COMPOSITIO MATHEMATICA

ROBERT M. HARDT
CLINT G. MCCRORY

Steenrod operations in subanalytic homology

Compositio Mathematica, tome 39, n^{o} 3 (1979), p. 333-371
<http://www.numdam.org/item?id=CM_1979__39_3_333_0>

COMPOSITIO MATHEMATICA, Vol. 39, Fasc. 3, 1979, pag. 333–371

STEENROD OPERATIONS IN SUBANALYTIC HOMOLOGY

Robert M. Hardt* and Clint G. McCrory**

Contents

§1. Introduction

We present here an investigation of the chain-level geometry of the Steenrod reduced squaring operations, including geometric proofs of their existence and uniqueness. These operations are constructed in *homology*, using the topology of mappings of cycles into Euclidean spaces. Roughly speaking, the i-th Steenrod homology operation assigns to a k cycle X a subcycle of double points for a map of the support of X to $\mathbf{R}^{k+i}$ or, alternately, a subcycle of branching points for a map of the support of X to $\mathbf{R}^{k+i-1}$. This operation is uniquely characterized in 4.2 as being a natural transformation of degree i on homology which satisfies a self-intersection axiom (4.1). We work throughout in the category of subanalytic sets ([5], [9], [10], [11], [12], [14], [15]), which includes both polyhedra and real analytic varieties. A similar construction for polyhedra occurs in [18].

The necessary preliminaries on the subanalytic category are presented in §2. Our principal geometric object, a **compact mod 2** **k-chain in $\mathbf{R}^m$**, X, is uniquely determined by its supporting set, $\underline{X}$,

*Research partially supported by NSF grant MPS 71-03036 A06.
**Research partially supported by NSF grant MCS 76-09817.

0010-437X/79/06/0333-39$00.20/0

which is a compact (purely) k dimensional subanalytic subset of $\mathbf{R}^m$. The boundary, sum, push-forward, and restriction of mod 2 k-chains are introduced, and various formulas involving these four constructions are discussed in 2.5. The resulting **subanalytic homology theory** $\mathbf{H}_*$ is naturally equivalent to ordinary singular homology theory on the subanalytic category (2.6). The theory of slicing ([6], [7], [8]) leads to an intersection product for mod 2 chains in Euclidean space.

A stable homology operation is a natural transformation on $\mathbf{H}_*$ which commutes with suspension. The crucial observation of §3 is that such an operation of degree i is equivalent to a function $\mathbf{G}$, called a **geometric homology operation**, that assigns to each compact mod 2 k-chain X, a homology class $\mathbf{G}(X) \in \mathbf{H}_{k-i}(\underline{X}, \underline{\partial X}; \mathbf{Z}_2)$ which is natural with respect to subanalytic homeomorphism, suspension, and restriction to certain open subanalytic subsets of $\underline{X}$.

In §4 is established the *uniqueness* of the Steenrod homology operations $\mathbf{\Phi}^i$. (This homology proof uses neither classifying spaces nor equivariant homology.) The self-intersection axiom for $\mathbf{\Phi}^i$ implies that $\mathbf{\Phi}^0$ is the identity, $\mathbf{\Phi}^1$ is the Bockstein operation, $\mathbf{\Phi}^i$ is stable, and $\mathbf{\Phi}^i(\alpha) = 0$ whenever $\alpha \in \mathbf{H}_k(A)$, where A is a subanalytic subset of $\mathbf{R}^m$ with $m \leq k + i$.

The *existence* of the Steenrod homology operations is established in either §6 or §7, which may be read independently. Both involve the **double point pair chain** $\mathscr{D}(f)$, which is defined in §5 for any subanalytic map $f : \underline{X} \to \mathbf{R}^{k+i}$ of a compact mod 2 k-chain X in $\mathbf{R}^m$ such that

$$\dim \underline{X}^2 \cap \{(x, y) : x \neq y, f(x) = f(y)\} \leq k - i \quad \text{and}$$
$$\dim(\underline{X} \times \underline{\partial X}) \cap \{(x, y) : x \neq y, f(x) = f(y)\} \leq k - i - 1.$$

These conditions are satisfied, for example, by the restriction to $\underline{X}$ of almost all linear maps from $\mathbf{R}^m$ to $\mathbf{R}^{k+i}$ (2.3). (They are also satisfied by many topologically "unstable" maps (8.1(3)(6)(7)(8)(9)).) The compact mod 2 $(k - i)$-chain $\mathscr{D}(f)$, which is defined by slicing, has support contained in $\operatorname{Clos}[\underline{X}^2 \cap \{(x, y) : x \neq y, f(x) = f(y)\}]$. The projection of $\underline{X}^2$ onto the first factor pushes $\mathscr{D}(f)$ forward to the **double point chain** $D(f)$, a compact mod 2 $(k - i)$-chain in $\mathbf{R}^m$. Thus, if f is an injection, then $D(f) = 0$, by definition. Alternately, taking the boundary of the image of $\mathscr{D}(f)$ in the symmetric product $\sigma(\underline{X}^2)$ leads to the **branching point chain** $B(f)$, a compact mod 2 $(k - i - 1)$-chain in $\mathbf{R}^m$. If f is locally an injection, then $B(f) = 0$, by definition. The homology class $\mathbf{B}^{i+1}(X)$ of $B(f)$ in $\mathbf{H}_{k-i-1}(\underline{X}, \underline{\partial X}; \mathbf{Z}_2)$ is independent of f, and $\mathbf{B}^{i+1}$ is a geometric homology operation whose corresponding stable homology operation satisfies the self-intersection axiom 4.1 for $i \geq 0$. Similar facts are true for $D(f)$, $\mathbf{D}^i(X)$, and $i \geq 1$ provided f satisfies the additional condition: $f^{-1}(\mathbf{R}_+^{k+i}) = \underline{X} \sim \partial \underline{X}$ (hence $f(\partial \underline{X}) \subset \mathbf{R}_0^{k+i}$) whenever $\partial X \neq 0$, where $\mathbf{R}_+^{k+i} = \mathbf{R}^{k+i} \cap \{(x_1, \ldots, x_{k+i}) : x_1 > 0\}$ and $\mathbf{R}_0^{k+i} = \mathbf{R}^{k+i} \cap \{(x_1, \ldots, x_{k+i}) : x_1 = 0\}$. A direct geometric argument (7.6(2)) shows that $\mathbf{B}^i = \mathbf{D}^i$ for all i.

In 8.2, we determine the action of the Steenrod homology operations on the homology of real projective space, by computing $D(f_i^k)$ for special functions f_i^k mapping the Veronese variety $\mathbf{P}^k$, into $\mathbf{R}^{k+i}$ for $i = 0, 1, \ldots, k$. In 8.1, we study the geometry of $D(f)$ and $B(g)$ for several maps $f : \mathbf{P}^2 \to \mathbf{R}^3$ and $g : \mathbf{P}^2 \to \mathbf{R}^2$.

In §9 it is shown that $\Phi^i(\alpha) = 0$ for any $\alpha \in \mathbf{H}_k(A, B; \mathbf{Z}_2)$ whenever there exists either a continuous injection of (A, B) into $(\operatorname{Clos} \mathbf{R}_+^{k+i}, \mathbf{R}_0^{k+i})$ or a continuous local injection of A into $\mathbf{R}^{k+i-1}$. Theorem 9.2(2) for $B = \emptyset$ is dual to Thom's nonembedding theorem [22, III, 1.5], while 9.2(2) for $B \neq \emptyset$ and 9.2(1) are apparently new.

Homology operations were first studied by Thom and Wu in the early fifties (cf. [24], [26]). They were also considered briefly by Steenrod. From the viewpoint of stable homotopy theory, stable homology operations are canonically dual to stable cohomology operations (cf. [18, §5]). It is shown in §10 that the operations Φ^i are dual in this way to the modulo 2 Steenrod cohomology operations Sq^i of [22].

One motivation for working in homology, rather than cohomology, is that characteristic classes for singular analytic varieties are naturally defined in homology (cf. [23], [19]). In fact, applying Steenrod homology operations to the modulo 2 fundamental class of a compact real analytic variety ([1, 3.7], [8, 7.1]) yields intrinsic homology classes which generalize the Stiefel-Whitney classes of the normal bundle of a manifold (10.3). Our branch point description of these normal classes generalizes Thom's description for smooth manifolds using singularities of maps [25, p. 80]. That the normal Stiefel-Whitney classes of a smooth manifold can also be described using double points follows from a theorem of Ronga [21]. The use of singular loci of maps to define intrinsic invariants of algebraic varieties has a long history (cf. [17]).

All of our proofs use only the basic properties of subanalytic chains described in §2. Complete details of many of these proofs have been included because of the relatively recent development of subanalytic theory. We have used triangulations in the proofs of 2.6(2)(4), 4.2, and 7.2, but not in our subanalytic constructions of $\mathbf{B}_i$ and $\mathbf{D}_i$.

The present work all carries over if the word "subanalytic" is replaced throughout by "semialgebraic". Much of the theory also carries over for the homology with infinite chains, $\mathbf{H}_*^{\mathrm{Inf}}$, defined as in 2.6(1) except with mod 2 chains having closed, possibly noncompact supports. Thus, any paracompact real analytic variety A, which has a fundamental class in $\mathbf{H}_k^{\mathrm{Inf}}(A;\mathbf{Z}_2)$ ([1, 3.7], [8, 7.1]) has "normal classes" in $\mathbf{H}_{k-i}^{\mathrm{Inf}}(A;\mathbf{Z}_2)$ for $0 \le i \le k$.

We wish to thank Thomas Banchoff for his help with the examples in 8.1 and his idea for the proof of 7.6(2). We also wish to thank Dennis Sullivan and Herbert Federer for leading us toward the points of view expressed in this paper.

§2. Subanalytic sets, maps, chains and homology

2.1. *Subanalytic sets and maps* ([14], [10])

The smallest class of subsets of paracompact real analytic spaces which contains singleton subsets and which is closed under the formation of locally finite unions, intersections, complements, connected components, inverse images by real analytic maps, and direct images by proper real analytic maps, is the class of **subanalytic subsets**. A continuous function between subsets of paracompact real analytic spaces M and N is a **subanalytic map** if its graph is a subanalytic subset of $M \times N$.

2.2. *Some notations*

Following [10, §2], we use, for any subanalytic subset A of a real analytic space, the symbols, $\operatorname{Clos} A$, $\operatorname{Fron} A$, and $\dim A$ to denote, respectively, the *closure of A*, *the frontier of A* ($[\operatorname{Clos} A] \sim A$), and the *dimension of A*. We shall often use the fact (See e.g. [10, §2]) that *any subanalytic subset A of M admits a locally-finite partition* $\mathcal{S}$ (called a **subanalytic stratification**) *into connected, subanalytic, real analytic submanifolds S, such that* $\dim R < \dim S$ *and* $R \subset \operatorname{Fron} S$ *for every* $R \in \mathcal{S}$ *which intersects* $\operatorname{Fron} S$. Since [11, Embedding Lemma] any finite dimensional subanalytic set B admits a proper subanalytic embedding into $\mathbf{R}^{1+2\dim B}$, *we will henceforth only consider subanalytic subsets of Euclidean space.*

For any nonempty subanalytic subset A of $\mathbf{R}^m$, we use the map

$$\zeta : \mathbf{R}\times\mathbf{R}^m \to \mathbf{R}\times\mathbf{R}^m, \quad \zeta(t,x) = (t,(1-|t|)x) \quad \text{for} \quad (t,x)\in\mathbf{R}\times\mathbf{R}^m,$$

to express the **suspension of** A, $\mathbf{S}A = \zeta([-1,1]\times A)$, and the **cone of** A, $\mathbf{C}A = \zeta([0,1]\times A)$, as $(1+\dim A)$-dimensional subanalytic subsets of

$\mathbf{R} \times \mathbf{R}^m$. For any subanalytic map f from A into a subanalytic subset B of Euclidean space, the function, $Sf : SA \to SB$, where

$$Sf(t, x) = (t, (1 - |t|)f[x/(1 - |t|)]) \quad \text{for} \quad (t, x) \in SA \sim \{(-1, 0), (1, 0)\},$$

$$Sf(-1, 0) = (-1, 0), \; Sf(1, 0) = (1, 0),$$

is a subanalytic map.

2.3. LEMMA: *For any bounded k and l dimensional subanalytic subsets A and B of $\mathbf{R}^m$ and integer n with $\sup\{k, l\} \leq n \leq k + l + 1$, the set,*

$$Q = \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^n) \cap \{p : \dim[(A + B) \cap \{(x, y) : x \neq y, p(x) = p(y)\}] > k + l - n\},$$

is a subanalytic subset of $\operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^n)$ *with* $\dim Q < mn$.

PROOF: The dimension of the subanalytic set,

$$Q' = [A \times B \times \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^n)] \cap \{(x, y, p) : x \neq y, p(x) = p(y)\},$$

does not exceed $k + l + mn - n$, because, for each $(x, y) \in A \times B$ with $x \neq y$,

$$\dim[\operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^n) \cap \{p : p(x - y) = 0\}] = mn - n.$$

Since the projection map g of Q' onto $\operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^n)$ may be stratified as in [9, 4.4],

$$Q = \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^n) \cap \{p : \dim g^{-1}\{p\} > (k + l + mn - n) - mn\}$$

is an at most $mn - 1$ dimensional subanalytic set.

2.4. *Integral k-chains*

We recall from [10, §2] that an oriented k dimensional subanalytic submanifold of an open subset M of $\mathbf{R}^m$ defines, by integration, a linear functional on the space $\mathscr{D}^k(M)$ of smooth k forms with compact support in M. An arbitrary sum T of integral multiples of such oriented subanalytic submanifolds is an **integral k-chain** in M. (The terminology "k dimensional subanalytic chain in M" is used in [10].) Then the support of T, spt T, is a closed (but not necessarily compact) subanalytic subset of M. From [10, §2], we recall that

$T + T'$ and $T - T'$ are integral k-chains in M whenever T' is another integral k-chain in M,

∂T, where $\partial T(\psi) = T(\mathrm{d}\psi)$ for $\psi \in \mathscr{D}^{k-1}(M)$, is an integral $(k-1)$-chain in M whenever $k \geq 1$,

$f_{\#}T$ is an integral k-chain in $\mathbf{R}^n$ whenever f is a subanalytic map into $\mathbf{R}^n$ with $\operatorname{spt} T \subset \operatorname{domain} f \subset M$ and $f \mid \operatorname{spt} T$ is proper,

$T \times S$ is an integral $(k+l)$-chain in $M \times N$ whenever S is an integral l-chain in N,

$T \llcorner A$ is an integral k-chain in M whenever A is a subanalytic subset of M.

For an open subanalytic subset U of M, an integral k-chain $T \mid U$ in U is well-defined by the condition, $(T \mid U)(\varphi) = T(\psi)$ whenever $\varphi \in \mathscr{D}^k(U)$, $\psi \in \mathscr{D}^k(m)$, and $\psi \mid U = \varphi$. Note that although evaluation of $T \llcorner U$ and $T \mid U$ on differential forms both involve restricting integration to $U \cap \operatorname{spt} T$, $T \llcorner U$ is an integral k-chain *in* M while $T \mid U$ is an integral k-chain *in* U; thus,

$$\operatorname{spt}(T \llcorner U) = M \cap \operatorname{Clos}(U \cap \operatorname{spt} T) \quad \text{while} \quad \operatorname{spt}(T \mid U) = U \cap \operatorname{spt} T.$$

2.5. Mod 2 *k-chains*, *compact* mod 2 *k-chains*

Two integral k-chains T and T' in M are said to be **congruent modulo 2** if $T - T' = 2T''$ for some integral k-chain T''. The resulting congruence classes are called **mod 2 k-chains** in M. (The terminology "k dimensional subanalytic chain modulo 2 in M" is used in [10].) Then the **support of** X,

$$\underline{X} = \bigcap_{T \in X} \operatorname{spt} T,$$

is a closed (not necessarily compact) subanalytic subset of M. (The notation, $\operatorname{spt}^2 X$, is used in [4, 4.2.26], [7], and [10].) We will call X a **compact mod 2 k-chain** whenever $\underline{X}$ is compact. By the stratification theorem of [10, §2], X may be represented by an integral k-chain T which has $\operatorname{spt} T = \underline{X}$ and which is the sum (as in 2.4) of disjoint oriented subanalytic submanifolds (of multiplicity one).

For any two mod 2 k-chains, X and X', in M, note that

$$X = X' \textit{ if and only if } \underline{X} = \underline{X'}.$$

It follows that *a k dimensional subanalytic subset G of M determines* *a unique* mod 2 *k-chain*, $\bar{G}$, *in* M *with* $(\underline{\bar{G}}) = G$ *if and only if*

$$G = \cup\{M \cap \operatorname{Clos} S : S \in \mathcal{S}, \dim S = k\}$$

for some (and hence any) subanalytic stratification $\mathcal{S}$ *of* G.

By [10, §2], the mod 2 chains,

$X + X' = X - X'$ (whenever X' is a mod 2 k-chain in M),
∂X (whenever $k \geq 1$),
$f_{\#}X$ (whenever $f \mid \underline{X}$ is a proper subanalytic map),
$X \times Y$ (whenever Y is a mod 2 l-chain in M),
$X \llcorner A$ (whenever A is a subanalytic subset of M),
$X \mid U$ (whenever U is an open subanalytic subset of M),

are well-defined because of the relation of the corresponding integral operations and multiplication by 2. For *notational convenience only* we also define the set

$$\underline{\partial X} = \emptyset \textit{ in case } X \textit{ is } 0 \textit{ dimensional.}$$

With ζ defined as in 2.2, $\mathbf{S}X = \zeta_{\#}(\overline{[-1, 1]} \times X)$ and $\mathbf{C}X = \zeta_{\#}(\overline{[0, 1]} \times X)$ are mod 2 $(k+1)$-chains in $\mathbf{R} \times M$.

Recall from [10, 4.5] and [7, §1] that the **slice** $\langle X, g, y \rangle$ is a mod 2 $(k-n)$-chain in M whenever $k \geq n$, g is a subanalytic map into $\mathbf{R}^n$, $\underline{X} \subset$ domain $g \subset M$, $\dim(\underline{X} \cap g^{-1}\{y\}) \leq k - n$, and $\dim(\underline{\partial X} \cap g^{-1}\{y\}) \leq k - n - 1$.

In case $n = 1$, the mod 2 $(k-1)$-chain,

$$\langle X, g, y+ \rangle = (\partial X) \llcorner \{x : g(x) > y\} - \partial(X \llcorner \{x : g(x) > y\}),$$

is defined for *all* $y \in \mathbf{R}$.

A mod 2 k-chain Y in M is said to **intersect suitably** with the mod 2 l-chain X if $k + l \geqslant m$, $\dim(\underline{X} \cap \underline{Y}) \leq k + l - m$, and $\dim[(\underline{X} \cap \underline{\partial Y}) \cup (\underline{Y} \cap \underline{\partial X})] \leq k + l - m - 1$, in which case, a mod 2 $(k + l - m)$-chain in M, called the **intersection modulo 2 of** X **and** Y, is defined by $X \cap^2 Y = r_{\#}\langle X \times Y, s, 0 \rangle$ where $r : (\mathbf{R}^m)^2 \to \mathbf{R}^m$, $s : (\mathbf{R}^m)^2 \to \mathbf{R}^m$, $r(x, y) = (x + y)/2$, and $s(x, y) = x - y$ for $(x, y) \in (\mathbf{R}^m)^2$.

Each of the following relationships is true *whenever both terms in the relationship are defined.*

(1) $\partial(\partial X) = 0$.
(2) $(f \circ g)_{\#}X = f_{\#}(g_{\#}X)$.
(3) $\partial(X \times Y) = (\partial X) \times Y + (X \times \partial Y)$.
(4) $\partial f_{\#}X = f_{\#}(\partial X)$, $\partial \mathbf{S}X = \mathbf{S}(\partial X)$.

(5) $\underline{f_{\#}X} \subset f(\underline{X})$ *with equality whenever* $\dim[\underline{X}^2 \cap \{(x, y): x \neq y, f(x) = f(y)\}] < \dim \underline{X}$.

(6) $\underline{X \times Y} = \underline{X} \times \underline{Y}$.

(7) $X = X \llcorner A + X \llcorner (M \sim A)$.

(8) $\underline{\mathbf{S}X} = \mathbf{S}\underline{X}$, $\underline{\mathbf{C}X} = \mathbf{C}\underline{X}$.

(9) $\underline{X \llcorner A} \subset M \cap \operatorname{Clos}(\underline{X} \cap A)$ *with equality whenever A is open.*

(10) $(f_{\#}X) \llcorner A = f_{\#}[X \llcorner f^{-1}(A)]$

(11) $(f_{\#}X) \mid U = [f \mid f^{-1}(U)]_{\#}[X \mid f^{-1}(U)]$.

(12) $(\partial X) \mid U = \partial(X \mid U)$.

(13) $\underline{(\partial X) \llcorner A} \subset \underline{\partial(X \llcorner A)}$ and $\underline{\partial(X \llcorner A) - (\partial X) \llcorner A} \subset \operatorname{Fron}(\underline{X \llcorner A})$ *whenever A is open.*

(14) $\underline{\langle X, g, y\rangle} \subset \underline{X} \cap g^{-1}\{y\}$.

(15) $\langle X, g, y\rangle \llcorner A = \langle X \llcorner A, g, y\rangle$.

(16) $\langle X, g, y\rangle \mid U = \langle X \mid U, g \mid U, y\rangle$.

(17) $\partial\langle X, g, y\rangle = \langle \partial X, g, y\rangle$.

(18) $\langle f_{\#}X, g, y\rangle = f_{\#}\langle X, g \circ f, y\rangle$.

(19) $\langle\langle X, g, y\rangle, p, z\rangle = \langle X, (g, p), (y, z)\rangle = \langle X, (p, g), (z, y)\rangle$.

(20) $\langle X, h \circ g, h(y)\rangle = \langle X, g, y\rangle$ *for any subanalytic homeomorphism* $h: \mathbf{R}^n \to \mathbf{R}^n$.

(21) $Y \times \langle X, g, y\rangle = \langle Y \times X, g \circ q, y\rangle$ *where q is the projection of* $Y \times X$ *onto X.*

(22) $\langle X, g, y\rangle = \langle\langle \cdots \langle\langle X, g_1, y_1+\rangle, g_2, y_2+\rangle \cdots\rangle, g_n, y_n+\rangle$ *where* $g = (g_1, g_2, \ldots, g_n)$ *and* $y = (y_1, y_2, \ldots, y_n)$.

(23) $\partial(X \cap^2 Y) = [(\partial X) \cap^2 Y] + (X \cap^2 \partial Y)$.

(24) $\mathbf{S}(X \cap^2 Y) = (\mathbf{S}X) \cap^2 (\mathbf{S}Y)$.

Properties (6) through (13) follow essentially from the definitions. Properties (2), (4), and (5) follow from [10, 4.4(4)(3)(2)]. Reference to [4, 4.1.6, 4.1.8, 4.3.11] is useful for properties (1), (3), (14), and (21). Property (23) involves properties (3) and (4), and property (24) involves properties (18), (20), and (21). Properties (14) through (21) are not as readily verified as (1) through (13) and involve the continuity of slicing (See [10, 4.5]).

To prove (22) in case $k = \dim \underline{X} = n$, it suffices to show that both sides of (22) agree near each point x of the locally finite set $\underline{X} \cap g^{-1}\{y\}$. For this purpose, we may assume $\underline{X}$ is compact. Since $\underline{\partial X} \cap g^{-1}\{y\} = \emptyset$, there are, by [7, 3.2, 4.1 Case 1], a closed ball B centered at y in $\mathbf{R}^n \sim g(\underline{\partial X})$ and a connected component A of $\underline{X} \cap g^{-1}(B)$ so that $A \cap g^{-1}\{y\} = \{x\}$ and $g_{\#}\overline{A} = i\overline{B}$ where $i = 1$ if $x \in \underline{\langle X, g, y\rangle}$ and $i = 0$ if $x \notin \underline{\langle X, g, y\rangle}$. Then, by (1) (4) (3),

$$\begin{aligned} g_{\#}\langle \overline{A}, g_1, y_1+\rangle &= [\partial g_{\#}(\overline{A} \llcorner \{x : g_1(x) > y_1\})] \llcorner B \\ &= i\overline{B \cap \{(z_1, \ldots, z_n) : z_1 = y_1\}}, \end{aligned}$$

$$g_\#\langle\langle \overline{A}, g_1, y_1+\rangle, g_2, y_2+\rangle = \overline{iB \cap \{(z_1, \ldots, z_n) : z_1 = y_1, z_2 = y_2\}},$$
$$g_\#\langle\langle \cdots \langle\langle \overline{A}, g_1, y_1+\rangle, g_2, y_2+\rangle, \cdots\rangle, g_n, y_n+\rangle = \overline{i\{y\}},$$

hence, $\langle\langle \cdots \langle\langle X, g_1, y_1+\rangle, g_2, y_2+\rangle, \cdots\rangle, g_n, y_n+\rangle \llcorner A = \langle X, g, y\rangle \llcorner A$.

In case $k > n$ and (22) is false, we may, by [7, 3.1, 3.2], choose a surjection $p = (p_{n+1}, \ldots, p_k) \in \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{k-n})$ and $z = (z_{n+1}, \ldots, z_k) \in \mathbf{R}^{k-n}$ so that

$$\dim(\underline{X} \cap g^{-1}\{y\} \cap p^{-1}\{z\}) \leq 0, \quad (\underline{\partial X}) \cap g^{-1}\{y\} \cap p^{-1}\{z\} = \emptyset, \text{ and}$$

$$\langle(\langle X, g, y\rangle - \langle\langle \cdots \langle\langle X, g_1, y_1+\rangle, g_2, y_2+\rangle, \cdots\rangle, g_n, y_n+\rangle), p, z\rangle \neq 0.$$

But applying (19) and the case $k = n$ twice then gives a contradiction to the last inequality.

2.6. *Subanalytic homology theory*

(1) Assuming $A \supset B$ are subanalytic subsets of $\mathbf{R}^m$ and k is a nonnegative integer, we recall from [10, 4.6] the $\mathbf{Z}_2$ vector spaces,

$$\mathscr{Z}_k(A, B) = \{X : X \text{ is a compact mod 2 } k\text{-chain}, \underline{X} \subset A, \text{ and } \underline{\partial X} \subset B\},$$
$$\mathscr{B}_k(A, B) = \{Y + \partial Z : Y \in \mathscr{Z}_k(B, B) \quad \text{and} \quad Z \in \mathscr{Z}_{k+1}(A, A)\},$$
$$\mathbf{H}_k(A, B) = \mathscr{Z}_k(A, B)/\mathscr{B}_k(A, B), \mathbf{H}_k(A) = \mathbf{H}_k(A, \emptyset),$$

(Thus we abbreviate the usual notations, $\mathbf{H}_k(A, B; \mathbf{Z}_2)$ and $\mathbf{H}_k(A; \mathbf{Z}_2)$ because we work here only with the coefficient group $\mathbf{Z}_2$.) For any subanalytic subsets $A' \supset B'$ of a Euclidean space and subanalytic map $f : (A, B) \to (A', B')$, $f_\#$ induces, by 2.5(4)(5), a homomorphism

$$\mathbf{H}_k(f) : \mathbf{H}_k(A, B) \to \mathbf{H}_k(A', B').$$

Also, for subanalytic sets $A \supset B \supset C$, the boundary operator induces, by 2.5(1), a homomorphism

$$\partial : \mathbf{H}_k(A, B) \to \mathbf{H}_{k-1}(B, C).$$

By the elementary arguments of [4, 4.4.1] (which use only properties 2.5(1) through 2.5(8), 2.5(10), and 2.5(13)), the operations $\mathbf{H}_k$ and ∂, on the category of subanalytic sets and maps, satisfy the *axioms of Eilenberg and Steenrod* for a homology theory with coefficient group $\mathbf{Z}_2$. We refer to these as, respectively, the *identity, naturality, boundary, exactness, homotopy, and dimension axioms.*

(2) *There is a natural equivalence between subanalytic homology and ordinary singular theory on the subanalytic category.* In fact, using the triangulation theorem ([11 Theorem 2] or [15]), one readily obtains a natural transformation from subanalytic theory to singular theory which induces an isomorphism on the homology of a singleton set (See [3, III, 10.1]).

(3) Here, for notational convenience only, we adopt the conventions,

$$\mathbf{S}\emptyset = \{(-1,0),(1,0)\} \subset \mathbf{R}\times\mathbf{R}^m, \mathbf{C}\emptyset = \{(1,0)\} \subset \mathbf{R}\times\mathbf{R}^m,$$

(when treating subsets of a fixed $\mathbf{R}^m$). Since $\mathbf{S}$ maps, by 2.5(4)(8), $\mathscr{Z}_j(A,B)$ into $\mathscr{Z}_{j+1}(\mathbf{S}A,\mathbf{S}B)$ and $\mathscr{B}_j(A,B)$ into $\mathscr{B}_{j+1}(\mathbf{S}A,\mathbf{S}B)$, $\mathbf{S}$ *induces a homomorphism*

$$\mathbf{S}:\mathbf{H}_j(A,B)\to\mathbf{H}_j(SA,SB),$$

which is by the exactness of a Mayer-Vietoris sequence [3, I, 5.6], an isomorphism for all $j\geq 0$. Similarly, $\mathbf{C}$ *induces a homomorphism,*

$$\mathbf{C}:\mathbf{H}_j(A,B)\to\mathbf{H}_j(\mathbf{C}A,(\{0\}\times A)\cup\mathbf{C}B),$$

which is, by the naturality and exactness axioms, an isomorphism for all $j\geq 0$.

(4) By use of the triangulation theorem ([11, Theorem 2] or [15]) and formation of the second barycentric subdivision [3, II, 9.9], *there exists, for any subanalytic pair $(E,F)\supset(A,B)$ in $\mathbf{R}^m$, an arbitrarily small, open subanalytic pair (U,V) so that (A,B) is a strong subanalytic deformation retraction of $(U\cap E, V\cap F)$. In particular, with $(E,F)=(A,B)$, the inclusion map ι of (A,B) into (U,V) induces an isomorphism, $\mathbf{H}_k(\iota):\mathbf{H}_k(A,B)\xrightarrow{\simeq}\mathbf{H}_k(U,V)$ for all k.*

(5) For any nonnegative integers k and l with $k+l\geq m$, the **homology intersection product in $\mathbf{R}^m$,**

$$.:\mathbf{H}_k(A,B)\times\mathbf{H}_l(A,B)\to\mathbf{H}_{k+l-m}(A,B),$$

is well-defined in [10, 4.6] by letting, for $\kappa\in\mathbf{H}_k(A,B)$ and $\lambda\in\mathbf{H}_l(A,B)$, $\kappa.\lambda$ equal $\mathbf{H}_{k+l-m}(\iota)^{-1}$ of the homology class in $\mathbf{H}_{k+l-m}(U,V)$ of $X\cap^2 Y$ for any $X\in\mathbf{H}_k(\iota)\kappa$ and $Y\in\mathbf{H}_l(\iota)\lambda$ which intersect suitably. From [10, 4.6] and 2.5(17)(18), it then follows that

$\kappa.\lambda$ equals $\mathbf{H}_{k+l-m}(\iota)^{-1}$ of the homology class in

$\mathbf{H}_{k+l-m}(U, V)$ *of* $r_{\#}\langle K\times L, s, a\rangle$ *where* $K\in\kappa$, $L\in\lambda$, $r:(\mathbf{R}^m)^2\to\mathbf{R}^m$, $s:(\mathbf{R}^m)^2\to\mathbf{R}^m$, $r(x,y)=(x+y)/2$, $s(x,y)=x-y$ *for* $(x,y)\in(\mathbf{R}^m)^2$, $a\in\mathbf{R}^m$,

$$|a|<\epsilon=\inf\{\mathrm{dist}(\underline{K}\cap\underline{L}, \mathrm{Fron}\ U), \mathrm{dist}([\underline{K}\cap\underline{\partial L}]\cup[\underline{L}\cap\underline{\partial K}], \mathrm{Fron}\ V)\},$$

$$\dim s^{-1}\{a\}\cap\underline{(K\times L)}\le k+l-m,\ \dim s^{-1}\{a\}\cap\underline{\partial(K\times L)}\le k+l-m-1.$$

By [10, §2], the latter conditions hold for all a off an at most $m-1$ dimensional subanalytic subset of $\mathbf{R}^m$.

2.7. *The homology of the support of a compact* mod 2 *k-chain*

Let X be a compact mod 2 k-chain in $\mathbf{R}^m$.

(1) The **fundamental class of** X is the homology class $\chi\in\mathbf{H}_k(\underline{X}, \underline{\partial X})$ of the cycle $X\in\mathscr{Z}_k(\underline{X}, \underline{\partial X})$.

(2) **The self-intersection class of** X **in** $\mathbf{R}^m$ is the homology class $\chi\cdot\chi\in\mathbf{H}_{2k-m}(\underline{X}, \underline{\partial X})$ where $2k\ge m$ and χ is the fundamental class of X.

Then $\chi\cdot\chi=0$ *whenever* $X\subset R^m\cap\{(x_1,\dots,x_m):x_1\ge 0\}$ *and* $\underline{\partial X}\subset\mathbf{R}^m\cap\{(x_1,\dots,x_m):x_1=0\}$. In fact if $Y=\gamma_{\#}(\overline{[0,1]}\times X)$ where $\gamma(t,(x_1,\dots,x_m))=(tx_1,x_2,\dots,x_m)$ for $(t,(x_1,\dots,x_m))\in[0,1]\times\underline{X}$ and r, s, U, V, ι, and $a=(a_1,\dots,a_m)$ are as in 2.6(5) with $A=\underline{X}$, $B=\underline{\partial X}$, $a_1>0$,

$$\dim s^{-1}\{a\}\cap(\underline{X}\times\underline{Y})\le 2k-m+1\quad\text{and}\quad\dim s^{-1}\{a\}\cap(\underline{X}\times\underline{\partial Y})\le 2k-m,$$

then $s^{-1}\{a\}\cap[\underline{X}\times(\underline{\partial Y-X})]=\emptyset$ and $\chi\cdot\chi$ is the image under $\mathbf{H}_{2k-m}(\iota)^{-1}$ of the homology class in $\mathbf{H}_{2k-m}(U, V)$ of

$$\begin{aligned} r_{\#}\langle X\times X, s, a\rangle &= r_{\#}\langle X\times\partial Y, s, a\rangle = r_{\#}\langle\partial(X\times Y), s, a\rangle - r_{\#}\langle(\partial X)\times Y, s, a\rangle \\ &= \partial r_{\#}\langle X\times Y, s, a\rangle - r_{\#}\langle(\partial X)\times Y, s, a\rangle\in\mathscr{B}_{2k-m}(U, V), \end{aligned}$$

by 2.5(3)(4)(5)(17).

(3) For any integral k-chain T representing the modulo 2 congruence class X with spt $T=\underline{X}$ as in 2.5 and integral $(k-1)$-chain S representing ∂X with spt $S=\underline{\partial X}$, there exists an integral $(k-1)$-chain R so that $\partial T-S=2R$. It follows that spt $R\subset\underline{X}$, that spt $\partial R\subset\underline{\partial X}$, and that the homology class in $\mathbf{H}_{k-1}(\underline{X}, \underline{\partial X})$ of the modulo 2 congruence class of R is independent of the choice of T and S. This class $\boldsymbol{\beta}(X)\in\mathbf{H}_{k-1}(\underline{X}, \underline{\partial X})$ is called the **Bockstein class of** X.

(4) For any open subanalytic subset U of $\mathbf{R}^m$ such that $\mathrm{Fron}(\underline{X} \cap U) \subset \underline{\partial(X \llcorner U)}$, the function sending a compact mod 2 j-chain W to $W \llcorner U$ maps, by 2.5(9)(13), $\mathscr{Z}_j(\underline{X}, \underline{\partial X})$ into $\mathscr{Z}_j[\underline{X \llcorner U}, \underline{\partial(X \llcorner U)}]$ and $\mathscr{B}_j(\underline{X}, \underline{\partial X})$ into $\mathscr{B}_j[\underline{X \llcorner U}, \underline{\partial(X \llcorner U)}]$; there is an induced homomorphism,

$$\llcorner U : \mathbf{H}_j(\underline{X}, \underline{\partial X}) \to H_j[\underline{X \llcorner U}, \underline{\partial(X \llcorner U)}],$$

whose value on $\alpha \in \mathbf{H}_j(\underline{X}, \underline{\partial X})$ we denote by $\alpha \llcorner U$.

§3. Geometric homology operations

3.1. DEFINITION: *For any integer i, a* ***stable homology operation*** $\boldsymbol{\theta}$ *of degree i on* $\mathbf{H}$ *is a sequence of natural transformations,* $\boldsymbol{\theta}_k : \mathbf{H}_k \to \mathbf{H}_{k-i}$ *for* $k \geq 0 \leq k - i$, *such that* $\mathbf{S}\boldsymbol{\theta}_k = \boldsymbol{\theta}_{k+1}\mathbf{S}$.

3.2. DEFINITION: *For any integer i, a* ***geometric homology operation*** $\mathbf{G}$ *of degree i on* $\mathbf{H}$ *is a function which assigns to each compact* mod 2 *k-chain X in Euclidean space with* $k \geq i$, *a homology class* $\mathbf{G}(X) \in \mathbf{H}_{k-i}(\underline{X}, \underline{\partial X})$ *such that*:

(1) $\mathbf{H}_{k-i}(h)\mathbf{G}(X) = \mathbf{G}[h_{\#}(X)]$ *whenever h is a subanalytic homeomorphism of X,*

(2) $\mathbf{G}(X) \llcorner U = \mathbf{G}(X \llcorner U)$ *whenever U is an open subanalytic subset of Euclidean space with* $\mathrm{Fron}(\underline{X} \cap U) \subset \underline{\partial(X \llcorner U)}$.

(3) $\mathbf{SG}(X) = \mathbf{G}(\mathbf{S}X)$.

3.3. THEOREM: *If* $k \in \{1, 2, \ldots, m\}$, $i \in \{1, 2, \ldots, k\}$, $\mathbf{G}$ *is a geometric homology operation of degree i on* $\mathbf{H}$, *and X, Y, and Z are compact* mod 2 *k-chains in* $\mathbf{R}^m$, *then*:

(1) $\mathbf{H}_{k-i}(f)\mathbf{G}(X) = \mathbf{H}_{k-i}(\iota_{f_{\#}X})\mathbf{G}(f_{\#}X)$ *whenever f is a subanalytic map of X onto f(X) in* $\mathbf{R}^n$ *and* $\iota_{f_{\#}X}$ *is the inclusion map of* $(\underline{f_{\#}X}, \underline{\partial f_{\#}X})$ *into* $(f(\underline{X}), f(\underline{\partial X}))$.

(2) $\partial\mathbf{G}(X) = \mathbf{G}(\partial X)$ *whenever* $k > i$.

(3) $\mathbf{CG}(X) = \mathbf{G}(\mathbf{C}X)$.

(4) $\mathbf{H}_{k-i}(\iota_Y)\mathbf{G}(Y) + \mathbf{H}_{k-i}(\iota_Z)\mathbf{G}(Z) = \mathbf{H}_{k-i}(\iota_{Y+Z})\mathbf{G}(Y+Z)$ *where* ι_Y, ι_Z, *and* ι_{Y+Z} *are the inclusion maps of, respectively,* $(\underline{Y}, \underline{\partial Y})$, $(\underline{Z}, \underline{\partial Z})$, *and* $(\underline{Y+Z}, \underline{\partial(Y+Z)})$ *into* $(\underline{Y} \cup \underline{Z}, \underline{\partial Y} \cup \underline{\partial Z})$.

PROOF OF (3): Apply 3.2(3) and 3.2(2) with the open set $(\mathbf{R} \times \mathbf{R}^m) \cap \{(t, x) : t > 0\}$.

PROOF OF (4) IN THE SPECIAL CASE $\partial\underline{Y} = \underline{Y} \cap \underline{Z} = \partial\underline{Z}$. Here we apply 3.2(2) after observing that

$$Y = (Y + Z) \llcorner (\mathbf{R}^m \sim Z),\ Z = (Y + Z) \llcorner (R^m \sim Y).$$

PROOF OF (2): Noting that

$$\partial(\{0\} \times X) = \partial \mathbf{C} \partial X,\ \partial \mathbf{G}(\partial X) = 0,\ \partial \mathbf{C}\mathbf{G}(\partial X) = \{0\} \times \mathbf{G}(\partial X),$$

we use the boundary axiom, 3.2(1), (3), and the special case of (4) with $Y = \overline{\{0\}} \times X$ and $Z = \mathbf{C}\partial X$.

PROOF OF (1): Here we use the map,

$$\lambda : [0, 1] \times \underline{X} \to \mathbf{R} \times \mathbf{R}^m \times \mathbf{R}^n,\ \lambda(t, x) = (t, (1 - t)x, tf(x)) \quad \text{for} \quad (t, x) \in [0, 1] \times \underline{X},$$

to define the *mapping cylinder* of f, $M_f = \lambda_{\#}(\overline{[0, 1]} \times X)$, observe that

$$\partial M_f = (\overline{\{0\}} \times X \times \overline{\{0\}}) + (\overline{\{1\}} \times \overline{\{0\}} \times f_{\#}X) + M_{f|\partial X},$$

and apply 3.2(1), (2), the special case of (4), and the identity and naturality axioms.

PROOF OF (4) IN THE GENERAL CASE: Noting that

$$\begin{aligned}\partial(&\overline{[-1, 0]} \times Y + \overline{[0, 1]} \times Z) \\ &= \overline{\{-1\}} \times Y + \overline{\{0\}} \times (Y + Z) + \overline{\{1\}} \times Z + \overline{[-1, 0]} \times \partial Y + \overline{[0, 1]} \times \partial Z,\end{aligned}$$

we here use 3.2(1), (2), the special case of (4), and the identity, naturality, and boundary axioms.

3.4. THEOREM: *Suppose G is a geometric homology operation of degree i on* $\mathbf{H}$.

(1) *If either $i < 0$ or $i = k > 0$, then $\mathbf{G}(X) = 0$ for all compact* mod 2 *k-chains X.*

(2) *If $i = 0$ and $\mathbf{G} \neq 0$, then $\mathbf{G}(X)$ equals the fundamental class of X for all compact* mod 2 *k-chains X.*

PROOF OF (2): Clearly $\mathbf{G}(X)$ is either 0 or the fundamental class of $\underline{X}$ in case $\underline{X}$ is a fixed singleton set in some Euclidean space. By 3.2(1)(3), the corresponding alternative holds in case $\underline{X}$ is any other singleton set or in case $\underline{X}$ is subanalytically homeomorphic to $[-1,1]^k$ for $k \in \{1,2,\ldots\}$. Finally, for an arbitrary compact mod 2 k-chain X in $\mathbf{R}^m$, this alternative also holds by 3.2(2), because there is, for each stratum S of a subanalytic stratification [10, §2] of $\underline{X}$, an open subset U or $\mathbf{R}^m$ so that $\underline{X \llcorner U}$ is subanalytically homeomorphic to $[-1,1]^k$.

PROOF OF (1): If $i<0$, then $k-i>k$ and $\mathscr{X}_{k-i}(\underline{X}, \underline{\partial X}) = \{0\}$.

Suppose now that $i=k>0$. To show that $\mathbf{G}(X)=0$, we may, by 3.2(2), assume that $\underline{X}$ is connected. In this case we may also assume that $\partial X = 0$, because, otherwise, $\mathbf{H}_0(\underline{X}, \underline{\partial X}) = 0$. Letting h be the homeomorphism projecting $\{0\} \times \underline{X}$ onto $\underline{X}$, we infer from 3.2(1) and 3.3(2) that

$$\begin{aligned} \mathbf{G}(X) &= \mathbf{H}_0(h)\mathbf{G}(\overline{\{0\}} \times X) = \mathbf{H}_0(h)\mathbf{G}(\partial \mathbf{C}(X)) \\ &= \mathbf{H}_0(h)\partial \mathbf{G}(\mathbf{C}X) = 0, \end{aligned}$$

because $\underline{\partial \mathbf{C}X}$ is connected and because the support of any compact Mod 2 0-chain which represents $\partial \mathbf{G}(\mathbf{C}X)$ has, by the triangulation theorem ([11, Theorem 2] or [15]), an even number of points.

3.5. *The groups of operations $\mathscr{O}$ and $\mathscr{G}$*

The set $\mathscr{O}$ of stable homology operations on $\mathbf{H}$ and the set $\mathscr{G}$ of geometric homology operations on $\mathbf{H}$ are graded abelian groups under addition, where, for each integer i, $\mathscr{O}_i$ and $\mathscr{G}_i$ are the operations of degree i.

THEOREM: (Compare [18, 1.3]) *The map*

$$\mathscr{I} : \mathscr{O} \to \mathscr{G},$$

where $(\mathscr{I}\boldsymbol{\theta})(X) = \boldsymbol{\theta}(\chi)$ *for any* $i \leq k$, $\boldsymbol{\theta} \in \mathscr{O}_i$, *and compact* mod 2 *$k$-chain X with fundamental class χ is a degree-preserving isomorphism whose inverse,*

$$\mathscr{J} : \mathscr{G} \to \mathscr{O},$$

is well-defined by choosing – for any $\mathbf{G} \in \mathscr{G}_i$, *subanalytic pair* (A,B), *and* $\alpha \in H_k(A,B)$ *with* $k \geq i$ *– a compact* mod 2 *k-chain* $X \in \mathscr{X}_k(A,B)$ *representing α and letting* $(\mathscr{J}\mathbf{G})(\alpha) = \mathbf{H}_{k-i}(\iota_X)\mathbf{G}(X)$, *where ι_X is the inclusion map of* $(\underline{X}, \underline{\partial X})$ *into* (A,B).

Proof: We prove the theorem in the following six steps:

(1) *$\mathscr{I}\boldsymbol{\theta} \in \mathscr{G}_i$ whenever $\boldsymbol{\theta} \in \mathscr{O}_i$*. Suppose k, X, h, and U are as in 3.2. If χ is the fundamental class of X, then $\mathbf{H}_k(h)(\chi)$, $\chi \llcorner U$, and $\mathbf{S}\chi$ are the fundamental classes of, respectively, $h_{\#}X$, $X \llcorner U$, and $\mathbf{S}X$. Thus, $\mathscr{I}\boldsymbol{\theta}$ satisfies 3.2(1) because $\boldsymbol{\theta}$ is natural and 3.2(3) because $\boldsymbol{\theta}$ is stable. To see that $\mathscr{I}\boldsymbol{\theta}$ also satisfies 3.2(2), we use the naturality of $\boldsymbol{\theta}$ and the commutative diagram

$$\begin{tikzcd}
\mathbf{H}_*(\underline{X}, \underline{\partial X}) \arrow[rr, "\mathbf{H}_*(\iota)"] \arrow[dr] & & \mathbf{H}_*(\underline{X}, \underline{\partial X} \cup (\underline{X} \sim U)) \\
& \mathbf{H}_*(\underline{X \llcorner U}, \underline{\partial(X \llcorner U)}) \arrow[ur, "\mathbf{H}_*(\kappa)"'] &
\end{tikzcd}$$

where ι and κ are inclusion maps, to verify that

$$\mathbf{H}_{k-i}(\kappa)[(\mathscr{I}\boldsymbol{\theta})(X) \llcorner U] = \mathbf{H}_{k-i}(\kappa)[(\mathscr{I}\boldsymbol{\theta})(X \llcorner U)],$$

and then note that $\mathbf{H}_{k-i}(\kappa)$ is, by 2.6(4), the excision axiom and [3, I, 12.2], an isomorphism.

(2) *$\mathscr{J}$ is well-defined* by 3.3(4)(2) and the boundary axiom.

(3) *$\mathscr{J}\mathbf{G} \in \mathscr{O}_i$ whenever $\mathbf{G} \in \mathscr{G}_i$*. The naturality of $\mathscr{J}\mathbf{G}$ follows from 3.3(1) and the naturality axiom. The stability of $\mathscr{J}\mathbf{G}$ follows from 3.2(3) and the naturality of $\mathbf{S}$.

(4) *$\mathscr{I}$ and $\mathscr{J}$ are clearly additive.*

(5) $\mathscr{I} \circ \mathscr{J} = \mathbf{id}_{\mathscr{G}}$ because any compact mod 2 k-chain is a representative of its own fundamental class.

(6) $\mathscr{J} \circ \mathscr{I} = \mathbf{id}_{\mathscr{O}}$ because any member of $\mathscr{O}$ is natural.

3.6. Corollary: *If $\boldsymbol{\theta}$ is a stable homology operation of degree i on* $\mathbf{H}$, *then*:

(1) *For $i < 0$ or for $i = k > 0$, $\boldsymbol{\theta}_k = 0$.*

(2) *For $i = 0$, $\boldsymbol{\theta}_k$ is either the zero operation for all k or the identity for all k.*

(3) *$\partial\boldsymbol{\theta}_k = \boldsymbol{\theta}_{k-1}\partial$ for $k \geq 1$.*

(4) *$\boldsymbol{\theta}_k$ is additive for all k.*

Proof: Combine 3.4, 3.3(2)(4) and 3.5. (Moreover, conclusions (3) and (4) follow directly from the stability of $\boldsymbol{\theta}$, and (1) and (4) are true without the assumption of stability.)

§4. The Steenrod homology operations

4.1. DEFINITION: *For each nonnegative integer i, a natural transformation Φ^i of degree i on* $\mathbf{H}$ *is called a* ***Steenrod homology operation*** *if $\Phi^i(\alpha) = \alpha . \alpha$ whenever $\alpha \in \mathbf{H}_k(A, B)$, $k \geq i$, and $A \supset B$ are subanalytic subsets of* $\mathbf{R}^{k+i}$.

4.2. THEOREM: (Uniqueness) *For each nonnegative integer i there is at most one Steenrod homology operation of degree i.*

PROOF: Suppose $0 \leq i \leq k$ are integers and $\boldsymbol{\theta} : \mathbf{H}_k \to \mathbf{H}_{k-i}$ and $\eta : \mathbf{H}_k \to H_{k-i}$ are two natural transformations such that $\boldsymbol{\theta}(\alpha) = \alpha . \alpha = \boldsymbol{\eta}(\alpha)$ for all α as in 4.1.

In case $i = 0$, $\boldsymbol{\theta}$ and $\boldsymbol{\eta}$ are both the identity map because $X \cap^2 X = X$ for any compact mod 2 k-chain X in $\mathbf{R}^k$.

In case $0 < i = k$, $\boldsymbol{\theta}$ and $\boldsymbol{\eta}$ are both, by naturality, the zero operation. In fact, for any subanalytic sets $B \subset A \subset \mathbf{R}^m$, we may let $\mathscr{E}$ be the family of connected components of A and choose one point $x_E \in E$ for each $E \in \mathscr{E}$ so that $x_E \in B$ whenever $E \cap B \neq \emptyset$. With $F = \{x_E : E \in \mathscr{E}\}$ and $G = B \cap F$, we define $f : (A, B) \to (F, G)$ so that $f(x) = x_E$ whenever $x \in E \in \mathscr{E}$ and deduce from the identity, naturality, and dimension axioms that

$$\boldsymbol{\theta}(A, B) = \mathbf{H}_0(f)^{-1} \circ \boldsymbol{\theta}(F, G) \circ \mathbf{H}_k(f) = \mathbf{H}_0(f)^{-1} \circ \boldsymbol{\theta}(F, G) \circ 0 = 0 = \boldsymbol{\eta}(A, B).$$

We now assume $0 < i < k$. To prove that $\boldsymbol{\theta} = \boldsymbol{\eta}$, it suffices, by 2.6 and naturality, to show that $\boldsymbol{\theta}(\chi) = \boldsymbol{\eta}(\chi)$ for the fundamental class χ of any compact mod 2 k-chain X in $\mathbf{R}^m$.

For this purpose, we may assume $\partial X = 0$, because, by naturality and the excision axiom (as in 3.5(1)),

$$\boldsymbol{\theta}(\chi) = \mathbf{H}_{k-i}(h)[\boldsymbol{\theta}(\mathscr{Y}) \llcorner U], \quad \boldsymbol{\eta}(\chi) = \mathbf{H}_{k-i}(h)[\boldsymbol{\eta}(\mathscr{Y}) \llcorner U],$$

and $\partial Y = 0$ where $U = (\mathbf{R} \times \mathbf{R}^m) \sim \mathbf{C}\, \underline{\partial X}$, $Y = \overline{\{0\}} \times X + \mathbf{C}\, \partial X$, $\mathscr{Y}$ is the fundamental class of Y and $h : (\{0\} \times \underline{X}, \{0\} \times \underline{\partial X}) \to (\underline{X}, \underline{\partial X})$ is the projection map. Finally, we may also assume, by naturality and [11, Theorem 2] or [15], that $\underline{X}$ is a compact polyhedron in $\mathbf{R}^m$.

Choose, by 2.3, a map $p \in \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{k+i})$ so that

$$\dim[\underline{X}^2 \cap \{(x, y) : x \neq y, p(x) = p(y)\}] \leq k - i,$$

and let $\mathscr{C}$ be a simplicial decomposition of $\underline{X}$ with respect to which the map $p \mid \underline{X}$ is simplicial. Let $b(C)$ denote the barycenter of C for each $C \in \mathscr{C}$,

$$K = \cup\{C : C \in \mathscr{C}, \dim C \leq k - i - 1\}$$
$$L = \cup\{\text{convex hull of } \{b(C_0), \ldots, b(C_j)\} : C_0, \ldots, C_j \in \mathscr{C},$$
$$C_0 \subset C_1 \subset \cdots \subset C_j, \text{ and } \dim C_0 \geq k - i\},$$
$$\Delta = (\mathbf{R}^m)^2 \cap \{(x, y) : x = y\}.$$

Since

$$\operatorname{Fron}[\underline{X}^2 \cap \{(x, y) : x \neq y, p(x) = p(y)\}] \subset K^2 \cap \Delta \subset (X \sim L)^2 \cap \Delta,$$

and $p \mid \underline{X}$ is simplicial, $p \mid L$ locally has a Lipschitz inverse, and there is an $\epsilon > 0$ so that

$$|p(x) - p(y)| \geq \epsilon|x - y| \text{ whenever } (x, y) \in \underline{X}^2 \text{ and } \operatorname{dist}((x, y), L^2 \cap \Delta) < \epsilon.$$

Since $\dim L = i < k$, there is, by 2.3, a map $q \in \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{k+i})$ so that $q \mid L$ is injective and $|q(v) - p(v)| \leq (\epsilon/2)|v|$ whenever $v \in \mathbf{R}^m$. Thus, $|q(x) - q(y)| \geq (\epsilon/2)|x - y|$ whenever $(x, y) \in \underline{X}^2$ and $\operatorname{dist}((x, y), L^2 \cap \Delta) < \epsilon$. Letting $D = \underline{X}^2 \cap \{(x, y) : x \neq y,\ q(x) = q(y)\}$, we infer that $L^2 \cap \operatorname{Clos} D = \emptyset$ because $L^2 \cap D = \emptyset$ and

$$L^2 \cap \operatorname{Fron} D \subset \operatorname{Clos}(D \cap \{(x, y) : \operatorname{dist}((x, y), L^2 \cap \Delta) < \epsilon\}) = \emptyset.$$

Inasmuch as K is a subanalytic deformation retract of $\underline{X} \sim L$ and $\dim K = k - i - 1$, there is an open subanalytic neighborhood V of L so that $(\operatorname{Clos} V)^2 \cap \operatorname{Clos} D = \emptyset$, $\operatorname{From}(\underline{X} \cap V) \subset \underline{\partial(X \llcorner V)}$, and $\mathbf{H}_{k-i}(\underline{X} \sim V) \cong \mathbf{H}_{k-i}(K) \cong 0$. Since $q \mid \underline{X \llcorner V}$ is thus injective, we may use the compact mod 2 k-chain $Z = q_\#(X \llcorner V)$ in $\mathbf{R}^{k+i}$ with fundamental class $\mathscr{Z}$ and the subanalytic homeomorphism $g = [q \mid (\underline{X \llcorner V})]^{-1}$ to conclude, by naturality and the excision axiom (as in 3.5(1)) and by the self-intersection hypothesis, that

$$\begin{aligned}\boldsymbol{\theta}(\chi) \llcorner V = \boldsymbol{\theta}(\chi \llcorner V) = \boldsymbol{\theta}[\mathbf{H}_k(g)\mathscr{Z}] &= \mathbf{H}_{k-i}(g)\boldsymbol{\theta}(\mathscr{Z}) \\ &= \mathbf{H}_{k-i}(g)(\mathscr{Z} . \mathscr{Z}) = \boldsymbol{\eta}(\chi) \llcorner V.\end{aligned}$$

Finally, the injectivity of the homology homeomorphism $\llcorner V$ follows by applying the exactness and excision axioms, 2.6(4), and [3, I, 12.2], to the commutative diagram

$$\begin{tikzcd}
0 \cong \mathbf{H}_{k-i}(\underline{X} \sim V) \arrow[r] & \mathbf{H}_{k-i}(\underline{X}) \arrow[r] \arrow[dr, "\llcorner V"'] & H_{k-i}(\underline{X}, \underline{X} \sim V) \arrow[d, "\cong"] \\
& & H_{k-i}(\underline{X \llcorner V}, \underline{\partial(X \llcorner V)}).
\end{tikzcd}$$

4.3. Corollary: $\mathbf{S}\boldsymbol{\Phi}^i = \boldsymbol{\Phi}^i\mathbf{S}$ *for any Steenrod homology operation* $\boldsymbol{\Phi}^i$.

Proof: Since $\mathbf{S}^{-1}\boldsymbol{\Phi}^i\mathbf{S}$ is a natural transformation, and, since, by 2.5(24),

$$(\mathbf{S}^{-1}\boldsymbol{\Phi}^i\mathbf{S})(\alpha) = \mathbf{S}^{-1}[(\mathbf{S}\alpha).(\mathbf{S}\alpha)] = (\mathbf{S}^{-1}\mathbf{S})(\alpha.\alpha) = \alpha.\alpha,$$

for any α as in 4.1, $\mathbf{S}^{-1}\boldsymbol{\Phi}^i\mathbf{S} = \boldsymbol{\Phi}^i$ by 4.2.

4.4. Theorem:

(0) *The identity operation on* $\mathbf{H}$ *is the Steenrod homology operation of degree* 0.

(1) *The Bockstein operation* $\mathscr{J}(\boldsymbol{\beta})$ (*see* 2.7(3) *and* 3.5) *is the Steenrod homology operation of degree* 1.

Proof of (0): We need only observe that $X \cap^2 X = X$ for any compact mod 2 k-chain X in $\mathbf{R}^k$.

Proof of (1): Using the integral analogues of 2.5(3)(4)(13) we readily verify that $\boldsymbol{\beta}$ is a geometric homology operation. By 3.5 and 4.2 it suffices to show that $\boldsymbol{\beta}(X) = \chi.\chi$ for any compact mod 2 k-chain X in $\mathbf{R}^{k+1}$ with fundamental class χ.

For this purpose, we use [8, 6.1] to choose U, V, and a as in 2.6(5) with $A = K = L = \underline{X}$, $B = \underline{\partial X}$, $|a| \neq 0$, and

$$\dim(\underline{X} \cap \{x + ta : x \in \underline{\partial X}, t \in \mathbf{R}\}) \leq k - 1,$$

and we let $Y = \tau_{a\#}X$ where $\tau_a(x) = x + a$ for $x \in \mathbf{R}^{k+1}$. For any open subanalytic subset F of $\mathbf{R}^{k+1}$, we let $\tilde{F} = \mathbf{R}^{k+1} \sim \operatorname{Clos} F$ and F_0 denote the integral $(k+1)$-chain in $\mathbf{R}^{k+1}$ obtained by giving F the standard orientation of $\mathbf{R}^{k+1}$ ($F_0 = \mathbf{E}^{k+1} \llcorner F$ in the notation of [F]). Since $\mathbf{H}_k(\mathbf{R}^{k+1}) \cong 0$, there is a unique bounded open subanalytic subset G of $\mathbf{R}^{k+1}$ so that $X - Y - f_\#(\overline{[0,1]} \times \partial X) = \partial\, \overline{\operatorname{Clos} G}$ where $f(t, x) = x + ta$

for $(t, x) \in \mathbf{R} \times \mathbf{R}^{k+1}$. Inasmuch as the support of the integral k-chain $T = (\partial G_0) \llcorner X$ is $\underline{X}$, it suffices, by 2.6(5) and 2.7(2)(3) to prove that *the modulo 2 congruence class of* $\frac{1}{2}(\partial T) \mid W$ *equals* $(X \cap^2 Y) \mid W$ *for any open ball* W *in* $\mathbf{R}^{k+1}$ *with center in* $\underline{X} \cap \{x : \operatorname{dist}(x, \underline{\partial X}) \geq |a|\}$ *and radius* $|a|$. To verify this we choose one of the two open subsets I of W for which $(\partial \overline{\operatorname{Clos} I}) \mid W = X \mid W$ and set

$$J = \operatorname{Int} \operatorname{Clos}[(G \cap W \sim I) \cup (I \sim G \cap W)];$$

hence, $(\partial \overline{\operatorname{Clos} J}) \mid W = Y \mid W$, observe that $(X \cap^2 Y) \mid W$ is the modulo 2 congruence class of $\partial(I_0 \mid W) \cap \partial(J_0 \mid W)$, and compute, using the integral analogue [6, 5.8(9)] of 2.5(23),

$$\begin{aligned} T \mid W = [(\partial I_0) \llcorner \tilde{J} - (\partial I_0 \llcorner J)] \mid W \\ = \partial(I_0 \mid W) \cap (\tilde{J}_0 \mid W) - \partial(I_0 \mid W) \cap (J_0 \mid W), \end{aligned}$$

$$(\partial T) \mid W = -2\partial(I_0 \mid W) \cap \partial(J_0 \mid W).$$

4.5. THEOREM: *If* $A \subset \mathbf{R}^m \cap \{(x_1, \ldots, x_m) : x_1 \geq 0\}$ *and* $B \subset \mathbf{R}^m \cap \{(x_1, \ldots, x_m) : x_1 = 0\}$ *are subanalytic subsets of* $\mathbf{R}^m$, $\alpha \in \mathbf{H}_k(A, B)$, *and* Φ^i *is the Steenrod homology operation of degree i, then* $\Phi^i(\alpha) = 0$ *whenever* $m - k \leq i \leq k$.

PROOF: Using the subanalytic homeomorphism, $h : (A, B) \to (h(A), h(B)) \subset (\mathbf{R}^{k+i}, \mathbf{R}^{k+i})$, $h(x_1, \ldots, x_m) = (x_1, \ldots, x_m, 0, \ldots, 0)$ for $(x_1, \ldots, x_m) \in A$, we may apply 2.7(2) to a representative of $\mathbf{H}_k(h)(\alpha)$.

Theorem 4.5 shows that, if they exist, the Steenrod homology operations provide obstructions to embedding subanalytic sets in low dimensional Euclidean spaces. This gives some motivation for the construction of the Steenrod homology operations, using double points, which follows in §5 and §7.

§5. The double point pair chain

Let m be a fixed positive integer,

$$\Gamma = (\mathbf{R}^m)^2 \cap \{(x, y) : x \neq y\}, \quad \Delta = (\mathbf{R}^m)^2 \cap \{(x, y) : x = y\},$$

$$\rho = \Gamma \to \Gamma, \ \rho(x, y) = (y, x) \quad \text{for } (x, y) \in \Gamma.$$

5.1. *The chain* $C(Z)$

Suppose Z is a mod 2 l-chain in Γ, and $\underline{Z}$ is subanalytic as a subset of (not only Γ, but also) $(\mathbf{R}^m)^2$. Then the closure of $\underline{Z}$ in $(\mathbf{R}^m)^2$ (which is subanalytic by the stratification theorem [10, §6]) supports, by 2.5, a unique mod 2 l-chain $C(Z)$. Since $\underline{\partial Z} = \Gamma \cap \underline{\partial C(Z)}$, $\underline{\partial Z}$ is also a subanalytic subset of $(\mathbf{R}^m)^2$.

LEMMA: *If $\rho_{\#}Z = Z$, then $\partial C(Z) = C(\partial Z)$.*

PROOF: Letting $s_i : (\mathbf{R}^m)^2 \to \mathbf{R}$, $s_i((x_1, \ldots, x_m), (y_1, \ldots, y_m)) = x_i - y_i$, we apply [7, 3.1] and [9, 4.4] with $M = (\mathbf{R}^m)^2$, $\mathscr{C} = \{\underline{Z}, \underline{\partial Z}, s_1^{-1}\{0\}, \ldots, s_m^{-1}\{0\}\}$ and $f = \rho \mid \underline{Z}$ to obtain a stratification $\mathscr{S}$ of $(\mathbf{R}^m)^2$ satisfying

$$C(Z) = \sum_{\underline{Z} \supset S \in \mathscr{S}, \dim S = k} C(\overline{\text{Clos } S}),$$

where, for each $S \in \mathscr{S}$ with $S \subset \underline{Z}$ and $\dim S = k$,

$$\rho(S) \in \mathscr{S},\ \rho(S) \subset \rho(\underline{Z}) = \underline{Z},\ \dim \rho(S) = k,$$
$$\partial(C[\overline{\Gamma \cap \text{Clos } \rho(S)}]) \llcorner \Delta = (\partial C[\overline{\Gamma \cap \text{Clos } S}]) \llcorner \Delta,$$

and $\rho(S) \neq S$ because sign $s_i \mid \rho(S)$ and sign $s_i \mid S$ are constant functions which differ for any $i \in \{1, 2, \ldots, m\}$ with $s_i \mid S \neq 0$. Thus, by cancellation modulo 2, $\partial C(Z) \llcorner \Delta = 0$,

$$\partial C(Z) = [\partial C(Z)] \llcorner \Gamma = C(\partial Z).$$

5.2. *The double point pair chain $\mathscr{D}(f)$*

For any $k \in \{1, 2, \ldots, m\}$, $j \in \{0, 1, \ldots, k\}$, compact mod 2 k-chain X in $\mathbf{R}^m$, and subanalytic map $f : \underline{X} \to \mathbf{R}^{k+j}$ satisfying the condition (see 2.3)

$$(*) \quad \begin{aligned} &\dim[\underline{X}^2 \cap \{(x, y) : x \neq y, f(x) = f(y)\}] \le k - j \quad \text{and} \\ &\dim[(\underline{X} \times \underline{\partial X}) \cap \{(x, y) : x \neq y, f(x) = f(y)\}] \le k - j - 1, \end{aligned}$$

the map $s_f : \underline{X}^2 \cap \Gamma \to \mathbf{R}^{k+j}$, $s_f(x, y) = f(x) - f(y)$ for $(x, y) \in \underline{X}^2 \cap \Gamma$, is subanalytic, and $\langle X^2 \mid \Gamma, s_f, 0\rangle$ is a mod 2 $(k-j)$-chain in Γ. Moreover $\underline{\langle X^2 \mid \Gamma, s_f, 0\rangle}$ is subanalytic as a subset of $(\mathbf{R}^m)^2$ because, for any subanalytic stratification [10, §2] $\mathscr{R}$ of $\underline{X}^2 \cap \Gamma \cap \{(x, y) : f(x) = f(y)\}$ compatible with $\{\underline{\partial X^2}, \underline{\langle X^2 \mid \Gamma, s_f, 0\rangle}\}$ is, by 2.5(3)(6)(14)(17) and the constancy theorem [7, 3.1], a union of sets $\Gamma \cap \text{Clos } R$ corresponding to certain $k - j$ dimensional strate $R \in \mathscr{R}$. The **double point pair chain** $\mathscr{D}(f)$ is the compact mod 2 $(k - j)$-chain in $(\mathbf{R}^m)^2$:

$$\mathscr{D}(f) = C\langle X^2 \mid \Gamma, s_f, 0\rangle.$$

THEOREM:

(0) $\partial\mathscr{D}(f) = C\langle(\partial X^2) \,|\, \Gamma, s_f, 0\rangle$ *whenever* $j < k$.

(1) $(h \times h)_\#\mathscr{D}(f) = \mathscr{D}(f \circ h^{-1})$ *for any subanalytic homeomorphism of* X.

(2) $\mathscr{D}(f) \llcorner U^2 = \mathscr{D}[f \,|\, \underline{X \llcorner U}]$ *for any subanalytic subset* U *of* $\mathbf{R}^m$ *such that* $f \,|\, \underline{X \llcorner U}$ *satisfies* $(*)$.

(3) $\mathscr{D}(\mathbf{S}f) = \delta_\#\mathbf{S}\mathscr{D}(f)$ *where* $\delta : \mathbf{R} \times (\mathbf{R}^m)^2 \to (\mathbf{R} \times \mathbf{R}^m)^2$, $\delta(t, (x, y)) = ((t, x), (t, y))$ *for* $(t, (x, y)) \in \mathbf{R} \times (\mathbf{R}^m)^2$.

(4) $\mathscr{D}(f) - \mathscr{D}(g) \in \mathscr{B}_{k-j}(\underline{X}^2, \underline{\partial X}^2)$ *for any other subanalytic map* $g : \underline{X} \to \mathbf{R}^{k+j}$ *satisfying* $(*)$.

PROOF OF (0): Apply 5.1 and 2.5(12)(17).

PROOF OF (1): Apply 2.5(18)(11)(5).

PROOF OF (2): Apply 2.5(9)(15).

PROOF OF (3): Let ζ be as in 2.2, let $\tilde{\Gamma}$ and $\tilde{C}$ be defined as Γ and C were in 5.1 with $\mathbf{R}^m$ replaced by $\mathbf{R} \times \mathbf{R}^m$, and let $\beta : \mathbf{R} \times \mathbf{R}^m \to \mathbf{R}^m$, $\beta(t, x) = x$, $\alpha : (\mathbf{R} \times \underline{X})^2 \to \mathbf{R}$, $\alpha((t, x), (u, y)) = t - u$ for $((t, x), (u, y)) \in (\mathbf{R} \times \underline{X})^2$. Using 2.5(7)(18)(19)(21)(16)(24), we compute

$$\begin{aligned}
\mathscr{D}(\mathbf{S}f) &= \tilde{C}\langle(\zeta \times \zeta)_\#(\overline{[-1, 1]} \times X)^2 \,|\, \tilde{\Gamma}, s_{\mathbf{S}f}, (0, 0)\rangle \\
&= (\zeta \times \zeta)_\#\tilde{C}\langle(\overline{[-1, 1]} \times X)^2 \,|\, \tilde{\Gamma}, (\alpha, \beta \circ s_{\mathbf{S}f} \circ (\zeta \times \zeta)), (0, 0)\rangle \\
&= (\zeta \times \zeta)_\#\tilde{C}\langle\langle(\overline{[-1, 1]} \times X)^2 \,|\, \tilde{\Gamma}, \alpha, 0\rangle, \beta \circ s_{\mathbf{S}f} \circ (\zeta \times \zeta), 0\rangle, \\
\delta_\#\mathbf{S}\mathscr{D}(f) &= (\zeta \times \zeta)_\#\delta_\#(\overline{[-1, 1]} \times C\langle X^2 \,|\, \Gamma, s_f, 0\rangle) \\
&= (\zeta \times \zeta)_\#\tilde{C}[\delta_\#\langle\overline{[-1, 1]} \times (X^2 \,|\, \Gamma), s_f \circ (\beta \times \beta) \circ \delta, 0\rangle \,|\, \tilde{\Gamma}] \\
&= (\zeta \times \zeta)_\#\tilde{C}\langle\delta_\#(\overline{[-1, 1]} \times X^2) \,|\, \tilde{\Gamma}, s_f \circ (\beta \times \beta), 0\rangle \\
&= (\zeta \times \zeta)_\#\tilde{C}\langle\langle(\overline{[-1, 1]} \times X)^2 \,|\, \tilde{\Gamma}, \alpha, 0\rangle, s_f \circ (\beta \times \beta), 0\rangle.
\end{aligned}$$

Moreover, $\langle\langle(\overline{[-1, 1]} \times X)^2 \,|\, \tilde{\Gamma}, \alpha, 0\rangle, b \circ s_{\mathbf{S}f} \circ (\zeta \times \zeta), 0\rangle$ equals $\langle\langle(\overline{[-1, 1]} \times X)^2 \,|\, \tilde{\Gamma}, \alpha, 0\rangle, s_f \circ (\beta \times \beta), 0\rangle$ by 2.5(22) because

$$\{((t, x), (t, y)) : v \cdot \beta \circ s_{\mathbf{S}f} \circ (\zeta \times \zeta)((t, x), (t, y)) > 0\}$$

$$= \{((t, x), (t, y) : v \circ s_f \circ (\beta \times \beta)((t, x), (t, y)) > 0\}$$

for every $t \in \mathbf{R}$ with $|t| < 1$ and every $v \in \mathbf{R}^{k+j} \cap \{(1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)\}$.

PROOF OF (4): Here we shall essentially use a subanalytic map

$\Theta:[0,1]\times\underline{X}\to\mathbf{R}^{k+j}$ for which $\Theta(0,\cdot)=g$ and $\Theta(t,\cdot)$ satisfies (*) for every $t\in[0,1]$. Although it may be impossible to obtain f as $\Theta(1,\cdot)$, we shall choose Θ so that $\mathscr{D}(f)-\mathscr{D}[\Theta(1,\cdot)]$ is the boundary (relative to $\partial\underline{X}^2$) of a $k-j+1$ dimensional slice determined by linear interpolation between f and $\Theta(1,\cdot)$. In detail, let

$$\iota:X\to\mathbf{R}^m\times\mathbf{R}^{k+j}\times\mathbf{R}^{k+j},\ \iota(x)=(x,f(x),g(x))\quad\text{for } x\in X,\ Y=\iota_{\#}X,$$

$$L=\operatorname{Hom}(\mathbf{R}^m\times\mathbf{R}^{k+j}\times\mathbf{R}^{k+j},\mathbf{R}^{k+j}),\ l=\dim L=(m+2k+2j)(k+j),$$

and note, by 2.3, that

$$J=L\cap\{p:\dim\underline{Y}^2\cap\{(u,v):u\neq v,\ s_p(u,v)=0\}>k-j\quad\text{or}$$

$$\dim(\underline{Y}\times\partial\underline{Y})\cap\{(u,v):u\neq v,\ s_p(u,v)=0\}>k-j-1\}$$

is an at most $l-1$ dimensional subanalytic subset of L. Moreover, the maps q and r belong to $L\sim J$, where $q(x,y,z)=y$ and $r(x,y,z)=z$ for $(x,y,z)\in\mathbf{R}^m\times\mathbf{R}^{k+j}\times\mathbf{R}^{k+j}$, because $q\circ\iota=f$ and $r\circ\iota=g$. The two sets

$$C=([0,1)\times\underline{Y}\times L)\cap\{(t,u,v,p):u\neq v,\ s_{(1-t)p+tq}(u,v)=0\},$$

$$D=([0,1)\times\underline{Y}\times\partial\underline{Y}\times L)\cap\{(t,u,v,p):u\neq v,\ s_{(1-t)p+tq}(u,v)=0\},$$

are subanalytic with $\dim C\leq k-j+l+1$ and $\dim D\leq k-j+l$ because, for each $(t,u,v)\in[0,1)\times\underline{Y}\times\underline{Y}$,

$$\dim L\cap\{p:p(u-v)=(1-t)^{-1}tq(v-u)\}=l-(k+j).$$

Since the restrictions to C and D of the projection map $\eta:[0,1]\times\underline{Y}\times\underline{Y}\times L\to L$ may be stratified as in [9, 4.3],

$$K=L\cap\{p:\dim C\cap\eta^{-1}\{p\}>k-j+1\quad\text{or}\quad\dim D\cap\eta^{-1}\{p\}>k-j\}$$

is an at most $l-1$ dimensional subanalytic subset of L. By the triangulation theorem, [11, Theorem 2] or [15], there exists a subanalytic curve $\theta:[0,1]\to L\sim J$ with $\theta(0)=r$ and $\theta(1)\in L\sim K$. Letting $p=\theta(1)$ and defining the maps

$$\lambda:[0,1]\times(\underline{X}^2\cap\Gamma)\to\underline{X}^2\cap\Gamma,\quad\lambda(t,(u,v))=(u,v),$$

$$\mu:[0,1]\times(\underline{X}^2\cap\Gamma)\to\mathbf{R}^{k+j},\quad\mu(t,u,v))=s_{(1-t)p+tq}[\iota(u),\iota(v)],$$

$$\nu : [0,1] \times (\underline{X}^2 \cap \Gamma) \to \mathbf{R}^{k+j}, \quad \nu(t, (u, v)) = s_{\theta(t)}[\iota(u), \iota(v)]$$

for $(t, (u, v)) \in [0, 1] \times (\underline{X}^2 \cap \Gamma)$, we see that

$$\dim \mu^{-1}\{0\} \leq k - j + 1, \quad \dim \mu^{-1}\{0\} \cap ([0, 1] \times (\underline{X} \times \underline{\partial X})) \leq k - j$$

because $p \in L \sim K$ and that

$$\dim \nu^{-1}\{0\} \leq k - j + 1, \quad \dim \nu^{-1}\{0\} \cap ([0, 1] \times (\underline{X} \times \underline{\partial X})) \leq k - j$$

because, for every $s \in [0, 1]$, $\theta(s) \in L \sim J$. For the slices

$$M = \lambda_{\#}\langle [0, 1] \times (X^2 \mid \Gamma), \mu, 0 \rangle, \quad N = \lambda_{\#}\langle [0, 1] \times (X^2 \mid \Gamma), \nu, 0 \rangle,$$

we recall 2.5(18)(21)(17)(4)(3)(14)(5) and apply 5.1 to M and N to conclude that

$$\begin{aligned}
\mathscr{D}(f) - \mathscr{D}(p \circ \iota) &= C(\lambda_{\#}\langle \{1\} \times (X^2 \mid \Gamma), \mu, 0 \rangle) - C(\lambda_{\#}\langle \{0\} \times (X^2 \mid \Gamma), \mu, 0 \rangle \\
&= C\lambda_{\#}(\partial \langle [0, 1] \times (X^2 \mid \Gamma), \mu, 0 \rangle \\
&\qquad - \langle [0, 1] \times \partial (X^2 \mid \Gamma), \mu, 0 \rangle) \\
&= C(\partial M) - C\lambda_{\#}\langle [0, 1] \times \partial (X^2 \mid \Gamma), \mu, 0 \rangle \\
&\qquad \in \mathscr{B}_{k-j}(\underline{X}^2, \underline{\partial X}^2), \\
\mathscr{D}(p \circ \iota) - \mathscr{D}(g) &= C(\lambda_{\#}\langle \{1\} \times (X^2 \mid \Gamma), \nu, 0 \rangle) - C(\lambda_{\#}\langle \{0\} \times (X^2 \mid \Gamma), \nu, 0 \rangle \\
&= C(\partial N) - C\lambda_{\#}\langle [0, 1] \times \partial (X^2 \mid \Gamma), \nu, 0 \rangle \\
&\qquad \in \mathscr{B}_{k-j}(\underline{X}^2, \underline{\partial X}^2).
\end{aligned}$$

(In the above proof, note that q and r may belong to different path components of $L \sim J$, in which case $(1 - t)p + tq \in J$ for some $0 < t < 1$).

§6. The branching point chain

Let m, Γ, Δ, and ρ be as in §5. Recalling from [4, 1.9] the Euclidean space $\mathbf{R}^m \odot \mathbf{R}^m \simeq \mathbf{R}^{\binom{m+1}{2}}$, we use the real analytic maps,

$$\sigma : (\mathbf{R}^m)^2 \to \mathbf{R}^m \times (\mathbf{R}^m \odot \mathbf{R}^m), \quad \tau : \mathbf{R}^m \times (\mathbf{R}^m \odot \mathbf{R}^m) \to \mathbf{R}^m,$$

$$\sigma(x, y) = (2^{-1}(x + y), (x - y) \odot (x - y)) \quad \text{for } (x, y) \in (\mathbf{R}^m)^2,$$

$$\tau(z, w) = z \quad \text{for} \quad (z, w) \in \mathbf{R}^m \times (\mathbf{R}^m \odot \mathbf{R}^m),$$

in expressing, for any subanalytic subset A of $\mathbf{R}^m$, the **two-fold symmetric product of** A, $\sigma(A^2)$, as a subanalytic subset of a Euclidean space. Thus, $\sigma^{-1}\{\sigma(x, y)\} = \{(x, y), (y, x)\}$, $\sigma(\Gamma) \cap \sigma(\Delta) = \emptyset$, $\sigma \mid \Delta$ and $\tau \mid \sigma(\Delta)$ are analytic diffeomorphisms, $\sigma \mid \Gamma$ is a two-sheeted analytic covering map, and $\tau[\sigma(A^2) \cap \sigma(\Delta)] = A$ (even though $\tau[\sigma(A^2)]$ is not necessarily contained in A).

6.1. Definition: For k, j, and f as in 5.2 with $j < k$, we observe that $\rho_\# \mathscr{D}(f) = \mathscr{D}(f)$ (hence $\sigma_\# \mathscr{D}(f) = 0$). Using the $\mathbf{Z}_2$ vector-space isomorphism,

$$\boldsymbol{\sigma} : \mathscr{Z}_{k+j}((\mathbf{R}^m)^2, (\mathbf{R}^m)^2) \cap \{Y : \rho_\# Y = Y\} \to \mathscr{Z}_{k+j}(\sigma(\mathbf{R}^m)^2, \sigma(\mathbf{R}^m)^2),$$

$$\boldsymbol{\sigma}(Y) = \overline{\sigma(\underline{Y})},$$

we note that $\underline{(\partial\boldsymbol{\sigma} - \boldsymbol{\sigma}\partial)\mathscr{D}(f)} \subset \sigma(\Delta)$ and define the **branching point chain** as the compact mod 2 $(k-j-1)$-chain in $\mathbf{R}^m$,

$$B(f) = \tau_\#(\partial\boldsymbol{\sigma} - \boldsymbol{\sigma}\partial)\mathscr{D}(f).$$

Theorem: *If X, f, g, h, and U are as in 5.2 and $j < k$, then*

(0) $\underline{\partial B(f)} \subset \underline{\partial X}$ *whenever* $j < k - 1$,
(1) $h_\# B(f) = B(f \circ h^{-1})$,
(2) $B(f) \llcorner U = B[f \mid \underline{X \llcorner U}]$,
(3) $\mathbf{S}B(f) = B(\mathbf{S}f)$,
(4) $B(f) - B(g) \in \mathscr{B}_{k-j-1}(\underline{X}, \underline{\partial X})$.

Proof of (0): Apply 2.5(4)(5) and 5.2(0).

Proof of (1): Apply 2.5(2)(4) and 5.2(1).

Proof of (2): Letting

$$E = \partial[\mathscr{D}(f) \llcorner U^2] - [\partial\mathscr{D}(f)] \llcorner U^2,$$

$$F = \partial(\boldsymbol{\sigma}[\mathscr{D}(f)] \llcorner \sigma(U^2)) - (\partial\boldsymbol{\sigma}[\mathscr{D}(f)]) \llcorner \sigma(U^2),$$

we infer that $\rho_\# E = E$ and that, by 2.5(7),

$$\boldsymbol{\sigma}(E) = \boldsymbol{\sigma}(E \llcorner \Gamma) + \boldsymbol{\sigma}(E \llcorner \Delta) = F \llcorner \sigma(\Gamma) + F \llcorner \sigma(\Delta) = F.$$

Thus, by 2.5(9) and 5.2(2),

$$\begin{aligned} B(f) \llcorner U &= \tau_{\#}[\partial(\boldsymbol{\sigma}[\mathscr{D}(f)] \llcorner \sigma(U^2)) - F - \boldsymbol{\sigma}(\partial[\mathscr{D}(f) \llcorner U^2]) + \boldsymbol{\sigma}(E)] \\ &= \tau_{\#}(\partial\boldsymbol{\sigma} - \boldsymbol{\sigma}\partial)[\mathscr{D}(f) \llcorner U^2] = B[f \mid (\underline{X \llcorner U})]. \end{aligned}$$

PROOF OF (3): Letting δ be as in 5.2(3) and letting $\tilde{\boldsymbol{\sigma}}$ and $\tilde{\tau}$ be defined as $\boldsymbol{\sigma}$ and τ were above with $\mathbf{R}^m$ replaced by $\mathbf{R} \times \mathbf{R}^m$, we observe that $\mathbf{S}\tau_{\#}\boldsymbol{\sigma} = \tilde{\tau}_{\#}\tilde{\boldsymbol{\sigma}}\delta_{\#}\mathbf{S} \mid \operatorname{domain}(\boldsymbol{\sigma})$ and use 2.5(2)(4) and 5.2(3) to compute

$$\begin{aligned} \mathbf{S}B(f) &= [\partial(\mathbf{S}\tau_{\#}\boldsymbol{\sigma}) - (\mathbf{S}\tau_{\#}\boldsymbol{\sigma})\partial]\mathscr{D}(f) \\ &= \tilde{\tau}_{\#}(\partial\tilde{\boldsymbol{\sigma}} - \tilde{\boldsymbol{\sigma}}\partial)\delta_{\#}\mathbf{S}\mathscr{D}(f) = B(\mathbf{S}f). \end{aligned}$$

PROOF OF (4): Letting λ, μ, ν, M, and N be as in the proof of 5.2(4) we conclude from 2.5(1)(4)(5)(6)(14) and 5.1 that

$$\begin{aligned} B(f) - B(g) &= \tau_{\#}(\partial\boldsymbol{\sigma} - \boldsymbol{\sigma}\partial)[\mathscr{D}(f) - \mathscr{D}(g)] \\ &= \tau_{\#}(\partial\boldsymbol{\sigma} - \boldsymbol{\sigma}\partial)[C(\partial M) + C(\partial N) \\ &\quad - C\lambda_{\#}\langle[\overline{0,1}] \times \partial(X^2 \mid \Gamma), \mu, 0\rangle - C\lambda_{\#}\langle[\overline{0,1}] \times \partial(X^2 \mid \Gamma), \nu, 0\rangle] \\ &= \partial\tau_{\#}(\partial\boldsymbol{\sigma} - \boldsymbol{\sigma}\partial)[C(M) + C(N)] \\ &\quad - \tau_{\#}(\partial\boldsymbol{\sigma} - \boldsymbol{\sigma}\partial)C\lambda_{\#}[\langle[\overline{0,1}] \times \partial(X^2 \mid \Gamma), \mu, 0\rangle \\ &\qquad + \langle[\overline{0,1}] \times \partial(X^2 \mid \Gamma), \nu, 0\rangle] \\ &\in \mathscr{B}_{k-j-1}(\tau[\sigma(\Delta) \cap \sigma(\underline{X^2})], \tau[\sigma(\Delta) \cap \sigma(\underline{\partial X^2})]) \\ &= \mathscr{B}_{k-j-1}(\underline{X}, \underline{\partial X}). \end{aligned}$$

6.2. DEFINITION: For any compact mod 2 k-chain X in $\mathbf{R}^m$, $\mathbf{B}^0(X) \in \mathbf{H}_k(\underline{X}, \underline{\partial X})$ is defined as the fundamental class of X and, for $i \in \{1, 2, \ldots, k\}$, $\mathbf{B}^i(X) \in \mathbf{H}_{k-i}(\underline{X}, \underline{\partial X})$ is well-defined as the homology class of $B(f)$ for any subanalytic map $f : \underline{X} \to \mathbf{R}^{k+i-1}$ satisfying 5.2(*) with $j = i - 1$ by 2.3 (which shows the existence of such f), 6.1(0) (which shows that $B(f) \in \mathscr{Z}_{k-i}(\underline{X}, \underline{\partial X})$, and 6.1(4) (which shows that independence of the choice of f).

THEOREM: $\mathbf{B}^i$ *is a geometric homology operation of degree* i *on* $\mathbf{H}$.

PROOF: This is clear for $i = 0$. For $i > 0$, properties 3.2(1) and 3.2(3) follow from 2.3, 6.1(1), and 6.1(3).

To verify 3.2(2), we use 2.3 to choose a subanalytic map f satisfying 5.2(*) with $j = i - 1$ as well as the estimate,

$$\dim[(\underline{X \llcorner U}) \times \underline{\partial(X \llcorner U)}] \cap \{(x, y) : x \neq y, f(x) = f(y)\} \leq k - i,$$

and then apply 6.1(2).

6.3. THEOREM: *For any compact* mod 2 *k-chain X in $\mathbf{R}^m$ with fundamental class χ:*
(0) $\mathbf{B}^0(X) = \chi$,
(1) $\mathbf{B}^1(X) = \boldsymbol{\beta}(X)$, *the Bockstein class of X,*
(2) $\mathbf{B}^{m-k}(X) = \chi \cdot \chi$ *whenever* $m - k \leq k$,
(3) $\mathbf{B}^i(X) = 0$ *whenever* $m - k < i \leq k$.

PROOF OF (0): See definition 6.2.

PROOF OF (3): For the subanalytic injection, $f : X \to \mathbf{R}^{k+i-1}$, $f(x_1, \ldots, x_m) = (x_1, \ldots, x_m, \; 0, \ldots, 0)$ for $(x_1, \ldots, x_m) \in X$, $B(f) = 0$ because $\mathscr{D}(f) = 0$.

PROOF OF (2): Choose U, V, $\mathbf{H}_{2k-m}(\iota)$, r, s, and a as in 2.6(4)(5) with $A = \underline{X}$, $B = \underline{\partial X}$, $K = X = L$, $a \neq 0$, and, by 2.3, $p \in \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{m-1})$ so that $p(a) = 0$ and $p \mid \underline{X}$ satisfies 5.2(*) with $j = m - k - 1$. Letting $q : \mathbf{R}^m \to \mathbf{R}$, $q(x) = (x \cdot a)/|a|$, $s_p(x, y) = p(x) - p(y)$, $s_q(x, y) = q(x) - q(y)$ for $(x, y) \in (\mathbf{R}^m)^2$, we note that $\tau \circ \sigma = r$ and that σ maps $G \cap \{(x, y) : s_q(x, y) > 0\}$ homeomorphically onto $\sigma(G)$ for any subanalytic subset G of $s_p^{-1}\{0\}$. Thus, by 2.5(2)(4)(5),

$$\begin{aligned} B(p \mid \underline{X}) &= \tau_\#(\partial \boldsymbol{\sigma} - \boldsymbol{\sigma} \partial)\mathscr{D}(p \mid \underline{X}) \\ &= \tau_\# \partial \sigma_\# [\mathscr{D}(p \mid \underline{X}) \llcorner \{(x, y) : s_q(x, y) > 0\}] \\ &\quad - \tau_\# \sigma_\# [\partial \mathscr{D}(p \mid \underline{X}) \llcorner \{(x, y) : s_q(x, y) > 0\}] \\ &= r_\# \langle \mathscr{D}(p \mid \underline{X}), s_q, 0+\rangle. \end{aligned}$$

Moreover, by 2.5(19)(20)(22),

$$\begin{aligned} r_\#\langle X^2, s, a\rangle &= r_\# C\langle X^2 \mid \Gamma, s, a\rangle = r_\# C\langle\langle X^2 \mid \Gamma, s_p, 0\rangle, s_q, |a|\rangle \\ &= r_\#\langle C\langle X^2 \mid \Gamma, s_p, 0\rangle, s_q, |a|\rangle = r_\#\langle \mathscr{D}(p \mid \underline{X}), s_q, |a| +\rangle. \end{aligned}$$

Conclusion (2) now follows from 2.6(5) because

$$\begin{aligned} B(p \mid \underline{X}) - r_\#\langle X^2, s, a\rangle &= r_\#[\partial \mathscr{D}(p \mid \underline{X}) \llcorner \{(x, y) : 0 < s_q(x, y) \leq |a|\}] \\ &\quad - r_\# \partial [\mathscr{D}(p \mid \underline{X}) \llcorner \{(x, y) : 0 < s_q(x, y) \leq |a|\}] \in \mathscr{B}_{2k-m}(U, V). \end{aligned}$$

PROOF OF (1): By 6.2, (2), 3.5, and 4.2 $\mathscr{I}(\mathbf{B}^1)$ is the unique Steenrod homology operation of degree 1, thus, $\mathbf{B}^1 = \boldsymbol{\beta}$ by 4.4(1).

6.4. Corollary (*Existence*): *The branching point operation* $\mathscr{J}(\mathbf{B}_i)$ *is the Steenrod homology operation of degree i for every nonnegative integer i.*

Proof: Combine 3.5, 4.2, 4.4, 6.2, and 6.3(2).

6.5. Remark: *One may verify* 6.3(1) *without appealing to the uniqueness theory of* §4. Choose first, by 2.3(3), $p \in \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^k)$ so that $f = p \mid \underline{X}$ satisfies 5.2(*) with $j = 0$, and second, a subanalytic stratification $\mathscr{S}$ of X as in [9, 4.4] with $g = p$, $L = \underline{X}$, and $\mathscr{C} = \{\underline{X}, \underline{\partial X}, \underline{B(f)}, \underline{\partial B(f)}\}$. Let T be the integral k-chain in $\mathbf{R}^m$ obtained by using, for each k dimensional $S \in \mathscr{S}$, the diffeomorphism $f \mid S$ to pull back to S the standard orientation of the open subset $f(S)$ of $\mathbf{R}^k$. Then $\boldsymbol{\beta}(X)$ is, by 2.7(3), the homology class in $\mathbf{H}_{k-1}(\underline{X}, \underline{\partial X})$ of the modulo 2 congruence class $A(f)$ of the integral k-chain $2^{-1}[(\partial T) \llcorner (\mathbf{R}^m \sim \underline{\partial X})]$. To show that $A(f) - B(f) \in \mathscr{B}_{k-1}(\underline{X}, \underline{\partial X})$, one computes the modulo 2 multiplicities of $A(f)$ and $B(f)$ at each $k-1$ dimensional $R \in \mathscr{S}$ with $R \cap \underline{\partial X} = \emptyset$ by counting the number of adjacent k dimensional $S \in \mathscr{S}$ which are mapped by f to each side of $f(R)$.

§7. The double point chain

Let m, k, j, X, f, s_f, and $\mathscr{D}(f)$ be as in 5.2 and $\pi : (\mathbf{R}^m)^2 \to \mathbf{R}^m$, $\pi(x, y) = x$ for $(x, y) \in (\mathbf{R}^m)^2$. The **double point chain** is the compact mod 2 $(k-j)$-chain in $\mathbf{R}^m$,

$$D(f) = \pi_{\#}\mathscr{D}(f).$$

7.1. Theorem: *If* m, k, j, X, f, g, h, *and* U *are as in* 5.2, $f = (f_1, \ldots, mf_{k+j})$, $j \geq 1$, *and*

$$f^{-1}(\mathbf{R}_+^{k+j}) = \underline{X} \sim \underline{\partial X} \text{ in case } \partial X \neq 0,$$

then:

(0) $\partial D(f) = D(f_\partial)$ *whenever* $j < k$ *where* $f_\partial = (f_2, \ldots, f_{k+j}) \mid \underline{\partial X}$,
(1) $h_{\#}D(f) = D(f \circ h^{-1})$,
(2) $D(f) \llcorner U = D[f \mid \underline{X \llcorner U}]$ *whenever* $f^{-1}[f(\underline{X} \cap U)] = \underline{X} \cap U$.
(3) $\mathbf{S}D(f) = D(\mathbf{S}f)$,
(4) $D(f) - D(g) \in \mathscr{B}_{k-j}(\underline{X}, \underline{\partial X})$ *whenever* $g^{-1}(\mathbf{R}_+^{k+j}) = \underline{X} \sim \underline{\partial X}$ for $\partial X \neq 0$.

PROOF OF (0): If $\partial X = 0$, then $\partial D(f) = \pi_{\#}\partial\mathscr{D}(f) = \pi_{\#}0 = 0$ by 2.5(4)(3) and 5.2(0). If $\partial X \neq 0$, then $\underline{\partial X} = f_1^{-1}\{0\}$ and, by 2.5(1)(3)(12),

$$\begin{aligned}\langle(\partial X^2) \mid \Gamma, s_{f_1}, 0+\rangle &= \partial([(\partial X^2) \mid \Gamma] \llcorner \{(x, y) : s_{f_1}(x, y) > 0\}) \\ &= \partial([(\partial X \times X) + (X \times \partial X) \mid \Gamma] \llcorner \{(x, y) : f_1(x) > f_1(y)\}) \\ &= \partial[(X \times \partial X) \mid \Gamma] = (\partial X)^2 \mid \Gamma.\end{aligned}$$

Using 5.1, 5.2(0), and 2.5(22), we compute that

$$\begin{aligned}\partial\mathscr{D}(f) &= C\langle(\partial X^2) \mid \Gamma, s_f, 0\rangle \\ &= C\langle\langle\cdots\langle\langle(\partial X^2) \mid \Gamma, s_{f_1}, 0+\rangle, s_{f_2}, 0+\rangle \cdots\rangle, s_{f_{k+j}}, 0+\rangle \\ &= C\langle\langle\cdots\langle(\partial X)^2 \mid \Gamma, s_{f_2}, 0+\rangle, \cdots\rangle, s_{f_{k+j}}, 0+\rangle \\ &= C\langle(\partial X)^2 \mid \Gamma, s_{f_\partial}, 0\rangle = \mathscr{D}(f_\partial).\end{aligned}$$

PROOF OF (1): Apply 2.5(2) and 5.2(1).

PROOF OF (2): Note that $\mathscr{D}(f) \llcorner \pi^{-1}(U) = \mathscr{D}(f) \llcorner U^2$ by 2.5(9) and apply 2.5(10) and 5.2(2).

PROOF OF (3): Letting δ be as in 5.2(3) and $\tilde{\pi}$ be defined as π was above with $\mathbf{R}^m$ replaced by $\mathbf{R} \times \mathbf{R}^m$, we observe that $\tilde{\pi}_{\#}\delta_{\#}\mathbf{S} = \mathbf{S}\pi_{\#}$ and then apply 5.2(3).

PROOF OF (4): Letting ι, Y, and λ be as in the proof of 5.2(4), we choose J, θ, μ, ν, M, and N as before with L replaced by

$$I = L \cap \{b : b(x, (u_1, \ldots, u_{k+j}), (v_1, \ldots, v_{k+j})) \in \operatorname{Clos} \mathbf{R}_+^{k+j}$$
$$\text{whenever } u_1 \geq 0 \text{ and } v_1 \geq 0\}.$$

Under the usual identification of L with $(m + 2k + 2j) \times (k + j)$ matrices, I becomes a closed quadrant in a Euclidean subspace of dimension $i = (m + 2k + 2j)(k + j - 1) + 2$. To obtain the crucial estimate, $\dim J \leq i - 1$, we repeat the proof of 2.3 (with $\operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^n)$, A, and B replaced by I, $\underline{Y}$, and either $\underline{Y}$ or $\underline{\partial Y}$) by computing that

$$\dim I \cap \{b : b[\iota(x) - \iota(y)] = 0\} \leq i - (k + j)$$

whenever $(x, y) \in \underline{X}^2$ and $x \neq y$. Noting that $\pi[\underline{C(M + N)}] \subset \underline{X}$ and that

$$(\pi \circ \lambda)[([0,1] \times \underline{\partial X^2}) \cap (\mu^{-1}\{0\} \cup \nu^{-1}\{0\})] \subset \underline{\partial X}$$

because $(b \circ \iota)^{-1}(\mathbf{R}_+^{k+j}) = \underline{X} \sim \underline{\partial X}$ for all $b \in I$, we conclude from 5.1, 2.5(4)(5)(6)(14), and the proof of 5.2(4) that

$$\begin{aligned} D(f) - D(g) &= \pi_\#[\mathscr{D}(f) - \mathscr{D}(g)] \\ &= -\pi_\# C\lambda_\#[\langle \overline{[0,1]} \times \partial(X^2 \mid \Gamma), \mu, 0\rangle \\ &\quad + \langle \overline{[0,1]} \times \partial(X^2 \mid \Gamma), \nu, 0\rangle \\ &\quad + \partial\pi_\# C(M+N) \in \mathscr{B}_{k-j}(\underline{X}, \underline{\partial X}). \end{aligned}$$

7.2. LEMMA: *For any compact* mod 2 *k-chain X in $\mathbf{R}^m$, $j \in \{0, 1, \ldots, k\}$, continuous function $g : (\underline{X}, \underline{\partial X}) \to (\operatorname{Clos} \mathbf{R}_+^{k+j}, \mathbf{R}_0^{k+j})$, and $\epsilon > 0$, there exists a subanalytic map $f : X \to \mathbf{R}^{k+j}$ satisfying* 5.2(*) *such that $f^{-1}(\mathbf{R}_+^{k+j}) = \underline{X} \sim \underline{\partial X}$ in case $\partial X \neq 0$, $|f(x) - g(x)| < \epsilon$, and $\dim f^{-1}\{f(x)\} = 0$ for all $x \in X$.*

PROOF: By the subanalytic triangulation theorem, ([11, Theorem 2] or [15]), we may assume that $\underline{X}$ and $\underline{\partial X}$ are compact polyhedra in $\mathbf{R}^m$. In this case, we may appeal to either [16, 4.8] or [28, Chapter 6, Theorem 18] to obtain first, a piecewise linear map $h : \underline{\partial X} \to \mathbf{R}_0^{k+j}$ so that

$\dim(\underline{\partial X})^2 \cap \{(x,y) : x \neq y, h(x) = h(y)\} \leq k - j - 1, \quad |h(x) - g(x)| < \epsilon$
and

$$\dim h^{-1}\{h(x)\} = 0 \quad \text{for all } x \in \underline{\partial X},$$

and

second, a piecewise linear map $f : \underline{X} \to \mathbf{R}^{k+j}$ so that $f \mid \underline{\partial X} = h$, $f^{-1}(\mathbf{R}_+^{k+j}) = \underline{X} \sim \underline{\partial X}$, $\dim \underline{X}^2 \cap \{(x,y) : x \neq y, f(x) = f(y)\} \leq k - j$, $|f(x) - g(x)| < \epsilon$ and $\dim f^{-1}\{f(x)\} = 0$ for all $x \in \underline{X}$.

7.3. DEFINITION: For any compact mod 2 k-chain X in $\mathbf{R}^m$, $\mathbf{D}^0(X) \in \mathbf{H}_k(\underline{X}, \underline{\partial X})$ is defined as the fundamental class of X and, for $i \in \{1, 2, \ldots, k\}$, $\mathbf{D}^i(X) \in \mathbf{H}_{k-i}(\underline{X}, \underline{\partial X})$ is well-defined as the homology class of $D(f)$ for any subanalytic map $f : \underline{X} \to \mathbf{R}^{k+i}$ satisfying 5.2(*) with $j = i$ and satisfying $f^{-1}(\mathbf{R}_+^{k+j}) = \underline{X} \sim \underline{\partial X}$ in case $\partial X \neq 0$ by 7.2 (which shows the existence of such f), 7.1(0) (which shows that $D(f) \in \mathscr{Z}_{k-i}(\underline{X}, \underline{\partial X})$), and 7.1(4) (which shows the independence of the choice of f).

THEOREM: *$\mathbf{D}^i$ is a geometric homology operation of degree i on* $\mathbf{H}$.

PROOF: This is clear for $i = 0$. For $i > 0$, properties 3.2(1) and 3.2(3) follow from 7.2, 7.1(1), and 7.1(3).

To verify 3.2(2), we recall the proof of 7.2 to choose a subanalytic map $f : \underline{X} \to \mathbf{R}^{k+i}$ such that $f^{-1}(\mathbf{R}_+^{k+i}) = \underline{X} \sim \underline{\partial X}$,

$$f^{-1}(\mathbf{R}^{k+i} \cap \{(x_1, \ldots, x_{i+1}) : x_1 > 0 < x_2\}) = (\underline{X \llcorner U}) \sim \underline{\partial(X \llcorner U)} \\ = (\underline{X} \sim \underline{\partial X}) \cap U.$$

For any subanalytic homeomorphism g mapping $\mathbf{R}^{k+i} \cap \{(x_1, \ldots, x_{i+1}) : x_1 \geq 0 \leq x_2\}$ onto $\mathbf{R}^{k+i} \cap \{(x_1, \ldots, x_{i+1}) : x_1 \geq 0\}$, $D(f) \llcorner U = D[f \mid (\underline{X \llcorner U})] = D[(g \circ f) \mid (\underline{X \llcorner U})]$, by 7.1(2) and 2.5(20); hence, $\mathbf{D}_i(X) \llcorner U = \mathbf{D}_i(X \llcorner U)$.

7.4. THEOREM: *For any compact* mod 2 k-chain X in $\mathbf{R}^m$ *with fundamental class* χ:

(0) $\mathbf{D}^0(X) = \chi$,

(1) $\mathbf{D}^1(X) = \boldsymbol{\beta}(X)$, *the Bockstein class of* X,

(2) $\mathbf{D}^{m-k}(X) = \chi \cdot \chi$ *whenever* $m - k \leq k$,

(3) $\mathbf{D}^i(X) = 0$ *whenever* $m - k < i \leq k$.

PROOF OF (0): See definition 7.3.

PROOF OF (3): For $x = (x_1, \ldots, x_m) \in \underline{X}$, let $u(x) = 1$ if $\partial X = 0$, $u(x) = \operatorname{dist}(x, \underline{\partial X})$ if $\partial X \neq 0$, and $f(x) = (u(x), x_1, x_2, \ldots, x_m, 0, \ldots, 0) \in \mathbf{R}^{k+i}$. Then $f^{-1}(\mathbf{R}_+^{k+i}) = \underline{X} \sim \underline{\partial X}$ and $D(f) = \pi_\# \mathscr{D}(f) = \pi_\# 0 = 0$.

PROOF OF (2): Let $\tilde{\Gamma}$ and $\tilde{\pi}$ be defined as Γ and π were in 5.0 and 7.0 with $\mathbf{R}^m$ replaced by $\mathbf{R} \times \mathbf{R}^m$, choose U, V, a and Y as in the proof of 4.4(1), and let

$$Z = g_\#([\overline{0, 1}] \times \partial X) \quad \text{where } g(t, x) = (t, x + ta) \quad \text{for } (t, x) \in \mathbf{R} \times \mathbf{R}^m,$$

$$W = (\{\bar{0}\} \times X) + (\{\bar{1}\} \times Y) + Z, \; G = (\mathbf{R} \times \mathbf{R}^m) \sim (\{1\} \times \underline{Y}) \sim \underline{Z},$$

$$f : W \to \mathbf{R}^m, \; f(t, x) = x \quad \text{for } (t, x) \in W, \text{ and } h = f \mid \{0\} \times \underline{X}.$$

Then $\dim \underline{W}^2 \cap \{(v, w) : v \neq w,\ f(v) = f(w)\} \leq 2k - m$, and $\partial W = 0$ by 2.5(3)(4)(5). Since $\{\bar{0}\} \times X = W \llcorner G$ and

$$\mathbf{D}^{m-k}(X) = \mathbf{H}_{2k-m}(h)\mathbf{D}^{m-k}(W \llcorner G) = \mathbf{H}_{2k-m}(h)\mathbf{D}^{m-k}(W) \llcorner G$$

by 7.3, it suffices, by 2.6(5), to show that the modulo 2 support of

$D(f) \llcorner G - \overline{\{0\}} \times (X \cap^2 Y)$ is contained in $\{0\} \times V$. To see this, we note, by 2.5(5), that $\underline{f_\# Z} \subset f(Z) \subset V$, and use 7.0, 5.2, and 2.5(10)(21)(18) to compute

$$D(f) \llcorner G = (\tilde{\pi}_\# C\langle W^2 \,|\, \tilde{\Gamma}, s_f, 0\rangle) \llcorner G$$
$$= \tilde{\pi}_\# C\langle([\overline{\{0\}} \times X] \times [(\overline{\{0\}} \times X) + (\overline{\{1\}} \times Y) + Z]) \,|\, \tilde{\Gamma}, s_f, 0\rangle$$
$$= \tilde{\pi}_\#(0) + \overline{\{0\}} \times (X \cap^2 Y) + \overline{\{0\}} \times (X \cap^2 f_\# Z).$$

Proof of (1): By 7.3, (2), 3.5, and 4.2, $\mathcal{J}(\mathbf{D}^1)$ is the unique Steenrod homology operation of degree 1; thus, $\mathbf{D}^1 = \boldsymbol{\beta}$ by 4.4(1).

7.5. Corollary: (*Existence*) *The double point operation* $\mathcal{J}(\mathbf{D}^i)$ *is the Steenrod homology operation of degree* i *for every nonnegative integer* i.

Proof: Combine 3.5, 4.2, 4.4, 7.3, and 7.4(2).

7.6. Remarks:

(1) *One may verify* 7.4(1) *without appealing to the uniqueness theory of* §4. As in the proof of 4.2, we may assume $\partial X = 0$. Choose first, by 2.3(4)(5), a subanalytic map $f: \underline{X} \to \mathbf{R}^{k+1}$ so that $\dim \underline{X}^2 \cap \{(x, y): x \neq y, f(x) = f(y)\} \leq k - 1$ and so that $\dim f^{-1}\{w\} \leqq 0$ whenever $w \in \mathbf{R}^{k+1}$, and second, a subanalytic stratification $\mathcal{S}$ of $\underline{X}$ as in [9, 4.4] with $\mathcal{C} = \{\underline{X}, D(f)\}$. Let $\mathcal{Q} = \mathcal{S} \cap \{S: \dim S = k\}$, $\mathcal{R} = \mathcal{S} \cap \{S: \dim S = k - 1\}$, and G be the unique bounded open subset of $\mathbf{R}^{k+1}$ with $\partial \overline{G} = f_\# X$. If $T = \Sigma_{R \in \mathcal{R}} (f \,|\, R)_\#^{-1}[(\partial G_0) \llcorner f(R)]$ where G_0 is as in the proof of 4.4(1), then it suffices to show that the modulo 2 congruence class of $\frac{1}{2}(\partial T) \llcorner (\underline{X} \sim \underline{\partial X})$ equals $D(f)$. This involves a local argument (similar to the proof of 4.4(1)) about (x, y) for each pair x, y of distinct points in $(\cup \mathcal{Q}) \sim \underline{\partial X}$ with $f(x) = f(y)$.

(2) *One may verify the equality* $\mathbf{B}^i = \mathbf{D}^i$ *for all* i *without appealing to the uniqueness theory of* §4 by recalling 7.2, 7.3, and 6.2 and by establishing the following:

Theorem: *If* k, j, X, *and* f *are as in* 7.1 *and if* $\dim f^{-1}\{w\} \leq 0$ *whenever* $w \in \mathbf{R}^{k+j}$, *then* $D(f) - B(p \circ f) \in \mathcal{B}_{k-j}(\underline{X}, \underline{\partial X})$ *for all* $p \in \mathrm{Hom}(\mathbf{R}^{k+j}, \mathbf{R}^{k+j-1})$ *such that* (*see* 2.3) $p(\mathbf{R}_0^{k+j}) = \mathbf{R}^{k+j-1}$, $\dim f(\underline{X})^2 \cap \{(v, w): v \neq w, p(v) = p(w)\} \leq k - j + 1$, *and* $\dim[f(\underline{X}) \times f(\underline{\partial X})] \cap \{(v, w): v \neq w, p(v) = p(w)\} \leq k - j$.

Proof: The function $p \circ f$ satisfies 5.2(*) with j replaced by $j - 1$

because f satisfies 5.2(*) and

$$\begin{aligned}&\underline{X}^2 \cap \{(x,y): x \neq y,\ (p\circ f)(x) = (p\circ f)(y)\}\\ &= [\underline{X}^2 \cap \{(x,y): x\neq y, f(x) = f(y)\}]\\ &\qquad\qquad \cup (f\times f)^{-1}\{(v,w): v \neq w, p(v) = p(w)\}.\end{aligned}$$

By replacing $\underline{X}$ by $\{(x, f(x)): x \in \underline{X}\} \cup \{(x, -f(x)): x \in \underline{X}\}$, applying 6.1(2) and 7.1(2) with $U = \mathbf{R}^m \times \mathbf{R}_+^{k+j}$, and applying 6.1(1) and 7.1(1) with $h(x) = (x, f(x))$ for $x \in \underline{X}$, we may assume $\underline{\partial X} = 0$.

Letting $Y = \{X^2 \mid \Gamma, s_{p\circ f}, 0\rangle \llcorner \{(x,y): s_{f_1}(x,y) > 0\}$ where $f = (f_1, \ldots, f_{k+j})$, we use 2.5(19)(20)(22) and 5.2 to deduce that $C(\partial Y) = \mathscr{D}(f)$. Letting σ, τ, and $\boldsymbol{\sigma}$ be as in 6.1, we note that $\underline{(\partial C - C\partial)Y} \subset \Delta$, $\pi \mid \Delta = (\tau\circ\sigma) \mid \Delta$, $\sigma_\# \mathscr{D}(f) = 0$, $\partial\mathscr{D}(p\circ f) = 0$, and $\sigma_\# CY = \boldsymbol{\sigma}\mathscr{D}(p\circ f)$, and we conclude, by 6.1, 7.0, and 2.5(2)(4)(5), that

$$\begin{aligned}B(p\circ f) - D(f) &= \tau_\#(\partial\boldsymbol{\sigma} - \boldsymbol{\sigma}\partial)\mathscr{D}(p\circ f) + \pi_\#\mathscr{D}(f)\\ &= \tau_\#\partial\boldsymbol{\sigma}\mathscr{D}(p\circ f) - 0 + \pi_\#\mathscr{D}(f)\\ &= \tau_\#\sigma_\#[\partial C(Y) - \mathscr{D}(f)] + \pi_\#\mathscr{D}(f)\\ &= \pi_\#(\partial C - C\partial)Y + \pi_\# C(\partial Y) = \partial\pi_\# C(Y)\\ &\qquad\qquad \in \mathscr{B}_{k-i}(\underline{X}).\end{aligned}$$

§8. Real projective space

For $(x_0, x_1, \ldots, x_k) \in \mathbf{R}^{k+1}$, let $[x_0, \ldots, x_k] \in \mathbf{R}^{\binom{k+1}{2}}$ be the point

$$(x_0^2, 2^{1/2}x_0x_1, \ldots, 2^{1/2}x_0x_k, x_1^2, 2^{1/2}x_1x_2, \ldots, 2^{1/2}x_1x_k, \ldots, 2^{1/2}x_{k-1}x_k, x_k^2).$$

Then $\mathbf{P}^k = \{[x_0, x_1, \ldots, x_k]: x_0^2 + \cdots + x_k^2 = 1\}$ is a k dimensional real analytic projective space, called the **Veronese variety**. For $i \in \{0, \ldots, k\}$, let ω_i^k be the nonzero element of $\mathbf{H}_i(\mathbf{P}^k) \simeq \mathbf{Z}_2$ ([2, 6.1.3]). This is the image of the fundamental class of $\mathbf{P}^i$ under the usual embedding of $\mathbf{P}^i$ into $\mathbf{P}^k$.

8.1. *Some specific projections of the Veronese surface*

Since $\omega_1^2 = \boldsymbol{\Phi}^1(\omega_2^2) = \mathbf{D}^1(\mathbf{P}^2) = \mathbf{B}^1(\mathbf{P}^2)$, ω_1^2 is generated either by $D(f)$ for any subanalytic map $f: \mathbf{P}^2 \to \mathbf{R}^3$ having a 1 dimensional double point pair set or by $B(g)$ for any subanalytic map $g: \mathbf{P}^2 \to \mathbf{R}^3$ having a 2 dimensional double point pair set.

(1) *Steiner's Roman surface* (cf. [13, p. 303]), $f: \mathbf{P}^2 \to \mathbf{R}^3$,

$f[x, y, z] = (2^{1/2}xy,\ 2^{1/2}xz,\ 2^{1/2}yz)$. Here $D(f)$ is the sum of the three cycles, $x = 0$, $y = 0$, and $z = 0$, in $\mathbf{P}^2$. The double point image of f is the collection of points (u, v, w) in $\mathbf{R}^3$ with at least two coordinates zero and $u^2 + v^2 + w^2 < 1/2$. $B(f)$ consists of the six points, $[0, 2^{1/2}/2, \pm 2^{1/2}/2]$, $[2^{1/2}/2, 0, \pm 2^{1/2}/2]$, $[2^{1/2}/2, \pm 2^{1/2}/2, 0]$. The images of these six points are "pinch points" of the image of f.

(2) *The crosscap.* (cf. [13, p. 315]), $f: \mathbf{P}^2 \to \mathbf{R}^3$, $f[x, y, z] = ((x^2 - y^2)/2^{1/2},\ 2^{1/2}xy,\ 2^{1/2}yz)$. Here $D(f)$ is the cycle, $y = 0$, in $\mathbf{P}^2$. The double point image of f is the line segment, $0 < u < 2^{1/2}/2$, $v = 0 = w$, in $\mathbf{R}^3$. $B(f)$ consists of the two points, $[1, 0, 0]$ and $[0, 0, 1]$, whose images are pinch points of the image of f.

(3) $f: \mathbf{P}^2 \to \mathbf{R}^3$, $f[x, y, z] = (x^2,\ 2^{1/2}xy,\ 2^{1/2}yz)$. Here $D(f)$ is again the cycle, $y = 0$. The cycle, $x = 0$, also maps to double points, but it occurs with zero multiplicity in $D(f)$ because the image of f is tangent to itself along the line segment, $u = 0 = v$, $-2^{1/2}/2 < w < 2^{1/2}/2$.

(4) $f: \mathbf{P}^2 \to \mathbf{R}^3$, $f[x, y, z] = (x^2, y^2, 2^{1/2}yz)$. Here $D(f)$ is not defined because the double point pair set of f has dimension two.

(5) $f: \mathbf{P}^2 \to \mathbf{R}^3$, $f[x, y, z] = (x^2, 2^{1/2}xy, 2^{1/2}xz)$. Here $D(f)$ is not defined because the double point pair set of r has dimension two. The circle, $x = 0$, is mapped to the origin; off of this circle, f is an embedding.

(6) $g: \mathbf{P}^2 \to \mathbf{R}^2$, $g[x, y, z] = (x^2, y^2)$. Here $B(g)$ is the sum of the cycles, $x = 0$, $y = 0$, and $z = 0$. The image of g is the closed triangle with vertices, $(0, 0)$, $(1, 0)$, and $(0, 1)$.

(7) $g: \mathbf{P}^2 \to \mathbf{R}^2$, $g[x, y, z] = (x^2, 2^{1/2}xy)$. Here $B(g)$ is the cycle, $z = 0$, along which g has a "fold".

(8) $g: \mathbf{P}^2 \to \mathbf{R}^2$, $g[x, y, z] = (x^2, 2^{1/2}yz)$. Here $B(g)$ is the sum of the cycles, $x = 0$ and $y = +z$. (The latter is a boundary.) The image of g is a closed triangle with vertices, $(1, 0)$, $(0, 2^{1/2}/2)$, and $(0, -2^{1/2}/2)$.

(9) $g: \mathbf{P}^2 \to \mathbf{R}^2$, $g[x, y, z] = (2^{1/2}xy, 2^{1/2}yz)$. Here $B(g)$ is the sum of the cycles, $y = 0$ and $y = \pm 2^{1/2}/2$. The image of g is the closed disc K of radius $2^{1/2}/2$ about the origin in $\mathbf{R}^2$. Moreover g maps the circle, $y = 0$, to the origin, and g has a fold along $y = \pm 2^{1/2}/2$. Over the interior of K minus the origin, g is a double covering.

Note that, in (7), (8), and (9), g is the composition of the map of (3) by the projection map of $\mathbf{R}^3$ onto one of the coordinate planes.

8.2. *Evaluation of Φ^i on* $\mathbf{H}_*(\mathbf{P}^k)$

Fix a generic $k \times k$ matrix (a_{lm}), and, for $[x_0, \ldots, x_k] \in \mathbf{P}^k$ and $i \in \{0, \ldots, k\}$, let

$$g_1^k[x_0, \ldots, x_k] = x_1(a_{11}x_0 + a_{12}x_2 + \cdots + a_{1k}x_k),$$

$$g_2^k[x_0, \ldots, x_k] = x_2(a_{21}x_0 + a_{22}x_1 + a_{23}x_3 + \cdots + a_{2k}x_k),$$

$$\vdots$$

$$g_k^k[x_0, \ldots, x_k] = x_k(a_{k1}x_0 + a_{k2}x_1 + \cdots + a_{kk}x_{k-1}),$$

$$f_i^k[x_0, \ldots, x_k] = (x_1^2, \ldots, x_k^2, g_1^k[x_0, \ldots, x_k], \ldots, g_i^k[x_0, \ldots, x_k]) \in \mathbf{R}^{k+i}.$$

By first enumerating possible double-point pairs $([x_0, \ldots, x_k], [y_0, \ldots, y_k])$ of f_0^k according to various differences in sign between x_jx_0 and y_jy_0 for $j \in \{1, \ldots, k\}$, we repeatedly form transverse slices in $(\mathbf{P}^k)^2 \cap \{(x, y) : x \neq y\}$ to find that f_k^k is injective and that, for $i \in \{1, \ldots, k-1\}$, $D(f_i^k)$ is the sum of certain $k-i$ dimensional linear subvarieties of $\mathbf{P}^k$. Since each such subvariety generates ω_{k-i}^k, our computations and 7.5 show that $\boldsymbol{\Phi}^i(\omega_i^k) = \mathbf{D}^i(\mathbf{P}^k) = m_i^k\omega_{k-i}^k$ where, with $m_0^k = 2^k - 1$ and $m_k^k = 0$, m_i^k is determined by the inductive relation, $m_i^k = m_{i-1}^k + m_i^{k-1}$. Pascal's identity implies that $m_i^k = \binom{i+k}{i}$ modulo 2. Since, for $0 \le i \le j \le k$, the usual embedding of $\mathbf{P}^j$ into $\mathbf{P}^k$ maps, in homology, ω_j^j to ω_j^k and ω_{j-i}^j to ω_{j-i}^k, the naturality of $\boldsymbol{\Phi}^i$ gives the formula

$$\boldsymbol{\Phi}^i(\omega_j^k) = \binom{i+j}{i}\omega_{j-i}^k.$$

This is consistent with results obtained in cohomology using [20, 4.2, 4.4], Thom's identity [20, p. 91], and 10.1 below.

§9. Embedding and immersion in Euclidean spaces

In this section we prove a nonembedding theorem for subanalytic sets which is dual to Thom's nonembedding theorem [3, III, 1.5]. In fact we extend Thom's theorem to pairs of spaces, and we prove an analogous non-immersion theorem, which is also new.

We pass from subanalytic maps to arbitrary continuous maps by the following result on maps which are "almost" immersions or embeddings. It generalizes a theorem of C. Weber [27, p. 134, Cor.].

9.1. THEOREM: *For any compact* mod 2 *k-chain X in $\mathbf{R}^m$ and $i \in \{1, 2, \ldots, k\}$, there is a positive number ϵ_X such that:*

(1) *If there exists a continuous map $g : X \to R^{k+i-1}$ and $0 < \epsilon < \epsilon_X$ for which*

$$\underline{X}^2 \cap \{(x, y) : |x - y| = \epsilon, g(x) = g(y)\} = \emptyset,$$

then $\mathbf{B}^i(X) = 0$.

(2) *If there exists a continuous map* $g:(\underline{X}, \underline{\partial X}) \to (\mathrm{Clos}\, \mathbf{R}_+^{k+i}, \mathbf{R}_0^{k+i})$ *for which*

$$\underline{X}^2 \cap \{(x,y): |x-y| \geq \epsilon_X,\ g(x)=g(y)\} = \emptyset,$$

then $\mathbf{D}^i(X) = 0$.

PROOF: Recalling the notations $\Delta, \mathscr{D}(f), \sigma, \tau, \pi$, and $\boldsymbol{\sigma}$ from §5, §6, and §7, we choose, by the reasoning of 2.6(4), relatively open subanalytic neighborhoods N of $\sigma(\Delta \cap \underline{X}^2)$ in $\sigma(\underline{X}^2)$ and N_∂ of $\sigma[\Delta \cap (\underline{\partial X})^2]$ in N so that the pair $(\sigma(\Delta \cap \underline{X}^2), \sigma[\Delta \cap (\underline{\partial X})^2])$ is a strong subanalytic deformation retract of (N, N_∂) with retraction r, such that $r[N \cap \sigma(\underline{\partial X^2})] \subset \sigma[\Delta \cap (\underline{\partial X})^2]$. Let

$$\epsilon_X = 2^{1/2} \operatorname{dist}[\Delta \cap \underline{X}^2, \sigma^{-1}(\mathrm{Bdry}\, U)].$$

PROOF OF (1): Choose ϵ as in (1) and, by the compactness of $\underline{X}$, a positive δ so that $|x-y| \neq \epsilon$ whenever $|g(x)-g(y)| < \delta$. Let f be as in 7.2 with j and ϵ replaced by $i-1$ and $\delta/2$. Then $N_\epsilon = \sigma(\underline{X}^2 \cap \{(x,y): |x-y| < \epsilon\}) \subset N$, and, by 2.5(13), the support of $\partial([\boldsymbol{\sigma}\mathscr{D}(f)] \llcorner N_\epsilon) - (\partial\boldsymbol{\sigma}\mathscr{D}(f)) \llcorner N_\epsilon$ is contained in $[\underline{\sigma\mathscr{D}(f)}] \cap \mathrm{Fron}\, N_\epsilon$, which is empty because $|g(x)-g(y)| < \delta$ whenever $f(x)=f(y)$. By 2.5(2)(4) and 6.1,

$$\begin{aligned} B(f) &= \tau_\#(\partial\boldsymbol{\sigma} - \boldsymbol{\sigma}\partial)\mathscr{D}(f) = \tau_\# r_\#[(\partial\boldsymbol{\sigma}\mathscr{D}(f)) \llcorner N_\epsilon - (\boldsymbol{\sigma}\partial\mathscr{D}(f)) \llcorner N_\epsilon] \\ &= \tau_\# r_\#[\partial([\boldsymbol{\sigma}\mathscr{D}(f)] \llcorner N_\epsilon) - (\boldsymbol{\sigma}\partial\mathscr{D}(f)) \llcorner N_\epsilon] \\ &= \partial[\tau_\# r_\#(\boldsymbol{\sigma}\mathscr{D}(f) \llcorner N_\epsilon)] - \tau_\# r_\#[(\boldsymbol{\sigma}\partial\mathscr{D}(f)) \llcorner N_\epsilon] \\ &\in \mathscr{B}_{k-i}(\underline{X}, \underline{\partial X}). \end{aligned}$$

PROOF OF (2): Choose by the compactness of $\underline{X}$, a positive γ so that $\sigma(x,y) \in N$ whenever $(x,y) \in \underline{X}^2$ and $|g(x)-g(y)| < \gamma$. Let f be as in 7.2 with j and ϵ replaced by i and $\gamma/2$. Then $\underline{\boldsymbol{\sigma}\mathscr{D}(f)} \subset N$ because $|g(x)-g(y)| < \gamma$ whenever $f(x)=f(y)$. Since (N, N_∂) deforms to $(\sigma(\Delta \cap \underline{X}^2), \sigma[\Delta \cap (\underline{\partial X})^2])$ and $\underline{\partial\boldsymbol{\sigma}\mathscr{D}(f)} \subset N_\partial$, there are subanalytic chains P, Q, and R so that

$$\underline{\partial\boldsymbol{\sigma}(f) - \partial P} \subset \sigma[\Delta \cap (\underline{\partial X})^2] \subset \sigma(\Delta \cap \underline{X}^2), \quad \underline{P} \subset N_\partial,$$

$$\boldsymbol{\sigma}\mathscr{D}(f) - P = Q + \partial R, \quad \underline{Q} \subset \sigma(\Delta \cap \underline{X}^2), \text{ and } \underline{R} \subset N.$$

Since σ is a two-sheeted covering map, branched along Δ, $\mathscr{D}(f) =$

$\boldsymbol{\sigma}^{-1}P + \partial\boldsymbol{\sigma}^{-1}R$. Thus, by 2.5(4),

$$D(f) = \pi_{\#}\mathscr{D}(f) = \pi_{\#}\boldsymbol{\sigma}^{-1}P + \partial\pi_{\#}\boldsymbol{\sigma}^{-1}R \in \mathscr{B}_{k-i}(\underline{X}, \partial\underline{X}).$$

9.2. By an **embedding** we mean any continuous injective map. By an **immersion** we mean any continuous locally injective map.

Recall from 6.4 and 7.5 that the Steenrod homology operation $\boldsymbol{\Phi}^i$ equals $\mathscr{J}(\mathbf{B}^i) = \mathscr{J}(\mathbf{D}^i)$ for any nonnegative integer i.

THEOREM: *Suppose $A \supset B$ are subanalytic sets and i, k, and n are positive integers.*

(1) *If A immerses in $\mathbf{R}^n$, then $\boldsymbol{\Phi}^i(\alpha) = 0$ whenever $\alpha \in \mathbf{H}_k(A, B)$ and $i > n - k$.*
(2) *If (A, B) embeds in $(\operatorname{Clos}\mathbf{R}^n_+, \mathbf{R}^n_0)$, then $\boldsymbol{\Phi}^i(\alpha) = 0$ whenever $\alpha \in \mathbf{H}_k(A, B)$ and $i \geq n - k$.*

PROOF: If $X \in \mathscr{X}_k(A, B)$ represents α and $\iota : (\underline{X}, \partial\underline{X}) \to (A, B)$ is the inclusion map, then by 3.5,

$$\boldsymbol{\Phi}^i(\alpha) = \mathbf{H}_{k-i}(\iota)\mathbf{B}^i(X) = \mathbf{H}_{k-i}(\iota)\mathbf{D}^i(X).$$

Since any immersion of A into $\mathbf{R}^n$ induces an immersion of $\underline{X}$ into $\mathbf{R}^{k+i-1}$ for $i > n - k$ and any embedding of (A, B) into $(\operatorname{Clos}\mathbf{R}^n_+, \mathbf{R}^n_0)$ induces an embedding of $(\underline{X}, \partial\underline{X})$ into $(\operatorname{Clos}\mathbf{R}^{k+i}_+, \mathbf{R}^{k+i}_0)$ for $i \geq n - k$, (1) and (2) follow from 9.1(1) and 9.1(2).

§10. Relation to cohomology operations

Let H_* be the modulo 2 singular homology functor, which is, by 2.6(2), naturally equivalent with modulo 2 subanalytic homology $\mathbf{H}_*$ on the subanalytic category. For each nonnegative integer i, let Φ^i be the operation on H_* corresponding to the Steenrod homology operation $\boldsymbol{\Phi}^i$ on $\mathbf{H}_*$. The following theorem describes the relation between the operations Φ^i and the Steenrod operations Sq^i on modulo 2 singular cohomology H^* [3].

Let $\langle\ ,\ \rangle$ be the Kronecker pairing between H^* and H_* and, for each nonnegative integer i, let $\overline{Sq}^i$ be the modulo 2 cohomology operation which is determined by the relations,

$$\overline{Sq}^0 = 1 \quad \text{and} \quad \Sigma_i \overline{Sq}^i Sq^{j-i} = 0 = \Sigma_i Sq^{j-i}\overline{Sq}^i \quad \text{for } j > 0,$$

(cf. [20, p. 136, 11E], [3, II, §4]).

10.1. THEOREM: *If $\alpha \in H^k(A, B)$ and $\beta \in H_l(A, B)$, then*

$$\langle \alpha, \Phi^i \beta \rangle = \langle \overline{Sq}^i \alpha, \beta \rangle.$$

PROOF: Following Milnor and Stasheff [20, p. 136], we define an action of Sq^i on *homology* by the identity $\langle \alpha, Sq^i \beta \rangle = \langle \overline{Sq}^i \alpha, \beta \rangle$. The naturality of this action follows directly from the naturality of the action of $\overline{Sq}^i$ on cohomology. Therefore, by the uniqueness theorem 4.2, we have only to check that the action of Sq^i on homology satisfies the self intersection axiom of 4.1.

Let $\alpha \in H_k(A, B)$ where $k \geq i$, $A \supset B$ are subanalytic subsets of $\mathbf{R}^m$, and $m = k + i$. To show that $sq^i(\alpha) = \alpha . \alpha$, we may assume that A and B are compact. Let M and N be compact m dimensional subanalytic manifolds with boundary in $\mathbf{R}^m$ such that N is contained in the interior of M, and M and N contain A and B, respectively, as strong subanalytic deformation retracts. Let $\iota : (A, B) \to (M, N)$ be the inclusion map and let $\mathcal{M}$ be the fundamental class of M. By Poincaré-Lefschetz duality [2, VIII, 7.2], there is a class $\beta \in H^{m-k}(M \sim N, \text{Fron}\, M)$ such that $H_k(\iota)\alpha = \beta \frown \mathcal{M}$ where $\frown$ is the cap product. Then $H_{k-i}(\iota)(\alpha \cdot \alpha) = (\beta \smile \beta) \frown \mathcal{M}$ where $\smile$ is the cup product. Thus, by the naturality of Sq^i, the parallelizability of M, the cap product formula of [20, p. 136, 11-F] (the proof of which uses the Cartan formula for Sq^i [3, I, 1(5)]), and the cup product axiom for Sq^i [3, I, 1(3)],

$$\begin{aligned} H_{k-i}(\iota) Sq^i \alpha &= Sq^i H_k(\iota)\alpha = Sq^i(\beta \frown \mathcal{M}) \\ &= \sum_j Sq^j \beta \frown Sq^{i-j} \mathcal{M} = Sq^i \beta \frown \mathcal{M} \\ &= (\beta \smile \beta) \frown \mathcal{M} = H_{k-i}(\iota)(\alpha \cdot \alpha); \end{aligned}$$

hence, $Sq^i \alpha = \alpha \cdot \alpha$.

10.2. COROLLARY: *If $\alpha \in H^k(A, B)$ and $\beta \in H_l(A, B \cup C)$, then, in $H_{l-k-i}(A, C)$,*

$$\Phi_i(\alpha \frown \beta) = \sum_j Sq^j \alpha \frown \Phi_{i-j} \beta.$$

This is just a restatement of the formula $Sq(\alpha \frown \beta) = Sq\, \alpha \frown Sq\, \beta$ of [20, p. 136, 11F].

REFERENCES

[1] A. BOREL and A. HAEFLIGER: La class d'homologie fondamentale d'un espace analytique. *Bull. Soc. Math. France 89* (1961) 461–513.

[2] A. DOLD: *Lectures on Algebraic Topology*. Berlin-Heidelberg-New York: Springer 1972.

[3] S. EILENBERG and N. STEENROD: *Foundations of Algebraic Topology*. Princeton: Princeton University Press 1952.

[4] H. FEDERER: *Geometric Measure Theory*. Berlin-Heidelberg-New York: Springer 1969.

[5] A.M. GABRIÉLOV: Projections of semi-analytic sets. *Funktsional' nyi analiz i ego prilozheniya 2*, no. 4 (1968) 18–30 [Functional analysis and its applications 2, no. 4, 282–291 (1968)].

[6] R.M. HARDT: Slicing and intersection theory for chains associated with real analytic varieties. *Acta Math. 129* (1972) 75–136.

[7] R.M. HARDT: Slicing and intersection theory for chains modulo ν associated with real analytic varieties. *Trans. Amer. Math. Soc. 183* (1973) 327–340.

[8] R.M. HARDT: Homology theory for real analytic and semianalytic sets. *Annali Sc. Norm. Sup. Pisa, s. 4,2,* n.1, (1975) 107–148.

[9] R.M. HARDT: Stratification of real analytic mappings and images. *Inventiones math. 28* (1975) 193–208.

[10] R.M. HARDT: Topological properties of subanalytic sets. *Trans. Amer. Math. Soc. 211* (1975) 57–70.

[11] R.M. HARDT: Triangulation of subanalytic sets and proper light subanalytic maps. Inventiones Math. 38 (1977) 207–217.

[12] R.M. HARDT: Homology and images of semianalytic sets. *Bull. Amer. Math. Soc. 80* (1974) 675–678.

[13] D. HILBERT and S. COHN-VOSSEN: *Geometry and the Imagination*. New York: Chelsea 1952.

[14] H. HIRONAKA: Subanalytic sets. In: *Number Theory, Algebraic Geometry, and Commutative Algebra in honor of Y. Akizuki*, pp. 453–493. Tokyo: Kinokuniya Publications 1973.

[15] H. HIRONAKA: Triangulations of algebraic sets. In: *Proceedings of Symposia in Pure Mathematics, Amer. Math. Soc. 29* (1975) 165–185.

[16] J.F.P. HUDSON: *Piecewise Linear Topology*. New York: Benjamin 1969.

[17] S. KLEIMAN: The enumerative geometry of singularities. In: *Real and Complex Singularities, Oslo* 1976, Sijthoff & Noordhoff International Publishers.

[18] C.G. MCCRORY: Geometric homology operations. *Studies in Algebraic Topology, Advances in Mathematics Supplementary Studies*, 5 (1978) 119–141.

[19] R. MACPHERSON: Chern classes for singular algebraic varieties. *Ann. of Math. 100* (1974) 423–432.

[20] J. MILNOR and J. STASHEFF: Characteristic Classes. *Annals of Math. Study 76.* Princeton: Princeton University Press 1974.

[21] F. RONGA: La classe duale aux points doubles d'une application. *Compositio Math. 27* (1973) 223–232.

[22] N. STEENROD and D. EPSTEIN: Cohomology Operations. *Annals of Math. Study 50.* Princeton: Princeton University Press 1962.

[23] D. SULLIVAN: Combinatorial invariants of analytic spaces. In: *Proceedings of the Liverpool Singularities Symposium I,* Springer Lecture Notes in Math. *192.* Berlin-Heidelberg-New York: Springer 1971.

[24] R. THOM: Espaces fibrés en sphères et carrés de Steenrod. *Annali. Sc. Norm. Sup. 69* (3), (1952) 109–181.

[25] R. THOM: Les singularités des applications différentiables. *Ann. Inst. Fourier 6,* (1956) 43–87.

[26] W.-T. Wu: *A Theory of Imbedding, Immersion, and Isotopy of Polytopes in a Euclidean Space.* Peking: Science Press 1965.

[27] C. Weber: ϵ-Applications dans une variété. *Comm. Math. Helv. 49* (1974) 125–135.

[28] E.C. Zeeman: *Seminar on Combinatorial Topology.* Inst. Hautes Études Sci., Bures-Sur-Yvette, and the University of Warwick at Coventry, 1963–66.

(Oblatum 3-I-1978)

Clint G. McCrory
Department of Mathematics
Brown University
Providence, Rhode Island 02912
USA

Robert M. Hardt
School of Mathematics
University of Minnesota
Minneapolis, Minnesota 55455
USA