

# COMPOSITIO MATHEMATICA

L. E. DOR

T. STARBIRD

**Projections of  $L_p$  onto subspaces spanned by independent random variables**

*Compositio Mathematica*, tome 39, n° 2 (1979), p. 141-175

[http://www.numdam.org/item?id=CM\\_1979\\_\\_39\\_2\\_141\\_0](http://www.numdam.org/item?id=CM_1979__39_2_141_0)

© Foundation Compositio Mathematica, 1979, tous droits réservés.

L'accès aux archives de la revue « Compositio Mathematica » (<http://http://www.compositio.nl/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme  
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

## PROJECTIONS OF $L_p$ ONTO SUBSPACES SPANNED BY INDEPENDENT RANDOM VARIABLES

L. E. Dor<sup>1,3</sup> and T. Starbird<sup>2</sup>

### Abstract

If the closed linear span of a sequence of independent random variables in  $L_p$ ,  $1 \leq p < \infty$ , is isomorphic to  $\ell_p$ , then it is complemented in  $L_p$ . Some generalizations to martingale-difference sequences are also discussed. Analytic conditions are given which determine whether a given sequence in  $L_p$  is equivalent to the usual  $\ell_p$ -basis. It is shown that if a modular sequence space is isomorphic to  $\ell_p$ , then it is identical with  $\ell_p$ .

### 1. Introduction

This paper is mainly concerned with subspaces of  $L_p$ ,  $1 \leq p < \infty$ , which are isomorphic to  $\ell_p$  and are spanned either by a sequence of independent random variables or by a martingale-difference sequence. One main result in this respect is

**THEOREM A:** *Let  $1 \leq p < \infty$ , and let  $(f_i)$  be a sequence of independent random variables in  $L_p$ . If the closed linear span  $[f_i]$  of  $(f_i)$  is isomorphic to  $\ell_p$ , then  $[f_i]$  is complemented in  $L_p$  (i.e. there is a bounded linear projection from  $L_p$  onto  $[f_i]$ ).*

<sup>1</sup>The contribution of the first-named author was partly supported by the National Science Foundation grants number MSP 73-08715 and MCS 76-07459.

<sup>2</sup>The contribution of the second-named author was partly supported by the National Science Foundation grant number MCS 77-04140.

<sup>3</sup>Current address: Wayne State University, Detroit, Michigan 48202.

For  $1 < p < \infty$ ,  $p \neq 2$ , none of the assumptions of Theorem A is redundant. Namely, there are uncomplemented subspaces of  $L_p$  which are isomorphic to  $\ell_p$  (see [21] for the case of  $2 < p < \infty$ , and [1] for  $1 < p < 2$ ; the case of  $1 < p < \frac{4}{3}$  was done earlier in [22] based on the results of [23]). Also, for  $1 \leq p < \infty$ ,  $p \neq 2$ , there are uncomplemented subspaces of  $L_p$  which are spanned by sequences of symmetric independent random variables. For  $2 < p < \infty$  such subspaces were constructed by H. P. Rosenthal [21, p. 284]. For  $1 \leq p < 2$  and any  $p < r < 2$ , there is a subspace  $E$  of  $L_p$  spanned by a sequence of independent random variables which is isomorphic to  $\ell_r$  (see [11], for example) and therefore can not be complemented there, since  $E^*$  does not embed isomorphically in  $L_p^*$  (see [11]).

It is an open problem whether every isomorph of  $\ell_1$  in  $L_1$  is complemented there.

In our work on Theorem A we obtained several structural results for unconditional basic sequences in  $L_p$ ,  $2 < p < \infty$ , which are interesting in their own right, and may be summarized as follows:

**THEOREM B:** *Let  $(f_i)$  be a normalized unconditional basic sequence in  $L_p$ ,  $2 < p < \infty$ . Assume that  $(f_i)$  is not equivalent to the usual  $\ell_p$ -basis. Then*

- (i) *for each  $\lambda > 1$  and  $n \in \mathbb{N}$  there are disjoint finite sets  $A_1, \dots, A_n$  of integers and elements  $x_1, \dots, x_n$  with  $x_i \in [f_j; j \in A_i]$  for  $i \leq n$ , so that  $(x_i, i \leq n)$  is  $\lambda$ -equivalent to the usual  $\ell_2^n$ -basis.*
- (ii) *If, in addition,  $(f_i)$  is a modular basis, then  $[f_i]$  contains a subspace isomorphic to  $\ell_2$ .*

Recall that a basic sequence  $(z_i)$  in a Banach space is called a modular basis if there is a sequence of Orlicz functions  $(\phi_i)$  such that for all sequences  $(\alpha_i)$  of scalars  $\sum_{i=1}^{\infty} \alpha_i z_i$  converges if and only if for some  $t > 0$   $\sum_{i=1}^{\infty} \phi_i(|\alpha_i|/t) < \infty$ . (Then the correspondence  $\sum_{i=1}^{\infty} \alpha_i z_i \rightarrow (\alpha_i)$  is an isomorphism of  $[z_i]$  onto the modular sequence space  $\ell_{(\phi_i)}$ ; see [11a] section 4.d, for example.) Two sequences  $(y_i), (z_i)$  in Banach spaces are said to be  $\lambda$ -equivalent if for all finitely non-zero sequences  $(\alpha_i)$  of scalars,

$$\lambda^{-1} \left\| \sum_i \alpha_i y_i \right\| \leq \left\| \sum_i \alpha_i z_i \right\| \leq \lambda \left\| \sum_i \alpha_i y_i \right\|.$$

Part (i) of Theorem B was obtained independently of us by W. B. Johnson, B. Maurey, G. Schechtman and L. Tzafriri [6], in the more general setting of Banach lattices satisfying a suitable convexity

condition. Both our proof of (i) and that given in [6] show first that the conclusion of (i) holds for some value of  $\lambda > 1$ . The extension to all values of  $\lambda > 1$  follows then from Krivine's theorem [10].

It follows in particular from Theorem B (ii) that for  $2 < p < \infty$ , every modular basis for  $\ell_p$  is equivalent to the usual basis of  $\ell_p$ . In fact, this result holds for all values of  $1 \leq p < \infty$ : for  $1 < p < 2$  this follows by duality from the case stated above, and for  $p = 1$  and  $p = 2$  this follows by default from the fact that all unconditional bases in  $\ell_1$  (respectively  $\ell_2$ ) are equivalent. This latter result for  $\ell_1$  was proved by J. Lindenstrauss and A. Pełczyński [12] as an application of Grothendieck's inequality. For  $\ell_2$  it follows easily from the parallelogram law (see also Fact 2.4 (iii) below).

The uniqueness of modular bases in  $\ell_p$  relates to the proof of Theorem A as follows: Let  $(f_i)$  be a sequence of mean zero independent random variables normalized in  $L_p$ ,  $1 \leq p < \infty$ . By a theorem of H. P. Rosenthal [20]  $(f_i)$  is then a modular basis for  $[f_i]$  in  $L_p$  (see also [2] for the identically distributed case). If  $[f_i]$  is isomorphic to  $\ell_p$ ,  $(f_i)$  is thus forced to be equivalent to the usual basis of  $\ell_p$ . The equivalence of a sequence  $(f_i)$  of independent random variables in  $L_p$  to the usual  $\ell_p$ -basis is characterized in analytical terms in Proposition 3.5. (Basically the condition is that a sufficient percentage of the norm of the  $f_i$ 's can be obtained on disjoint sets.) These analytical conditions are then used to produce a projection onto  $[f_i]$ .

This last component of the proof can be extended to show that a weaker version of Theorem A holds for martingale differences: if  $(f_i)$  is a martingale-difference sequence in  $L_p$ ,  $1 < p < \infty$ , and if  $[f_i]$  is equivalent to the usual  $\ell_p$ -basis, then  $[f_i]$  is complemented in  $L_p$ . This is the content of Theorem 5.1 below. In the first draft of this paper this result was proved only for  $2 < p < \infty$ . The proof for  $1 < p < 2$  is due to G. Schechtman. We thank Professor Schechtman for permission to reproduce his proof here.

We now describe in greater detail the organization of the paper. Section 2 contains notation and some known preliminary facts.

Section 3 contains three characterizations of sequences in  $L_p$  which are equivalent to the usual  $\ell_p$ -basis. Theorem 3.3 is identical with part (i) of Theorem B stated above. It is local-isomorphic in character, and concerns  $2 < p < \infty$ . A preliminary result is Proposition 3.1, which states that for every  $1 \leq p < \infty$ , any normalized unconditional basic sequence in  $L_p$  whose elements are positive functions is equivalent to the usual  $\ell_p$ -basis. Our second characterization is analytic in character. We show in Proposition 3.4 that for  $2 < p < \infty$  the normalized unconditional basic sequence  $(f_i)$  in  $L_p$  is equivalent to the usual basis

of  $\ell_p$  if and only if  $\sum_{i=1}^{\infty} \|g_i\|_2^{2p/(p-2)} < \infty$  for every sequence  $(g_i)$  which can be obtained from  $(f_i)$  by an isometric automorphism of  $L_p$  (i.e. by a change of density). We also list in that proposition the corresponding condition for  $1 \leq p < 2$ , which is the existence of disjoint sets  $E_i$  such that  $\inf_i \int_{E_i} |f_i| > 0$ . This part follows from known results of W. B. Johnson and E. Odell [7] and of the first-named author [4]. Finally in Proposition 3.5 we show that the analytic conditions in Proposition 3.4 may be weakened, in both cases, if we assume that the  $f_i$  are independent random variables.

In Section 4 we prove results on modular sequence spaces. In Theorem 4.2 we show that for  $2 < p < \infty$ , every modular basic sequence in  $\ell_p$  is equivalent to the usual  $\ell_p$ -basis. This result together with a result of W. B. Johnson and E. Odell [7] implies part (ii) of Theorem B. The formulation of our result as given in Theorem B (ii) was pointed out to us by H. P. Rosenthal. As outlined above, Theorem 4.2 and known results imply that for  $1 \leq p < \infty$ ,  $\ell_p$  has a unique modular basis. This fact is stated as Corollary 4.3. As a preliminary to Theorem 4.2 we present in Proposition 4.1 a characterization of the containment  $\ell_{(\phi_i)} \subseteq \ell_{(\psi_i)}$  between two modular spaces, in terms of a certain parameter relating each of the functions  $\phi_i$  to the corresponding function  $\psi_i$ . Several results of this type with various degrees of generality are available (see [25] and [27]). We found the formulation in Proposition 4.1 somewhat more concise and more convenient for our proofs.

In Section 5 we prove Theorem A, and present the generalization to the case of martingale-difference sequences (Theorem 5.1) referred to earlier. We also present in Proposition 5.2 a related result of A. Pełczyński and H. P. Rosenthal [16]. The main reason for including this result here is that it is not stated explicitly in [16]. Our proof of this result is different from that of [16].

## 2. Notation and preliminaries

We consider here the spaces  $L_p = L_p[0, 1]$ ,  $1 \leq p \leq \infty$ , where  $[0, 1]$  is endowed with the Lebesgue measure  $A \rightarrow |A|$  on the  $\sigma$ -algebra  $\mathcal{L}$  of the Lebesgue measurable sets. However, the results easily extend to  $L_p$  on an arbitrary probability space. The scalar field is either  $\mathbb{R}$  or  $\mathbb{C}$ .  $q$  will denote the conjugate exponent of  $p$ ,  $(1/p + 1/q = 1)$ .

If  $\mathcal{F}$  is a family of measurable functions,  $\mathcal{A}(\mathcal{F})$  will denote the smallest  $\sigma$ -algebra with respect to which all  $f \in \mathcal{F}$  are measurable. If  $\mathcal{A}$  is a  $\sigma$ -subalgebra of  $\mathcal{L}$ ,  $\mathcal{E}_{\mathcal{A}}$  will denote the conditional expectation

operator with respect to  $\mathcal{A}$  (defined by the relationship  $\int_A \mathcal{E}_{\mathcal{A}} f = \int_A f$ , all  $f \in L_1$ ,  $A \in \mathcal{A}$ ).  $\mathcal{E}_{\mathcal{A}}$  is a contractive projection in all  $L_p$ ,  $1 \leq p \leq \infty$ . A sequence  $(f_i)$  in  $L_1$  is called a martingale-difference sequence if for all  $m < n$ ,  $\mathcal{E}_{\mathcal{A}_m} f_n = 0$ , where  $\mathcal{A}_m = \mathcal{A}(f_1, \dots, f_m)$ . If  $P$  and  $Q$  are expressions denoting real numbers and  $K > 0$ , then  $P \stackrel{K}{\sim} Q$  will be used to abbreviate the relation  $P/K \leq Q \leq KP$ .

We use standard Banach space notation as in [11]. In particular for a set  $S$  in a Banach space,  $[S]$  denotes its closed linear span.  $[x_i]$  abbreviates  $\{x_i, i \in \mathbb{N}\}$ . A projection means a bounded linear idempotent operator. Two basic sequences  $(x_i)$  and  $(y_i)$  in some Banach spaces are called  $K$ -equivalent (denoted  $(x_i) \stackrel{K}{\sim} (y_i)$ ) if for all  $n$  and all scalars  $\alpha_1, \dots, \alpha_n$ ,  $\|\sum_1^n \alpha_i x_i\| \stackrel{K}{\sim} \|\sum_1^n \alpha_i y_i\|$ , and are called equivalent (denoted  $(x_i) \sim (y_i)$ ) if they are  $K$ -equivalent for some  $K < \infty$ . A basic sequence  $(x_i)$  is called  $K$ -unconditional if for all choices of  $\epsilon_i = \pm 1$ ,  $(x_i) \stackrel{K}{\sim} (\epsilon_i x_i)$ .  $(x_i)$  is called semi-normalized if  $\inf_i \|x_i\| > 0$  and  $\sup_i \|x_i\| < \infty$ .

For easy reference, we assemble here several well-known facts, which will be used later. The first three are easy to verify. Fact 2.4 is a standard consequence of Khinchine's inequalities and underlies many of the known results on the isomorphic structure of  $L_p$ . We include a simple proof probably known in folklore, which is however hard to extract from existing literature.

**FACT 2.1:** *Let  $X$  be a Banach space, and let  $(x_i)$  be a basic sequence in  $X$ , equivalent to the usual  $\ell_p$ -basis,  $1 \leq p \leq \infty$ . Then  $[x_i]$  is complemented in  $X$  if and only if there is a sequence  $(x_i^*) \subseteq X^*$  such that  $x_i^*(x_j) = \delta_{ij}$  for all  $i, j$ , and such that there is  $K < \infty$  satisfying  $\|\sum_{i=1}^n \alpha_i x_i^*\| \leq K (\sum_{i=1}^n |\alpha_i|^q)^{1/q}$  for all  $n$  and all  $(\alpha_i)$ .*

**FACT 2.2:** *Let  $X$  be a Banach space,  $Y$  and  $E$  subspaces of  $X$  with  $E$  finite-dimensional. Then  $Y$  is complemented in  $X$  if and only if  $Y + E$  is.*

**FACT 2.3:** *Let  $1 \leq p < \infty$  and let  $(f_i)$  be a sequence of mean zero independent random variables in  $L_p$ . Then  $(f_i)$  is an unconditional basic sequence in  $L_p$ .*

Let us recall that the Rademacher functions  $(r_i)$  are defined as independent, symmetric,  $\{-1, 1\}$ -valued random variables on  $[0, 1]$ . Khinchine's inequalities say that for all  $0 < p < \infty$ , there are  $0 < A_p \leq$

$B_p < \infty$  such that for all  $n \in \mathbb{N}$  and all scalars  $\alpha_1, \dots, \alpha_n$ ,

$$(2.1) \quad A_p \left( \sum_{i=1}^n |\alpha_i|^2 \right)^{1/2} \leq \left\| \sum_{i=1}^n \alpha_i r_i \right\|_{L_p} \leq B_p \left( \sum_{i=1}^n |\alpha_i|^2 \right)^{1/2}.$$

The right-hand inequality for  $0 < p < 2$  (the left-hand inequality for  $2 < p < \infty$ ) follows immediately from Hölder's inequality and the orthonormality of  $(r_i)$  in  $L_2$ , with  $B_p = 1$  (respectively  $A_p = 1$ ). For a proof see [26], for example.

**FACT 2.4:** *Let  $(g_i)$  be a sequence in  $L_p$ ,  $1 \leq p < \infty$ . Then*

(i) *for any  $n \in \mathbb{N}$*

$$\begin{aligned} A_p^p \int_0^1 \left( \sum_{i=1}^n |g_i(t)|^2 \right)^{p/2} dt &\leq \int_0^1 \left\| \sum_{i=1}^n r_i(s) g_i \right\|_{L_p}^p ds \\ &\leq B_p^p \int_0^1 \left( \sum_{i=1}^n |g_i(t)|^2 \right)^{p/2} dt. \end{aligned}$$

*If, moreover,  $(g_i)$  is  $K$ -unconditional, then*

(ii) *for any  $n \in \mathbb{N}$*

$$\begin{aligned} K^{-p} A_p^p \int_0^1 \left( \sum_{i=1}^n |g_i(t)|^2 \right)^{p/2} dt &\leq \left\| \sum_{i=1}^n g_i \right\|^p \\ &\leq K^p B_p^p \int_0^1 \left( \sum_{i=1}^n |g_i(t)|^2 \right)^{p/2} dt, \text{ and} \end{aligned}$$

(iii) *for any  $n \in \mathbb{N}$*

*if  $1 \leq p \leq 2$  then*

$$(2.2) \quad A_p K^{-1} \left( \sum_{i=1}^n \|g_i\|^2 \right)^{1/2} \leq \left\| \sum_{i=1}^n g_i \right\| \leq B_p K \left( \sum_{i=1}^n \|g_i\|^2 \right)^{1/2}$$

*and if  $2 \leq p < \infty$ , then*

$$(2.3) \quad A_p K^{-1} \left( \sum_{i=1}^n \|g_i\|^p \right)^{1/p} \leq \left\| \sum_{i=1}^n g_i \right\| \leq B_p K \left( \sum_{i=1}^n \|g_i\|^2 \right)^{1/2}.$$

**PROOF:** (i) follows from pointwise application of (2.1) to the right-hand side of the equality

$$\begin{aligned} \int_0^1 \left\| \sum_{i=1}^n r_i(s) g_i \right\|^p ds &= \int_0^1 \int_0^1 \left| \sum_{i=1}^n r_i(s) g_i(t) \right|^p dt ds \\ &= \int_0^1 \left\| \sum_{i=1}^n g_i(t) r_i \right\|_{L_p}^p dt, \end{aligned}$$

which is, of course, justified by the Fubini-Tonelli theorem.

To prove (ii), note that since  $(g_i)$  is unconditional,  $\|\sum_{i=1}^n r_i(s)g_i\| \leq \|\sum_{i=1}^n g_i\|$  for all  $0 \leq s \leq 1$ , and consequently,

$$(2.4) \quad \left\| \sum_{i=1}^n g_i \right\| \sim^K \left( \int_0^1 \left\| \sum_{i=1}^n r_i(s)g_i \right\|^p ds \right)^{1/p}.$$

Now, in view of (ii), (iii) will follow if we establish that: for  $1 \leq p \leq 2$ ,

$$(2.5) \quad \left( \sum_{i=1}^n \|g_i\|^2 \right)^{1/2} \leq \left( \int_0^1 \left( \sum_{i=1}^n |g_i(t)|^2 \right)^{p/2} dt \right)^{1/p} \\ \leq \left( \sum_{i=1}^n \|g_i\|^p \right)^{1/p}$$

while for  $2 \leq p < \infty$ ,

$$(2.6) \quad \left( \sum_{i=1}^n \|g_i\|^p \right)^{1/p} \leq \left( \int_0^1 \left( \sum_{i=1}^n |g_i(t)|^2 \right)^{p/2} dt \right)^{1/p} \\ \leq \left( \sum_{i=1}^n \|g_i\|^2 \right)^{1/2}.$$

Fix  $2 \leq p < \infty$ . Then the left-hand side inequality in (2.6) follows from the elementary inequality  $(\sum_{i=1}^n |\alpha_i|^p)^{1/p} \leq (\sum_{i=1}^n |\alpha_i|^2)^{1/2}$  i.e.  $\sum_{i=1}^n |\alpha_i|^p \leq (\sum_{i=1}^n |\alpha_i|^2)^{p/2}$  applied pointwise with  $\alpha_i = g_i(t)$  and integrated over  $t$ . The right-hand side inequality is immediate from the triangle inequality in  $L_{p/2}$  applied to the sum of the functions  $|g_1|^2, \dots, |g_n|^2 \in L_{p/2}$ .

Now fix  $1 \leq p \leq 2$ . The right-hand side of (2.5) follows, as before, from the inequality  $(\sum_{i=1}^n |\alpha_i|^2)^{1/2} \leq (\sum_{i=1}^n |\alpha_i|^p)^{1/p}$ . The other inequality in (2.5) comes from the triangle inequality, this time in  $\ell_{2/p}$ : We may assume that  $(g_i)$  are simple functions, i.e. there is a measurable partition  $E_1, \dots, E_m$  of  $[0, 1]$  and scalars  $(c_{ij})$  such that  $g_i = \sum_{j=1}^m c_{ij} \mathbf{1}_{E_j}$ ,  $i = 1, \dots, n$ . Putting  $b_{ij} = |c_{ij}|^p |E_j|$  we get:

$$\left( \sum_{i=1}^n \|g_i\|^2 \right)^{p/2} = \left\{ \sum_{i=1}^n \left( \sum_{j=1}^m |c_{ij}|^p |E_j| \right)^{2/p} \right\}^{p/2} \\ = \left\{ \sum_{i=1}^n \left( \sum_{j=1}^m b_{ij} \right)^{2/p} \right\}^{p/2} \\ \leq \sum_{j=1}^m \left( \sum_{i=1}^n |b_{ij}|^{2/p} \right)^{p/2} \\ = \sum_{j=1}^m \left( \sum_{i=1}^n |c_{ij}|^2 \right)^{p/2} |E_j| \\ = \int_0^1 \left( \sum_{i=1}^n |g_i(t)|^2 \right)^{p/2} dt,$$

which is the required inequality. The triangle inequality was applied



here to the sums of the vectors  $y_1, \dots, y_m$  in  $\ell_{2/p}^n$ , where  $y_j = (b_{1j}, b_{2j}, \dots, b_{nj})$ .

REMARK: All the lower estimates in Fact 2.4 are due to W. Orlicz [15]. His proof in (i) and (ii) is identical with the one given here and may be used to show also the upper estimates in (i) and (ii), which were not stated there.

The upper estimates in (iii) are due to Kadec [8].

FACT 2.5: *Let  $(x_i)$  and  $(y_i)$  be two minimal sequences in a Banach space  $X$ . If  $(x_i)$  is equivalent to the usual  $\ell_p$ -basis,  $1 < p < \infty$ , and if  $\sum_{i=1}^\infty \|x_i - y_i\|^q < \infty$ , then  $(y_i)$  is also equivalent to the usual  $\ell_p$ -basis. A sequence  $(z_i)$  is called minimal if there are  $z_i^*$  in the dual such that  $z_i^*(z_j) = \delta_{ij}$ , all  $i, j$ .*

PROOF: By assumption, there are  $a, b > 0$  such that for all  $m \in \mathbb{N}$  and all scalars  $\alpha_1, \dots, \alpha_m$  we have

$$a \left( \sum_{i=1}^m |\alpha_i|^p \right)^{1/p} \leq \left\| \sum_{i=1}^m \alpha_i x_i \right\| \leq b \left( \sum_{i=1}^m |\alpha_i|^p \right)^{1/p}.$$

Let  $z_i = y_i - x_i, i \in \mathbb{N}$ . Omitting a finite number of terms in the sequence if necessary, we may assume that  $(\sum_{i=1}^\infty \|z_i\|^q)^{1/q} \equiv c < a$ . Then  $\|\sum_{i=1}^m \alpha_i z_i\| \leq (\sum_{i=1}^m |\alpha_i|^p)^{1/p} (\sum_{i=1}^m \|z_i\|^q)^{1/q} \leq c (\sum_{i=1}^m |\alpha_i|^p)^{1/p}$ , and so  $(a - c)(\sum_{i=1}^m |\alpha_i|^p)^{1/p} \leq \|\sum_{i=1}^m \alpha_i y_i\| \leq (b + c)(\sum_{i=1}^m |\alpha_i|^p)^{1/p}$  for all  $m$  and all scalars  $\alpha_1, \dots, \alpha_m$ .

REMARK: This fact was first observed by Gohberg and Markus [5].

### 3. Sequences equivalent to the usual $l_p$ -basis

We present in this section three characterizations of sequences in  $L_p$  equivalent to the usual  $l_p$ -basis. These results will be used in the other sections of the paper. Theorem 3.3 is identical with Theorem B(i) stated in the introduction.

Fix  $2 < p < \infty$ . Two well-known results describe the fact that  $L_p$  has a rather small variety of subspaces. The Kadec-Pelczyński Theorem [9] says that a subspace of  $L_p$  either embeds isomorphically in  $\ell_2$  or contains a subspace isomorphic to  $\ell_p$ , while the Johnson–Odell Theorem [7] says that every subspace of  $L_p$  either embeds in  $\ell_p$  or contains a subspace isomorphic to  $\ell_2$ . Our first characterization of

sequences equivalent to the usual basis of  $\ell_p$ , given in Theorem 3.3, gives a local version of the dichotomy expressed in the Johnson–Odell Theorem. We begin with a preliminary result, which may be of independent interest.

**PROPOSITION 3.1:** *Let  $1 \leq p < \infty$ , and let  $(g_i)$  be a sequence of positive functions in  $L_p$ ,  $K > 0$ .*

(i) *If for all choices of  $n \in \mathbb{N}$  and  $\epsilon_1, \dots, \epsilon_n = \pm 1$*

$$(3.1) \quad \left\| \sum_{i=1}^n g_i \right\| \leq K \left\| \sum_{i=1}^n \epsilon_i g_i \right\|$$

*then we have for all  $n$ ,*

$$(3.2) \quad \left( \sum_{i=1}^n \|g_i\|^p \right)^{1/p} \leq \left\| \sum_{i=1}^n g_i \right\| \leq (2KB_p)^2 \left( \sum_{i=1}^n \|g_i\|^p \right)^{1/p}$$

(ii) *If  $(g_i)$  is  $K$ -unconditional and normalized then  $(g_i)$  is  $(2KB_p)^2$ -equivalent to the usual  $\ell_p$ -basis.*

**PROOF (i):** The left-hand inequality in (3.2) is easily proved: for all  $t$ ,

$$\sum_{i=1}^n |g_i(t)|^p \leq \left( \sum_{i=1}^n |g_i(t)| \right)^p = \left( \sum_{i=1}^n g_i(t) \right)^p,$$

so

$$\sum_{i=1}^n \|g_i\|^p = \int_0^1 \sum_{i=1}^n |g_i(t)|^p dt \leq \int_0^1 \left( \sum_{i=1}^n g_i(t) \right)^p dt = \left\| \sum_{i=1}^n g_i \right\|^p.$$

To see the right-hand inequality in (3.2), let first  $1 \leq p \leq 2$ . Then by Fact 2.4(iii), we have

$$\left\| \sum_{i=1}^n g_i \right\| \leq KB_p \left( \sum_{i=1}^n \|g_i\|^p \right)^{1/p}$$

which implies (3.2).

Now let  $2 < p < \infty$ . We have

$$(3.3) \quad \int_0^1 \left( \sum_{i=1}^n g_i(t) \right)^p dt = \left\| \sum_{i=1}^n g_i \right\|^p \leq (KB_p)^p \int_0^1 \left( \sum_{i=1}^n g_i(t)^2 \right)^{p/2} dt$$

by (3.1) and Fact 2.4 (ii). Let  $\lambda = 2KB_p$  and  $E = \{t; \sum_{i=1}^n g_i(t) \leq$

$\lambda(\sum_{i=1}^n g_i(t)^2)^{1/2}$ . The idea of the proof now is that a large part of the time (namely on  $E$ ),  $\sum_{i=1}^n g_i$  is comparable with  $(\sum_{i=1}^n |g_i|^2)^{1/2}$ , since by (3.3) these two functions are comparable on the average. But then  $\sum_{i=1}^n g_i$  is also comparable with  $(\sum_{i=1}^n |g_i|^p)^{1/p}$  which brings us to the extreme right-hand side expression in (3.2). In detail:

$$\begin{aligned} & \int_{\sim E} \left( \sum_{i=1}^n g_i(t)^2 \right)^{p/2} dt \\ & \leq \lambda^{-p} \int_{\sim E} \left( \sum_{i=1}^n g_i(t) \right)^p dt \quad \text{by the definition of } E \\ & \leq \frac{1}{2} \int_0^1 \left( \sum_{i=1}^n g_i(t)^2 \right)^{p/2} dt \quad \text{by (3.3).} \end{aligned}$$

Therefore we have

$$(3.4) \quad \int_E \left( \sum_{i=1}^n g_i(t)^2 \right)^{p/2} \geq \frac{1}{2} \int_0^1 \left( \sum_{i=1}^n g_i(t)^2 \right)^{p/2}.$$

Now for any  $t \in E$  we have

$$\begin{aligned} \sum_{i=1}^n g_i(t)^2 &= \sum_{i=1}^n g_i(t)^{2-q} g_i(t)^q \\ &\leq \left( \sum_{i=1}^n g_i(t) \right)^{2-q} \left( \sum_{i=1}^n g_i(t)^p \right)^{q/p} \\ &\leq \lambda^{2-q} \left( \sum_{i=1}^n g_i(t)^2 \right)^{(2-q)/2} \left( \sum_{i=1}^n g_i(t)^p \right)^{q/p}. \end{aligned}$$

The first inequality is Hölder's inequality with exponents  $1/(2-q)$  and  $p/q$  (note that  $2-q+q/p=1$ ); the other one comes from the definition of  $E$ . Transferring sides we have on  $E$ :

$$\begin{aligned} & \left( \sum_{i=1}^n g_i(t)^2 \right)^{q/2} \leq \lambda^{2-q} \left( \sum_{i=1}^n g_i(t)^p \right)^{q/p}, \text{ or} \\ (3.5) \quad & \left( \sum_{i=1}^n g_i(t)^2 \right)^{p/2} \leq \lambda^{p-2} \sum_{i=1}^n g_i(t)^p. \end{aligned}$$

Putting things together

$$\begin{aligned} \sum_{i=1}^n \|g_i\|^p &\geq \int_E \sum_{i=1}^n g_i(t)^p dt \\ &\geq \lambda^{2-p} \int_E \left( \sum_{i=1}^n g_i(t)^2 \right)^{p/2} dt \quad \text{by (3.5)} \end{aligned}$$

$$\begin{aligned} &\geq \lambda^{2-p} 2^{-1} \int_0^1 \left( \sum_{i=1}^n g_i(t)^2 \right)^{p/2} dt && \text{by (3.4)} \\ &\geq \lambda^{2-2p} \int_0^1 \left( \sum_{i=1}^n g_i(t) \right)^p dt && \text{by (3.3), i.e.} \\ \left\| \sum_{i=1}^n g_i \right\| &\leq \lambda^{(2p-2)/p} \left( \sum_{i=1}^n \|g_i\|^p \right)^{1/p} \\ &\leq \lambda^2 \left( \sum_{i=1}^n \|g_i\|^p \right)^{1/p} \end{aligned}$$

completing the proof of (i).

(ii). Let  $\alpha_1, \dots, \alpha_n$  be scalars; then replacing  $g_i$  in (3.2) by  $|\alpha_i|g_i$ , we obtain:

$$\begin{aligned} K^{-1} \left( \sum_{i=1}^n |\alpha_i|^p \right)^{1/p} &= K^{-1} \left( \sum_{i=1}^n \|\alpha_i g_i\|^p \right)^{1/p} \\ &\leq K^{-1} \left\| \sum_{i=1}^n |\alpha_i| g_i \right\| \leq \left\| \sum_{i=1}^n \alpha_i g_i \right\| \\ &\leq \left\| \sum_{i=1}^n |\alpha_i| g_i \right\| \leq (2KB_p)^2 \left( \sum_{i=1}^n |\alpha_i|^p \right)^{1/p}, \end{aligned}$$

the inequalities being justified by (3.2), the unconditionality of  $(g_i)$ , the triangle inequality for scalars, and (3.2), respectively.

REMARK: After the first version of this paper was completed, a simpler proof of Proposition 3.1 was suggested by G. Schechtman. Here is a modification of his proof which also improves the constant in (3.2) from  $(2KB_p)^2$  to  $(KB_p)^{2/q}$ :

$$\begin{aligned} (KB_p)^{-p} \int \left( \sum_i g_i \right)^p &\leq \int \left( \sum_i g_i^2 \right)^{p/2} && \text{by (3.3)} \\ &= \int \left( \sum_i g_i^{2-q} g_i^q \right)^{p/2} \\ &\leq \int \left\{ \left( \sum_i g_i \right)^{2-q} \left( \sum_i g_i^q \right)^{q/p} \right\}^{p/2} && \text{by Hölder} \\ &\leq \left\{ \int \left( \sum_i g_i \right)^p \right\}^{(2-q)/2} \left\{ \int \sum_i g_i^q \right\}^{q/2} && \text{by Hölder.} \end{aligned}$$

Transferring sides and raising to the power  $2/(qp)$  we obtain from here

$$\left\{ \int \left( \sum_i g_i \right)^p \right\}^{1/p} \leq (KB_p)^{2/q} \left\{ \int \sum_i g_i^q \right\}^{1/p}.$$

We will find it more convenient to use Proposition 3.1 (i) in the following contrapositive form:

**COROLLARY 3.2:** *Given  $1 \leq p < \infty$  and  $\delta > 0$  there is  $M = M(p, \delta)$  such that for all  $n$  and all sequences  $g_1, \dots, g_n \geq 0$  in  $L_p$ , if*

$$\left\| \sum_{i=1}^n g_i \right\| \geq M \left( \sum_{i=1}^n \|g_i\|^p \right)^{1/p}$$

*then there is a choice of signs  $\epsilon_i = \pm 1$ , ( $i \leq n$ ), for which*

$$\left\| \sum_{i=1}^n \epsilon_i g_i \right\| \leq \delta \left\| \sum_{i=1}^n g_i \right\|.$$

We are now ready for our first characterization.

**THEOREM 3.3** (see also [6]): *Let  $2 < p < \infty$ , and let  $(f_i)$  be a normalized unconditional basic sequence in  $L_p$ . Then either*

- (i)  *$(f_i)$  is equivalent to the usual basis of  $\ell_p$ , or else*
- (ii) *there is  $\lambda \geq 1$  such that for any  $n \in \mathbb{N}$  there are disjoint finite sets  $A_1, \dots, A_n$  of integers and linear combinations  $h_i = \sum_{j \in A_i} c_j f_j$  for which  $(h_1, \dots, h_n)$  is  $\lambda$ -equivalent to the usual basis of  $\ell_2^n$ .*

*These possibilities are mutually exclusive. Moreover, by Krivine’s Theorem ([10], see also [17]), we may replace the phrase “there is  $\lambda \geq 1$ ” in the statement (ii) by the phrase “for all  $\lambda > 1$ ”.*

**PROOF:** Assume that (i) fails. Fix an integer  $n$  and  $\delta > 0$ , and let

$$(3.6) \quad M > nM(p/2, \delta/2n)$$

in the notation of Corollary 3.2. Since (i) fails, we can, in view of Fact 2.4 (iii) find an integer  $m$  and scalars  $\alpha_1, \dots, \alpha_m$  such that

$$(3.7) \quad \left\| \sum \alpha_j f_j \right\|_p > B_p K M^{1/2} \left( \sum_{j=1}^m |\alpha_j|^p \right)^{1/p}.$$

where  $K$  is the unconditionality constant of  $(f_j)$  in  $L_p$ . Set  $r = p/2$ , and for  $j = 1, \dots, m$ , let  $g_j = |f_j|^2$ ,  $\beta_j = |\alpha_j|^2$ . Then we have

$$\begin{aligned}
\left\| \sum_{j=1}^m \beta_j g_j \right\|_r^{1/2} &= \left( \int_0^1 \left( \sum_{j=1}^m |\alpha_j f_j|^2 \right)^{p/2} dt \right)^{1/p} \\
&\geq B_p^{-1} K^{-1} \left\| \sum_{j=1}^m \alpha_j f_j \right\|_r && \text{by Fact 2.4 (ii)} \\
&\geq M^{1/2} \left( \sum_{j=1}^m |\alpha_j|^p \right)^{1/p}, && \text{by (3.7), i.e.}
\end{aligned}$$

(3.8) 
$$\left\| \sum_{j=1}^m \beta_j g_j \right\|_r \geq M \left( \sum_{j=1}^m \beta_j^r \right)^{1/r}$$

To conclude the proof we shall need the following simple

**LEMMA:** *Let  $x$  and  $y$  be nonzero elements in a Banach space, and let  $\epsilon > 0$ . If  $\|x - y\| \leq \epsilon \|x + y\|$ , then  $\left\| \frac{x}{\|x\|} - \frac{x+y}{\|x+y\|} \right\| \leq 2\epsilon$ , and consequently  $\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq 4\epsilon$ .*

**PROOF OF LEMMA:**

$$\begin{aligned}
&\left\| \frac{x}{\|x\|} - \frac{x+y}{\|x+y\|} \right\| \\
&= \left\| \frac{2x}{\|2x\|} - \frac{x+y}{\|x+y\|} \right\| \leq \frac{\|2x - (x+y)\|}{\|x+y\|} \\
&+ \left| \frac{1}{\|2x\|} - \frac{1}{\|x+y\|} \right| \|2x\| \\
&= \frac{\|x-y\|}{\|x+y\|} + \frac{\| \|x+y\| - \|2x\| \|}{\|x+y\|} \\
&\leq \frac{2\|x-y\|}{\|x+y\|} \leq 2\epsilon.
\end{aligned}$$

**PROOF OF THE THEOREM, concluded:** Let

$$u = \sum_{j=1}^m \beta_j g_j / \left\| \sum_{j=1}^m \beta_j g_j \right\|_r.$$

*Claim.* For each  $k \leq n$  it is possible to find a partition of the set  $\{1, \dots, m\}$  into  $k$  disjoint sets  $A_1, \dots, A_k$  so that if we set  $x_k = \sum_{j \in A_i} \beta_j g_j$  for  $i = 1, \dots, k$ , then

$$(3.9) \quad \|x_i / \|x_i\|_r - u\|_r < k \cdot \delta / n, \text{ all } i \leq k.$$

We shall prove the claim by induction on  $k \leq n$ . Assume it holds for a

certain value of  $k < n$ , and let  $A_i, x_i$  be as in the claim. We have

$$\begin{aligned} \sum_{i=1}^k \left\| \sum_{j \in A_i} \beta_j g_j \right\|_r &\geq \left\| \sum_{j=1}^m \beta_j g_j \right\|_r \\ &\geq M \left( \sum_{j=1}^m \beta_j^r \right)^{1/r} \quad \text{by (3.8), so} \end{aligned}$$

for some value of  $i \leq k$ , which we shall fix now,

$$\sum_{j \in A_i} \|\beta_j g_j\|_r \geq \frac{M}{k} \left( \sum_{j=1}^m \beta_j^r \right)^{1/r} \geq \frac{M}{k} \left( \sum_{j \in A_i} \beta_j^r \right)^{1/r}.$$

Now by (3.6),  $M/k \geq M(r, \delta/2n)$  so by Corollary 3.2 there are subsets  $B$  and  $C = A_i \setminus B$  of  $A_i$  such that

$$\left\| \sum_{j \in B} \beta_j g_j - \sum_{j \in C} \beta_j g_j \right\|_r \leq \frac{\delta}{2n} \left\| \sum_{j \in A_i} \beta_j g_j \right\|_r$$

i.e. putting  $y = \sum_{j \in B} \beta_j g_j, z = \sum_{j \in C} \beta_j g_j$

we have  $y + z = x_i$ , and  $\|y - z\|_r < (\delta/2n)\|x_i\|_r$ . Thus by the Lemma we have

$$\|y/\|y\|_r - x_i/\|x_i\|_r\|_r \leq \delta/n$$

which by the induction hypothesis implies that  $\|y/\|y\|_r - u\|_r \leq (k + 1)\delta/n$ . A similar assertion holds for  $z$ . Thus replacing  $A_i$  by the two sets  $B$  and  $C$  and  $x_i$  by  $y$  and  $z$ , we obtain the objects postulated in the claim for  $k + 1$ .

Let  $A_1, \dots, A_n$  be as in claim for  $k = n$ , let  $x_i = \sum_{j \in A_i} \beta_j g_j$ , and let

$$y_i = x_i/\|x_i\|_r = \sum_{j \in A_i} \gamma_j g_j$$

Then for any positive  $b_1, \dots, b_n$ ,

$$\begin{aligned} \left| \left\| \sum_{i=1}^n b_i y_i \right\|_r - \sum_{i=1}^n b_i \right| &\leq \left\| \sum_{i=1}^n b_i (y_i - u) \right\|_r \\ &\leq \delta \sum_{i=1}^n b_i \text{ by (3.9), and so} \\ (3.10) \quad (1 - \delta) \sum_{i=1}^n b_i &\leq \left\| \sum_{i=1}^n b_i y_i \right\|_r \leq \sum_{i=1}^n b_i. \end{aligned}$$

Now let  $c_j = \sqrt{\gamma_j}$  for all  $j \leq m$ , and let  $h_i = \sum_{j \in A_i} c_j f_j$ . Then for any scalars  $a_1, \dots, a_n$  we have

$$\begin{aligned} \left\| \sum_{i=1}^n a_i h_i \right\|_p &= \left\| \sum_{i=1}^n \sum_{j \in A_i} a_i c_j f_j \right\|_p \\ &\geq K^{-1} \left( \int_0^1 \left( \sum_{i=1}^n \sum_{j \in A_i} |a_i c_j f_j(t)|^2 \right)^{p/2} dt \right)^{1/p} \\ &= K^{-1} \left\| \sum_{i=1}^n |a_i|^2 y_i \right\|_r^{1/2} \\ &\geq (1 - \delta) K^{-1} \left( \sum_{i=1}^n |a_i|^2 \right)^{1/2}. \end{aligned}$$

The first inequality holds by the unconditionality of  $(f_j)$  and Fact 2.4 (ii), while the second one comes from (3.10). Similarly we get

$$\left\| \sum_{i=1}^n a_i h_i \right\| \leq KB_p \left( \sum_{i=1}^n |a_i|^2 \right)^{1/2}.$$

We have thus shown that statement (ii) of the Theorem holds with  $\lambda$  equal to  $KB_p$ , where  $K$  is the unconditional constant of  $(f_i)$  in  $L_p$  (recall that  $A_p = 1$ ).

The other assertions of the Theorem are clear.

The proof of Theorem 3.3 shows also the following quantitative version of it:

**THEOREM 3.3\*:** *Let  $2 < p < \infty$ , and let  $1 \leq K < \infty$ . Then for every  $n \in \mathbb{N}$  there is  $0 < M < \infty$  so that whenever  $(f_i)$  is a finite or infinite normalized basic sequence in  $L_p$  with unconditional constant  $\leq K$ , then either  $\|\sum_i \alpha_i f_i\| \leq M(\sum_i |\alpha_i|^p)^{1/p}$  for all choices of scalars  $(\alpha_i)$ , or else there are disjoint sets  $A_1, \dots, A_n$  of indices and functions  $h_i \in [f_j; j \in A_i]$ ,  $i = 1, 2, \dots, n$ , such that  $(h_1, h_2, \dots, h_n)$  is  $KB_p$ -equivalent to the usual basis of  $\ell_2^n$ .*

In basis theoretic terminology, Theorem 3.3 says that every normalized unconditional basic sequence in  $L_p$ ,  $2 < p < \infty$ , which is not equivalent to the usual  $\ell_p$ -basis has a permutation in which the usual basis of  $\ell_2$  is block-finitely represented (see [17]). It is not known whether it is necessary to use a permutation in this statement. Namely, the following question is open:

**PROBLEM 3.A:** *Let  $2 < p < \infty$ , and let  $(f_i)$  be a normalized un-*



conditional basic sequence in  $L_p$  which is not equivalent to the usual  $\ell_p$ -basis. Is there  $\lambda > 1$  such that for each  $n$  there are disjoint intervals  $A_1, \dots, A_n$  in  $\mathbb{N}$  and functions  $h_i \in [f_j; j \in A_i]$  such that  $(h_1, \dots, h_n)$  is  $\lambda$ -equivalent to the usual basis of  $\ell_2^n$ ?

We now pass to an analytic characterization of sequences equivalent to the usual  $\ell_p$ -basis. In the case of  $1 \leq p < 2$ , this is the union of the results of Johnson–Odell [7] and of the first-named author [4]. Let us explain the background and notation of our result for  $2 < p < \infty$ . If  $\phi \in L_p = L_p(\lambda)$  and  $\phi > 0$  almost everywhere, we set  $d\lambda_\phi = \phi^p d\lambda$ , and define for  $f \in L_p(\lambda)$ ,  $U_\phi f = f/\phi$ . By the Radon–Nikodým theorem  $U_\phi$  is an isometry of  $L_p(\lambda)$  onto  $L_p(\lambda_\phi)$ . Such an isometry is called a change of density. One should keep in mind that isomorphic and isometric properties of subspaces and subsets of  $L_p$  are invariant under changes of density, while the shape of functions and their  $L_r$  norms for  $r \neq p$ , are not.

**PROPOSITION 3.4:** *Let  $(f_i)$  be a semi-normalized unconditional basic sequence in  $L_p = L_p(\lambda)$ ,  $1 \leq p < \infty$ ,  $p \neq 2$ . Then  $(f_i)$  is equivalent to the usual basis of  $\ell_p$  if and only if*

- (i)  $1 \leq p < 2$ , and there are disjoint measurable subsets  $(E_i)$  of  $[0, 1]$ , and  $\delta > 0$  such that  $\int_{E_i} |f_i(t)|^p dt \geq \delta^p$  for all  $i \in \mathbb{N}$ .

or

- (ii)  $2 < p < \infty$ , and for every  $\phi > 0$  with  $\int_0^1 \phi(t)^p dt = 1$ , we have

$$C_\phi \equiv \left\{ \sum_{i=1}^{\infty} \|U_\phi f_i\|_{L_2(\lambda_\phi)}^{2p/(p-2)} \right\}^{(p-2)/2p} < \infty.$$

Moreover, for  $2 < p < \infty$ , if (ii) holds then  $C \equiv \sup\{C_\phi; \phi > 0, \int_0^1 \phi(t)^p dt = 1\} < \infty$  and for all  $n$  and all  $\alpha_1, \dots, \alpha_n$ ,

$$\left\| \sum_{i=1}^n \alpha_i f_i \right\| \leq KB_p C \left( \sum_{i=1}^n |\alpha_i|^p \right)^{1/p}$$

where  $K$  is the unconditional constant of  $(f_i)$ .

**PROOF:** For  $1 \leq p < 2$ , the “if” statement of the Proposition is the content of Lemma 2 in [7], while the “only if” statement follows from Theorem B of [4].

For  $2 < p < \infty$ , Fact 2.4 (iii) implies that  $(f_i)$  has a lower  $\ell_p$ -estimate.

By Fact 2.4 (ii),  $(f_i)$  will have an upper  $\ell_p$ -estimate if and only if there is  $M < \infty$  such that for all  $n$  and all  $\alpha_1, \dots, \alpha_n$ ,

$$\int_0^1 \left( \sum_{i=1}^n |\alpha_i f_i(t)|^2 \right)^{p/2} \leq M^p \sum_{i=1}^n |\alpha_i|^p.$$

Also, if such  $M$  exists we will have by Fact 2.4 (ii) that

$$\left\| \sum_{i=1}^n \alpha_i f_i \right\| \leq KB_p M \left( \sum_{i=1}^n |\alpha_i|^p \right)^{1/p}$$

for all  $n$  and all  $\alpha_1, \dots, \alpha_n$ . Setting  $g_i = |f_i|^2$ ,  $r = p/2$ ,  $s = p/(p - 2)$  we see that such finite  $M$  exists if and only if  $\sup M_n < \infty$  where  $M_n^2 = \sup\{(\int_0^1 |\sum_{i=1}^n \beta_i g_i(t)|^r dt)^{1/r}; \sum_{i=1}^n |\beta_i|^r = 1\}$ ;  $\sup_n M_n$  is the smallest admissible value of  $M$ .

Fix  $n \in \mathbb{N}$ , and let  $T_n: \ell_r \rightarrow L_r$  be defined by  $T_n((\alpha_i)) = \sum_{i=1}^n \alpha_i g_i$ . Then

$$\begin{aligned} M_n^2 = \|T_n\| &= \|T_n^*\| = \sup\{\|T_n^* \psi\|; \psi > 0, \|\psi\|_s = 1\} \quad \text{since } T_n \text{ is positive} \\ &= \sup\left\{ \left( \sum_{i=1}^n \left( \int_0^1 g_i(t) \psi(t)^s \right)^{1/s}; \psi > 0, \int_0^1 \psi^s = 1 \right) \right\} \\ &= \sup\left\{ \left( \sum_{i=1}^n \left( \int_0^1 |f_i(t)|^2 \phi(t)^{p-2} dt \right)^{s/2}; \phi > 0, \int_0^1 \phi^p dt = 1 \right) \right\} \\ &= \sup\left\{ \left( \sum_{i=1}^n \|U_\phi f_i\|_{L_2(\lambda_\phi)}^{2p/(p-2)} \right)^{(p-2)/p}; \phi > 0, \int_0^1 \phi^p dt = 1 \right\}. \end{aligned}$$

(We made the substitution  $\phi^{p-2} = \psi$ ). Thus we have  $\sup_n M_n = C$ , which shows that  $(f_i)$  is equivalent to the usual  $\ell_p$ -basis if and only if  $C < \infty$ , and proves the “moreover” statement of the Proposition. To complete the proof it is enough to note that for  $\phi^{p-2} = \psi$ ,  $C_\phi^2 = \sup_n \|T_n^* \psi\|$ , and therefore if  $C_\phi < \infty$  for all  $\phi > 0$  in  $L_p$ , then for all  $\psi \in L_s$ ,  $\sup_n \|T_n^* \psi\| < \infty$  and so by the uniform boundedness principle,  $C = \sup_{\|\phi\|=1} C_\phi = \sup_n \|T_n^*\|^{1/2} < \infty$ .

**REMARK:** It is interesting to compare part (ii) of the Proposition with the following result, which is a simple consequence of results of H. P. Rosenthal [19] and B. Maurey [13] (apply Théorème 2 and Théorème 10 of [13] and Fact 2.4 (ii): *Let  $1 \leq p < \infty$ , and let  $(f_i)$  be a semi-normalized unconditional basic sequence in  $L_p$ . Then  $(f_i)$  is equivalent to the usual  $\ell_2$ -basis if and only if there is  $0 < \phi \in L_p$  such*

that

$$\sup_i \|U_\phi f_i\|_{L_2(\lambda_\phi)} < \infty \text{ and}$$

$$\inf_i \|U_\phi f_i\|_{L_2(\lambda_\phi)} > 0.$$

(Of course, for each fixed  $p$  one of these conditions is always satisfied for semi-normalized  $(f_i)$ ).

In the final result of this section we give an analytic condition which is necessary and sufficient for a sequence of independent random variables in  $L_p$  to be equivalent to the usual  $\ell_p$ -basis. This condition is satisfied by every sequence in  $L_p$  which is equivalent to the usual  $\ell_p$ -basis. For independent random variables in  $L_p$ ,  $2 < p < \infty$ , the essential part of Proposition 3.5 is due to H. P. Rosenthal [21].

**PROPOSITION 3.5:** *Fix a value of  $p$ ,  $1 \leq p < \infty$ ,  $p \neq 2$ , and let  $(f_i)$  be a sequence in  $L_p$ . Suppose that, among the following conditions, all those which are pertinent to the given value of  $p$  are satisfied:*

- (i) *if  $1 \leq p < \infty$ ,  $(f_i)$  is a semi-normalized sequence of independent non-constant random variables,*
- (ii) *if  $1 \leq p < 2$ , there exist  $\delta > 0$  and a sequence  $(E_i)$  of subsets of  $[0, 1]$  such that*

$$(3.11) \quad \sum_{i=1}^{\infty} |E_i| < \infty, \text{ and}$$

$$(3.12) \quad \int_{E_i} |f_i(t)|^p dt \geq \delta^p \text{ for all } i \in \mathbb{N},$$

- (iii) *if  $2 < p < \infty$ , then*

$$\sum_{i=1}^{\infty} \|f_i\|_2^{2p/(p-2)} < \infty$$

- (iv) *if  $1 < p < \infty$ , then*

$$\sum_{i=1}^{\infty} \left| \int_0^1 f_i(t) dt \right|^q < \infty.$$

*Then  $(f_i)$  is a basic sequence equivalent to the usual basis of  $\ell_p$ . Conversely, if  $1 \leq p < \infty$ ,  $p \neq 2$ , and  $(f_i)$  is a sequence in  $L_p$  which is*

equivalent to the usual basis of  $\ell_p$ , then statements (ii), (iii) and (iv) hold. Moreover, statement (ii) is actually true for all values of  $p$ ,  $1 \leq p < \infty$ ,  $p \neq 2$ , and the sets in (ii) may be taken to be of the form

$$(3.13) \quad E_i = [|f_i| \geq \beta_i]$$

for suitable constants  $\beta_i > 0$ .

PROOF: We shall first prove the converse statement of the Proposition. Suppose that  $1 \leq p < \infty$ ,  $p \neq 2$ , and  $(f_i)$  is a sequence in  $L_p$  which is equivalent to the usual basis of  $\ell_p$ . By a result of the first named author [4], there are  $\delta > 0$  and a sequence  $(E_i)$  of disjoint measurable subsets of  $[0, 1]$  such that (3.12) is satisfied. Thus (ii) is valid for all  $1 \leq p < \infty$ ,  $p \neq 2$ , in a stronger form. To see that  $(E_i)$  may be replaced by a sequence satisfying (3.13) as well as (3.11) and (3.12) (with a possibly smaller value of  $\delta > 0$ ), we need only show

LEMMA: Let  $(f_i)$  be a bounded sequence in  $L_p$ ,  $1 \leq p < \infty$ . If  $\delta > 0$  and  $(E_i)$  is a sequence of subsets of  $[0, 1]$ , such that (3.11) and (3.12) hold, then there are  $\beta_i \geq 0, i \in \mathbb{N}$  such that the sets  $F_i = [|f_i| \geq \beta_i]$  satisfy  $\sum_{i=1}^{\infty} |F_i| < \infty$  and  $\int_{F_i} |f_i(t)|^p dt \geq \delta^p/2$  for all  $i \in \mathbb{N}$ .

PROOF OF THE LEMMA: For  $i \in \mathbb{N}$ , let  $\beta_i$  be the unique number for which

$$(3.14) \quad \int_{|f_i| \geq \beta_i} |f_i(t)|^p dt \geq \delta^p/2, \text{ while } \int_{|f_i| > \beta_i} |f_i(t)|^p dt \leq \delta^p/2.$$

Then we have

$$\begin{aligned} |E_i| \beta_i^p &\geq \int_{E_i \cap \{|f_i| \leq \beta_i\}} |f_i(t)|^p dt \\ &\geq \int_{E_i} |f_i(t)|^p dt - \int_{|f_i| > \beta_i} |f_i(t)|^p dt > \delta^p/2 \end{aligned}$$

by (3.12) and the second inequality in (3.14). On the other hand, setting  $K = \sup_i \|f_i\|$ , and  $F_i = [|f_i| \geq \beta_i]$  for  $i \in \mathbb{N}$ , we have

$$|F_i| \beta_i^p \leq \int_{F_i} |f_i(t)|^p dt \leq K^p.$$

Putting these two inequalities together we have  $|F_i|/|E_i| \leq 2K^p/\delta^p$ , and

so  $\sum_{i=1}^\infty |F_i| < \infty$  by (3.11), which concludes the proof of the Lemma in view of the first inequality in (3.14).

We now return to the proof of the converse assertion of the Proposition and so resume our assumption that  $(f_i)$  is equivalent to the usual basis of  $\ell_p$ . Statement (iii) for  $2 < p < \infty$  follows immediately from Proposition 3.4 (ii) with  $\phi = 1$ . To prove statement (iv), let  $1 < p < \infty$ . Consider the map  $T: \ell_p \rightarrow L_p$  given by  $T((\alpha_i)) = \sum_{i=1}^\infty \alpha_i f_i$ . Then we have  $T^*: L_q \rightarrow \ell_q$  and so

$$\left( \sum_{i=1}^\infty \left| \int_0^1 f_i(t) dt \right|^q \right)^{1/q} = \|T^*1\|_{\ell_q} \leq \|T\| < \infty.$$

We now prove the direct implication asserted in the Proposition. Assume that  $(f_i)$  is a sequence in  $L_p$  satisfying (i)–(iv). For each  $i$ , let  $h_i$  be an  $\mathcal{A}(f_i)$ -measurable function in  $L_q$  with  $\int_0^1 h_i(t) dt = 0$  and  $\int_0^1 f_i(t) h_i(t) dt = 1$ . Then  $(h_i)$  is a biorthogonal sequence to  $(f_i)$ . Hence, in order to show that  $(f_i)$  is equivalent to the usual  $\ell_p$ -basis we may discard finitely many of the  $f_i$ 's and show that the remaining sequence is equivalent to the usual  $\ell_p$ -basis.

We first make the additional assumption that every  $f_i$  has mean 0. If  $2 < p < \infty$ , Theorem 3 in Rosenthal's paper [21] renders then:

$$\begin{aligned} \left\| \sum_{i=1}^n \alpha_i f_i \right\|_p &\sim \left( \sum_{i=1}^n \|\alpha_i f_i\|_p^p \right)^{1/p} + \left( \sum_{i=1}^n \|\alpha_i f_i\|_2^2 \right)^{1/2} \\ &\sim \left( \sum_{i=1}^n \|\alpha_i f_i\|_p^p \right)^{1/p} \sim \left( \sum_{i=1}^n |\alpha_i|^p \right)^{1/p}. \end{aligned}$$

(The second equivalence follows from Hölder's inequality and (iii), while the third one follows from the fact that  $(f_i)$  is semi-normalized.)

Let now  $1 \leq p < 2$ , and assume that (ii) holds. By the Lemma we may assume that the sets  $(E_i)$  are of the form  $[|f_i| \geq \beta_i]$ , so  $E_i \in \mathcal{A}(f_i)$ , for all  $i \in \mathbb{N}$ , which makes them independent events. Discarding finitely many values of  $i$ , if necessary, we may assume that  $|E_i| < 1$  for all  $i$ , and so (3.11) implies that

$$c \equiv \prod_{i=2}^\infty (1 - |E_i|) > 0.$$

For each  $i$ , let  $G_i = E_i \setminus \cup_{j>i} E_j$ . Then  $G_i$  are disjoint measurable sets, and by the independence of  $f_i$  we have, for all  $i$ ,

$$(3.15) \quad \int_{G_i} |f_i(t)|^p dt = \prod_{j>i} (1 - |E_j|) \int_{E_i} |f_i(t)|^p dt \geq c \delta^p.$$

$(f_i)$  is an unconditional basic sequence by Fact 2.3. Therefore by (3.15) and Lemma 2 of [7] (or, equivalently, Proposition 3.4 (i)),  $(f_i)$  is equivalent to the usual  $\ell_p$ -basis.

The Proposition is now proved for  $1 \leq p < \infty, p \neq 2$  under the assumption that all  $f_i$  have mean 0. We now drop that assumption. Let  $c_i = \int_0^1 f_i(t)dt$ , and let  $g_i = f_i - c_i, i \in \mathbb{N}$ . Then, for  $1 \leq p < \infty, p \neq 2$ ,  $(g_i)$  is a semi-normalized sequence of independent, non-constant, mean 0 random variables in  $L_p$ . If  $1 \leq p < 2$ , note that

$$\left( \int_{E_i} |c_i|^p \right)^{1/p} = |c_i| |E_i|^{1/p} \leq \|f_i\| |E_i|^{1/p} \rightarrow 0$$

as  $i \rightarrow \infty$ . So, after discarding finitely many values of  $i$ , statement (ii) holds for  $(g_i)$  with a smaller value of  $\delta$ . If  $2 < p < \infty$ , note that  $|c_i| \leq \|f_i\|_1 \leq \|f_i\|_2$ , so  $\|g_i\|_2 \leq 2\|f_i\|_2$  and therefore (iii) holds for  $(g_i)$  as it did for  $(f_i)$ . By the case already proved, we may conclude, for  $1 \leq p < \infty, p \neq 2$ , that  $(g_i)$  is equivalent to the usual  $\ell_p$ -basis.

Now note that  $f_i = g_i + c_i$ . If  $1 < p < \infty, p \neq 2$ , then by (iv),

$$\sum_{i=1}^{\infty} \|f_i - g_i\|^q = \sum_{i=1}^{\infty} |c_i|^q < \infty,$$

so by Fact 2.5,  $(f_i)$  is equivalent to the usual  $\ell_p$ -basis. If, on the other hand,  $p = 1$ , then there is a constant  $a > 0$  such that for any  $n$  and any  $\alpha_1, \dots, \alpha_n$ ,

$$\begin{aligned} a \sum_{i=1}^n |\alpha_i| &\leq \left\| \sum_{i=1}^n \alpha_i g_i \right\| \leq \left\| \sum_{i=1}^n \alpha_i f_i \right\| + \left| \sum_{i=1}^n \alpha_i c_i \right| \\ &= \left\| \sum_{i=1}^n \alpha_i f_i \right\| + \left| \int_0^1 \sum_{i=1}^n \alpha_i f_i \right| \leq 2 \left\| \sum_{i=1}^n \alpha_i f_i \right\| \\ &\leq 2 \sup_i \|f_i\| \sum_{i=1}^n |\alpha_i|, \end{aligned}$$

which completes the proof.

**REMARK:** For arbitrary normalized unconditional basic sequences  $(f_i)$  in  $L_p, 1 \leq p < \infty, p \neq 2$  the conditions (ii), (iii) and (iv) are not sufficient to ensure that  $(f_i)$  is equivalent to the usual  $\ell_p$ -basis. For example, let  $A_1, A_2, \dots$  be disjoint subsets of  $[0, 1]$  with  $\sum_{n=1}^{\infty} n|A_n| < \infty$ . Given  $1 \leq p < \infty$ , there are  $K_p < \infty$  and functions  $(f_{n,i}; n \in \mathbb{N}, i \leq n)$  such that each  $f_{n,i}$  is mean 0 and supported in  $A_n$  and  $(f_{n,i}; n \in \mathbb{N}, i \leq n)$  is  $K_p$ -equivalent to the usual basis of  $(\sum_{n=1}^{\infty} \oplus \ell_2^n)_p$ . This is im-

mediate from Khinchine's inequality and the fact that  $L_p(A_n)$  is isometric to  $L_p$ . Thus  $(f_{n,i})$  is unconditional and not equivalent to the usual  $\ell_p$ -basis. But  $(f_{n,i})$  satisfies (ii) (with  $E_{n,i} = A_n$ , and  $\delta = 1$ ), and if  $p > 2$ , it satisfies also (iii), by an application of Hölder's inequality as in (5.3) below.

#### 4. Modular sequence spaces

The main result of this section is Theorem 4.2 which states that for  $2 < p < \infty$ ,  $\ell_p$  is, up to normalization, the only modular sequence space that embeds isomorphically in  $\ell_p$ . As a consequence of this and known results we obtain (Corollary 4.3) that for all  $1 \leq p < \infty$  a modular sequence space which is isomorphic to  $\ell_p$  must be identical with  $\ell_p$  (again, up to normalization). Our interest in this result here stems from the fact, proved by H. P. Rosenthal [20], that for any sequence of mean 0 independent random variables  $(f_i)$  in  $L_p$  the set of all sequences of scalars  $(\alpha_i)$  for which  $\sum_{i=1}^{\infty} \alpha_i f_i$  converges in norm is a modular sequence space. (In the case when the  $f_i$  are, in addition, identically distributed, this result was proved earlier by J. Bretagnolle and D. Dacunha-Castelle [2]). We shall use Theorem 4.2 in conjunction with this fact in our proof of Theorem A presented in the next section.

We shall use the term  $\phi$ -function to denote a strictly increasing continuous function from  $\mathbb{R}^+$  to  $\mathbb{R}^+$  which is 0 at 0. If  $(\phi_i)$  is a sequence of  $\phi$ -functions then the modular sequence space  $\ell_{(\phi_i)}$  is the linear space of all sequences  $(\alpha_i)$  of scalars which for some  $t > 0$  satisfy  $\sum_{i=1}^{\infty} \phi_i(|\alpha_i|/t) < \infty$ .  $\ell_{(\phi_i)}$  is made into a complete linear metric space by the metric:  $d((\alpha_i), 0) \leq \epsilon \Leftrightarrow \sum_{i=1}^{\infty} \phi_i(|\alpha_i|/\epsilon) \leq \epsilon$ . A convex  $\phi$ -function is called an Orlicz function. If  $\phi_i$  are all Orlicz functions then the metric  $d$  is equivalent to the norm:  $\|(\alpha_i)\| \leq \epsilon \Leftrightarrow \sum_{i=1}^{\infty} \phi_i(|\alpha_i|/\epsilon) \leq 1$ .

We start with a criterion for one modular sequence space to be a subset of another one. A criterion for equality of two modular sequence spaces given by J. Woo [25] is similar in spirit to ours, but is somewhat more restrictive.

**PROPOSITION 4.1:** *Let  $(\phi_i)$  and  $(\psi_i)$  be two sequences of  $\phi$ -functions. Then the following two statements are equivalent:*

- (i)  $\ell_{(\phi_i)} \subseteq \ell_{(\psi_i)}$
- (ii) *For any  $a > 0$  (equivalently, for some  $a > 0$ ), there is  $C > 0$  such that*

$$(4.1) \quad \sum_{j=1}^{\infty} \epsilon(\phi_j, \psi_j, C, a) < \infty$$

where we define for functions  $\phi$  and  $\psi$ :

$$(4.2) \quad \epsilon(\phi, \psi, C, a) = \sup\{\psi(u) : a \geq \psi(u) \geq C\phi(Cu)\}.$$

PROOF: Let us first explain the motivation behind the rather technical condition (ii). It is clear that if there is  $C < \infty$  such that for all  $j$  and all  $u$ ,  $\psi_j(u) \leq C\phi_j(Cu)$ , then

$$(4.3) \quad \sum_{j=1}^{\infty} \phi_j(\alpha_j) < \infty \Rightarrow \exists t < \infty. \exists. \sum_{j=1}^{\infty} \psi_j(\alpha_j/t) < \infty$$

for all  $\alpha_j \geq 0$ , which in turn is equivalent to (i). Set  $\epsilon_j = \epsilon(\phi_j, \psi_j, C, a)$ . Definition (4.2) says that  $\epsilon_j$  is the smallest number such that for all  $u$  with  $\psi_j(u) \in [\epsilon_j, a]$ , the condition  $\psi_j(u) \leq C\phi_j(Cu)$  does hold. The values of  $u$  with  $\psi_j(u) < \epsilon_j$  are exempted from this requirement, since they are already summable in view of (4.1), while the values where  $\psi_j(u) > a$  turn out not to matter.

(i)  $\Rightarrow$  (ii). Fix  $a > 0$  and assume that (i) holds while  $\sum_{j=1}^{\infty} \epsilon(\phi_j, \psi_j, C, a) = \infty$  for all values of  $C < \infty$ . Since each term of this series is bounded by  $a$ , we can find disjoint finite sets of integers  $(B_k)$  such that

$$(4.4) \quad a \leq \sum_{j \in B_k} \epsilon(\phi_j, \psi_j, k^2, a) \leq 2a.$$

Now by definition we can for each  $j \in B_k$  find  $\alpha_j$  satisfying

$$(4.5) \quad \epsilon(\phi_j, \psi_j, k^2, a)/2 \leq \psi_j(\alpha_j) \leq \epsilon(\phi_j, \psi_j, k^2, a)$$

and

$$(4.6) \quad \psi_j(\alpha_j) \geq k^2 \phi_j(k^2 \alpha_j).$$

Let  $\beta_j = k^2 \alpha_j$  if  $j \in B_k$  for some  $k$ , and  $\beta_j = 0$  otherwise. Then

$$\begin{aligned} \sum_{j=1}^{\infty} \phi_j(\beta_j) &= \sum_{k=1}^{\infty} \sum_{j \in B_k} \phi_j(k^2 \alpha_j) \\ &\leq \sum_{k=1}^{\infty} \sum_{j \in B_k} \psi_j(\alpha_j)/k^2 \quad \text{by (4.6)} \\ &\leq \sum_{k=1}^{\infty} 2a/k^2 < \infty \quad \text{by (4.4) and (4.5).} \end{aligned}$$



On the other hand, given any  $t < \infty$  we have with  $k_0 = [t] + 1$ ,

$$\begin{aligned} \sum_{j=1}^{\infty} \psi_j(\beta_j/t) &\geq \sum_{k=k_0}^{\infty} \sum_{j \in B_k} \psi_j(\beta_j/k^2) \\ &= \sum_{k=k_0}^{\infty} \sum_{j \in B_k} \psi_j(\alpha_j) \geq \sum_{k=k_0}^{\infty} a/2 = \infty \end{aligned}$$

the last inequality holding by the left-hand inequalities in (4.4) and (4.5). Thus  $(\beta_j) \in \ell_{(\phi_j)} \setminus \ell_{(\psi_j)}$ , contradicting (i).

(ii)  $\Rightarrow$  (i). Let  $C, a$  be such that  $\sum_{j=1}^{\infty} \epsilon_j < \infty$  where  $\epsilon_j = \epsilon(\phi_j, \psi_j, C, a)$ . Assume that  $\alpha_j \geq 0$ , all  $j$  and  $\sum_{j=1}^{\infty} \phi_j(\alpha_j) < \infty$ . Let

$$\begin{aligned} I &= \{j; \psi_j(\alpha_j/C) < C\phi_j(\alpha_j)\} \\ J &= \{j; a \geq \psi_j(\alpha_j/C) \geq C\phi_j(\alpha_j)\} \\ K &= \{j; \psi_j(\alpha_j/C) \geq \max(a, C\phi_j(\alpha_j))\}. \end{aligned}$$

Then  $\sum_{j \in I} \psi_j(\alpha_j/C) \leq C \sum_{j \in I} \phi_j(\alpha_j) < \infty$ , and  $\sum_{j \in J} \psi_j(\alpha_j/C) \leq \sum_{j \in J} \epsilon_j < \infty$  by the definition of  $\epsilon(\cdot)$  and by (4.1). Thus to complete the proof we need to show that  $K$  is a finite set. Assume  $K$  is infinite. Since  $\phi_j(\alpha_j) \rightarrow 0$ , there are infinitely many  $j \in K$  satisfying

$$(4.7) \quad C\phi_j(\alpha_j) \leq a/2.$$

For each such  $j$ ,  $\psi_j(\alpha_j/C) \geq a$  by definition of  $K$ , and since  $\psi_j$  is continuous, there is  $\beta_j$  satisfying  $0 < \beta_j \leq \alpha_j/C$ , and  $a/2 \leq \psi_j(\beta_j) \leq a$ . This, together with (4.7) implies that

$$C\phi_j(C\beta_j) \leq C\phi_j(\alpha_j) \leq a/2 \leq \psi_j(\beta_j) \leq a,$$

and so by (4.2),

$$\epsilon_j = \epsilon(\phi_j, \psi_j, C, a) \geq \psi_j(\beta_j) \geq a/2.$$

Since this happens for infinitely many values of  $j$ ,  $\sum_{j=1}^{\infty} \epsilon_j = \infty$ , contradicting our assumption.

**REMARK:** After this paper was accepted for publication Professor L. Drewnowski brought to our attention the work of I. V. Šragin [27] in which inclusion relationships between sequence spaces are studied in greater generality than in Theorem 4.1 here.

The following question seems natural in this context:

**PROBLEM 4.A:** Let  $(\phi_i)$  be a sequence of  $\phi$ -functions, such that  $\ell_{(\phi_i)}$  is isomorphic to a Banach space. Are there Orlicz functions  $(\psi_i)$  such that  $\ell_{(\phi_i)} = \ell_{(\psi_i)}$ ?

In the case when  $\phi_i = \phi_1$  for all  $i$  (i.e. for Orlicz sequence spaces) the answer is affirmative.

We can now prove the main result of this section.

**THEOREM 4.2:** Let  $2 < p < \infty$ , and let  $(x_i)$  be a semi-normalized basic sequence in  $\ell_p$ . Assume that  $(x_i)$  is a modular basis, that is, there is a sequence  $(\phi_i)$  of  $\phi$ -functions such that for all sequences  $(\alpha_i)$  of scalars

$$(4.8) \quad \sum_{i=1}^{\infty} \alpha_i x_i \text{ converges if and only if } (\alpha_i) \in \ell_{(\phi_i)}.$$

Then  $(x_i)$  is equivalent to the usual  $\ell_p$ -basis.

**PROOF:** Since  $(\alpha_i) \in \ell_{(\phi_i)}$  if and only if  $(|\alpha_i|) \in \ell_{(\phi_i)}$ ,  $(x_i)$  is an unconditional basic sequence. Assume that  $(x_i)$  is not equivalent to the usual  $\ell_p$ -basis. By fact 2.4 (iii) there is  $c > 0$  such that for all  $n$  and all  $(\alpha_i)$ ,  $\|\sum_{i=1}^n \alpha_i x_i\| \geq c(\sum_{i=1}^n |\alpha_i|^p)^{1/p}$ . Thus, Theorem 3.3\* implies that there is  $\lambda > 1$  such that for every  $n$  there is  $M_n$  with the property that for every  $A \subseteq \mathbb{N}$  either  $(x_i, i \in A)$  is  $M_n$ -equivalent to the usual basis of  $\ell_p^{|A|}$ , or there are combinations  $y_1, \dots, y_n$  of disjoint subsets of  $\{x_i; i \in A\}$  such that  $(y_i, i \leq n)$  is  $\lambda$ -equivalent to the usual  $\ell_2^n$ -basis. We may assume that  $(M_n)$  increases.

We claim that there are disjoint finite subsets  $(A_n)$  of  $\mathbb{N}$  such that

1°. for all  $n$ ,  $(x_i, i \in A_n)$  fails to be  $M_n$ -equivalent to the usual basis of  $\ell_p^{|A_n|}$ , and

2°. if  $z_n \in [x_i, i \in A_n]$  for  $n = 1, 2, \dots$  then

$$\left\| \sum_{n=1}^{\infty} z_n \right\| \lesssim \left( \sum_{n=1}^{\infty} \|z_n\|^p \right)^{1/p}.$$

To obtain 2° it is enough to get disjoint finite sets  $(E_n)$  of integers such that for all  $n$  and all  $z \in [x_i, i \in A_n]$ ,  $\|z - z_{|E_n}\| \leq \|z\|/K2^{n+1}$ , where  $K$  is the unconditional constant of  $(x_i)$ . The construction of sets  $A_n$  and  $E_n$  is a standard gliding hump argument: If  $A_i, E_i$  have been constructed for  $i = 1, 2, \dots, n-1$ , let  $E = \bigcup_{i < n} E_i$ ,  $A = \bigcup_{i < n} A_i$ . Since the basis  $x_i$  is shrinking there is  $m$  such that for all  $z \in [x_i, i \geq m]$ ,  $\|z_{|E}\| < \|z\|/K2^{n+2}$ . Now, since  $(x_i, i \geq m)$  is not equivalent to

the usual basis of  $\ell_p$ , there is a finite set  $A_n \subseteq \mathbb{N} \cap [m, \infty)$  which satisfies 1°. Finally, since the unit ball of  $[x_i, i \in A_n]$  is compact, there is a finite set  $F \subseteq \mathbb{N}$  such that for all  $z \in [x_i, i \in A_n]$ ,  $\|z - z|_F\| < \|z\|/K2^{n+2}$ . Setting  $E_n = F \setminus E$  completes the induction step of the construction.

If  $B_k$  are disjoint subsets of  $\mathbb{N}$ ,  $z_k = \sum_{i \in B_k} \gamma_i x_i$  and  $\psi_k(u) = \sum_{i \in B_k} \phi_i(|\gamma_i|u)$  for all  $u \geq 0$ , then by (4.8)  $\sum_{k=1}^\infty \alpha_k z_k$  converges if and only if  $(\alpha_k) \in \ell_{(\psi_k)}$ . Define

$$\Psi_n = \left\{ \psi; \psi(u) = \sum_{i \in A_n} \phi_i(|\gamma_i|u), \left\| \sum_{i \in A_n} \gamma_i x_i \right\| \leq \lambda \right\},$$

and let

$$\epsilon_n(C) = \sup\{\epsilon(\phi, \psi, C, 1); \psi \in \Psi_n\}$$

where  $\phi(u) = u^p$  for  $u \geq 0$ . Then there is  $C < \infty$  such that

$$(4.9) \quad \sum_{n=1}^\infty \epsilon_n(C) < \infty.$$

In fact, if (4.9) failed for all  $C < \infty$  there would be  $C_n \nearrow \infty$  such that  $\sum_{n=1}^\infty \epsilon_n(C_n) = \infty$ , and so we could find  $\psi_n \in \Psi_n$  with  $\sum_{n=1}^\infty \epsilon(\phi, \psi_n, C_n, 1) = \infty$ . Since  $\epsilon(\phi, \psi, C, 1)$  is a decreasing function of  $C$ , this would imply that for all  $C$ ,  $\sum_{n=1}^\infty \epsilon(\phi, \psi_n, C, 1) = \infty$ . On the other hand,  $\psi_n(u) = \sum_{i \in A_n} \phi_i(|\gamma_i|u)$  and the elements  $z_n = \sum_{i \in A_n} \gamma_i x_i$  have norm  $\leq \lambda$ . If  $\sum_{n=1}^\infty |\alpha_n|^p < \infty$ , then  $\sum_{n=1}^\infty \alpha_n z_n$  converges by 2°, and therefore  $(\alpha_n) \in \ell_{(\psi_n)}$  by the introductory remark of this paragraph. Thus by Proposition 4.1, there is  $C < \infty$  such that  $\sum_{n=1}^\infty \epsilon(\phi, \psi_n, C, 1) < \infty$ , a contradiction.

Passing to a subsequence, if necessary, we may assume that

$$(4.10) \quad \sum_{n=1}^\infty n \epsilon_n(C) < \infty,$$

and that 1° is still satisfied. By 1° and the defining property of the  $M_n$ , there are disjoint subsets  $B_j^n \subseteq A_n$  for  $j \leq n, n = 1, 2, \dots$  and linear combinations  $y_j^n = \sum_{i \in B_j^n} \gamma_i^n x_i$  such that for each  $n$ ,  $(y_j^n, j \leq n)$  is  $\lambda$ -equivalent to the usual  $\ell_2^n$ -basis. Let  $\psi_j^n(u) = \sum_{i \in B_j^n} \phi_i(|\gamma_i^n|u)$ , all  $u \geq 0$ ,  $n$ , and  $j \leq n$ . Then  $\psi_j^n \in \Psi_n$ , and so by (4.10),

$$\sum_{n=1}^\infty \sum_{j \leq n} \epsilon(\phi, \psi_j^n, C, 1) \leq \sum_{n=1}^\infty n \epsilon_n(C) < \infty.$$

Thus, putting  $\psi_j^n$  into one sequence,  $\ell_p \subseteq \ell_{(\psi_j^n)}$ , by Proposition 4.1. So, if  $\sum_{n=1}^\infty \sum_{j \leq n} |\alpha_j^n|^p < \infty$ , then  $(\alpha_j^n) \in \ell_{(\psi_j^n)}$ , which is equivalent to the convergence of  $\sum_{n=1}^\infty \sum_{j \leq n} \alpha_j^n y_j^n$ . On the other hand, this latter condition is equivalent, by  $2^\circ$  and by the defining property of  $y_j^n$ , to the condition  $\sum_{n=1}^\infty (\sum_{j \leq n} |\alpha_j^n|^2)^{p/2} < \infty$ . Thus  $\sum_{n=1}^\infty \sum_{j \leq n} |\alpha_j^n|^p < \infty$  implies that  $\sum_{n=1}^\infty (\sum_{j \leq n} |\alpha_j^n|^2)^{p/2} < \infty$ , which is false, as one easily checks. This contradiction proves the Theorem.

REMARKS. (a): Theorem 4.2 fails if we drop the assumption that  $2 \leq p < \infty$  or if we replace the assumption that  $(x_i)$  is a modular basis by the assumption that  $(x_i)$  is an unconditional basis. To see the first fact, fix  $1 \leq p < 2$ . One checks that if  $p(n)$  decreases to  $p$  fast enough and if for each  $n$ ,  $g_n$  is a  $p(n)$ -stable random variable normalized in  $L_p$ , then the sequence  $(g_n)$  satisfies the conditions (ii) and (iv) of Proposition 3.5. With such a sequence  $(p(n))$  fixed, choose a sequence of integers  $k(n) \rightarrow \infty$  such that  $d(\ell_{p(n)}^{k(n)}, \ell_p^{k(n)}) > n$  for all  $n$ . If  $A_1, A_2, \dots$  is a partition of  $\mathbb{N}$  with  $|A_n| = k(n)$  for all  $n$ , and if we take  $(f_i)$  to be a sequence of independent random variables normalized in  $L_p$  and such that for each  $i \in A_n$ ,  $f_i$  is  $p(n)$  stable, then  $[f_i]$  is isomorphic to  $(\sum_{n=1}^\infty \oplus \ell_{p(n)}^{k(n)})_p$  by Proposition 3.5 and thus embeds in  $\ell_p$ , and  $(f_i)$  is a modular basis by H. P. Rosenthal's result [20]. But by our choice of  $k(n)$ ,  $(f_i)$  is not equivalent to the usual basis of  $\ell_p$ .

On the other hand, for each  $2 < p < \infty$  there exists an unconditional basic sequence  $(x_i)$  in  $\ell_p$  such that  $[x_i]$  is not isomorphic to  $\ell_p$ . This follows from the results of A. Szankowski [24], as was pointed out to us by L. Tzafriri.

(b). As part of our proof of Theorem 4.2 we show that the usual basis of  $(\sum_{n=1}^\infty \oplus \ell_2^n)_p$  is not a modular basis if  $2 < p < \infty$ . This proof could be generalized to show that the usual basis of  $(\sum_{n=1}^\infty \oplus \ell_r^n)_p$  is not a modular basis if  $p \neq r$ . A weaker form of the latter result was proved earlier by H. P. Rosenthal (unpublished), namely he showed that if  $p \neq r$  then the usual basis of  $(\sum_{n=1}^\infty \oplus \ell_r)_p$  is not a modular basis.

We now show that the conclusion of Theorem 4.2 is valid for all  $1 \leq p < \infty$  provided  $(x_i)$  is a basis for all of  $\ell_p$ , and the  $\phi_i$  are Orlicz functions.

**COROLLARY 4.3:** *Let  $1 \leq p < \infty$ , and let  $(x_i)$  be a semi-normalized basis for  $\ell_p$ . If there is a sequence of Orlicz functions  $(\phi_i)$  such that for all sequences  $(\alpha_i)$  of scalars  $\sum_{i=1}^\infty \alpha_i x_i$  converges if and only if  $(\alpha_i) \in \ell_{(\phi_i)}$ , then  $(x_i)$  is equivalent to the usual basis of  $\ell_p$ .*

**PROOF:** Since  $(\alpha_i) \in \ell_{(\phi_i)}$  if and only if  $(|\alpha_i|) \in \ell_{(\phi_i)}$ ,  $(x_i)$  is un-

conditional. Thus for  $p = 1$  or  $2$ , the assertion of the Theorem follows from the fact that all the semi-normalized unconditional bases for  $\ell_p$  are equivalent. For  $p = 2$  this is immediate from Fact 2.4 (iii), while for  $p = 1$  this result was proved by Lindenstrauss and Pełczyński [12] as a consequence of Grothendieck's inequality.

If  $2 < p < \infty$ , our assertion follows immediately from Theorem 4.2. Finally, for  $1 < p < 2$  we obtain the result by duality: Let  $(x_i^*)$  be the sequence in  $\ell_q$  which is biorthogonal to  $(x_i)$ . Then  $(x_i^*)$  is an unconditional semi-normalized basis for  $\ell_q$ . Now the map  $T: (\alpha_i) \rightarrow \sum_{i=1}^{\infty} \alpha_i x_i$  is an isomorphism of  $\ell_{(\phi_i)}$  onto  $\ell_p$  by the closed graph theorem. As is well known (see [25], for example), there is a sequence of Orlicz functions  $(\psi_i)$  such that  $\ell_{(\phi_i)}^* = \ell_{(\psi_i)}$  with the natural identification. Consequently,  $\sum_{i=1}^{\infty} \beta_i x_i^*$  converges if and only if  $(\beta_i) \in \ell_{(\psi_i)}$ , and so by Theorem 4.2,  $(x_i^*)$  is equivalent to the usual  $\ell_q$ -basis.  $(x_i)$  must then be equivalent to the usual  $\ell_p$ -basis.

Theorem B (ii) follows immediately from Theorem 4.2 and the theorem of W. B. Johnson and E. Odell [7]:

**PROOF OF THEOREM B (ii):** Let  $(f_i)$  be a modular normalized basic sequence in  $L_p$ ,  $2 < p < \infty$ . If  $\ell_2$  fails to embed isomorphically in  $[f_i]$ , then by the Johnson–Odell Theorem [7],  $[f_i]$  embeds isomorphically in  $\ell_p$ . But then by Theorem 4.2,  $(f_i)$  is equivalent to the usual  $\ell_p$ -basis, contrary to the assumption of Theorem B.

## 5. Projections onto subspaces of $L_p$

**PROOF OF THEOREM A:** Let  $(f_i)$  be a sequence of independent random variables in  $L_p$  with  $[f_i]$  isomorphic to  $\ell_p$ . If  $c_i = \int f_i$  and  $g_i = f_i - c_i$ , then  $[1, (f_i)] = [1, (g_i)]$ ; hence by Fact 2.2 it is enough to prove that  $[g_i]$  is complemented. Since  $[g_i]$  is isomorphic to  $\ell_p$ , we may assume without loss of generality that all  $f_i$  have mean 0. We may also assume that  $(f_i)$  is normalized and that no  $f_i$  is constant.

We first show that  $(f_i)$  is equivalent to the usual  $\ell_p$ -basis. For  $p = 1$  this follows from Fact 2.3 and the deep fact that  $\ell_1$  has a unique unconditional basis, (see [12]). For  $2 < p < \infty$  it follows from Theorem 4 and Lemma 7 of [21] that either  $(f_i)$  is equivalent to the usual  $\ell_p$ -basis or else  $[f_i]$  contains a subspace isomorphic to  $\ell_2$ , which is not the case here. Finally if  $1 < p < 2$ , a result of H. P. Rosenthal (Theorem 4.2 of [20]) shows that there are  $\phi$ -functions  $(\phi_i)$  such that  $\sum_{i=1}^{\infty} \alpha_i f_i$  converges in  $L_p$  if and only if  $\sum_{i=1}^{\infty} \phi_i(|\alpha_i|/t) < \infty$  for some  $t < \infty$ . Moreover, the functions  $\phi_i$  have the property that  $\phi_i(t)/t$

increases with  $t > 0$ . Setting  $\psi_i(t) = \int_0^t \phi_i(u)/u \, du$ , each  $\psi_i$  is an Orlicz function, and  $\phi_i(t/2) \leq \psi_i(t) \leq \phi_i(t)$  for all  $i$  and  $t$ . So  $\sum_{i=1}^\infty \alpha_i f_i$  converges in  $L_p$  if and only if  $(\alpha_i) \in \ell_{(\psi_i)}$ , and so by Corollary 4.3,  $\ell_{(\psi_i)} = \ell_p$ , i.e.  $\sum_{i=1}^\infty \alpha_i f_i$  converges if and only if  $\sum_{i=1}^\infty |\alpha_i|^p < \infty$ .

We now proceed to show that  $[f_i]$  is complemented in  $L_p$ . Let first  $p = 1$ . By Proposition 3.5 there are sets  $E_i \in \mathcal{A}(f_i)$  and a number  $\delta > 0$  satisfying

$$(5.1) \quad \sum_{i=1}^\infty |E_i| < \infty \text{ and } \int_{E_i} |f_i(t)| dt \geq \delta, \text{ for all } i.$$

By discarding finitely many values of  $i$  we may assume that  $|E_i| < 1$ , all  $i$ , and so  $c \equiv \prod_{i=1}^\infty (1 - |E_i|) > 0$ , by (5.1). Discarding some more, if necessary, we may assume that  $\sum_{i=1}^\infty |E_i| < c\delta$ . Let  $G_i = E_i \setminus \bigcup_{j>i} E_j$ . Then  $G_i$  are disjoint measurable sets and for  $i \neq j$  we have:

$$\int_{G_i} |f_i(t)| dt \leq \int_{E_i} |f_i(t)| dt = |E_i| \int_0^1 |f_i(t)| dt = |E_i|$$

and so

$$\begin{aligned} \sum_{j \neq i} \int_{G_j} |f_i(t)| dt &\leq \sum_{i=1}^\infty |E_i| < c\delta \\ &\leq c \int_{E_i} |f_i(t)| dt \leq \int_{G_i} |f_i(t)| dt. \end{aligned}$$

(the last inequality follows from the fact that  $G_i = E_i \cap \bigcap_{j>i} \sim E_j$ , and from the independence of the events  $E_j$ ). Thus we have proved that  $(f_i)$  are relatively disjoint in the sense of [18], and so  $[f_i]$  is complemented (see [18]).

Now let  $1 < p < \infty$ ,  $p \neq 2$ . By Proposition 3.5, there are sets  $E_i \in \mathcal{A}(f_i)$  and  $\delta > 0$  satisfying  $\sum_{i=1}^\infty |E_i| < \infty$  and  $\int_{E_i} |f_i(t)|^p dt \geq \delta^p$ , all  $i$ . Let  $\psi_i$  be the Hahn–Banach functional of  $f_i|_{E_i}$ , i.e.  $\psi_i$  is supported on  $E_i$ ,  $\|\psi_i\|_q = 1$  and  $d_i \equiv \int_0^1 \psi_i(t) f_i(t) dt = (\int_{E_i} |f_i(t)|^p dt)^{1/p} \geq \delta$ , all  $i$ . Each  $\psi_i$  is a non-constant  $\mathcal{A}(f_i)$ -measurable function. Thus, putting  $\phi_i = \psi_i/d_i$ , we have

$$(5.2) \quad \int_0^1 f_i(t) \phi_j(t) dt = \delta_{ij}, \text{ all } i, j,$$

since the  $f_i$  are mean 0.

Now  $(\phi_i)$  are non-constant independent random variables, and

$1 \leq \|\phi_i\|_q \leq \delta^{-1}$  for all  $i$ . For each  $i$ ,

$$\left| \int_0^1 \phi_i(t) dt \right| \leq \|\phi_i\|_q |E_i|^{1/p} \leq \delta^{-1} |E_i|^{1/p}$$

by Hölder's inequality, so

$$\sum_{i=1}^{\infty} \left| \int_0^1 \phi_i(t) dt \right|^p \leq \delta^{-1} \sum_{i=1}^{\infty} |E_i| < \infty.$$

Thus  $(\phi_i)$  satisfy conditions (i), (ii) and (iv) of Proposition 3.5 with  $p$  replaced by  $q$ . Moreover, if  $2 < q < \infty$ , then

$$(5.3) \quad \begin{aligned} \|\phi_i\|_2^2 &= \int_{E_i} |\phi_i(t)|^2 dt \leq \left( \int_{E_i} |\phi_i(t)|^q dt \right)^{2/q} |E_i|^{(q-2)/q} \\ &\leq \delta^{-2} |E_i|^{(q-2)/q} \end{aligned}$$

by Hölder's inequality, and so

$$\sum_{i=1}^{\infty} \|\phi_i\|_2^{2q/(q-2)} \leq \delta^{-2q/(q-2)} \sum_{i=1}^{\infty} |E_i| < \infty$$

which shows that  $(\phi_i)$  satisfies also condition (iii) of Proposition 3.5. Thus in any case  $(\phi_i)$  is equivalent to the usual  $\ell_q$ -basis, and so by Fact 2.1 and (5.2),  $[f_i]$  is complemented in  $L_p$ .

Theorem A implies in particular that a sequence of independent random variables in  $L_p$  which is equivalent to the usual  $\ell_p$ -basis has a complemented span. For  $1 < p < \infty$  this statement generalizes to the case of martingale differences as follows.

**THEOREM 5.1** (obtained jointly with G. Schechtman): *Let  $1 < p < \infty$ , and let  $(f_i)$  be a martingale-difference sequence in  $L_p$ . If  $(f_i)$  is equivalent to the usual  $\ell_p$ -basis, then  $[f_i]$  is a complemented subspace of  $L_p$ .*

**PROOF:** Let  $(f_i)$  be a martingale-difference sequence in  $L_p$  which is equivalent to the usual  $\ell_p$ -basis. Since the conclusion of the Theorem is obvious for  $p = 2$  we assume that  $1 < p < 2$  or  $2 < p < \infty$ . Then by a result of the first-named author (Theorem B of [4]) there are disjoint measurable sets  $A_1, A_2, \dots$  and a constant  $c > 0$  so that for all  $i$ ,

$$\left( \int_{A_i} |f_i(t)|^p dt \right)^{1/p} \geq c.$$

Let  $\psi_1, \psi_2, \dots$  be functions in  $L_q$ , such that for each  $i$ ,  $\psi_i$  is supported on  $A_i$ ,  $\|\psi_i\| \leq c^{-1}$ , and

$$(5.4) \quad \int_0^1 f_i(t)\psi_i(t)dt = 1.$$

Let  $\mathcal{E}_i$  denote the conditional expectation operator  $\mathcal{E}_{\mathcal{B}_i}$  with respect to the  $\sigma$ -algebra  $\mathcal{B}_i = \mathcal{A}(f_1, f_2, \dots, f_i)$  generated by  $f_1, \dots, f_i$ . Finally introduce  $\phi_i = (\mathcal{E}_i - \mathcal{E}_{i-1})\psi_i$ ,  $i = 1, 2, \dots$  (here  $\mathcal{E}_0 = 0$ ). Thus each  $\phi_i$  is  $\mathcal{B}_i$ -measurable, and  $\mathcal{E}_{i-1}\phi_i = 0$ , which implies that  $(\phi_i)$ , as well as  $(f_i)$ , is a martingale-difference sequence with respect to the  $\sigma$ -algebras  $\mathcal{B}_1, \mathcal{B}_2, \dots$ .

For all  $i \neq j$  we thus have that  $\int_0^1 f_i(t)\phi_j(t)dt = 0$ , while for all  $i$  we have

$$\begin{aligned} \int_0^1 f_i(t)\phi_i(t)dt &= \int_0^1 f_i(t)\{(\mathcal{E}_i\psi_i)(t) - (\mathcal{E}_{i-1}\psi_i)(t)\}dt \\ &= \int_0^1 \mathcal{E}_i(f_i\psi_i)(t)dt - \int_0^1 (\mathcal{E}_{i-1}f_i)(\mathcal{E}_{i-1}\psi_i)(t)dt \\ &= \int_0^1 f_i(t)\psi_i(t)dt = 1 \end{aligned} \quad \text{by (5.4).}$$

The second equality here follows from the fact that  $f_i$  is  $\mathcal{B}_i$ -measurable (for the first term) and from the fact that  $\mathcal{E}_{i-1}\psi_i$  is  $\mathcal{B}_{i-1}$ -measurable, (for the second term). Thus  $(f_i), (\phi_i)$  is a biorthogonal system.

By a theorem of D. L. Burkholder (Theorem 9 of [3]),  $(\phi_i)$  is unconditional with a constant  $K$  (actually  $K$  depends on  $p$  only). Therefore, fixing any  $n \in \mathbb{N}$  and scalars  $\alpha_1, \dots, \alpha_n$ , we have by Fact 2.4 (ii) that

$$\begin{aligned} \left\| \sum_{i=1}^n \alpha_i \phi_i \right\|_q &\leq KB_q \left\{ \int_0^1 \left( \sum_{i=1}^n |\alpha_i \phi_i(t)|^2 \right)^{q/2} dt \right\}^{1/q} \\ &= KB_q \left\| \left( \sum_{i=1}^n |\alpha_i \phi_i|^2 \right)^{1/2} \right\|_q. \end{aligned}$$

Now, since  $(\mathcal{E}_i)$  is an increasing sequence of conditional expectations, Stein's inequality (Theorem 8 of [28]) implies that

$$\left\| \left( \sum_{i=1}^n |\mathcal{E}_i(\alpha_i \psi_i)|^2 \right)^{1/2} \right\|_q \leq C_q \left\| \left( \sum_{i=1}^n |\alpha_i \psi_i|^2 \right)^{1/2} \right\|_q$$



where  $C_q$  is a constant dependent on  $q$  alone. Similarly, we have

$$\left\| \left( \sum_{i=1}^n |\mathcal{E}_{i-1}(\alpha_i \psi_i)|^2 \right)^{1/2} \right\|_q \leq C_q \left\| \left( \sum_{i=1}^n |\alpha_i \psi_i|^2 \right)^{1/2} \right\|_q .$$

Combining the last three inequalities we obtain

$$\begin{aligned} \left\| \sum_{i=1}^n \alpha_i \phi_i \right\|_q &\leq KB_q \left\| \left( \sum_{i=1}^n |\alpha_i \phi_i|^2 \right)^{1/2} \right\|_q \\ &\leq KB_q \left\| \left( \sum_{i=1}^n |\mathcal{E}_i(\alpha_i \psi_i)|^2 \right)^{1/2} \right\|_q \\ &\quad + KB_q \left\| \left( \sum_{i=1}^n |\mathcal{E}_{i-1}(\alpha_i \psi_i)|^2 \right)^{1/2} \right\|_q \\ &\leq 2KB_q C_q \left\| \left( \sum_{i=1}^n |\alpha_i \psi_i|^2 \right)^{1/2} \right\|_q \\ &= 2KB_q C_q \left\| \left( \sum_{i=1}^n |\alpha_i \psi_i|^q \right)^{1/q} \right\|_q \\ &= 2KB_q C_q \left( \sum_{i=1}^n \|\alpha_i \psi_i\|_q^q \right)^{1/q} \\ &\leq 2KB_q C_q c^{-1} \left( \sum_{i=1}^n |\alpha_i|^q \right)^{1/q} . \end{aligned}$$

Here the second inequality follows from the triangle inequality in  $\ell_2^n$  and from the definition of  $\phi_i$ , while the first equality follows from the fact that the functions  $\psi_1, \psi_2, \dots, \psi_n$  are disjointly supported.

Now  $(f_i)$  and  $(\phi_i)$  satisfy the assumptions of Fact 2.1, so  $[f_i]$  is complemented in  $L_p$ .

**REMARKS (a):** In the case when  $2 < p < \infty$  the proof can be significantly simplified, and does not require either Stein’s inequality or the result of [4].  $\psi_i$  may be taken to be the Hahn–Banach functional of  $f_i$  in  $L_q([0, 1], \mathcal{B}_i, dt)$ , for each  $i$ , and  $(\phi_i)$  may be defined as in the proof. Then  $(\phi_i)$  will be biorthogonal to  $(f_i)$  and unconditional in  $L_q$ , and so the upper estimate needed in Fact 2.1 will follow from Fact 2.4 (iii).

(b). Theorem 5.1 could be used as an alternative to the last two paragraphs in the proof of Theorem A, for the case  $1 < p < \infty$ .

Unlike a sequence of independent random variables, a martingale-difference sequence in  $L_p$  ( $1 \leq p < \infty, p \neq 2$ ), can span a subspace isomorphic to  $\ell_p$  without being equivalent to the usual  $\ell_p$ -basis. For  $1 < p < \infty, p \neq 2$ , one can easily construct a martingale-difference

sequence equivalent to the usual basis of  $(\sum_{n=1}^{\infty} \ell_2^n)_p$ , as in the Remark following Proposition 3.5.  $(\sum_{n=1}^{\infty} \ell_2^n)_p$  is isomorphic to  $\ell_p$  by a well-known result of A. Pełczyński (see [11a]). In  $L_1$  the sequence  $(f_n)$  where for each  $n$ ,  $f_n$  is the first function on the  $n$ 'th level of the Haar system, is a martingale-difference sequence; clearly  $[f_n]$  is isomorphic to  $\ell_1$  and one easily checks that  $(f_n)$  is a conditional basis.

While the examples of martingale differences referred to in the last paragraph span complemented subspaces, there is for each  $2 < p < \infty$  a subspace  $X$  of  $L_p$  spanned by a martingale-difference sequence such that  $X$  is isomorphic to  $\ell_p$  and uncomplemented in  $L_p$ . In fact, each of the subspaces of  $L_p$  constructed by H. P. Rosenthal in [21] to be isomorphic to  $\ell_p$  and uncomplemented in  $L_p$  is most naturally realized as the span of a martingale-difference sequence.

These facts leave then two open problems:

**PROBLEM 5.A:** *Let  $1 \leq p < 2$ , and let  $(f_i)$  be a martingale-difference sequence in  $L_p$ . If  $[f_i]$  is isomorphic to  $\ell_p$ , must  $[f_i]$  be complemented in  $L_p$ ?*

**PROBLEM 5.B:** *Let  $(f_i)$  be a martingale-difference sequence in  $L_1$ . If  $(f_i)$  is equivalent to the usual  $\ell_1$ -basis, is  $[f_i]$  complemented in  $L_1$ ?*

The proof of Theorem 5.1 can be used to prove also the following result of Pełczyński and Rosenthal [16]. (This result is stated implicitly in the proof of Theorem 3.1 there).

**PROPOSITION 5.2** [16]: *Let  $(f_i)$  be a martingale-difference sequence in  $L_p$ ,  $1 < p < \infty$ . If  $[f_i]$  is isomorphic to  $\ell_2$  then it is complemented in  $L_p$ .*

**PROOF:** If  $2 \leq p < \infty$ , every isomorphic copy of  $\ell_2$  in  $L_p$  is complemented there by [9]. Let  $1 < p < 2$ . By the result of Burkholder mentioned above [3]  $(f_i)$  is unconditional, and so by Fact 2.4 (iii),  $(f_i)$  is equivalent to the usual  $\ell_2$ -basis, provided we assume, as we may, that  $(f_i)$  is normalized. We now find a bounded martingale-difference sequence  $(\varphi_i)$  in  $L_q$ , biorthogonal to  $(f_i)$ , as in the proof of Theorem 5.1 (using  $A_i = [0, 1]$ ). To conclude that  $[f_i]$  is complemented we use Fact 2.1 (with  $p$  and  $q$  replaced by 2 there), noting that the necessary inequality involving  $(\varphi_i)$  is guaranteed by Burkholder's result [3] and Fact 2.4 (iii).

**REMARK:** The result stated here as Proposition 5.2 was used by

Pelczyński and Rosenthal [16] to show that every subspace of  $L_p$ ,  $1 < p < \infty$ , isomorphic to  $\ell_2$  contains a subspace complemented in  $L_p$ . This part of the result of [16] was proved earlier by V. Milman [14] using a different method. However [14] does not give the quantitative estimates given in Theorem 3.1 of [16].

### Acknowledgments

The authors thank Professors I. D. Berg, P. Enflo and H. P. Rosenthal for several stimulating discussions concerning the contents of this paper. The authors are particularly grateful to Professor G. Schechtman for significantly improving one of the results.

### REFERENCES

- [1] G. BENNETT, L. E. DOR, V. GOODMAN, W. B. JOHNSON, C. M. NEWMAN: On uncomplemented subspaces of  $L_p$ ,  $1 < p < 2$ . *Israel J. Math.* 26 (1977) 178–187.
- [2] J. BRETAGNOLLE, D. DACUNHA-CASTELLE: Application de l'étude de certaines formes linéaires aléatoires au plongement d'espaces de Banach dans des espaces  $L^p$ . *Ann. Sci. Ecole Normale Supérieure 4<sup>e</sup> ser.* 2 (1969) 437–480.
- [3] D. L. BURKHOLDER: Martingale transforms. *Annals of Math. Stat.* 37 (1966) 1494–1504.
- [4] L. E. DOR: On projections in  $L_1$ . *Annals of Math.* 102 (1975) 463–474.
- [5] I. T. GOHBERG, A. S. MARKUS: On the stability of bases in Banach and Hilbert spaces. *Izv. Akad. Nauk Mold. SSR* 5 (1962) 17–35.
- [6] W. B. JOHNSON, B. MAUREY, G. SCHECHTMAN, L. TZAFRIRI: *Symmetric structures in Banach spaces*.
- [7] W. B. JOHNSON, E. ODELL: Subspaces of  $L_p$  which embed into  $\ell_p$ . *Compositio Math.* 28 (1974) 37–49.
- [8] M. J. KADEC: On conditionally convergent series in the space  $L^p$ . *Uspehi Mat. Nauk* 9:1 (1954) 107–109.
- [9] M. I. KADEC, A. PELCZYŃSKI: Bases, lacunary sequences, and complemented subspaces of  $L_p$ . *Studia Math.* 21 (1962) 161–176.
- [10] J.-L. KRIVINE: Sous-espaces de dimension finie des espaces de Banach réticulés. *Ann. of Math.* 104 (1976) 1–29.
- [11] J. LINDENSTRAUSS, L. TZAFRIRI: Classical Banach spaces. *Lecture Notes in Mathematics*. Springer Verlag, Berlin 1973.
- [11a] J. LINDENSTRAUSS, L. TZAFRIRI: Classical Banach spaces I, *Ergebnisse der Mathematik Grenzgebiete* 92. Springer Verlag, Berlin 1977.
- [12] J. LINDENSTRAUSS, A. PELCZYŃSKI: Absolutely summing operators in  $\mathcal{L}_p$  spaces and their applications. *Studia Math.* 29 (1968) 275–326.
- [13] B. MAUREY: Théorèmes de factorisation pour les opérateurs linéaires à valeurs dans les espaces  $L^p$ . *Asterisque* 11, Paris 1974.
- [14] V. D. MILMAN: Geometric theory of Banach spaces. Part I, theory of basic and minimal systems. *Uspehi Mat. Nauk* 25:3 (1970) 113–174 (Russian), English translation: *Russian Math. Surveys* 25:3 (1970) 111–170.
- [15] W. ORLICZ: Über unbedingte Konvergenz in Funktionenraumen I/II. *Studia Math* 4 (1933) 33–37, 41–47.

- [16] A. PELCZYŃSKI, H. P. ROSENTHAL: Localization techniques in  $L_p$  spaces. *Studia Math.* 52 (1975) 263–289.
- [17] H. P. ROSENTHAL: On a theorem of J. L. Krivine concerning block finite representability of  $\ell_p$  in general Banach spaces. *J. Functional Analysis.*
- [18] H. P. ROSENTHAL: On relatively disjoint families of measures, with some applications to Banach space theory. *Studia Math.* 37 (1970) 13–36.
- [19] H. P. ROSENTHAL: On subspaces of  $L_p$ . *Annals of Math.* 97 (1973) 344–373.
- [20] H. P. ROSENTHAL: On the span in  $L^p$  of sequences of independent random variables (II). *Proceedings VI Berkeley Symp. Math. Stat. Prob. Vol. II* (1970/1) 149–167.
- [21] H. P. ROSENTHAL: On the subspaces of  $L^p$  ( $p > 2$ ) spanned by sequences of independent random variables. *Israel J. Math.* 8 (1970) 273–303.
- [22] H. P. ROSENTHAL: Projections onto translation-invariant subspaces of  $L^p(G)$ . *Memoirs A.M.S.* 63 (1966).
- [23] W. RUDIN: Trigonometric series with gaps. *J. Math. Mech.* 9 (1960) 203–227.
- [24] A. SZANKOWSKI: A Banach lattice without the approximation property. *Israel J. Math.* 24 (1976) 329–337.
- [25] J. Y. T. WOO: On modular sequence spaces. *Studia Math.* 48 (1973) 271–289.
- [26] A. ZYGMUND: *Trigonometric Series. vol. I*, 2nd. edition, Cambridge Univ. Press, Cambridge, England, 1959.
- [27] I. V. ŠRAGIN: Conditions for the embedding of classes of sequences and their consequences. *Mat. Zametki* 20 (1976) 681–692. English translation: *Math. Notes Acad. Sci. USSR* 20 (1976) 942–948.
- [28] E. STEIN: Topics in Harmonic Analysis related to the Littlewood–Paley theory. *Annals of Math. Studies No. 63*, Princeton 1970.

(Oblatum 5-X-1977 & 30-V-1978)

L. E. Dor  
University of Illinois  
Urbana, IL 61801  
U.S.A.

T. Starbird  
Ohio State University  
Columbus, Ohio 43210  
U.S.A.