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TOWARDS THE JANTZEN CONJECTURE II

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Abstract

A recently developed additivity principle for Goldie rank is combined with the results of the first paper of this series to establish a very slightly weaker form of the Jantzen conjecture for the primitive spectrum of a complex simple Lie algebra of type A_n. Precise information on the primitive spectra in other simple Lie algebras and on the Goldie ranks of the associated quotient algebras is also obtained.

1. Introduction

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $U(\mathfrak{g})$ its enveloping algebra and Prim $U(\mathfrak{g})$ the set of primitive ideals of $U(\mathfrak{g})$. The classification of Prim $U(\mathfrak{g})$ for $\mathfrak{g}$ simple of type A_{n-1} (Cartan notation) for $n \leqslant 6$ (and several other low rank cases) was given by Borho and Jantzen [2], [3], and from their results Jantzen [1], 5.9 guessed its solution for general n. In [8], 8.2, 11.8, we suggested what form this conjecture should take for an arbitrary semisimple Lie algebra and indicated how this should be related to the Goldie ranks of the quotient algebras $U(\mathfrak{g})/I: I \in \text{Prim } U(\mathfrak{g})$. In the present work we show how the results of [8] combined with a specially developed additivity principle for Goldie rank [9] lead to a proof of a very slightly weaker form of Jantzen's conjecture and also to some quite precise results in the general case. A further result which obtains from 5.3 and [8], 7.7 is that for regular central characters the anni-

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hilators of the simple subquotients of a given principal series module for $SL(n, \mathbb{C})$ are pairwise disjoint. The notation is that of [8].

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2. Multiplicities

2.1 Fix a triangular decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ for $\mathfrak{g}$, set $\mathfrak{b} := \mathfrak{n} \oplus \mathfrak{h}$ and for each $\lambda \in \mathfrak{h}^*$, let $M(\lambda) := U(\mathfrak{g}) \otimes_{U\mathfrak{b}} \mathbb{C}_{\lambda-\rho}$ denote the associated Verma module [8], 3.1. Given M a finitely generated left $U(\mathfrak{g})$ module define the Gelfand–Kirillov dimension $\mathrm{d}(M)$ (resp. multiplicity $e(M)$) of M as in [8], 2.1. Given left $U(\mathfrak{g})$ modules, M, N define $\mathrm{Hom}_{\mathbb{C}}(M, N)$ *as a left* $U := U(\mathfrak{g}) \otimes U(\mathfrak{g})$ module ([8], 3.2) and let $L(M, N)$ denote the U submodule of ad $\mathfrak{g}$ finite elements. (We recall in particular that each $X \in \mathfrak{n}^-$ defines a locally nilpotent derivation ad X of $L(M, N)$). Set $n = \dim \mathfrak{n}^-$ and let r be the smallest positive integer such that $(\mathrm{ad}\, \mathfrak{n}^-)^r \mathfrak{g} = 0$.

LEMMA: *Fix $\lambda, \mu \in \mathfrak{h}^*$ and let M (resp. N) be a subquotient of $M(\lambda)$ (resp. $M(\mu)$). Suppose* $\mathrm{d}(L(M, N)) \geq \mathrm{d}(M) + \mathrm{d}(N)$. *Then equality holds and*

$$\mathrm{e}(L(M, N)) \leq \binom{2n}{n} r^{2n} \mathrm{e}(M)\mathrm{e}(N).$$

This is very modest generalization of [8], 6.4 and for which the proof needs only the following trivial changes. Replace $\mathfrak{m}^-$ by $\mathfrak{n}^-$ and $V_{B'}(\lambda)$ (resp. $V_{B'}(\mu)$) by any $\mathfrak{b}$ stable finite dimensional generating subspace M^0 (resp. N^0) of M (resp. N).

2.2 Relative to the given triangular decomposition of $\mathfrak{g}$, let R (resp. R^+) denote the set of all non-zero (resp. positive) roots and $B \subset R^+$ the set of simple roots. Let s denote the largest coefficient of a simple root that can occur in a positive root and set $t = sn : n = \dim \mathfrak{n}^-$. Given V a locally finite $\mathfrak{h}$ module, let $V_\mu : \mu \in \mathfrak{h}^*$ denote its weight subspace of weight μ.

LEMMA: *Let M be a simple subquotient of a Verma module and set* $I = \mathrm{Ann}\, M$. *Then*

(i) $\mathrm{d}(U(\mathfrak{g})/I) = 2\, \mathrm{d}(M)$.

(ii) $(\mathrm{e}(M))^2 \leq (1 + t)^{2n}\, \mathrm{e}(U(\mathfrak{g})/I)$.

The first assertion is just [7], 2.7 and the second a refinement of it.

Let T (resp. T_-, E) denote the image of $\mathfrak{n}\oplus\mathbb{C}$ (resp. $\mathfrak{n}^-\oplus\mathbb{C}$, $\mathfrak{g}\oplus\mathbb{C}$) in $U(\mathfrak{n})/(I\cap U(\mathfrak{n}))$ (resp. $U(\mathfrak{n}^-)/(I\cap U(\mathfrak{n}^-))$, $U(\mathfrak{g})/I$). Let e denote the canonical generator of M which we suppose has weight λ and for all $k\in\mathbb{N}$ set $M^k := T_-^k\mathrm{e} = E^k\mathrm{e}$. Define an ordering $\geq$ on $\mathbb{Z}B$ through $\mu\geq\nu$ given $\mu-\nu\in\mathbb{N}B$. For each $k\in\mathbb{N}$, M^k is a finite direct sum of its weight subspaces $(M^k)_{\lambda-\nu}$ of weight $\lambda-\nu$: $\nu\in\mathbb{Z}B$ and we set $\Omega_k=\{\nu\in\mathbb{Z}B: (M^k)_{\lambda-\nu}\neq 0\}$. Choose a maximal element $\mu\in\Omega_k$. Since M is a simple $U(\mathfrak{g})$ module there exists for each $m\in(M^k)_{\lambda-\mu}$ an element $a\in U(\mathfrak{g})$ such that $am=\mathrm{e}$. From the decomposition $U(\mathfrak{g})=U(\mathfrak{n}^-)\otimes U(\mathfrak{n})\otimes U(\mathfrak{h})$ and the fact that M^0 is a one dimensional highest weight space for M of weight λ we can suppose that $a\in U(\mathfrak{n})_\mu$. Then for all $\nu\in\Omega_k\backslash\{\mu\}$ we have $a(M^k)_{\lambda-\nu}\subset M_{\lambda+\mu-\nu}=0$ where the last step holds because $\mu-\nu\in\mathbb{N}B$. Now M^k is $\mathfrak{b}$ stable so by induction on $\geq$ it follows easily that $\mathrm{Hom}(M^k, M^0)\subset\bigoplus_{\nu\in\Omega_k} U(\mathfrak{n})_\nu|_{M^k}$. Conversely from the relation $M^k = T_-^k\mathrm{e}$, we obtain $\mathrm{Hom}(M^0, M^k)\subset T_-^k|_{M^0}$ and

$$\Omega_k\subset\left\{\sum_{\alpha\in R^+} r_\alpha\alpha : r_\alpha\in\{0,1,2,\ldots,k\}\right\}.$$

For each $\nu\in\mathbb{N}B$, $\alpha\in B$, let r_α^ν denote the coefficient of α in ν. Recalling the definition of t, it follows for each $\nu\in\Omega_k$ that

$$\sum_{\alpha\in B} r_\alpha^\nu\leq kt.$$

On the other hand if $a\in S(\mathfrak{n})$ is homogeneous of degree m and of weight ν, then

$$\sum_{\alpha\in B} r_\alpha^\nu\geq m.$$

Hence $\mathrm{Hom}(M^k, M^0)\subset T^{kt}|_{M^k}$ and so $\mathrm{Hom}(M^k, M^k)\subset T_-^k T^{kt}|_{M^k}\subset E^{k(1+t)}|_{M^k}$, which gives

$$\dim E^{k(t+1)}\geq(\dim M^k)^2=\left(\frac{\mathrm{e}(M)}{\mathrm{d}(M)!}\right)^2 k^{2\,\mathrm{d}(M)}+0(k^{2\,\mathrm{d}(M)-1}).$$

Yet by definition

$$\dim E^{k(1+t)}=\frac{\mathrm{e}(U(\mathfrak{g})/I)(k(1+t))^{2\,\mathrm{d}(M)}}{(2\,\mathrm{d}(M))!}+0(k^{2\,\mathrm{d}(M)-1}).$$

Equating powers of k gives (ii).

2.3 Let M be the subquotient of a Verma module. By [2], 3.16, there exists a positive integer u depending only on $\mathfrak{g}$ such that the length of M is less than u. Define n, r, t as in 2.2.

PROPOSITION: *Let M be a subquotient of a Verma module. Set $I = \text{Ann}\, M$. Then*

(i) $\mathrm{d}(U(\mathfrak{g})/I) = \mathrm{d}(L(M, M)) = 2\,\mathrm{d}(M)$.

(ii) $\mathrm{e}(L(M, M)) \leq (4rtu)^{2n}\, \mathrm{e}(U(\mathfrak{g})/I)$.

Since $L(M, M)$ contains $U(\mathfrak{g})/I$ as a $U(\mathfrak{g})$ submodule, we obtain (i) from 2.2 (i) and the first part of 2.1. From the definition of u and say [8], 2.2 we can choose a simple subquotient L of M satisfying $\mathrm{d}(L) = \mathrm{d}(M)$ and $u\,\mathrm{d}(L) \geq \mathrm{d}(M)$. Since $I \subset \text{Ann}\, L$, we obtain (ii) from 2.1 and 2.2 (ii).

3. Goldie Rank

3.1 Let B be a prime (left and right) Noetherian ring. Recall that the Goldie rank rk B is just the maximum number of direct summands of left (or right) ideals of B. Given K a left ideal of B let rk K denote the maximum number of direct summands of left ideals of B in K. Given e an idempotent of B we remark that eBe is a prime Noetherian subring of B and rk eBe = rk Be.

3.2 Let $\mathscr{A}, \mathscr{B}$ be simple Artinian rings with $\mathscr{A}$ a subring of $\mathscr{B}$ and containing the identity 1 of $\mathscr{B}$. Then $z := (\text{rk}\, \mathscr{B})/(\text{rk}\, \mathscr{A}) \in \mathbb{N}^+$ (see [5], Prop. 2, p. 137). Given $b \in \mathscr{B}$, set $\ell(b) = \{a \in \mathscr{A}: ab = 0\}$. The following is a straightforward exercise in idempotent manipulations using [5], Thm. 2, p. 47 and Prop. 4, p. 51.

LEMMA: *There exists a set $\{h_1, h_2, \ldots, h_z\}$ of pairwise orthogonal idempotents in $\mathscr{B}$ satisfying $\ell(h_i) = 0$ for all $i = 1, 2, \ldots, z$.*

3.3 Let A, B be prime Noetherian rings with A a subring of B. Assume that the set S of regular elements of A is a subset of the regular elements of B and is an Ore subset for both A and B and such that $S^{-1}A$ and $S^{-1}B$ are simple, Artinian. Suppose further that A and B are finitely generated left $U(\mathfrak{g})$ modules.

COROLLARY: *If* $\mathrm{d}(A) = \mathrm{d}(B)$, *then*

$$(\text{rk}\, B)/(\text{rk}\, A) \leq \mathrm{e}(B)/\mathrm{e}(A).$$

Apply 3.2 with $\mathscr{A} = S^{-1}A$, $\mathscr{B} = S^{-1}B$. Choose $s \in S$ such that $h_i s \in B$ for all i. Then $Ah_1 + Ah_2 \dotplus \cdots + Ah_z$ is a direct sum of $U(\mathfrak{g})$ submodules of B each isomorphic to A. By say [8], 2.2 this gives the required assertion.

4. The additivity principle

4.1 Let F be a commutative field, $U(\mathfrak{a})$ the enveloping algebra of a finite dimensional F-Lie algebra $\mathfrak{a}$. Let A be a quotient of $U(\mathfrak{a})$ and assume that A embeds in a prime ring B with identity $1 \in A$. Assume that B is finitely generated as a left and a right $U(\mathfrak{a})$ module and that each $b \in B$ is locally ad $\mathfrak{a}$ finite (c.f. [8], 2.3). Let $\{P_1, P_2, \ldots, P_r\}$ be the set of minimal primes of A. By [9], 3.9 there exist $z_i^A \in \mathbb{N}^+$ such that

$$\sum_{i=1}^{r} z_i^A \operatorname{rk}(A/P_i) = \operatorname{rk} B.$$

Here we obtain further information on the z_i^A.

4.2 Let S be the set of regular elements of A. By [9], 3.5, S is contained in the set of regular elements of B and by [9], 3.7 it is an Ore subset in both A and B and $\mathcal{A} := S^{-1}A$ is Artinian and $\mathcal{B} := S^{-1}B$ simple Artinian. Given C a subring of $\mathcal{B}$ and T a subset of $\mathcal{B}$ we set $\ell_C(T) = \{c \in C: cT = 0\}$, $r_C(T) = \{c \in C: Tc = 0\}$. When $C = A$ we drop the subscript.

Given P a minimal prime of A, then $P \cap S = \emptyset$ and so $S^{-1}P = PS^{-1}$ is a (minimal) prime of $\mathcal{A}$ (see for example [10], 2.5–2.10). Set $B_1 = \ell_B(P)$, $B_2 = r_B(B_1)$, $B_3 = \{b \in B: B_1 b \subset B_1\}$. These are all $U(\mathfrak{a}) \otimes U(\mathfrak{a})$ submodules of B. Also $B_2 B \subset B_2$, $B_1 B_2 = 0$, $B_3 \supset B_1$ and so $B_1/(B_1 \cap B_2)$, B_3/B_1 have natural F-algebra structures. Again $BB_1 \subset B_1$ and so $S^{-1}B_1$ is a left ideal of the simple Artinian ring $\mathcal{B}$ and so we may write $S^{-1}B_1 = \mathcal{B}e$ for some idempotent $e \in \mathcal{B}$. Again $B_3 \supset A$ and $A \cap B_1 = \ell(P)$ and so $A/\ell(P)$ embeds in B_3/B_1.

LEMMA:

(i) $\ell_{\mathcal{B}}(S^{-1}P) = \mathcal{B}e$.

(ii) $S^{-1}B_j \cap B = B_j$: $j = 1, 2, 3$.

(iii) $S^{-1}B_j = B_j S^{-1}$: $j = 1, 2, 3$.

(iv) $B_1/(B_1 \cap B_2)$ *(resp.* B_3/B_1*) is a prime Noetherian ring and* Fract $B_1/(B_1 \cap B_2)$ *(resp.* B_3/B_1*) is naturally isomorphic to* $e\mathcal{B}e$ *(resp.* $(1-e)\mathcal{B}(1-e)$*). In particular* $\operatorname{rk} B_1/(B_1 \cap B_2) + \operatorname{rk} B_3/B_1 = \operatorname{rk} B$.

(ii) is clear for $j = 1$. Again $S^{-1}B_1 = S^{-1}\ell_B(P) = \ell_{\mathcal{B}}(P) = \ell_{\mathcal{B}}(PS^{-1}) = \ell_{\mathcal{B}}(S^{-1}P)$. This gives (i) and the inclusion $S^{-1}B_1 \supset B_1 S^{-1}$. Conversely given $s \in S$, $b \in B_1$ we can choose $t \in S$, $c \in B$ such that $s^{-1}b = ct^{-1}$. Then $ct^{-1}P = s^{-1}bP = 0$ and so $c \in B_1$. This establishes (iii) for $j = 1$. Similarly (using (ii) for $j = 1$) we obtain $S^{-1}B_2 = B_2 S^{-1} = (1-e)\mathcal{B}$ and $S^{-1}B_3 = B_3 S^{-1} = \mathcal{B}e + (1-e)\mathcal{B}(1-e)$. This gives (ii) and (iii). Given $x \in S^{-1}B_1 \cap S^{-1}B_2$, then by the Ore condition there exists $s \in S$ such that $sx \in B \cap S^{-1}B_1 \cap S^{-1}B_2 = B_1 \cap B_2$ by (ii). A similar computation on the right and (iii) gives $S^{-1}(B_1 \cap B_2) = S^{-1}B_1 \cap S^{-1}B_2 = (1-e)\mathscr{B}e = B_1S^{-1} \cap B_2S^{-1} = (B_1 \cap B_2)S^{-1}$.

Given $s \in S$, $b + (B_1 \cap B_2) \in B_1/(B_1 \cap B_2)$ (resp. $b + B_1 \in B_3/B_1$) such that $sb \in B_1 \cap B_2$ (resp. $sb \in B_1$) then $b \in S^{-1}B_1 \cap S^{-1}B_2 \cap B = B_1 \cap B_2$ (resp. $b \in S^{-1}B_1 \cap B = B_1$). A similar result holds for right multiplication and combined with our previous observations establishes (iv).

4.3 Retain the notation and hypotheses of 4.1 and 4.2. Take an ordering of the minimal primes of A so that $\ell(P_1) \neq 0$ (this is possible by say [9], 2.4) and take $P = P_1$ in 4.2. Recall 4.2 (iv).

PROPOSITION:

(i) $z_1^A - z_1^{A/\ell(P_1)} = \operatorname{rk} B_1/(B_1 \cap B_2)/\operatorname{rk} A/P_1$.

(ii) $z_i^A - z_i^{A/\ell(P_1)} = 0$: $i = 2, 3, \ldots, r$.

Let N denote the nilradical of A and recall ([9], 2.7) that $\mathscr{N} := S^{-1}N$ is the nilradical of $\mathscr{A}$. Again (c.f. [9], 3.9) $\{Q_i := S^{-1}P_i : i = 1, 2, \ldots, r\}$ is the set of primes of the Artinian ring $\mathscr{A}$ and Fract $A/P_i = \mathscr{A}/Q_i$, up to isomorphism. Further recall ([9], 3.8) that there exists a set $\{e_1, e_2, \ldots, e_r\}$ of pairwise orthogonal idempotents for $\mathscr{A}$ satisfying $\Sigma e_i = 1$ and such that $Q_i = (1 - e_i)\mathscr{A} + \mathscr{N}$. Taking $i = 1$ and applying 4.2 (i) we obtain $e = ee_1$. This gives $ee_i = ee_1e_i = 0$, for all $i = 2, 3, \ldots, r$.

Now recall that $A/\ell(P_1)$ identifies as a subalgebra of B_3/B_1. By 4.2 (iv) we may apply [9], 3.8 (i) to both $A/\ell(P_1)$ and A. This gives for $i > 1$

$$\begin{aligned} z_i^{A/\ell(P_1)} \operatorname{rk} A/P_i &= \operatorname{rk}((1-e)\mathscr{B}(1-e)e_i) \text{ (considered as a left } (1-e)\mathscr{B}(1-e) \text{ module)} \\ &= \operatorname{rk}((\mathscr{B}(1-e)e_i) \qquad \text{(considered as a left } \mathscr{B} \text{ module)} \\ &= \operatorname{rk} \mathscr{B}e_i = z_i^A \operatorname{rk} A/P_i. \end{aligned}$$

This gives (ii). (i) follows from (ii), 4.2 (iv) and [9], 3.8 (ii).

4.4 Retain the notation and hypotheses of 4.3. Let M be a faithful B module. Recall (c.f. [9], 2.4) that some product of the P_i must vanish and let $\mathscr{L}(A)$ be the length of a shortest product. Given P a minimal prime of A such that $\ell(P) \neq 0$, then $\mathscr{L}(A/\ell(P)) < \mathscr{L}(A)$. Consider M as a $U(\mathfrak{a})$ module and B as a $U(\mathfrak{a}) \otimes U(\mathfrak{a})$ module.

THEOREM: *M admits a normal series $M = M_1 \supsetneq M_2 \supsetneq \cdots \supsetneq M_{t+1} = 0$ such that for all $s = 1, 2, \ldots, t$ and all $i = 1, 2, \ldots, r$*

(i) Ann L_s, where $L_s := M_s/M_{s+1}$, *is a minimal prime of A.*

(ii) *B admits a subquotient $B_{(s)}$ naturally isomorphic to a prime Noetherian subring of* $\mathrm{Hom}_F(L_s, L_s)$.

(iii)

$$z_i^A = \sum \{\mathrm{rk}\, B_{(s)}\colon \mathrm{Ann}\, L_s = P_i\}/\mathrm{rk}\, A/P_i.$$

Choose a minimal prime P of A such that $\ell(P) \neq 0$ and set $M_2 = \{m \in M\colon B_1 m = 0\}$. Then $M_2 \supset PM$ and since M is a faithful B module it follows that $B_1 = \{b \in M\colon bM_2 = 0\}$. Now $B_1 B_2 M = 0$, so $B_2 M \subset M_2$ and since M is a faithful B module it follows that $B_2 = \{b \in B\colon bM \subset M_2\}$. Hence $B_{(1)} := B_1/(B_1 \cap B_2)$ (which is a subquotient of B) identifies with a subring of $\mathrm{Hom}_F(L_1, L_1)$. By 4.2 (iv) it is prime Noetherian.

Now $\ell(P) M_2 \subset B_1 M_2 = 0$ and so $\ell(P)$ Ann $L_1 = 0$. Since $\ell(P) \neq 0$ by hypothesis, it follows from [9], 2.2 (ii), 2.6 (i), 3.1, 3.6, that $\mathrm{d}(A/\mathrm{Ann}\, L_1) = \mathrm{d}(A) = \mathrm{d}(A/P)$. Yet Ann $L_1 \supset P$ and so by [9], 2.5 (i) equality must hold. Again since M is a faithful B module, it follows that $B_3 M_2 \subset M_2$ and so B_3/B_1 (which is a subquotient of B) identifies with a subring of $\mathrm{Hom}_F(M_2, M_2)$. By 4.2 (iv) it is prime Noetherian and we recall that it contains $A/\ell(P)$ as a subring. Hence (i) and (ii) obtain by induction on $\mathscr{L}(A)$ and then (iii) follows from 4.3.

5. Main theorem

5.1 Let W be the group generated by the $s_\alpha\colon \alpha \in R$. Given $B' \subset B$, set $R' = \mathbb{Z}B' \cap R$ and define $p_{B'} \in S(\mathfrak{h})$ as in [8], 6.6. (Up to a scalar, $p_{B'}$ is the product of the roots in $R' \cap R^+$ and $2 \deg p_{B'} = \mathrm{card}\, R'$.) Let $P_{\mathscr{B}'}$ be the simple W submodule of $S(\mathfrak{h})_m$ generated by $p_{B'}$, with $m = \deg p_{B'}$ (c.f. [8], 8.6) and let Ω_R (resp. Ω_R^m) denote the subset of $\hat{W}$ defined by the $P_{\mathscr{B}'}\colon B' \subset B$ (resp. $P_{\mathscr{B}'} \subset S(\mathfrak{h})_m\colon B' \subset B$).

5.2 Given $\lambda \in \mathfrak{h}^*$ define B_λ, W_λ as in [8], 3.3. Recall that W_λ is again a Weyl group so we may define corresponding subsets $\Omega_{R,\lambda}$, $\Omega_{R,\lambda}^m$ of $\hat{W}_\lambda$. Let $L(\lambda)$ denote the unique simple subquotient of $M(\lambda)$ and set $I(\lambda) = \mathrm{Ann}\, L(\lambda)$. Set $\hat{\lambda} = W\lambda$, $\mathscr{X}_{\hat{\lambda}} = \{I(\mu)\colon \mu \in W\lambda\}$ and for each $m \in \mathbb{N}$, set $\mathscr{X}_{\hat{\lambda}}^m = \{I \in \mathscr{X}_{\hat{\lambda}}\colon \mathrm{d}(U(\mathfrak{g})/I) = \mathrm{card}\, R - 2m\}$. Call λ regular if $(\lambda, \alpha) \neq 0$, for all $\alpha \in R$.

THEOREM: *Assume $\lambda \in \mathfrak{h}^*$ regular and that there exists $w \in W$ such that $wB_\lambda \subset B$. Then for all $m \in \mathbb{N}$ one has*

$$\text{card}\, \mathscr{X}_{\hat{\lambda}}^m \geq \sum_{\sigma \in \Omega_{R,\lambda}^m} \dim \sigma.$$

By [6], 4.2 we can assume $B_\lambda \subset B$ without loss of generality. Then for each $B' \subset B_\lambda$, let $M := M_{B'}(\lambda)$ be the induced module defined by B', λ and set $I = I_{B'}(\lambda) := \text{Ann}\, M_{B'}(\lambda)$. By [8], 4.3 (ii), 4.8, 5.10, it follows that $L(M, M)$ is a prime Noetherian ring of locally ad $\mathfrak{g}$ finite elements, is finitely generated as a left and as a right $U(\mathfrak{g})$ module and $\text{rk}\, L(M, M) = p_{B'}(\lambda)$. Set $m = \deg p_{B'}$. We apply 4.4 with $A = U(\mathfrak{g})/I$, $B = L(M, M)$. Let $\{P_1, P_2, \ldots, P_r\}$ denote the set of minimal primes of A. We have $P_i \in \mathscr{X}_{\hat{\lambda}}$ and by 2.3 (i) and [9], 3.9 (i) that $P_i \in \mathscr{X}_{\hat{\lambda}}^m$, for all i. By [9], 3.9 (ii) there exist $z_i \in \mathbb{N}^+$ such that

$$(*) \qquad \sum_{i=1}^{r} z_i \,\text{rk}\, A/P_i = p_{B'}(\lambda).$$

By 2.3 (ii), 3.3, 4.4 (iii), [2], 3.16, there exists a positive integer $c(\mathfrak{g})$ depending only on $\mathfrak{g}$ (in particular independent of λ) such that $z_i \leq c(\mathfrak{g})$. Then by [8], 6.5, it follows that the assertion of the theorem holds for at least some $\lambda \in \mathfrak{h}^*$. By [2], 2.12 it must then hold for all λ regular.

Remarks: The inequality can be strict. For example take $\mathfrak{g}$ simple of type D_4 with $m = 7$ (see [8], 11.7). Using [2], 2.14, the technical assumption $wB_\lambda \subset B$ can be weakened: for example it is enough that every strict subset of B_λ can be conjugated into B. Finally for λ non-regular recall [2], 2.12.

5.3 Assume that the simple factors of $\mathfrak{g}$ are all of type A_n. Let Σ_λ denote the set of involutions of W_λ.

Corollary: *For all $\lambda \in \mathfrak{h}^*$ regular one has*

$$\text{card}\, \mathscr{X}_{\hat{\lambda}} = \text{card}\, \Sigma_\lambda.$$

It suffices to recall that $\Omega_{R,\lambda} = \hat{W}_\lambda$ in this case (c.f. [8], 8.4) and to apply [4], Prop. 9 to 5.2.

Remarks: Note in particular that this establishes [8], 10.3. This gives a natural partition of $\mathscr{X}_{\hat{\lambda}}$ into disjoint subsets $(\mathscr{X}_{\hat{\lambda}})_\sigma$: $\sigma \in \hat{W}_\lambda$ satisfying $\text{card}(\mathscr{X}_{\hat{\lambda}})_\sigma = \dim \sigma$ for λ regular. Again for λ regular and each $I \in (\mathscr{X}_{\hat{\lambda}})_\sigma$ there exist exactly $\dim \sigma$ elements of W_λ such that $I = I(w\lambda)$: $w \in W_\lambda$ (fixing say $-\lambda$ dominant). Thus to establish the Jantzen conjecture [1], 5.9, it remains to show that for each $I \in (\mathscr{X}_{\hat{\lambda}})_\sigma$ the zero variety of $\text{gr}\, I$ is the closure of the appropriate Richardson orbit [8], 8.2. We remark that this orbit can also be viewed as the image of σ under the Springer map (see [8], 11.8). For this it suffices to establish [8], 9.1 – that is to show that $\operatorname{gr} J$ is primary for every induced ideal J. So far we have only been able to show that the dimension of the zero variety of $\operatorname{gr} I$ has the appropriate value [7], 4.1.

5.4 For type A_n the lower bound 5.2 on card $\mathscr{X}_{\hat{\lambda}}^m$ is in fact its exact value. This is very nearly true in general as may be seen by examining the upper bound implied by [6], 5.1 and [7], 3.1. Even aside from the combinatorial questions involving the Weyl group, we do not yet have enough information to give as complete a solution as in type A_n. The main difficulty arises from the presence of non-Richardson-like orbits. Nevertheless we can extend the low rank computations of Borho and Jantzen (c.f. [7], Sect. 5 and [8], Sect. 11.8) to type B_4 (or C_4). In this case the upper and lower bounds for card $\mathscr{X}_{\hat{\lambda}}^m$: $-\lambda \in P(R)^{++}$, coincide and give card $\mathscr{X}_{\hat{\lambda}} = 50$: $-\lambda \in P(R)^{++}$. In type D_4 the upper and lower bounds also coincide for m values corresponding to Richardson orbits and the total number of such ideals is 32. One further obtains card $\mathscr{X}_{\lambda}^4 = 2$ or α, card $\mathscr{X}_{\lambda}^7 = 4$: $-\lambda \in P(R)^{++}$, where the latter corresponds to the almost maximal ideals (c.f. [2], 4.5 and [8], 11.8). By [3] the total number of ideals is 36. Finally we remark that the upper bound on the z_i in (*) implies that they are constant in some Zariski dense subset of $P(R)^{++}$. Consequently if the upper and lower bounds coincide for a given value of m, then by (*) the Goldie ranks of the corresponding quotient algebras are defined in this subset by a basis of $\bigoplus\{P_{\mathscr{B}'}: B' \subset B: \operatorname{card} R' = 2m\}$. This establishes a weak version of [8], 11.1.

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