alpha_n} \frac{\partial}{\partial x_i}, Y \right)(p) = \chi \left(x_1^{\alpha_1} \ldots x_n^{\alpha_n} \frac{\partial}{\partial x_i}, j^1Y(p) \right)(p).$$

$\chi(Z, \bar{Y})(0) = 0$ for every $Z \in \chi(\mathbf{R}^n)$, because $\chi(Z, \bar{Y})(q_s) = 0$ and $q_s \to 0$. Furthermore for $a = (a_1, \ldots, a_n)$ $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ can be considered as a polynomial in $x_1 - a_1, \ldots, x_n - a_n$ and using lemma 1:

$$\chi \left(x_1^{\alpha_1} \ldots x_n^{\alpha_n} \frac{\partial}{\partial x_i}, j^tY(p_s) \right)(p_s) = \chi \left(x_1^{\alpha_1} \ldots x_n^{\alpha_n} \frac{\partial}{\partial x_i}, j^1Y(p_s) \right)(p_s)$$

for every $t \in \mathbf{N}$.

But, for every s, $j^1Y(p_s)$ is a linear combination of

$$\frac{\partial}{\partial x_j}, x_k \frac{\partial}{\partial x_l}, j, k, l = 1, 2, \ldots, n.$$

Since any linear operator on a finite dimensional vectorspace is continuous:

$$\chi\left(x_1^{\alpha_1}\dots x_n^{\alpha_n}\frac{\partial}{\partial x_i}, j^1Y(p_s)\right)(p_s)\to\chi\left(x_1^{\alpha_1}\dots x_n^{\alpha_n}\frac{\partial}{\partial x_i}, j^1Y(0)\right)(0),$$

for $s\to\infty$. This together implies that the limit of:

$$\chi\left(x_1^{\alpha_1}\dots x_n^{\alpha_n}\frac{\partial}{\partial x_i}, Y\right)(p_s)=\chi\left(x_1^{\alpha_1}\dots x_n^{\alpha_n}\frac{\partial}{\partial x_i}, \bar{Y}\right)(p_s)$$
$$+\chi\left(x_1^{\alpha_1}\dots x_n^{\alpha_n}\frac{\partial}{\partial x_i}, j^1Y(p_s)\right)(p_s)$$

is

$$\chi\left(x_1^{\alpha_1}\dots x_n^{\alpha_n}\frac{\partial}{\partial x_i}, j^1Y(0)\right)(0).$$

Translation gives (2) for any $p\in\mathbf{R}^n$. Therefore:

$$\chi(X, Y)(p_s)=\chi(\bar{X}, \bar{Y})(p_s)+\chi(j^1X(p_s), j^1Y(p_s))(p_s)$$

which goes to $\chi(j^1X(0), j^1Y(0))(0)$ for $s\to\infty$.

Compare this with the proof of the continuity of local operators on vectorfields in [2], XVIII. 13. problem 1.

LEMMA 4: *There are constants $\gamma_1,\dots,\gamma_4$ such that*

$$\chi(X, Y)(0)=\gamma_1\cdot\nabla_XY(0)+\gamma_2\cdot\nabla_YX(0)+\gamma_3\cdot((\operatorname{div} Y)(0))\cdot X(0)$$
$$+\gamma_4\cdot((\operatorname{div} X)(0))\cdot Y(0).$$

PROOF: Lemmas 1 and 3 imply that χ can be written as:

$$\chi(X, Y)(0)=(M_1(\mathrm{d}Y(0)))\cdot X(0)+(M_2(\mathrm{d}X(0)))\cdot Y(0)\tag{3}$$

with M_i linear maps from the $(n\times n)$-matrices to the $(n\times n)$-matrices. The lemma is proved when I show that for certain constants γ_1, γ_3:

$$M_1(A)=\gamma_1\cdot A+\gamma_3\cdot\operatorname{Tr}(A)\cdot I.\tag{4}$$

for all matrices A.

Now take $Y(0)=0$, $A=\mathrm{d}Y(0)$, $f(x)=L\cdot x$ (L a linear invertible map) and use naturality (1) in equation (3). This implies:

$$M_1(L^{-1}\cdot A\cdot L)=L^{-1}\cdot M_1(A)\cdot L.\tag{5}$$

Let L run over all diagonal and permutation matrices and deduce from (5) that there exist constants γ_1, γ_3 such that (4) is true for all diagonal matrices A. Therefore (4) is true for all diagonalisable matrices A. But every matrix is a sum of diagonalisable matrices. This proves (4).

Proof of the theorem

a) The constants in lemma 4 satisfy $\gamma_1 = -\gamma_2$, $\gamma_3 = -\gamma_4$. That is:

$$\chi(X, Y) = \gamma_1 \cdot [X, Y] + \gamma_3 \cdot ((\operatorname{div} Y)X - (\operatorname{div} X)Y). \tag{6}$$

To show this, it is now sufficient to prove χ is antisymmetric, i.e. that $\chi(X, X) = 0 \ \forall X \in \mathscr{X}(\mathbb{R}^n)$.

If $X(0) = 0$, then lemmas 1 and 3 give $\chi(X, X)(0) = 0$.

If $X(0) \neq 0$ the flow-box theorem [1] gives a local diffeomorphism $\varphi: (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that

$$\varphi_* X(0) = \frac{\partial}{\partial x_1}: \quad \varphi_* \chi(X, X)(0) = \chi\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1}\right)(0) = 0$$

and again $\chi(X, X)(0) = 0$.

b) For $n = 1$: $(\operatorname{div} Y)X - (\operatorname{div} X)Y = [X, Y]$ and we are done.

c) If $n \geq 2$: The operator $(X, Y) \to (\operatorname{div} Y)X - (\operatorname{div} X)Y$ does not commute with every diffeomorphism φ and therefore $\gamma_3 = 0$ in equation (6). To see this, take

$$X = \sum_{i=1}^{\alpha} X_i(x_1, \ldots, x_\alpha) \frac{\partial}{\partial x_i}, \ Y = \sum_{i=\alpha+1}^{n} Y_i(x_{\alpha+1}, \ldots, x_n) \frac{\partial}{\partial x_i}$$

and a (non-measure preserving) local diffeomorphism $\tilde{\varphi}_1: (\mathbb{R}^\alpha, 0) \to (\mathbb{R}^\alpha, 0)$ such that $\operatorname{div} X(0) = \operatorname{div} Y(0) = 0$, $Y(0) \neq 0$ and $\operatorname{div}((\varphi_1)_* X) \neq 0$.

Define $\varphi(x_1, \ldots, x_n) = (\tilde{\varphi}_1(x_1, \ldots, x_\alpha), x_{\alpha+1}, \ldots, x_n)$, then $\operatorname{div}(\varphi_* X)(0) \neq 0$, $\operatorname{div}(\varphi_* Y) = 0$, that is $(\operatorname{div} Y)X - (\operatorname{div} X)Y$ does not commute with this φ.

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