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HOMOTOPY THEORY OF SIMPLICIAL SCHEMES

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Introduction

If one reads the current literature on simplicial schemes, one finds that it is divided into two seemingly different kinds of theory. Let us briefly recall them.

First, there is the homotopy-type approach, due to Friedlander [8] and Hoobler-Rector [9]. The main idea is essentially the following: given a simplicial scheme $X.$, consider bisimplicial schemes $U..$ so that each $U_{p.}$ is an étale hypercovering of X_p. Then the simplicial sets $\Delta\pi(U..)$ (π means connected components, Δ means diagonal) form a pro-object (which is defined to be the homotopy type of $X.$) in the homotopy category. While one can get good theorems out of this approach (see [8], [9] and §IV.3), the definition has a bit of an ad-hoc flavor, and there are several loose ends. (For example, how does the cohomology of the homotopy type relate to $X.$?)

And then there is the more cohomological theory of simplicial schemes due to Deligne and Illusie. It is developed in massive generality in [11, Ch. VI] and [14, V bis and VI.7] (one should also read [6, §5] for a much more elementary and readable exposition). This theory deals mainly with the cohomology of $X.$ (meaning either cohomology groups or higher direct images) and its relation to the cohomology of each X_n.

The aim of this paper is to create a homotopy theory for simplicial schemes which agrees with Friedlander's and yet uses all of the available machinery – which means not only the above mentioned theory of Deligne and Illusie, but also the work of Artin and Mazur [3].

There is an initial problem in trying to do this. In the above approach to the homotopy type of $X.$, one is working in a category

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we call $Et(X.)$, the category of maps $Y. \to X.$ where each $Y_n \to X_n$ is étale. But one cannot apply Illusie's theory to $Et(X.)$ – it is not big enough (in the specific sense that the maps $X_n \times \Delta[n] \to X.$ are not in general in $Et(X.)$). Our solution is to enlarge $Et(X.)$ to a category we call $C(X.)$ (which consists of all maps $Y. \to X.$ where each $Y_n \to X_n$ is locally of finite type). Then, with suitable restrictions on $X.$ (see §I.3), we can apply the machinery of both Illusie and Artin-Mazur to $C(X.)$, and we get a homotopy type for $X.$ which agrees (up to weak equivalence) with the one outlined above.

The paper has four chapters, and here is a summary of their main points.

In Chapter I, we discuss the two notions of sheaf on a simplicial scheme (Def.'s I.2 and I.4). The first kind of sheaf has the best properties (see the examples of §I.4 and also Prop. II.2), while the second kind turns out to be sheaves on $C(X.)$ (this is Prop. I.5). There are sheaves (called local systems – Def. I.7) for which the two notions agree, and the important fact for us is that locally constant sheaves are local systems (Prop. I.9). We also show that $C(X.)$ and $Et(X.)$ at least have the same locally constant sheaves (Prop. I.10).

Chapter II deals with cohomology. All that we do is plug $C(X.)$ into Illusie's machine (so in particular, one should skip all the proofs). The most important fact is Proposition II.3, which opens up the machinery of Proposition II.2 (a spectral sequence) for computing the cohomology of a local system in $C(X.)$.

Chapter III is the heart of the paper. We define the étale homotopy type of a simplicial scheme $X.$ to be the Artin-Mazur homotopy type of $C(X.)$ (Def. III.5). The main theorem (Thm. III.8) shows that our definition is weakly equivalent to the others in the literature.

In Chapter IV, we give some applications. We determine the homotopy type of a hypercovering (Thm. IV.2) and examine what happens to homotopic maps (Thm. IV.6) (both of these proofs use the machinery of Chapter I and II). We also prove a comparison theorem (Thm. IV.8) for simplicial schemes over C (and this proof is essentially independent of the first two chapters – it uses only the ad-hoc definition of the homotopy type). Finally, we apply some of these results to the study of classifying spaces of algebraic groups.

There are two appendices. We use a nonstandard notion of sheaf, and this is explained in Appendix A. Appendix B is more technical – it deals with the construction of certain adjoint functors.

This paper is partially based on material contained in the author's dissertation. And the author would like to thank Eric Friedlander for many useful conversations.

I. SHEAVES ON SIMPLICIAL SCHEMES

§1. Simplicial objects

Let Δ be the category whose objects are the non-empty finite ordered sets $[n]=\{0,1,\ldots,n\}$ and whose morphisms are order preserving maps.

If C is any category, then the category of simplicial objects in C, denoted simpl(C), is the functor category $\mathrm{Hom}(\Delta^0, C)$.

An object of simpl(C) is denoted $X.$, and it consists of the following data:

1) $\forall$ integer $n \geq 0$, an object $X_n \in C$
2) $\forall$ morphism $f:[n]\to[m]$ in Δ, a map $X(f): X_m \to X_n$ in C
3) (Compatibility) If we have $f:[n]\to[m]$ and $g:[m]\to[k]$ in Δ, then $X(g\circ f)=X(f)\circ X(g)$.

A map between simplicial objects is denoted $f.: X. \to Y.$ and consists of maps $f_n: X_n \to Y_n \forall n \geq 0$ (satisfying obvious compatibility conditions).

One can also describe simplicial objects and maps between them using boundary and degeneracy maps. See May [12].

If X is an object of C, we have the constant simplicial object on X, $K(X,0)$, where for all $n\geq 0$, $K(X,0)_n = X$, and all the $K(X,0)(f)$ are just 1_X.

§2. Schemes and sheaves

All schemes that appear in this paper are assumed to be locally noetherian.

Let S be a scheme. Then S_{et} will denote the usual étale site of S. We will also use the site S_{ft}, which is the category of maps $f: X\to S$ where f is locally of finite type, and we give it the étale topology.

Sheaves on S_{et} (resp. S_{ft}) are denoted $S_{et}^{\sim}$ (resp. $S_{ft}^{\sim}$). Since we work mostly with $S_{ft}^{\sim}$, the phrase "étale sheaf on S" will mean an element of $S_{ft}^{\sim}$. We call a sheaf on S_{et} a "small étale sheaf on S".

The reader should consult Appendix A to see the relation between $S_{et}^{\sim}$ and $S_{ft}^{\sim}$.

§3. Simplicial schemes

A simplicial scheme $X.$ is just a simplicial object in the category of schemes. (Remember that we're assuming that each X_n is locally noetherian.)

All simplicial schemes that appear in this paper are assumed to have the following property:

DEFINITION I.1: A simplicial scheme $X.$ is *locally of finite type* if for every map $f:[n]\to[m]$ in Δ, the map $X(f):X_m\to X_n$ is locally of finite type.

§4. Etale sheaves on a simplicial scheme – 1st definition

We have the following definition of sheaf on a simplicial scheme (see [6, 5.1.7], [11, VI.5] and [14, VI.7]):

DEFINITION I.2: Let $X.$ be a simplicial scheme. Then a sheaf $F.$ on $X.$ consists of the following data:

1) $\forall$ integer $n\geq 0$, an étale sheaf F_n on X_n
2) $\forall$ morphism $f:[n]\to[m]$ in Δ, a map $F(f):X(f)^*F_n\to F_m$ of sheaves on X_m
3) (Compatibility) If we have maps $f:[n]\to[m]$ and $g:[m]\to[k]$ in Δ, then $F(g\circ f)=F(g)\circ X(g)^*(F(f))$

A map between sheaves, $a.:F.\to G.$, consists of the following data:

1) $\forall$ integer $n\geq 0$, a map $a_n:F_n\to G_n$ of sheaves on X_n
2) (Compatibility) $\forall$ map $f:[n]\to[m]$ in Δ, $a_m\circ F(f)=G(f)\circ X(f)^*(a_n)$.

To distinguish these sheaves from the ones of §5, we call a sheaf of Def. I.2 an $(X.)_{ft}$ sheaf. The category of all $(X.)_{ft}$ sheaves is called $(X.)_{ft}^{\sim}$ (see Prop. I.3).

Some examples of $(X.)_{ft}$ sheaves are:

1) $\mathcal{O}_{X.}$: On each X_n, we have $\mathcal{O}_{X_n}$, and each map $f:[n]\to[m]$ in Δ gives us $X(f):X_m\to X_n$ which induces a map $X(f)^*\mathcal{O}_{X_m}\to\mathcal{O}_{X_n}$, and this is $\mathcal{O}_{X.}(f)$.
2) Given a map $\epsilon:X.\to K(S,0)$, we get $\Omega^1_{X./S}$: on each X_n we have $\Omega^1_{X_n/S}$, and $f:[n]\to[m]$ in Δ gives $X(f):X_m\to X_n$ which by [7, IV.16.4.19] induces a map $X(f)^*\Omega^1_{X_m/S}\to\Omega^1_{X_n/S}$, and this is $\Omega^1_{X./S}(f)$.
3) Given a map $\epsilon:X.\to K(S,0)$ and a sheaf F on S, we get ϵ^*F on $X.$: We use ϵ_n^*F on X_n, and for $f:[n]\to[m]$ in Δ, $\epsilon^*F(f)$ is

$$X(f)^*\epsilon_n^*F\xrightarrow{\sim}(\epsilon_n\circ X(f))^*F=\epsilon_m^*F.$$

We want to identify $(X.)_{ft}$ sheaves as actual sheaves on some site associated to $X.$. This is done as follows:

Let $(X.)_{ft}$ be the category whose objects are maps $U\to X_n$ which are locally of finite type, and whose morphisms are commutative squares:

$$\begin{array}{ccc} V & \longrightarrow & U \\ \downarrow & & \downarrow \\ X_m & \xrightarrow{X(f)} & X_n \end{array}$$

The topology on $(X.)_{ft}$ is the smallest topology so that the following families becomes coverings:

$$\begin{array}{ccc} U_i & \longrightarrow & U \\ \downarrow & & \downarrow \\ X_n & \xrightarrow{1X_n} & X_n \end{array} \quad i \in I$$

where $\{U_i \to U\}_{i \in I}$ is an étale cover of U.

PROPOSITION I.3. *$(X.)_{ft}$ is a site, and its category of sheaves, $(X.)_{ft}^{\sim}$, is equivalent to the category of $(X.)_{ft}$ sheaves.*

PROOF: See [14, VI.7.4.7].

We'll discuss global sections of $(X.)_{ft}$ sheaves in §II.2.

And given a simplicial map $u.: X. \to Y.$, we get inverse image $(u.^*)$ and direct image $(u._*)$ functors which are easy to describe.

§5. Etale sheaves on a simplicial scheme – 2nd definition

Here is another definition of sheaf on a simplicial scheme:

DEFINITION I.4: Let $X.$ be a simplicial scheme. Then a sheaf $F.$ on $X.$ consists of the following data:

1) $\forall$ integer $n \geq 0$, an étale sheaf F_n on X_n
2) $\forall$ morphism $f:[n] \to [m]$ in Δ, a map $F(f): F_m \to X(f)^* F_n$ of sheaves on X_m
3) (Compatability) If we have maps $f:[n] \to [m]$ and $g:[m] \to [k]$ in Δ, then $F(g \circ f) = X(g)^*(F(f)) \circ F(g)$.

A map of sheaves $a.: F. \to G.$ is a collection of maps $a_n: F_n \to G_n$ satisfying obvious compatabilities.

We call a sheaf of Definition I.4 a $C(X.)$ sheaf (see Prop. I.5). And the category of $C(X.)$ sheaves is denoted $C(X.)^{\sim}$ (again, see Prop. I.5).

Some examples of $C(X.)$ sheaves are:

1) $\mathcal{O}_{X.}$: On each X_n we have $\mathcal{O}_{X_n}$, and for $f:[n]\to[m]$ in Δ, $X(f):X_m\to X_n$ is locally of finite type, so that $X(f)^*\mathcal{O}_{X_m}\to\mathcal{O}_{X_n}$ is an isomorphism (see §4 of Appendix A). It's inverse is $\mathcal{O}_{X.}(f)$. (Note that if $X.$ is not locally of finite type, $\mathcal{O}_{X.}$ in general will not exist.)
2) Given a map $\epsilon: X.\to K(S,0)$ and F a sheaf on S, we construct ϵ^*F from the sheaves ϵ_n^*F and the isomorphisms $\epsilon_m^*F = (\epsilon_n\circ X(f))^*F \xrightarrow{\sim} X(f)^*\epsilon_n^*F$.

As in section 4, we want to identify the $C(X.)$ sheaves as sheaves on some site. This is done as follows:

Let $C(X.)$ be the full subcategory of simpl$(Sch)/X.$ consisting of maps $f.: Y.\to X.$ where each $f_n: Y_n\to X_n$ is locally of finite type (note this automatically implies that each Y_n is locally noetherian and that $Y.$ is locally of finite type).

We give $C(X.)$ the smallest topology where the following two kinds of families are coverings:

*) $\{V.^i\to V.\}_{i\in I}$, where $\forall n\geq 0$, $\{V_n^i\to V_n\}_{i\in I}$ is an étale cover of V_n

**) $\{V_n\times\Delta[n]\to V.\}_{n\geq 0}$, where the maps $V_n\times\Delta[n]\to V.$ are the adjunction maps described in Appendix B.

(In the proof of Prop. I.5, we'll see that this topology can be described as the topology generated by the families $\{W_{i,n}\times\Delta[n]\to V.\}_{n\geq 0, i\in I_n}$ where for each $n\geq 0$ $\{W_{i,n}\to V_n\}_{i\in I_n}$ is an étale cover, and each $W_{i,n}\to V_n$ gives us a map $W_{i,n}\times\Delta[n]\to V.$ by adjointness.)

PROPOSITION I.5: *The sheaves on the site $C(X.)$ are equivalent to the category of $C(X.)$ sheaves* (Def. I.4).

PROOF: Let $\mathcal{N}$ be the category whose objects are (locally noetherian) schemes and whose morphisms are all maps which are locally of finite type. We make $\mathcal{N}$ into a site by giving it the étale topology.

We will use $\mathcal{N}$ to analyze both the sheaves on $C(X.)$ and $C(X.)$ sheaves.

First, $C(X.)$: Since $X.$ is locally of finite type, we see that $C(X.) = \text{simpl}(\mathcal{N})/X.$. Furthermore, if we give simpl$(\mathcal{N})$ the topology generated by the families *) and **) above, then $C(X.)^{\sim} \simeq \text{simpl}(\mathcal{N})^{\sim}/X.$.

Next, the sheaves of Definition I.4: First, note that $\forall n$, $\mathcal{N}/X_n = (X_n)_{ft}$ (see §2) so that $\mathcal{N}^{\sim}/X_n \simeq (X_n)_{ft}^{\sim}$. Then one sees easily that the data giving a $C(X.)$ sheaf is equivalent to a simplicial map $F.\to X.$ in simpl$(\mathcal{N}^{\sim})$.

Thus, $C(X.)^{\sim} \simeq \text{simpl}(\mathcal{N})^{\sim}/X.$, and the category of $C(X.)$ sheaves is $\text{simpl}(\mathcal{N}^{\sim})/X.$. So we need only prove that $\text{simpl}(\mathcal{N})^{\sim} \simeq \text{simpl}(\mathcal{N}^{\sim})$.

We have a natural inclusion $\text{simpl}(\mathcal{N}) \subseteq \text{simpl}(\mathcal{N}^{\sim})$, and $\text{simpl}(\mathcal{N}^{\sim})$ is a topos (see [10, I.2.3.1]). So by [13, IV.1.2.1], $\text{simpl}(\mathcal{N}^{\sim})$ is the category of sheaves on $\text{simpl}(\mathcal{N})$ if the following two things are true:

1) The topology on $\text{simpl}(\mathcal{N})$ determined by *) and **) is the same as the induced topology from $\text{simpl}(\mathcal{N}^{\sim})$ (where we use the topology generated by epimorphic families).

2) $\text{Simpl}(\mathcal{N})$ is a generating family for $\text{simpl}(\mathcal{N}^{\sim})$.

The second of these is easy to prove. Take any $F. \in \text{simpl}(\mathcal{N}^{\sim})$. For any $n \geq 0$, the family of maps $V \to F_n$, where $V \in \mathcal{N}$, is epimorphic by the Yoneda Lemma. By adjointness we get the family $\{V \times \Delta[n] \to F.\}_{n \geq 0, V \to F_n}$ which is easily seen to be epimorphic. Since $V \times \Delta[n] \in \text{simpl}(\mathcal{N})$, we see that $\text{simpl}(\mathcal{N})$ is a generating family.

Now, to prove 1). We need to understand the induced topology on $\text{simpl}(\mathcal{N})$, which is done as follows:

A family $\{V.^i \to V.\}_{i \in I}$ in $\text{simpl}(\mathcal{N})$ covers in the induced topology $\Leftrightarrow \{V.^i \to V.\}_{i \in I}$ covers in $\text{simpl}(\mathcal{N}^{\sim}) \Leftrightarrow \{V.^i \to V.\}_{i \in I}$ is epimorphic in $\text{simpl}(\mathcal{N}^{\sim}) \Leftrightarrow \forall n \geq 0$, $\{V_n^i \to V_n\}_{i \in I}$ is epimorphic in $\mathcal{N}^{\sim} \Leftrightarrow \forall n \geq 0$, $\{V_n^i \to V_n\}_{i \in I}$ covers in $\mathcal{N} \Leftrightarrow \forall n \geq 0$, $\{V_n^i \to V_n\}_{i \in I}$ is refined by an étale cover of V_n.

So we see instantly that any family of the form *) covers in the induced topology. To see that the families $\{V_n \times \Delta[n] \to V.\}_{n \geq 0}$ of **) cover in the induced topology, note that $\{(V_n \times \Delta[n]_m \to V_m\}_{n \geq 0}$ is refined by the étale cover $1_{V_m}: V_m \to V_m$ (since 1_{V_m} factors $V_m \to (V_m \times \Delta[m])_m \to V_m$).

Finally, let's take a family $\{V.^i \to V.\}_{i \in I}$ which covers in the induced topology and show it covers in the topology generated by *) and **).

By the above, for every $n \geq 0$, we have an étale cover $\{W_{j,n} \to V_n\}_{j \in J_n}$ which refines $\{V_n^i \to V_n\}_{i \in I}$. This means $\forall j \in J_n$, there is $i \in I$ so that $W_{j,n} \to V_n$ factors $W_{j,n} \to V_n^i \to V_n$. By adjointness, we get maps $W_{j,n} \times \Delta[n] \to V.$ and for each $n, j \in J_n$, there is $i \in I$ so that $W_{j,n} \times \Delta[n] \to V.$ factors $W_{j,n} \times \Delta[n] \to V.^i \to V.$. Thus $\{W_{j,n} \times \Delta[n] \to V.\}_{n \geq 0, j \in J_n}$ refines $\{V.^i \to V.\}_{i \in I}$.

Since $\{W_{j,n} \to V_n\}_{j \in J_n}$ is an étale cover, $\{W_{j,n} \times \Delta[n] \to V_n \times \Delta[n]\}_{j \in J_n}$ is a family of the form *), and $\{V_n \times \Delta[n] \to V.\}_{n \geq 0}$ is of the form **). So axiom 3 of topology (see [1]) says that $\{W_{j,n} \times \Delta[n] \to V.\}_{n \geq 0, j \in J_n}$ covers in the topology generated by *) and **). Since this family refines $\{V.^i \to V.\}_{i \in I}$, the latter also covers in this topology. Q.E.D.

Here is the explicit relationship between Definition I.4 and sheaves on $C(X.)$ (no proof is given):

COROLLARY I.6: *Let $F.$ be a sheaf on $X.$. Then:*

1) *For $U \to X_n$ locally of finite type, adjointness gives us $U \times \Delta[n] \to X.$ in $C(X.)$, and $F_n(U) = F.(U \times \Delta[n])$*

2) *For $Y. \to X.$ in $C(X.)$, $F.(Y.)$ is defined by the exact sequence of sets:*

$$F.(Y.) \to \prod_{n \geq 0} F_n(Y_n) \overset{a}{\underset{b}{\rightrightarrows}} \prod_{f:[n]\to[m] \text{ in } \Delta} X(f)^* F_n(Y_m)$$

where a and b are defined as follows:

i) *For every map f of Δ, we have $F(f): F_m \to X(f)^* F_n$. The map a comes from $F(f)(Y_m): F_m(Y_m) \to X(f)^* F_n(Y_m)$*

ii) *For every map f of Δ, we have a map $F_n \to X(f)_* X(f)^* F_n$ which gives us a map $F_n(Y_n) \to X(f)^* F_n(Y_n \times_{X_n} X_m)$. But the map $Y. \to X.$ gives us a map $Y_m \to Y_n \times_{X_n} X_m$. The map b comes from the composition*

$$F_n(Y_n) \to X(f)^* F_n(Y_n \times_{X_n} X_m) \to X(f)^* F_n(Y_m).$$

Finally, given a simplicial map $u.: Y. \to X.$, we get $u.^*$ and $u._*$, but we cannot describe them explicitly unless $u.$ satisfies certain strong conditions.

§6. Local systems

The difference between Definition I.2 and Definition I.4 is that the arrows in condition 2) get reversed. So if the arrows are isomorphisms, the two notions of sheaf coincide. To make this precise, we have:

DEFINITION I.7: A local system $F.$ on a simplicial scheme $X.$ consists of the following data:

1) $\forall$ integer $n \geq 0$, an étale sheaf F_n on X_n

2) $\forall$ morphism $f:[n] \to [m]$ in Δ, an isomorphism

$F(f): X(f)^* F_n \xrightarrow{\sim} F_m$ of sheaves on X_m

3) (Compatibility) as usual.

We've already seen two examples of locally systems. First, there's $\mathcal{O}_{X.}$ (but only because $X.$ is locally of finite type). And there are the sheaves $\epsilon^* F$ (see Example 3 of §4). In the next section we'll discuss other important examples.

Here is another way to look at local systems:

PROPOSITION I.8: *The category of local systems on $X.$ is equivalent to the following category of descent data*:

Objects: pairs (F_0, φ) where F_0 is an étale sheaf on X_0 and φ is an isomorphism $\varphi\colon d_0^* F_0 \xrightarrow{\sim} d_1^* F_0$ on X_1 satisfying the cocycle condition $d_1^*\varphi = d_0^*\varphi \circ d_2^*\varphi$ on X_2.

Morphisms: Usual morphisms of descent data (see [3, §10]).

PROOF: Given a local system $F.$, the maps $\partial_0, \partial_1 : [0] \to [1]$ in Δ give us maps $d_i = X(\partial_i) : X_1 \to X_0$ and isomorphisms $F(\partial_i) : F_1 \to d_i^* F_0$.

Set $\varphi = F(\partial_1) \circ F(\partial_0)^{-1} : d_0^* F_0 \to d_1^* F_0$. One easily verifies the cocycle condition, so that (F_0, φ) is descent data.

The proof that $F. \to (F_0, \varphi)$ is the desired equivalence is in §II.1.

Q.E.D.

§7. Locally constant sheaves

A $C(X.)$ sheaf $F.$ is locally constant if there is a covering $\{U.^i \to X.\}_{i\in I}$ such that $F.$ is constant on each $U.^i$.

PROPOSITION I.9: *Every locally constant $C(X.)$ sheaf is a local system.*

PROOF: First, note that a constant sheaf is obviously a local system.

Now, suppose we have $F.$ and a covering $\{U.^i \to X.\}_{i\in I}$ so that $F.$ is constant on each $U.^i$. Then, for each $f : [n] \to [m]$ in Δ, the map $F(f) : F_m \to X(f)^* F_n$ on X_m pulls back to an isomorphism on each U^i_m. But in the proof of Prop. I.5, we saw that the family $\{U^i_m \to X_m\}_{i\in I}$ is refined by an étale cover of X_m.

So $F(f)$ pulls back to isomorphisms on some étale cover of X_m, which implies $F(f)$ is on isomorphism. Q.E.D.

REMARK: One can prove a similar proposition about locally constant $(X.)_{ft}$ sheaves (see [5, Ch. One, Thm. I.9]).

§8. Small sheaves on simplicial schemes

In Definitions I.2 and I.4, we could have used small étale sheaves rather than étale sheaves (see §2). Let's see what happens when we do this.

Using Definition I.2 with small étale sheaves, we get a category of sheaves, which we call $(X.)_{et}^{\sim}$, and we can use the proof of Prop. I.3 to show that these are the sheaves on the site $(X.)_{et}$ (which is the full subcategory of $(X.)_{ft}$ consisting of étale maps $U \to X_n$).

But we have trouble if we try to redo §5 using small étale sheaves. Let $Et(X.)$ be the category of all maps $Y. \to X.$ where each map $Y_n \to X_n$ is étale, and we give $Et(X.)$ the topology determined by the families *) of §5 (note that in general **) is not even in $Et(X.)$). We would want to identify sheaves on $Et(X.)$ with the sheaves of Definition I.4 (using small sheaves), but the proof of Prop. I.5 does not work in this case (however, if all of the maps $X(f): X_m \to X_n$ are étale, then everything works beautifully – see [5, §I.4]).

We will make no use of Definition I.4 for small étale sheaves. But the site $Et(X.)$ will play an important role in Chapters III and IV. Here is a result which will be useful there:

PROPOSITION I.10: *The natural inclusion* $i: Et(X.) \hookrightarrow C(X.)$ *induces an equivalence on the categories of locally constant sheaves.*

PROOF: i gives a map of topoi $(i^*, i_*): Et(X.)^{\sim} \to C(X.)^{\sim}$. Obviously i^* gives us a fully faithful functor between the categories of representable sheaves on each site, and if F is representable on $Et(X.)$, then $i_* i^* F = F$. Then note that all of this is true for the categories of locally representable sheaves. Finally, note that any locally constant sheaf on $Et(X.)$ is locally representable.

So i^* gives us a fully faithful map between the categories of locally constant sheaves. i^* will be essentially surjective if we can show that any locally constant sheaf on $C(X.)$ can be trivialized by a covering of $X.$ in $Et(X.)$.

Let $F.$ be a locally constant sheaf on $C(X.)$. We can write $X.$ as a disjoint union of connected simplicial schemes $X.^i$ (see Prop. III.4) and it suffices to trivialize each $F._{|X.^i}$ in $Et(X.^i)$. So we can assume that $X.$ is connected.

Since $X.$ is connected, all the fibers of $F.$ are the same, i.e. there is a set S and covering $\{Y.^i \to X.\}_{i \in I}$ in $C(X.)$ so that $F._{|Y.^i}$ is represented by $Y.^i \times S$.

In particular, we have the covering $Y = \amalg_{i \in I} Y_0^i \to X_0$ in $(X_0)_{ft}$ so that $F_0|_Y$ is represented by $Y \times S$. If we take an étale cover $Z \to X_0$ which refines $Y \to X_0$, then $F_0|_Z$ is represented by $Z \times S$. The functor β_0 of Appendix B gives us a covering $\eta: \beta_0(Y) \to \beta_0(X_0) = X.$ in $Et(X.)$ (since β_0 clearly takes covers to covers). Note also that $\beta_0(Y_0) = Y.$

Replacing $\beta_0(Y)$ and $\eta^*F.$ by $X.$ and $F.$, we can assume that F_0 is represented by $X_0 \times S$.

By Theorem I.8 and Proposition I.9, $F.$ is determined by descent data $\varphi : d_0^*F_0 \to d_1^*F_0$, i.e. a map $\varphi : X_1 \times S \to X_1 \times S$, which is uniquely determined by the map $\pi(\varphi): \pi(X_1) \times S \to \pi(X_1) \times S$ (π means connected components). Thus we get descent data on $\pi(X.)$ which determines $F.$ uniquely. More precisely, $\pi(\varphi)$ determines a simplicial covering space $L. \to \pi(X.)$ with fiber S, and we get $F.$ from the cartesian diagram:

$$\begin{array}{ccc} F. & \longrightarrow & L. \\ \downarrow & & \downarrow \\ X. & \longrightarrow & \pi(X.) \end{array}$$

Let $M.$ be the P.H.S. for $Aut(S)$ associated to $L.$ (So that $L. = M. \times_{Aut(S)} S$). $M. \to \pi(X.)$ is an $Aut(S)$-torseur.

Define $Y. \to X.$ by the cartesian diagram:

$$\begin{array}{ccc} Y. & \longrightarrow & M. \\ \downarrow & & \downarrow \\ X. & \longrightarrow & \pi(X.) \end{array}$$

Then $Y.$ is an $Aut(S)$-torseur for $X.$, and $Y. \times_{Aut(S)} S$ represents $F.$. Since $Y. \times_{X.} Y. \simeq Y. \times Aut(S)$, $F.$ trivializes on $Y.$.

But it is obvious that $Y. \to X.$ is in $Et(X.)$, so we are done. Q.E.D.

§9. Points

A point of $C(X.)$ is a morphism of topoi $\xi : \text{Sets} \to C(X.)^{\sim}$, $\xi = (\xi^*, \xi_*)$. An equivalent notion is that of a "fiber functor" $\xi^* : C(X.)^{\sim} \to \text{Sets}$ where ξ^* is exact and commutes with direct limits (see [13, IV.6]).

It's easy to construct points of $C(X.)$:

Let $Y \to X_n$ be locally of finite type. Take $y \in Y$ and chose an embedding $\xi : k(y) \hookrightarrow \Omega$, Ω algebraically closed. Then we get the usual fiber functor from $Y_{et}^{\sim}$ to Sets, $F \mapsto F_{\xi}$.

Then define $\xi^* : C(X.)^{\sim} \to \text{Sets}$ by $\xi^*(F.) = ((F_n)_Y)_{\xi}$ (see Appendix A). Such a point is called ordinary. It is easy to see that there are "enough" ordinary points of $C(X.)$ (in the sense that $F. \to G.$ is an isomorphism $\Leftrightarrow \xi^*F. \to \xi^*G.$ is an isomorphism for all ordinary points).

DEFINITION I.11: A pointed simplicial scheme is a simplicial scheme $X.$ together with an ordinary point corresponding to some $x \in X_0$.

II. COHOMOLOGY OF SIMPLICIAL SCHEMES

§1. Relation to Illusie's Work

All of the proof in this chapter are based on the results of Chapter VI of Illusie's Springer Notes [11]. So we need to translate our two notions of sheaf into Illusie's language.

Let $X.$ be a simplicial scheme. Then we get a fibered topos above Δ° (see [11, VI.5.1]), called X, as follows:

1) For each $n \geq 0$, we have the topos $(X_n)_{ft}^{\sim}$
2) For each map $f:[n]\to[m]$ in Δ, the map of schemes $X(f): X_m \to X_n$ induces a map of topoi $(X(f)_*, X(f)^*): (X_m)_{ft}^{\sim} \to (X_n)_{ft}^{\sim}$

Furthermore, since $X(f): X_m \to X_n$ is locally of finite type, the functor $X(f)^*: (X_n)_{ft}^{\sim} \to (X_m)_{ft}^{\sim}$ has a left adjoint $X(f)_!$ (since $X(f): X_m \to X_n$ is in $(X_n)_{ft}$, $(X_m)_{ft}^{\sim} = (X_n)_{ft}^{\sim}/X_m$, and $X(f)_!$ is just the forgetful functor $(X_n)_{ft}^{\sim}/X_m \to (X_n)_{ft}^{\sim}$ which sends $F \to X_m$ to F – see [13, IV.5.2.2]). Thus X is a "good" fibered topos [11, VI.5.4].

The one checks that the category of $(X.)_{ft}$ sheaves is what Illusie calls Top(X) (see [11, VI.5.2.1]). And the category of $C(X.)$ sheaves is Top°(X) (see [11, VI.5.2.2]).

Local systems on $X.$ are just the equivalent categories Cart(Top(X)) $\simeq$ Cart(Top°(X)) (see [11, VI.6.5.1]). Furthermore, the category of descent data of Prop. I.8 is called $B'X$ in [11, VI.8.1.7] (and the proof of Prop. I.8 is just the large diagram of [11, VI.8.1.7]).

§2. Cohomology groups of simplicial schemes

Let $F.$ be an abelian $(X.)_{ft}$ sheaf. Then it becomes an abelian sheaf on the site $(X.)_{ft}$, so that we get cohomology groups $H^q((X.)_{ft}, F.)$, which we call $H^q(X., F.)$. First, let's see how to compute $H^0(X., F.) = \Gamma(X., F.)$:

PROPOSITION II.1: $\Gamma(X., F.) = \mathrm{Hom}_{(X.)_{ft}^{\sim}}(X., F.) = \mathrm{Ker}(F_0(X_0) \rightrightarrows F_1(X_1))$, *where* $X.$ *is the* $(X.)_{ft}$ *sheaf which is the representable sheaf* $\mathrm{Hom}_{(X_n)_{ft}^{\sim}}(\ , X_n)$ *on each* X_n.

PROOF: One sees that $X.$ is the final object of $(X.)_{ft}^{\sim}$, which gives us the first equality.

And the global section functors $\Gamma(X_n,\ ):(X_n)_{ft}^{\sim}\to$ Sets give us a morphism of fibered topoi $\Gamma(X,\ ):X\to$ Sets [11, VI.5.5]. This gives a direct image map $\Gamma(X,\ )_*:\mathrm{Top}(X)\to$ Sets [11, VI.5.8], which is just $\Gamma(\mathrm{Top}(X),\ )=\Gamma(X.,\ )$, and this is computed by the formula $\Gamma(X.,F.)=\varprojlim_{\Delta^\circ} F_n(X_n)$ (see [11, VI.5.8.1, ex. ii] for all this). And it is well-known that this inverse limit is just $\mathrm{Ker}(F_0(X_0)\rightrightarrows F_1(X_1))$.

Q.E.D.

Thus, $H^q(X.,F.)=R^q\Gamma(X,\ )_*(F.)$, so by [11, VI.6.2.3.2] we get:

PROPOSITION II.2: *There is a spectral sequence, functorial in X. and F.:*

$$E_1^{p,q}=H^q(X_p,F_p)\Rightarrow H^{p+q}(X.,F.).$$

Furthermore, if $X.\in\mathrm{simpl}(S_{ft})$ *and F is a sheaf on S, then* $H^q(X.,\epsilon^*F)$ *is the hypercohomology group* $\mathbf{H}^q(X.,F)=\mathbf{Ext}^q(\mathbf{Z}_{X.},F)$, *and the above is the usual hypercohomology spectral sequence for* $\mathbf{Z}_{X.}$.

REMARK: If $F.$ is a sheaf on $(X.)_{et}$ (see §1.8), we get cohomology groups $H^q((X.)_{et},F.)$, and the analogues of Propositions II.1 and II.2 are true. We also have the following relation with the cohomology of $(X.)_{ft}$ sheaves:

The inclusion $i:(X.)_{et}\to(X.)_{ft}$ gives us a morphism of topoi (i^*,i_*): $(X.)_{ft}^{\sim}\to(X.)_{et}^{\sim}$. If $F.$ (resp. $G.$) is a sheaf on $(X.)_{ft}^{\sim}$ (resp. $(X.)_{et}^{\sim}$), then $H^q((X.)_{ft},F.)=H^q((X.)_{et},i_*F.)$ and $H^q((X.)_{et},G.)=H^q((X.)_{ft},i^*G.)$. (To show this, use the proof of [14, VII.4.1].)

Now, let $F.$ be an abelian $C(X.)$ sheaf. Then we get cohomology groups $H^q(C(X.),F.)$, which we call $H^q(X.,F.)$.

If $F.$ is an abelian local system on $X.$, then our notation seems ambiguous. But this is not the case:

PROPOSITION II.3: *If F. is an abelian local system on X., then the two cohomologies defined above are cannonically isomorphic.*

PROOF: From Proposition II.1, we have $\Gamma(X,\ ):X\to$ Sets, and we saw that $H^q(X.,F.)$ is $R^q\Gamma(X,\ )_*(F.)$ when we regard $F.$ as an $(X.)_{ft}$ sheaf.

Now $\Gamma(X,\ )$ also gives us $\Gamma(X,\ )_*:\mathrm{Top}^\circ(X)=C(X.)^{\sim}\to$ Sets, and we compute $\Gamma(X,\ )_*$ by the formula of [11, VI.5.8.2]. By Corollary

I.6, we see that $\Gamma(X, F.)_* = F.(X.) = \Gamma(X., F.)$, so that $H^q(X., F.)$ is just $R^q\Gamma(X, \ )_*(F.)$ when we regard $F.$ as a $C(X.)$ sheaf.

So all we have to do is show that the conditions of [11, VI.6.5.3] are satisfied. But for every map $f:[n]\to[m]$ in Δ, $X(f): X_m \to X_n$ is in $(X_n)_{ft}$, so that $X(f)^*: (X_n)_{ft}^{\sim} \to (X_n)_{ft}^{\sim}/X_m = (X_m)_{ft}^{\sim}$ takes flasques to flasques (see [14, V.4.11]). Thus, by [11, VI.6.1.4, third paragraph], $F.$ has a flasque resolution (I.)* where each J_n^i (and thus each $X(f)^*I_n^i$) is flasque. This is what [11, VI.6.5.3] requires. Q.E.D.

III. HOMOTOPY TYPE OF SIMPLICIAL SCHEMES

§1. Review of Artin-Mazur theory

We need to recall the definition of the homotopy type of a site.

DEFINITION III.1: An Artin-Mazur site is a category C with a grothendieck topology, where C has the following properties:

1) Finite inverse limits and disjoint sums exist in C
2) C is locally connected and distributive
3) C has a point p

(See [3, §9] for a discussion of this.)

If X is a locally noetherian scheme, then both X_{et} and X_{ft} are Artin-Mazur sites (once we chose a point).

Every Artin-Mazur site C has a connected component functor $\pi: C \to \text{Sets}$, where for $X \in C$, $\pi(X)$ is the set of connected components of X.

We need to recall hypercoverings:

DEFINITION III.2: A hypercovering of C is a simplicial object $U.$ of C which satisfies:

1) $U_0 \to e$ is a covering (e is the final object of C)
2) For all n, the map $U_{n+1} \to (\text{cosk}_n\, U.)_{n+1}$ is a covering
3) There is a distinguished element $x_0 \in p^*(U_0)$ (the point of $U.$)

And $HR(C)$ is the homotopy category of hypercoverings (with point preserving maps).

All of this is in [3, §8]. Remember that $HR(C)^0$ is a filtering category.

Now, it's easy to define the homotopy type of C. The functor $\pi: C \to \text{Sets}$ gives a functor $simpl(\pi): \text{simpl}(C) \to \text{simpl}(\text{Sets}) = \text{Simp-}$

licial Sets. If we pass to homotopy categories and restrict to hypercoverings, we get a functor (which we call π):

(1) $\pi : HR(C) \to \mathscr{H}$

where $\mathscr{H}$ is the homotopy category of pointed simplicial sets. Since $HR(C)$ is filtering we get an object of Pro-$\mathscr{H}$.

DEFINITION III.3: The homotopy type of C, denoted $\{C\}_{ht}$, is the pro-object (1) (often written $\{\pi(U.)\}_{U.\in HR(C)}$).

Now, let $f : C \to C'$ be a morphism of Artin-Mazur sites (so that f commutes with finite inverse limits, disjoint sums, takes coverings to coverings and preserves the point).

Then f induces a map $\{f\}_{ht} : \{C'\}_{ht} \to \{C\}_{ht}$. To see this, first, note that for any $X \in C$, there is a natural map $f_* : \pi(f(X)) \to \pi(X)$. (Write $X = \coprod_{U\in\pi(X)} U$, so that $f(X) = \coprod_{U\in\pi(X)} f(U)$. Then each connected component V of $f(X)$ must lie in some $f(U)$, and we get $f_*(V) = U$.) Also note that $U. \in HR(C)$ implies $f(U.)$ is in $HR(C')$.

So for any $U. \in HR(C)$, we have a map $f_* : \pi(f(U.)) \to \pi(U.)$ with $f(U.) \in HR(C')$. This gives a map $\{f\}_{ht} : \{C'\}_{ht} \to \{C\}_{ht}$.

§2. The homotopy type of a simplicial scheme

Let $X.$ we a pointed simplicial scheme. Remember that we assume $X.$ to be locally of finite type and each X_n to be locally noetherian.

PROPOSITION III.4: *$C(X.)$ is an Artin-Mazur site.*

PROOF: Clearly, we need only show that $C(X.)$ is locally connected. Take any $Y. \in C(X.)$, and we get the simplicial set $\pi(Y.)$ whose n-simplicies are the connected components of Y_n (this gives us a functor π from $C(X.)$ to simplicial sets).

Write $\pi(Y.) = \coprod_{i\in I} Z.^i$, where each $Z.^i$ is a connected simplicial set. Then, for a fixed i in I, take all components of the Y_n's that are in $Z.^i$. These form a simplicial scheme $Y.^i$ and one sees easily that $Y. = \coprod_{i\in I} Y.^i$ in $C(X.)$.

Since π preserves disjoint sums and $\pi(Y.^i) = Z.^i$ is connected, $Y.^i$ must be connected. So $Y.$ is a sum of connected objects. Q.E.D.

The connected component functor of $C(X.)$ will be called π_0 (so $\pi_0(Y.) = \{\text{the connected components of } Y.\} = \pi_0(\pi(Y.)))$.

Applying §1 to $C(X.)$, we get:

DEFINITION III.5: *$\{X.\}_{et}$, the étale homotopy type of $X.$, is the object $\{C(X.)\}_{ht}$ of* Pro-$\mathcal{H}$.

And if $f.: X. \to Y.$ is a pointed simplicial map, it induces a morphism of sites $C(f.): C(Y.) \to C(X.)$ and this, by §1, gives us a map $\{f.\}_{et}: \{X.\}_{et} \to \{Y.\}_{et}$ (often denoted just f_{et}) in Pro-$\mathcal{H}$.

§3. Hypercoverings of a simplicial scheme

A hypercovering of $C(X.)$ will be called a hypercovering of $X.$, and their homotopy category will be called $HR(X.)$.

Note that hypercoverings of $X.$, being objects $(U.).$ of simpl$(C(X.))$, can be regarded as bisimplicial schemes $U..$. For us, the outer dot is "external" and the inner one "internal", (in the sense that $U._q$ is in $C(X.)$ for all q).

We have the following characterization of hypercoverings:

PROPOSITION III.6: *$U..$ is a hypercovering of $X.$ iff $U_p.$ is a hypercovering of X_p (in $(X_p)_{ft}$) for all p.*

PROOF: $U..$ is a hypercovering of $X.$ iff all the maps $U._0 \to X.$ and $U._{n+1} \to (\operatorname{cosk}_n U..)_{n+1}$ are coverings in $C(X.)$. By the topology we gave $C(X.)$, this is true iff for all p, all the maps $U_{p,0} \to X_p$ and $U_{p,n+1} \to ((\operatorname{Cosk}_n U..)_{n+1})_p$ are coverings in $(X_p)_{ft}$.

The functor $V. \mapsto V_p$ commutes with inverse limits since it has a left adjoint (see Appendix B). Thus $((\operatorname{cosk}_n U..)_{n+1})_p = (\operatorname{cosk}_n U_p.)_{n+1}$, so that the above is equivalent to the condition that for all p, the maps $U_{p,0} \to X_p$ and $U_{p,n+1} \to (\operatorname{cosk}_n U_p.)_{n+1}$ are coverings. But this is the definition of a hypercovering. Q.E.D.

In particular, this proposition gives us a functor from $HR(X.)$ to $HR(X_p)$. And we have:

PROPOSITION III.7: *The functor $HR(X.) \to HR(X_p)$ is cofinal.*

PROOF: By Appendix B, the functor from $C(X.)$ to $(X_p)_{ft}$ which sends $Y.$ to Y_p has a right adjoint $\beta_p: (X_p)_{ft} \to C(X.)$. This gives us a functor (which we'll simply call β) $\beta: \operatorname{simpl}((X_p)_{ft}) \to \operatorname{simpl}(C(X.))$.

If we can show that β takes hypercoverings to hypercoverings, then we are done. For if $\beta(U.) = U..$, then the adjunction map $U_{p.} \to U.$ satisfies a) of [3, A.1.5]. And b) of [3, A.1.5] also follows easily from the adjunction maps.

The formula for β_p in Appendix B shows that β_p takes coverings to coverings. Also, β_p, being a right adjoint, commutes with inverse limits.

Now, let $U.$ be a hypercovering in $(X_p)_{ft}$. Then $U_0 \to X_p$ covers, so that $\beta(U.)_0 = \beta_p(U_0) \to \beta_p(X_p) = X.$ also covers. And since $U_{n+1} \to (\operatorname{cosk}_n U.)_{n+1}$ covers, so does $\beta(U.)_{n+1} = \beta_p(U_{n+1}) \to \beta_p((\operatorname{cosk}_n U.)_{n+1}) = \operatorname{cosk}_n \beta(U.))_{n+1}$. Thus, $\beta(U.)$ is a hypercovering in $C(X.)$. Q.E.D.

§4. Another way to define homotopy type

Recall that $Et(X.)$ is the site consisting of all maps $Y. \to X.$ where each $Y_n \to X_n$ is étale (and we use the topology determined by the families *) of §5 – see also §8). Note that $Et(X.)$ is an Artin-Mazur site.

We will show that one can define the homotopy type of $X.$ using the site $Et(X.)$ rather than $C(X.)$ (this is good because $Et(X.)$ is so much smaller than $C(X.)$). However, we won't take the homotopy type of the site $Et(X.)$, instead, we do the following construction:

Let $U.. \in HR(Et(X.))$. If we take the connected components of each U_{pq}, we get a bisimplicial set $\pi(U..)$. Taking it's diagonal, we get the simplicial set $\Delta\pi(U..)$. This gives us the object $\{\Delta\pi(U..)\}_{U..\in HR(Et(X.))}$ of Pro-$\mathcal{H}$.

THEOREM III.8: *$\{X.\}_{et}$ is weakly isomorphic to $\{\Delta\pi(U..)\}_{U..\in HR(Et(X.))}$ in* Pro-$\mathcal{H}$.

PROOF: The proof is in two parts.

First, lets show that $\{X.\}_{et}$ is isomorphic to $\{\Delta\pi(U..)\}_{U..\in HR(C(X.))}$. Recall that $\{X.\}_{et}$ is the pro-object $\{\pi_0^h(\pi(U..))\}_{U..\in HR(C(X))}$, where $\pi_0^h(\pi(U..))_q = \pi_0(\pi(U._q)) = \pi_0(U._q)$ (the h stands for horizontal).

For any bisimplicial set $L..$, the maps $L_{pq} \to \pi_0(L._q)$ give us a map $L.. \to K(\pi_0^h(L..), 0)$. Taking diagonals gives us a map $\Delta L.. \to \pi_0^h(L..) = \Delta K(\pi_0^h(L..), 0)$. Applying this to the $\pi(U..)$'s, we get a map $\{\Delta\pi(U..)\}_{U..\in HR(C(X.))} \to \{\pi_0^h(\pi(U..))\}_{U..\in HR(C(X.))} = \{X.\}_{et}$.

Now, assume that $U..$ is in $HR(C(X.))$ and for every q, $U._q = \amalg_n W_{q,n} \times \Delta[n]$, where each $W_{q,n}$ is some scheme. Then $\pi_0(\pi(U._q)) = \amalg_n \pi(W_{q,n})$, and the map $\pi(U._q) = \amalg_n \pi(W_{q,n}) \times \Delta[n] \to \amalg_n K(\pi(W_{q,n}), 0) = K(\pi_0(\pi(U._q)), 0)$ is a homotopy equivalence since

$\Delta[n]$ is contractible. Thus, by [4, XII 4.2 and 4.3], the map $\Delta\pi(U..) = \underrightarrow{\text{holim}}\ \pi(U..) \to \underrightarrow{\text{holim}}\ K(\pi_0^h(\pi(U..)), 0) = \pi_0^h(\pi(U..))$ is a homotopy equivalence.

If we can show that $U..$'s as above are cofinal in $HR(C(X.))$, then the map $\{\Delta\pi(U..)\}_{U..\in HR(C(X.))} \to \{X.\}_{et}$ will be an isomorphism in Pro-$\mathcal{H}$. But for any $V.$ in $C(X.)$, the map $\amalg_n V_n \times \Delta[n] \to V.$ is a covering, and then using [3, §8], one can easily show that $U..$'s where each $U._q$ is of the form $\amalg_n W_{q,n} \times \Delta[n]$ are cofinal in $HR(C(X.))$.

The second part of the proof is to show that the obvious map $\{\Delta\pi(U..)\}_{U..\in HR(C(X.))} \to \{\Delta\pi(U..)\}_{U..\in HR(Et(X.))}$ is a weak isomorphism in Pro-$\mathcal{H}$. We will use the Artin-Mazur-Whitehead Theorem of [3, §4] (it is no loss of generality to assume that $X.$ is connected).

First, we have to show that the π_1's are isomorphic. To do this, note that as above, we get a map $\{\Delta\pi(U..)\}_{U..\in HR(Et(X.))} \to \{Et(X.)\}_{ht}$, and in fact we have a commutative diagram in Pro-$\mathcal{H}$:

$$\begin{array}{ccc}
\{\Delta\pi(U..)\}_{U..\in HR(C(X.))} & \to & \{C(X.)\}_{ht} = \{X.\}_{et} \\
\downarrow & & \downarrow \\
\{\Delta\pi(U..)\}_{U..\in HR(Et(X.))} & \to & \{Et(X.)\}_{ht}
\end{array}$$

The top arrow is an isomorphism by what we've just done, and the arrow on the right induces an isomorphism on π_1 by Proposition I.10 (this follows from [3, §10]). So we need only show that the map $\{\pi_1(\Delta\pi(U..))\}_{U..\in HR(Et(X.))} \to \pi_1(Et(X.))$ is an isomorphism. Regarding these pro-objects as functors, we need only show that for every group G, the map $\text{Hom}(\pi_1(Et(X.)), G) \to \underset{U..\in HR(Et(X.))}{\underrightarrow{\lim}} \text{Hom}(\pi_1(\Delta\pi(U..)), G)$ is an isomorphism.

But elements of the first set are G-torseurs of $Et(X.)$ (this is by [3, Cor. 10.7] and the fact that locally constant sheaves on $Et(X.)$ are representable). Also, $\text{Hom}(\pi_1(\Delta\pi(U..)), G)$ is the set of simplicial G-torseurs on $\Delta\pi(U..)$. And the above map can be explicitly described as follows: Let $P.$ be a G-torseur of $Et(X.)$, and let $U..$ be a hypercovering of $Et(X.)$ where $P.$ becomes trivial over $U._0$. Then $P.$ maps to the simplicial G-torseur $\Delta\pi(U.. \times_{X.} P.)$ over $\Delta\pi(U..)$ (see [3, Cor. 10.6]). To show that this map is an isomorphism, we will need the following lemma:

LEMMA III.9: *Let $L..$ be a bisimplicial set. There is an equivalence of categories between bisimplicial covering spaces of $L..$ and simplicial covering spaces of $\Delta L..$.*

PROOF: If $V..$ is a bisimplicial covering space of $L..$, then obviously $\Delta V..$ is a simplicial covering space of $\Delta L..$. We need to construct a quasi-inverse to this functor.

Let $V.$ be a simplicial covering space of $\Delta L..$. Then $V.$ is a local system on $\Delta L..$, and hence by Prop. I.8, $V.$ is determined by descent data (V_0, φ) where $\varphi : d_1^{h*} d_1^{v*} V_0 = d_1^{v*} d_1^{h*} V_0 \to d_0^{h*} d_0^{v*} V_0 = d_0^{v*} d_0^{h*} V_0$ over L_{11} (h means horizontal, v means vertical in $L..$).

Then $s_0^{v*}\varphi : d_1^{h*} V_0 \xrightarrow{\sim} d_0^{h*} V_0$ over L_{10} gives us descent data $(V_0, s_0^{v*}\varphi)$ with respect to $L._0$, which determines a local system $V._0$ over $L._0$ (where $V_{00} = V_0$).

Note that $s_0^{h*}\varphi : d_1^{v*} V_0 \xrightarrow{\sim} d_0^{v*} V_0$ is compatible with the descent data of $d_1^{v*} V._0$ and $d_0^{v*} V._0$, and so gives an isomorphism $\tilde{\varphi} : d_1^{v*} V._0 \xrightarrow{\sim} d_0^{v*} V._0$. $\tilde{\varphi}$ satisfies the cocycle condition so that $(V._0, \tilde{\varphi})$ is descent data for $L.. = (L.). \in$ simpl(Simplicial sets). This gives us a local system $V..$ on $(L.).$.

It is easy to see that $V..$ is a bisimplicial covering space of $L..$, and this construction gives the desired quasi-inverse. Q.E.D.

Now, assume that we have G-torseurs $P.$ and $P.'$ in $Et(X.)$ which map to the same thing in $\varinjlim_{U.. \in HR(Et(X.))} \mathrm{Hom}(\pi_1(\Delta\pi(U..), G)$. Then there is a hypercovering $U..$ in $Et(X.)$ so that $P.$ and $P.'$ become trivial on $U._0$ and $\Delta\pi(U.. \times_{X.} P.) \simeq \Delta\pi(U.. \times_{X.} P.')$. Since $\pi(U.. \times_{X.} P.)$ and $\pi(U.. \times_{X.} P.')$ are bisimplicial covering spaces of $\pi(U..)$, the above lemma says that $\pi(U.. \times_{X.} P.) \simeq \pi(U.. \times_{X.} P.')$ over $\pi(U..)$. Since $U.. \times_{X.} P.$ and $U.. \times_{X.} P.'$ are trivial over $U..$, this says that $U.. \times_{X} P. \simeq U.. \times_{X.} P.'$ over $U..$. This gives us descent data for an isomorphism $P. \simeq P.'$, which is effective since $U..$ is a hypercovering of $X.$. So $P. \simeq P.'$.

And given a simplicial G-torseur $V.$ of $\Delta\pi(U..)$, the above lemma says that $V. \cong \Delta V..$, where $V..$ is a bisimplicial G-torseur of $\pi(U..)$. Then each $V_p.$ is a simplicial G-torseur of $\pi(U_p.)$, and $U_{p.}$ is a hypercovering of X_p (by Proposition II.6), so that we get a G-torseur P_p of X_p using [3, Cor. 10.6]. These fit together to give a G-torseur $P.$ of $X.$. Now $P.$ might not become trivial on $U._0$, so find a map $U..' \to U..$ in $HR(Et(X.))$ so that $P.$ becomes trivial on $U._0'$. Then one easily checks that $P.$ maps to $V. \times_{\Delta\pi(U..)} \Delta\pi(U..')$, so that

$$\mathrm{Hom}(\pi_1(Et(X.)), G) \to \varinjlim_{U.. \in HR(Et(X.))} \mathrm{Hom}(\pi_1(\Delta\pi(U..)), G)$$

in onto.

Thus, the map on π_1's is an isomorphism. Next, we have to show that we get an isomorphism on cohomology with local coefficients. For simplicity, we'll just do the case of constant coefficients – the general case is much the same (except that the notation is messier – see the proof of Theorem IV.8).

Given an abelian group G, the map $\{\pi(U..)\}_{U..\in HR(C(X.))}\to \{\pi(U..)\}_{U..\in HR(Et(X.))}$ induces a map of spectral sequences:

$$\begin{array}{ccc} E_1^{p,q} = \varinjlim_{U..\in HR(Et(X.))} H^q(\pi(U_p.), G) & \to & 'E_1^{p,q} = \varinjlim_{U..\in HR(C(X.))} H^q(\pi(U_p.), G) \\ \Downarrow & & \Downarrow \\ \varinjlim_{U..\in HR(Et(X.))} H^{p+q}(\Delta\pi(U..), G) & \to & \varinjlim_{U..\in HR(C(X))} H^{p+q}(\Delta\pi(U..), G) \end{array}$$

To show that the bottom line is an isomorphism, we need only show that we have an isomorphism at E_1.

Fix some $p\geq 0$. Note that the functors from $HR(Et(X.))$ to $HR((X_p)_{et})$ (and from $HR(C(X.))$ to $HR((X_p)_{ft})$) which send $U.$ to $U_p.$ are cofinal by Proposition III.7. Thus, the map $E_1^{p,q}\to 'E_1^{p,q}$ can be written as

$$\varinjlim_{U_p\in HR((X_p)_{et})} H^q(\pi(U_p), G) \to \varinjlim_{U_p\in HR((X_p)_{ft})} H^q(\pi(U_p), G).$$

But the inclusion $HR((X_p)_{et})\to HR((X_p)_{ft})$ is cofinal, so that the map is an isomorphism.

So we're done! Q.E.D.

§5. Some further results

First, there is one case when $\{Et(X.)\}_{ht}$ does give the homotopy type of $X.$:

PROPOSITION III.10: *If $X.$ is an étale simplicial scheme (meaning that all the maps $X(f): X_m\to X_n$ are étale), then $\{X.\}_{et}\simeq\{Et(X.)\}_{ht}$ in* Pro-$\mathcal{H}$.

PROOF: Take $V.\in Et(X.)$ and let $V.'\to V.$ be a covering in $C(X.)$. Then the proof of Prop. I.5 gives us, for each n, an étale, cover $\{W_{i,n}\to V_n\}_{i\in I_n}$ which refines $V_n'\to V_n$. So we get a covering $\amalg_{n,I_n} W_{i,n}\times \Delta[n]\to V.$ which refines $V.'\to V.$. Since $W_{i,n}\to V_n\to X_n$ is étale and $X.$ is étale, one sees that $\amalg_{n,I_n} W_{i,n}\times\Delta[n]$ is in $Et(X.)$.

Thus, any cover of $V.$ in $C(X.)$ can be refined by a cover in $Et(X.)$. Then the techniques of [3, §8] enable us to show that the functor $HR(Et(X.))\to HR(C(X.))$ is cofinal. Thus $\{Et(X.)\}_{ht}\simeq\{C(X.)\}_{ht} = \{X.\}_{et}$. (Q.E.D.

And if $X.$ is étale, the first part of the proof of Thm. III.8 shows that $\{X.\}_{et}$ is actually isomorphic to $\{\Delta\pi(U..)\}_{U..\in HR(Et(X.))}$. An application of this is:

PROPOSITION III.11: *Let L. be a simplicial set and let k be an algebraically closed field. Then* $L.\times \mathrm{Spec}(k)$ *is a simplicial scheme, and* $\{L.\times \mathrm{Spec}(k)\}_{et} \simeq$ *the homotopy type of L. in* Pro-$\mathscr{H}$.

PROOF: Since k is algebraically closed, any $U..$ in $HR(Et(L.\times \mathrm{Spec}\, k))$ is of the form $L..\times \mathrm{Spec}(k)$, where $L..$ is some bisimplicial set. Note that $\Delta\pi(L..\times \mathrm{Spec}(k)) = \Delta L..$, and we will show that $\Delta L.. \approx L.$.

$L..\times \mathrm{Spec}(k)$ is a hypercovering of $L.\times \mathrm{Spec}(k)$, so that by Proposition III.7, each $L_p.$ is a hypercovering of L_p. By [3, 8.5(a)], the map $L_p.\to K(L_p, 0)$ is a weak equivalence. Arguing as we did in the first part of the proof of Theorem III.8, we see that the map $L..\to K(L., 0)$ induces a weak equivalence $\Delta L..\to L.$. Q.E.D.

Finally, we can compare $\{X.\}_{et}$ to the rigid étale homotopy type of $X.$ (denoted $(X.)_{ret}$) defined by Friedlander [8, §3] (and this is also the homotopy type used by Rector and Hoobler [9, §3]).

PROPOSITION III.12: *If each* X_n *is quasi-projective, then* $\{X.\}_{et}$ *is weakly isomorphic to* $(X.)_{ret}$ *in* Pro-$\mathscr{H}$.

PROOF: We will use the notation of [8, §3].

We have the functor from $RC(X.)$ to $HR(Et(X.))$ which sends $U.$ to $\mathrm{cosk}_0\, U.$ $(= N_X(U.)$ in [8]), and this gives us a map $\{\Delta\pi(U..)\}_{U..\in HR(Et(X.))}\to \{\Delta\pi(\mathrm{cosk}_0\, U.)\}_{U.\in RC(X.)} \overset{\mathrm{def}}{=} (X.)_{ret}$ in Pro-$\mathscr{H}$.

If we can show that this map is a weak equivalence, then we're done by Theorem III.8. Again, we'll use the Artin-Mazur-Whitehead Theorem.

The above map gives an isomorphism on π_1 if for every group G, we have an isomorphism

$$\varinjlim_{U.\in RC(X.)} \mathrm{Hom}(\pi_1(\Delta\pi(\mathrm{Cosk}_0\, U.)), G)\to$$

$$\varinjlim_{U..\in HR(Et(X.))} \mathrm{Hom}(\pi_1(\Delta\pi(U..)), G) \simeq \{G \text{ torseurs in } Et(X.)\}$$

(the last isomorphism is in the proof of Theorem III.8).

We can construct an inverse to this map as follows: Let $P.$ be a G-torseur in $Et(X.)$, and find $U.$ in $RC(X.)$ so that $U_0 \times_{X_0} P_0$ is constant over U_0 (this can be done since $RC(X.) \to RC(X_0)$ is cofinal [8, Lemma 3.4]). Since $U. \times_{X.} P.$ is locally constant over $U.$, it's a local system, so that each $U_n \times_{X_n} P_n$ is constant over U_n. Then it's easy to see that $\Delta\pi((\mathrm{Cosk}_0\, U.) \times_{X.} P.)$ is a simplicial G torseur over $\Delta\pi(\mathrm{Cosk}_0\, U.)$, hence an element of $\mathrm{Hom}(\pi_1(\Delta\pi(\mathrm{Cosk}_0\, U.)), G)$. The verification that this is the inverse is left to the reader.

To show that we have an isomorphism on cohomology is easy. One uses the spectral sequence technique of the second part of the proof of Theorem III.8 together with the cofinality of the functors $HR(Et(X.)) \to HR((X_p)_{et})$ (Prop. III.7) and $RC(X.) \to RC(X_p)$ (one also must use Prop. 3.2 of [8] – details are left to the reader). Q.E.D.

IV. MAIN RESULTS

§1. The homotopy type of a hypercovering

In this section we assume that X is a locally noetherian scheme and that $U.$ is a hypercovering of X in X_{ft}. We want to compute $\{U.\}_{et}$.

LEMMA IV.1: $\{K(X, 0)\}_{et} = \{X\}_{et}$.

PROOF: Let $V..$ be a hypercovering of $K(X, 0)$. Then $V._0$ is a hypercovering of X. We use $V_0.$ to construct the bisimplicial scheme $W.. = K(V_0, 0)$, where $W_{pq} = V_{0q}$ (so that $W._q = K(V_{0q}, 0)$). Each $W_p.$ is just $V_0.$, so by Proposition III.6, $W..$ is a hypercovering of $K(X, 0)$ and we have a map $W.. \to V..$ which takes $W_{p,q} = V_{0,q}$ to V_{pq} via the unique degeneracy $(s_0)^p$.

Thus, hypercoverings of the form $K(V., 0)$, where $V.$ is a hypercovering of X, are cofinal in $HR(K(X, 0))$. So $\{K(X, 0)\}_{et} = \{\pi_0(K(V., 0))\}_{V.\in HR(X)}$.

But $\pi_0(K(V., 0))_q = \pi_0(K(V_q, 0)) = \pi(V_q)$, so $\pi_0(K(V., 0)) = \pi(V.)$, and then $\{K(X, 0)\}_{et} = \{\pi(V.)\}_{V.\in HR(X)} = \{X\}_{et}$. Q.E.D.

Now, given a hypercovering $U.$ of X, the natural map $\epsilon : U. \to K(X, 0)$ gives us a map $\epsilon_{et} : \{U.\}_{et} \to \{K(X, 0)\}_{et} = \{X\}_{et}$.

THEOREM IV.2: *If X is connected, then $\epsilon_{et} : \{U.\}_{et} \to \{X\}_{et}$ is a $\#$-isomorphism in* Pro-$\mathscr{H}$ *(see* [3, 4.2]).

PROOF: We will use the Whitehead theorem of Artin-Mazur [3, 4.3]. But first we need:

LEMMA IV.3: *If $U.$ is a hypercovering of X, then the category of locally constant $C(U.)$ sheaves is equivalent to the category of locally constant sheaves on X.*

PROOF: If F is a locally constant sheaf on X, ϵ^*F is certainly a locally constant sheaf on $U.$.

Now, let $F.$ be a locally constant $C(U.)$ sheaf. By Proposition I.9, it's a local system, so by Prop. I.8, $F.$ is equivalent to the descent data (F_0, φ) on $U.$. But $U.$ is a hypercovering, so this descent data is effective. Thus $F. \cong \epsilon^*F$ for some sheaf F on X. Since $F.$ is locally constant, one sees that F must also be locally constant. Q.E.D.

With this lemma, it is easy to show that the map $\pi_1(\{U.\}_{et}) \to \pi_1(\{K(X, 0)\}_{et})$ is an isomorphism. Namely, ϵ induces a map ϵ^*: locally constant sheaves on $K(X, 0) \to$ locally constant sheaves on $U.$. Since both U and $K(X, 0)$ are hypercoverings of X, the above lemma implies that ϵ^* is an equivalence. Then it follows easily from [3, Cor. 10.7] that the map on π_1's is an isomorphism.

Next, we have to show that for any local coefficient system Γ on $\{K(X, 0)\}_{et}$, the map $H^q(\{K(X, 0)\}_{et}, \Gamma) \to H^q(\{U.\}_{et}, \epsilon^*\Gamma)$ is an isomorphism.

By [3, Cor. 10.8], Γ corresponds to an abelian locally constant $C(K(X, 0))$ sheaf $F.$ (and by Lemma IV.3, $F. = K(F, 0)$ for some locally constant sheaf F on X) and the above map is just the map $H^q(K(X, 0), K(F, 0)) \to H^q(U., \epsilon^*F)$.

But $K(F, 0)$ (resp. ϵ^*F) are local systems and thus may be regarded as $(K(X, 0))_{ft}$ (resp. $(U.)_{ft}$) sheaves. Furthermore, Proposition II.3 says that the above cohomologies are the cohomologies of $K(F, 0)$ (resp. ϵ^*F) as $(K(X, 0))_{ft}$ (resp. $(U.)_{ft}$) sheaves. Thus, we can use Proposition II.2 to write the above map as a map of hypercohomologies $\mathbf{H}^q(K(X, 0), F) \to \mathbf{H}^q(U., F)$.

Since $K(X, 0)$ and $U.$ are hypercoverings of X, both of the cohomology groups are just $H^q(X, F)$ (see [3, 8.14]) and the map between them is an isomorphism. Q.E.D.

§2. Homotopic maps of simplicial schemes

We want to prove that homotopic simplicial maps induce the same maps on homotopy types.

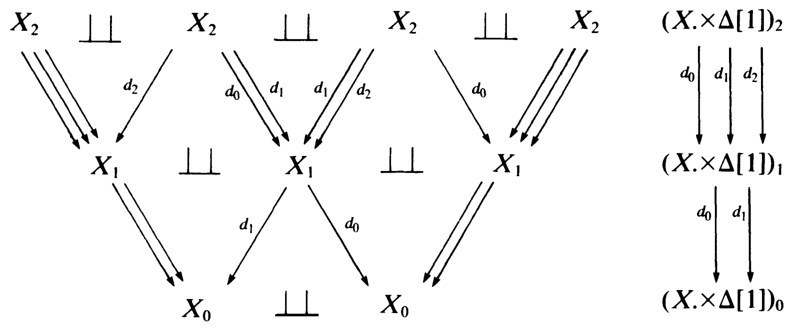
PROPOSITION IV.4: *For a connected simplicial scheme $X.$, the projection $p: X.\times\Delta[1]\to X.$ induces a # isomorphism $p_{et}:\{X.\times\Delta[1]\}_{et}\to\{X.\}_{et}$ in* Pro-$\mathscr{H}$.

PROOF: Again, we use the Artin-Mazur Whitehead Theorem.

I claim that every locally constant sheaf on $X.\times\Delta[1]$ is of the form $p^*F.$, where $F.$ is a locally constant sheaf on $X.$.

By Proposition I.9 and Prop. I.8, we can represent a locally constant sheaf $G.$ on $X.\times\Delta[1]$ as a sheaf G on $(X.\times\Delta[1])_0$ and an isomorphism $\varphi: d_1^*G \xrightarrow{\sim} d_0^*G$ on $(X.\times\Delta[1])_1$, where φ satisfies the cocycle condition $d_1^*\varphi = d_0^*\varphi\circ d_2^*\varphi$ on $(X.\times\Delta[1])_2$.

Let's draw the lower part of $X.\times\Delta[1]$:

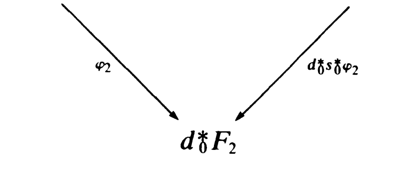


So G is really a pair (F_1, F_2) of locally constant sheaves on X_0. And φ is really three maps $\varphi_1, \varphi_2, \varphi_3$ – one on each copy of X_1 – and the cocycle condition gives us four compatibilities – again, one on each copy of X_2.

Then φ_1 and φ_3 and the outer compatabilities give us descent data for F_1 and F_2: $\varphi_1: d_1^*F_1\simeq d_0^*F_1$ and $\varphi_3: d_1^*F_2\simeq d_0^*F_2$.

And φ_2, on the middle copy of X_1, gives us an isomorphism $\varphi_2: d_1^*F_1\xrightarrow{\sim} d_0^*F_2$. Apply the degeneracy $s_0: X_0\to X_1$ to this, we get an isomorphism $s_0^*\varphi_2: F_1\xrightarrow{\sim} F_2$ (since $d_0s_0 = d_1s_0 = 1_{X_0}$).

If we take the cocycle condition, restrict it to the copy of X_2 which is second from the left (resp. third from the left) and apply the degeneracy s_1^* (resp. s_0^*), we get the diagram:



(resp., the diagram:

$$
\begin{array}{ccc}
d_1^*F_1 & \xrightarrow{\;d_1^* s_0^* \varphi_2\;} & d_1^*F_2 \\
 & {\scriptstyle \varphi_2}\searrow \quad \swarrow{\scriptstyle \varphi_3} & \\
 & d_0^*F_2 &
\end{array}
$$

Putting these diagrams together along the φ_2 edge, we get:

$$
\begin{array}{ccc}
d_1^*F_1 & \xrightarrow{\;\varphi_1\;} & d_0^*F_1 \\
{\scriptstyle d_1^* s_0^* \varphi_2}\downarrow & & \downarrow{\scriptstyle d_0^* s_0^* \varphi_2} \\
d_1^*F_1 & \xrightarrow{\;\varphi_3\;} & d_0^*F_2
\end{array}
$$

Thus, $s_0^*\varphi_2$ is an isomorphism of descent data $s_0^*\varphi_2 : (F_1, \varphi_1) \to (F_2, \varphi_3)$, and using this diagram and the first one, we see that (G, φ) is isomorphic to $p^*(F_1, \varphi_1)$. (F_1, φ_1) determines a locally constant sheaf $F.$ on $X.$ (using Theorem I.8), so $G. \cong p^*F.$, and my claim is proved.

Thus p^* is an equivalence of the categories of locally constant sheaves. As in the proof of Theorem IV.2, this shows that p_{et} induces an isomorphism $\pi_1(\{X. \times \Delta[1]\}_{et}) \to \pi_1(\{X\}_{et})$.

And the proof of Theorem IV.2 also shows that we need only prove that the map $H^q(X., F.) \to H^q(X. \times \Delta[1], p^*F.)$ is an isomorphism. But we can regard $F.$ (resp. $p^*F.$) as a $(X.)_{ft}$ (resp. $(X. \times \Delta[1])_{ft}$) sheaf, and we get the same cohomology (Proposition II.3).Then Proposition II.2 gives us a map of spectral sequences:

$$
\begin{array}{ccc}
E_1^{p,q} = H^q(X_p, F_p) & \longrightarrow & 'E_1^{p,q} = H^q((X. \times \Delta[1])_p, (p^*F.)_p) \\
\Downarrow & & \Downarrow \\
H^{p+q}(X., F.) & \longrightarrow & H^{p+q}(X. \times \Delta[1], p^*F.)
\end{array}
$$

But $(X. \times \Delta[1])_p = \amalg_{\Delta[1]_p} X_p$ and $(p^*F.)_p$ is just F_p on each copy of X_p. Thus

$$H^q((X. \times \Delta[1])_p, (p^*F.)_p) = \prod_{\Delta[1]_p} H^q(X_p, F_p) = \mathrm{Hom}(\Delta[1]_p, H^q(X_p, F_p)).$$

To make sense out of this, we need some notation. If $C^{\cdot}$ is a cosimplicial group and $K.$ a simplicial set, we get a cosimplicial group

$\mathrm{Hom}(K., C^{\cdot})$ where $\mathrm{Hom}(K., C^{\cdot})^p = \mathrm{Hom}(K_p, C^p)$. And the map $K_0 \to pt$ (pt is the trivial simplicial set) gives us a map $C^{\cdot} = \mathrm{Hom}(pt, C^{\cdot}) \to \mathrm{Hom}(K., C^{\cdot})$.

Then $E_1^{*,q}$ is a cosimplicial group (using the simplicial structure of $X.$ and regarding $F.$ as a $(X.)_{ft}$ sheaf), as is $'E_1^{*,q}$. Furthermore, we see that $'E_1^{*,q} = \mathrm{Hom}(\Delta[1], E_1^{*,q})$, and the map $E_1^{*,q} \to {}'E_1^{*,q}$ is just the above map $E_1^{*,q} \to \mathrm{Hom}(\Delta[1], E_1^{*,q})$.

So to show that we have an isomorphism from E_2 on (which we will prove the proposition), we need only show that the map $E_1^{*,q} \to \mathrm{Hom}(\Delta[1], E_1^{*q})$ is a quasi-isomorphism. This is done by:

Lemma IV.5: *If $K.$ is contractible, the map $C^{\cdot} \to \mathrm{Hom}(K., C^{\cdot})$ is a quasi-isomorphism.*

Proof: $\mathrm{Hom}(K., C^{\cdot})$ is actually a bicosimplicial object where $\mathrm{Hom}(K., C^{\cdot})^{p,q} = \mathrm{Hom}(K_p, C^q)$. So we get an Eilenberg-Zilber spectral sequence $H^p(K., H^q(C^{\cdot})) \Rightarrow H^{p+q}(\mathrm{Hom}(K., C^{\cdot}))$. Applying this to the map $K. \to pt$, we get a diagram:

$$\begin{array}{c} E_2^{p,q} = H^p(pt, H^q(C^{\cdot})) \to H^p(K., H^q(C^{\cdot})) = E_2^{p,q} \\ \Downarrow \qquad\qquad \Downarrow \\ H^{p,q}(C^{\cdot}) \to H^{p+q}(\mathrm{Hom}(K., C^{\cdot})) \end{array}$$

where the bottom arrow comes from the map $C^{\cdot} \to \mathrm{Hom}(K., C^{\cdot})$. But we obviously have an isomorphism at the E_2 level. Q.E.D.

Now, everything else is easy:

Theorem IV.6: *Let $f., g.: X. \to Y.$ be homotopic point preserving maps, where $X.$ is connected. Then $f_{et}^{\#} = g_{et}^{\#}: \{X.\}_{et}^{\#} \to \{Y\}_{et}^{\#}$ in* Pro-$\mathscr{H}$ (*see* [3, §4]).

Proof: We have $a_0, a_1: X. \to X. \times \Delta[1]$, the inclusion of $X.$ into the two ends of $X. \times \Delta[1]$. Then $p \circ a_i = 1_{X.}$, $i = 0, 1$, where p is as in Proposition IV.4. Thus $p_{et} \circ (a_0)_{et} = p_{et} \circ (a_1)_{et} = 1_{\{X.\}_{et}}$. Since $p_{et}^{\#}$ is an isomorphism, we see that $(a_0)_{et}^{\#} = (a_1)_{et}^{\#}$ is Pro-$\mathscr{H}$.

We can assume $f.$ and $g.$ are strongly homotopic, so that there is a map $H.: X. \times \Delta[1] \to Y.$ with $f. = H. \circ a_0$ and $g. = H. \circ a_1$. Then $f_{et}^{\#} = H_{et}^{\#} \circ (a_0)_{et}^{\#} = H_{et}^{\#} \circ (a_1)_{et}^{\#} = g_{et}^{\#}$. Q.E.D.

A corollary of the proofs of Proposition IV.4 and Theorem IV.6 is:

Corollary IV.7: *Let $f., g.: X. \to Y.$ be homotopic maps, and let $F.$*

be a local system on $Y.$. Then:

1) $f.^*F. \simeq g.^*F.$

2) *The maps* $f.^*, g.^*: H^q(Y., F.) \to H^q(X., f.^*F.) \simeq H^q(Y., f.^*F.)$ *are the same.*

3) *The maps* $E_r(f.^*)$, $E_r(g.^*)$ $(1 \le r \le \infty)$ *of spectral sequences*

$$\begin{array}{ccc} E_1^{p,q} = H^q(Y_p, F_p) & \to & E_1^{p,q} = H^q(X_p, f_p^*F_p) \simeq H^q(X_p, g_p^*F_p) \\ \Downarrow & & \Downarrow \\ H^{p+q}(Y., F.) & \to & H^{p+q}(X., f.^*F.) \simeq H^{p+q}(X., g.^*F.) \end{array}$$

agree from E_1 on.

§3. The comparison theorem

Let $X.$ be a simplicial scheme of finite type over $\boldsymbol{C}$ (which means that each X_n is of finite type over $\boldsymbol{C}$). Then we get a simplicial analytic space $X.^{an}$. We want to compare $\{X.\}_{et}$ with the usual homotopy type of the simplicial space $X.^{an}$ (and we'll denote this by $cl(X.^{an})$).

THEOREM IV.8: *There is a natural weak isomorphism in* Pro-$\mathscr{H}$ *between* $\{X.\}_{et}^{\wedge}$ *and* $cl(X.^{an})^{\wedge}$ (*where* $^{\wedge}$ *denotes pro-finite completion*).

PROOF: First, we need to recall the definition of $cl(X.^{an})$. Rather than the usual "geometric realization" approach, we'll use the simplicial methods of [4]. We have the singular functor Sin: Spaces $\to$ Simplicial Sets, so that $\text{Sin}\, X.^{an}$ is a bisimplicial set. Then we set $cl(X.^{an}) = \Delta\ \text{Sin}\, X.^{an}$.

Now, let $CL(X.^{an})$ be the category of all maps $f.: Y. \to X.^{an}$, where $Y.$ is a simplicial topological space and each f_n is a local homeomorphism. With the obvious topology, $CL(X.^{an})$ becomes an Artin-Mazur site, and we have:

PROPOSITION IV.9: $cl(X.^{an}) \simeq \{\Delta\pi(U..)\}_{U.. \in HR(CL(X.^{an}))}$ *in* Pro-$\mathscr{H}$.

PROOF: Let $U..$ be a member of $HR(CL(X.^{an}))$ with the following property: for all p and q, U_{pq} is a disjoint union of open contractible subsets of X_p^{an}. Then we construct the bisimplicial set $K..$ where $K_{pq} = \text{Sin}_p\, U_{qp}$ ($K..$ can be regarded as the functor from Δ° to simplicial sets which sends $[q]$ to Δ Sin $U_{q\cdot}$).

The natural maps $U_{qp} \to X_q^{an}$ give us maps $K_{pq} = \text{Sin}_p\, U_{qp} \to \text{Sin}_p\, X_q^{an}$, so that we get a map of bisimplicial sets $a..: K.. \to \text{Sin}\, X.^{an}$.

Also, the obvious maps $K_{pq} = \mathrm{Sin}_p\, U_{qp} \to \pi(U_{qp})$ give us a map of bisimplicial sets $\beta..: K.. \to \pi(U..)$.

If we fix p, then [3, Thm. 12.1] shows that the maps $a._p: K._p = \Delta\ \mathrm{Sin}\ U_p. \to \mathrm{Sin}\ X_p^{an}$ and $\beta._p: K._p = \Delta\ \mathrm{Sin}\ U_p. \to \pi(U_p.)$ are isomorphisms in $\mathscr{H}$. Thus, by [8, XII.4.2], the maps $\underrightarrow{\mathrm{holim}}\, K.. \to \underrightarrow{\mathrm{holim}}\ \mathrm{Sin}\ X.^{an}$ and $\underrightarrow{\mathrm{holim}}\, K.. \to \underrightarrow{\mathrm{holim}}\ \pi(U..)$ are isomorphisms in $\mathscr{H}$. But by [8, XII.3.4], $\underrightarrow{\mathrm{holim}}\ \mathrm{Sin}\ X.^{an} = \Delta\ \mathrm{Sin}\ X.^{an} = cl(X.^{an})$ and $\underrightarrow{\mathrm{holim}}\ \pi(U..) = \Delta\pi(U..)$, so that $\Delta\pi(U..) \simeq cl(X.^{an})$ in $\mathscr{H}$.

Using the techniques of [3, §8], one can easily prove that $U..$'s as above are cofinal in $HR(CL(X.^{an}))$. This proves the proposition.

Q.E.D.

Now, for the proof of the theorem. There is an obvious functor $Et(X.) \to CL(X.^{an})$ which gives a map in Pro-$\mathscr{H}$:

$$cl(X.^{an}) \approx \{\Delta\pi(U..)\}_{U..\in HR(CL(X.^{an}))} \longrightarrow \{\Delta\pi(U..)\}_{U..\in HR(Et(X.))} \overset{\#}{\approx} \{X.\}_{et}$$

(the #-isomorphism at the right is by Thm. III.8). We can assume that $X.$ is connected and we'll use the Artin-Mazur-Whitehead Theorem to show that this map induces an isomorphism after pro-finite completion.

By arguments similar to the π_1 part of Thm. III.8, we see that $\pi_1(\mathrm{cl}(X.^{an})) \simeq \pi_1(\{CL(X.^{an})\}_{ht})$. Thus, to show that we have an isomorphism on $\hat{\pi_1}$'s, we need only show that the functor $Et(X.) \to CL(X.^{an})$ induces an equivalence on the categories of G-torseurs for every finite group G (this follows from [3, Cor. 10.7]). But it is easy to construct a quasi-inverse. Given a G-torseur $P.$ of $X.^{an}$, each P_n is a G-torseur of X_n^{an}. Since G is finite, the Grauert-Remmert Theorem (see [15, XI 4.3(iii)]) says that $P_n = Q_n^{an}$ where Q_n is an étale G-torseur. These fit together to form a G-torseur $Q.$ in $Et(X.)$. The desired quasi-inverse is the functor which sends $P.$ to $Q.$. So we have proved that $\pi_1(cl(X.^{an}))\hat{} \simeq \pi_1(\{X.\}_{et})\hat{}$.

Now, for cohomology. A finite local system on $\{\Delta\pi(U..)\}_{U..\in HR(Et(X.))}$ is a map $\Gamma: \pi_1(\{\Delta\pi(U..)\}_{U..\in HR(Et(X.))}) \to \mathrm{Aut}(M)$, where M is a finite group. This gives a local system $\Gamma.$ on $\Delta\pi(U..)$, for some $U.. \in HR(Et(X.))$. To finish the proof, we need to show that the map

$$\underset{a: U'.. \to U..\text{ in } HR(Et(X.))}{\underrightarrow{\lim}} H^q(\Delta\pi(U'..), a^*\Gamma.) \to \quad ①$$

$$\underset{a: U'.. \to U..^{an}\text{ in } HR(CL(X.^{an}))}{\underrightarrow{\lim}} H^q(\Delta\pi(U'..), a^*\Gamma.) \quad ②$$

is an isomorphism.

By Lemma III.9, $\Gamma.$ is isomorphic to $\Delta\Gamma..$, where $\Gamma..$ is a local system over $\pi(U..)$. Note that for any a (from either ① or ② above), we have a functorial spectral sequence $E_1^{p,q} = H^q(\pi(U'_{p\cdot}), a^*\Gamma_{p\cdot}) \Rightarrow H^{p+q}(\Delta\pi(U'_{\cdot\cdot}), a^*\Gamma.)$. Thus, we are reduced to showing that the map

$$(1) \qquad \varinjlim_{a \text{ as in } ①} H^q(\pi(U'_{p\cdot}), a^*\Gamma_{p\cdot}) \to \varinjlim_{a \text{ as in } ②} H^q(\pi(U'_{p\cdot}), a^*\Gamma_{p\cdot})$$

is an isomorphism.

To relate this to X_p, first let's recall that $\Gamma..$ arises as follows (see the proof of Thm. III.8): there is a locally constant abelian sheaf $F. \to X.$ which becomes trivial on $U._0$. Then $\Gamma.. = \pi(U.. \times_{X.} F.)$, and in particular, $a^*\Gamma_{p\cdot} = (U'_{p\cdot} \times_{X_p} F_p)$ (a as in case ①) and $a^*\Gamma_{p\cdot} = \pi(U'_{p\cdot} \times_{X_p^{an}} F_p^{an})$ (a as in case ②). Note also (using the proof of Prop. III.7) that the functors $HR(Et(X.)) \to HR((X_p)_{et})$ and $HR(CL(X.^{an})) \to HR(CL(X_p^{an}))$ which send $U..$ to $U_{p\cdot}$ are cofinal. (It's obvious what $CL(X_p^{an})$ is.) Thus, we can write (1) as:

$$\varinjlim_{a: U'_\cdot \to U_{p\cdot} \text{ in } HR((X_p)_{et})} H^q(\pi(U'_\cdot), \pi(U'_\cdot \times_{X_p} F_p)) \to$$
$$\varinjlim_{a: U'_\cdot \to U_{p\cdot}^{an} \text{ in } HR(CL(X_p^{an}))} H^q(\pi(U'_\cdot), \pi(U'_\cdot \times_{X_p^{an}} F_p^{an})).$$

Then the Verdier Theorem [3, Cor. 10.8] show that this map is just the map $H^q(X_p, F_p) \to H^q(X_p^{an}, F_p^{an})$, which is an isomorphism by [15, XI 4.4(iii)].

So the theorem is proved. Q.E.D.

§4. The classifying space of an algebraic group

In this section we apply the techniques and results of §§1–3 to the study of group schemes.

Let X be a locally noetherian scheme, and let G be a group scheme of finite type over X. B_G, the classifying topos of G, will denote the category of G-objects in $X_{\widetilde{ft}}$ (see [13, IV.2.4]).

Let $W(G)$ denote the simplicial scheme $\operatorname{cosk}_0 G$. And $\bar{W}(G)$ is the simplicial scheme where $\bar{W}(G)_0 = X$, $\bar{W}(G)_n = G \times_X G \times \cdots \times_X G$ (n times), and the boundary and degeneracy maps are defined symbolically as follows:

$$d_0(g_1, \ldots, g_n) = (g_2, \ldots, g_n)$$

$$d_i(g_1, \ldots, g_n) = (g_1, \ldots, g_{i-1}, g_{i+1} \cdot g_i, g_{i+2}, \ldots, g_n) \quad 0 < i < n$$

$d_n(g_1, \ldots, g_n) = (g_1, \ldots, g_{n-1})$

$s_i(g_1, \ldots, g_n) = (g_1, \ldots, g_i, 1, g_{i+1}, \ldots, g_n) \quad 0 < i < n$

Note that $W(G)$ lies naturally in simpl(B_G) (using the diagonal action), and that the quotient of $W(G)$ by G is just $\bar{W}(G)$. The map $W(G) \to \bar{W}(G)$ can be described very explicitly (see [8, §2] for all this).

$\bar{W}(G)$ can be called the "simplicial classifying scheme" of G, and we want to relate it to the Artin-Mazur homotopy type of B_G (which will be discussed below):

THEOREM IV.10: *There is a canonical weak equivalence between* $\{B_G\}_{ht}$ *and* $\{\bar{W}(G)\}_{et}$ *in* Pro-$\mathscr{H}$.

PROOF: First, how do we get a homotopy type for B_G?

Let's start with the connected component functor of $X_{\tilde{ft}}$. Note that for any F in $X_{\tilde{ft}}$, there is an epimorphism $U \to F$ where U is in X_{ft}. From this, one can construct a hypercovering $U.$ of F where each U_n is in X_{ft}. Then set $\pi(F) = \pi_0(\pi(U.))$. One easily proves that this doesn't depend on $U.$, and that it is the connected component functor of $X_{\tilde{ft}}$.

Then we get the functor $\pi_G : B_G \to \text{Sets}$ where $\pi_G(F) = \pi(F/G)$ (F/G is the quotient of F by G in $X_{\tilde{ft}}$). Again, one easily proves that this is the connected component functor of B_G. So to define $\{B_G\}_{ht}$, we need only show that $HR(B_G)$ has a cofinal small subcategory.

First, note that any F in B_G/G has a covering $U \to F$ where $U \in X_{ft}$ (for by [13, IV.5.8], $F \simeq H \times_X G$ in B_G, where H has trivial G-action, so that a covering $U \to H$ with U in X_{ft} gives us the covering $U \times_X G \to H \times_X G = F$). Note also that the map $G \to e$ covers in B_G, so that hypercoverings $U.$, where U_0 factors through G, are cofinal in $HR(B_G)$. These two observations, combined with the tehniques of [3, §8], prove that hypercoverings $U.$, where each U_n is in X_{ft}, are cofinal in $HR(B_G)$.

(In a similar way, we get homotopy types for simpl$(B_G)/W(G)$ and $C(\bar{W}(G))^{\sim}$, and obviously $\{C(\bar{W}(G))^{\sim}\}_{ht} \simeq \{C(\bar{W}(G))\}_{ht} = \{\bar{W}(G)\}_{et}$.)

Now, we get to the actual proof.

Since $G \to e$ covers in X_{ft}, $W(G) = \text{cosk}_0\, G$ is a hypercovering in $X_{\tilde{ft}}$. Since $W(G)$ is in simpl(B_G), it is a hypercovering of B_G (since B_G has the induced topology and the map $B_G \to X_{\tilde{ft}}$ preserves inverse limits, in particular cosk). Thus, by an argument which is step-by-step the same as Theorem IV.2, we see that $\{B_G\}_{ht} \simeq \{\text{simpl}(B_G)/W(G)\}_{ht}$.

I claim we have equivalences of categories simpl$(B_G)/W(G) \simeq$

$\text{simpl}(X_{\widetilde{ft}})/\bar{W}(G) \simeq C(\bar{W}(G))^{\sim}$. (the last equivalence is trivial). Note that this will prove the theorem!

We have a functor from $\text{simpl}(B_G)/W(G)$ to $\text{simpl}(X_{\widetilde{ft}})/\bar{W}(G)$ which sends $Y. \to W(G)$ to $Y./G \to W(G)/G \simeq \bar{W}(G)$ ($Y./G$ is the quotient of $Y.$ by G). Going the other way, we map $Z. \to \bar{W}(G)$ in $\text{simpl}(X_{\widetilde{ft}})/\bar{W}(G)$ to the projection $Z. \times_{\bar{W}(G)} W(G) \to W(G)$ in $\text{simpl}(B_G)/W(G)$. I claim that these functors are quasiinverses of each other.

First, note that if F is in B_G and $a: F \to G$ is a map in B_G, then $F \simeq (F/G) \times_X G$ in B_G (where the isomorphism depends on a). This is by [13, IV.5.8].

To show that $Y. \simeq (Y./G) \times_{\bar{W}(G)} W(G)$, we need to show that the diagram:

$$\begin{array}{ccc} Y. & \longrightarrow & W(G) \\ \downarrow & & \downarrow \\ Y./G & \longrightarrow & \bar{W}(G) \end{array} \tag{1}$$

is cartesian. Given n, projection on the first factor gives us a map from $W(G)_n$ to G. Thus, using the above remark, we see that level n, (1) looks like:

$$\begin{array}{ccc} Y_n \simeq (Y_n/G) \times_X G & \longrightarrow & W(G)_n \simeq \bar{W}(G)_n \times_X G \\ \downarrow & & \downarrow \\ Y_n/G & \longrightarrow & \bar{W}(G)_n \end{array}$$

which is obviously cartesian. So (1) is also cartesian.

Finally, the projection map $Z. \times_{\bar{W}(G)} W(G) \to Z.$ gives a map $(Z. \times_{\bar{W}(G)} W(G))/G \to Z.$ (since G acts trivially on $Z.$). Again picking a map $W(G)_n \to G$, we see that $Z_n \times_{\bar{W}(G)_n} W(G)_n \simeq Z_n \times_{\bar{W}(G)_n} (\bar{W}(G)_n \times_X G) \simeq Z_n \times_X G$, and the quotient is obviously Z_n.

This is true for all n, so that $(Z. \times_{\bar{W}(G)} W(G))/G \xrightarrow{\sim} Z.$.

This finishes the proof. Q.E.D.

If we take $X = \text{Spec}(C)$, then G is nothing but a group variety over C. From Theorems IV.8 and IV.10 we get:

COROLLARY IV.11: *We have the following weak isomorphisms in* Pro-$\mathcal{H}$:

$$\{B_G\}^{\wedge}_{ht} \simeq \{\bar{W}(G)\}^{\wedge}_{et} \simeq cl(\bar{W}(G^{an}))^{\wedge} \simeq (B_{G^{an}})^{\wedge}$$

($B_{G^{an}}$ is the classifying space of the topological group G^{an}, and the last isomorphism is well-known).

Note that this, together with [3, Cor. 10.8], proves Theorems 3.7 and 3.8 of [9].

Appendix A

We want to examine various aspects of sheaves on X_{ft}, and also we want to see how they relate to sheaves on X_{et} (we call these small étale sheaves – see §I.2). We will assume that the reader is familiar with small étale sheaves. (Given a map of schemes $f: X \to Y$, we will use the notation f^*_{et}, $R^q f_{et*}$ for the usual functors.)

§1. First, we need to mention some of the set-theoretical difficulties involved. Namely, the sites X_{ft} and $C(X.)$, as defined, are too big. But in every case, we can restrict ourselves to a small subcategory which is large enough for our purposes. We will always assume that this has been done.

§2. One easily verifies that a sheaf F on X_{ft} consists of the following data:

1) For every map $f: U \to X$ in X_{ft}, a small étale sheaf $F_{f: U \to X}$ on U (which we often denote F_U).

2) For every map $g: U \to V$ in X_{ft}, a map $f(g): g^*_{et} F_V \to F_U$ in $U^{\sim}_{et}$, and $F(g)$ is an isomorphism if g is étale.

3) (Compatability) Given maps $g: U \to V$ and $h: V \to W$ in X_{ft}, then $F(h \circ g) = F(g) \circ g^*_{et}(F(h))$.

Note that this is analogous to both Definition I.2 and the definition of a sheaf on the crystalline site.

An example is the structure sheaf $\mathcal{O}_X$ of X_{ft}. It is defined by the formula $\mathcal{O}_X(U \to X) = \Gamma(U, \mathcal{O}_U)$. In terms of 1), 2) and 3) above, $(\mathcal{O}_X)_U = \mathcal{O}_U$ (where this means the usual structure sheaf of U_{et}), with the usual transition maps.

§3. We have the functor $u_*: X^{\sim}_{ft} \to X^{\sim}_{et}$ defined by $u_*(F) = F_X$, and the functor $u^*: X^{\sim}_{et} \to X^{\sim}_{ft}$, defined as follows: given G in $X^{\sim}_{et}$ and $g: U \to X$ in X_{ft}, then $(u^*G)_U = g^*_{et} G$. u_* and u^* have all of the usual properties (see [2, VI, Prop. 1.6]).

Note that those X_{ft} sheaves for which all the $F(g)$'s are isomorphisms (analogous to the local systems of Def. I.7) are just the sheaves on X_{et} (via u_* and u^*). Note also that $u^*\mathcal{O}_X$ is not $\mathcal{O}_X$.

§4. Given a map of schemes $f: X \to Y$, we get an inverse image functor $f^*_{ft}: Y^{\sim}_{ft} \to X^{\sim}_{ft}$. In all cases we will consider, f will be locally of finite type, so that f^*_{ft} is just a restriction functor. More precisely, given $F: Y_{ft} \to \text{Sets}$ in $Y_{\widetilde{ft}}$, then $f^*_{ft}F$ is the composition $X_{ft} \xrightarrow{\sim} Y_{ft}/X \to Y_{ft} \to \text{Sets}$. (In terms of §2, given $U \to X$ in X_{ft}, then $(f^*_{ft}F)_U = F_U$).

In particular, note that $f^*_{ft}\mathcal{O}_Y = \mathcal{O}_X$ (see §I.5, Ex. 1).

Appendix B

Let C be a category with disjoint sums and fibered products. Take $X.$ in simpl(C). Then for each $n \geq 0$, we have the functor p_n: $\text{simpl}(C)/X. \to C/X_n$ which sends $Y. \to X.$ to $Y_n \to X_n$. We will construct left and right adjoints to p_n.

First, let's do the left adjoint. Recall that the standard n-simplex $\Delta[n]$ is the simplicial set $\text{Hom}_\Delta(\ ,[n]): \Delta^\circ \to \text{Sets}$. Given X in C, we construct the simplicial object $X \times \Delta[n]$ as follows:

1) $(X \times \Delta[n])_m = \coprod_{f \in \text{Hom}_\Delta([m],[n])} X_f$, where each X_f is a copy of X.

2) Given a map $g:[m] \to [k]$ in Δ, we get a map from $(X \times \Delta[n])_k = \coprod_{f \in \text{Hom}_\Delta([k],[n])} X_f$ to $(X \times \Delta[n])_m = \coprod_{h \in Hom_\Delta([m],[n])} X_h$ where X_f is mapped (via the identity) to $X_{f \circ g}$.

Also, we have a map from $Y_n \times \Delta[n]$ to $Y.$ described as follows: given $m \geq 0$, each $f \in \text{Hom}_\Delta([m],[n])$ gives a map from $Y_n = (Y_n)_f$ to Y_m, so that together these give a map from $(Y_n \times \Delta[n])_m = \coprod_{f \in \text{Hom}_\Delta([m],[n])} (Y_n)_f$ to Y_m.

Then the left adjoint to p_n is the functor which sends $Y \to X_n$ in C/X_n to the composition $Y \times \Delta[n] \to X_n \times \Delta[n] \to X.$ in $\text{simpl}(C)/X.$. The map $Y_n \times \Delta[n] \to Y.$ gives one adjunction map, and the other is easy to describe.

Next, let's construct the right adjoint (which we denote by β_n) of p_n. Given an object $Y \to X_n$ of C/X_n, we construct the simplicial object $\beta_n(Y \to X_n)$ as follows:

1) $(\beta_n(Y \to X_n))_m = \underset{f \in \text{Hom}_\Delta([n],[m])}{\times_{X_m}} (X_m \times_{X_n} Y_f)$, where $X_m \times_{X_n} Y_f$ is determined by the cartesian diagram:

$$\begin{array}{ccc} X_m \times_{X_n} Y_f & \longrightarrow & X_m \\ \downarrow & & \downarrow {\scriptstyle X(f)} \\ Y & \longrightarrow & X_n \end{array}$$

2) Take a map $g:[m] \to [k]$ in Δ. Let f be in $\text{Hom}_\Delta([n],[m])$. Then we get the maps $pr_{g \circ f}: (\beta_n(Y \to X_n))_k \to X_k \times_{X_n} Y_{g \circ f}$ and $X(g) \times_{1_{X_n}} 1_Y: X_k \times_{X_n} Y_{g \circ f} \to X_m \times_{X_n} Y_f$. The composition is a map from $(\beta_n(Y \to X_n))_k$ to $X_m \times_{X_n} Y_f$, and together these describe a map from $(\beta_n(Y \to X_n))_k$ to $(\beta_n(Y \to X_n))_m$.

There is a natural map from $(\beta_n(Y \to X_n))_m$ to X_m, and hence a map from $\beta_n(Y \to X_n)$ to $X.$. Thus β_n is a functor from C/X_n to simpl $(C)/X$.

The adjunction maps are easy to describe, but since we don't need to know them explicitly, we'll omit them.

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