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UPPER BOUNDS FOR AN IWASAWA INVARIANT

Frank Gerth III*

1. Introduction

Let k_0 be a finite extension of $\mathbf{Q}$, the field of rational numbers, and let K be a $\mathbf{Z}_\ell$-extension of k_0 (that is, K is a Galois extension of k_0, and $\mathrm{Gal}(K/k_0)$ is topologically isomorphic to the additive group of the ℓ-adic integers $\mathbf{Z}_\ell$). Let the intermediate fields be denoted by $k_0 \subset k_1 \subset k_2 \subset \cdots \subset k_n \subset \cdots \subset K$, where k_n/k_0 is a cyclic extension of degree ℓ^n. Let σ be a topological generator of $\mathrm{Gal}(K/k_0)$, and let $\sigma_n = \sigma \mid k_n$. Then $\mathrm{Gal}(k_n/k_0)$ is generated by σ_n, and we shall sometimes denote $\mathrm{Gal}(k_n/k_0)$ by $\langle\sigma_n\rangle$. We define $\tau_n = \sigma_n^{\ell^{n-1}}$, and we note that $\mathrm{Gal}(k_n/k_{n-1}) = \langle\tau_n\rangle$.

Next we let A_n denote the ℓ-class group of k_n for all n (that is, A_n is the Sylow ℓ-subgroup of the ideal class group of k_n). For any finite group C, we let $|C|$ denote the order of C. Then from the theory of $\mathbf{Z}_\ell$-extensions (see [4]), $|A_n| = \ell^{e_n}$ with

$$e_n = \mu\ell^n + \lambda n + \nu \tag{1}$$

for n sufficiently large, where μ, λ, ν are integers (called the Iwasawa invariants of K/k_0) with $\mu \geq 0$ and $\lambda \geq 0$. In general it is difficult to compute μ, λ, ν, and we usually do not know how large n must be in order for equation 1 to be valid. Our goal in this paper is to specify an upper bound for μ based on the number of ramified primes in K/k_0 and on $|A_{n_1}|$ for a particular n_1. Before stating our main theorem, we recall some facts about $\mathbf{Z}_\ell$-extensions (cf. [4]). It is known that only finitely many primes ramify in K/k_0 (in fact, all the ramified primes are above the rational prime ℓ). If s_n denotes the number of primes of

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k_n that are ramified over k_0, then $s_{n+1} \geq s_n$ for all n, and there is an integer n_0 such that $s_n = s_{n_0}$ for all $n \geq n_0$. We let s denote s_{n_0}, and we observe that the s primes of k_{n_0} which are ramified over k_0 are also totally ramified in K/k_{n_0}. We are now ready to state our main theorem.

THEOREM 1.: *Let K/k_0 be a $\mathbf{Z}_\ell$-extension, where k_0 is a finite extension of $\mathbf{Q}$. Let s be defined as above, and let n_1 be chosen so that $l^{n_1} > s-1$. If μ is the Iwasawa invariant of K/k_0, then*

$$\mu \leq e_{n_1}/(l^{n_1} - s + 1),$$

where $|A_{n_1}| = \ell^{e_{n_1}}$, and A_{n_1} is the ℓ-class group of k_{n_1}.

REMARK: In [3] Greenberg shows that μ is "locally bounded" on certain $\mathbf{Z}_\ell$-extensions of k_0. Our Theorem 1 can be used to provide explicit local bounds for μ.

2. Preliminary results

We let the notation be the same as in section 1. We shall use multiplicative notation in each A_n, and the action of $\mathrm{Gal}(k_n/k_0)$ shall be expressed by exponentiation. For $j > i \geq 0$, we define $N_{i,j} : A_j \to A_i$ to be the map induced by the norm map from k_j to k_i, and we define $J_{j,i} : A_i \to A_j$ to be the map induced by the inclusion map of k_i into k_j. For each $n > 0$, we define

$$T_n = 1 + \sigma_n^{\ell^{n-1}} + \cdots + \sigma_n^{(\ell-1)\ell^{n-1}}. \qquad (2)$$

We note that $a^{T_n} = J_{n,n-1}(N_{n-1,n}(a))$ for each $a \in A_n$. We now list several facts that are proved in [2].

LEMMA 1: $T_n \equiv (1-\sigma_n)^{(\ell-1)\ell^{n-1}} \bmod(\ell\mathbf{Z}_\ell[\langle\sigma_n\rangle])$.

LEMMA 2: *Let $R = \mathbf{Z}_\ell[\langle\sigma_n\rangle]/(T_n)$. Then R is a principal ideal domain. Next let A be a finite abelian ℓ-group, and define* rank $A = \dim_{\mathbf{F}_\ell}(A/A^\ell)$, *where $\mathbf{F}_\ell$ is the finite field with ℓ elements. If A is also a cyclic R module, and* rank $A = r < (\ell-1)\ell^{n-1}$, *then A is an elementary abelian ℓ-group with* rank $= r$.

LEMMA 3: *For $j > i \geq n_0$, $N_{i,j}$ is surjective.*

The following lemma is a consequence of [5], Lemma 4.

LEMMA 4: *Let* $A_n^{\langle\tau_n\rangle} = \{a \in A_n \mid a^{\tau_n} = a\}$. *Then*

$$|A_n^{\langle\tau_n\rangle}| \leq |A_{n-1}| \cdot \ell^{s-1}.$$

3. Proof of Theorem 1

Let the notation be the same as in sections 1 and 2. Let $V_n = A_n/A_n^\ell$. We note that V_n may be viewed as a finite dimensional vector space over $\mathbf{F}_\ell$, and $(1-\sigma_n)$ is a nilpotent linear transformation on V_n. In fact

$$(1-\sigma_n)^{\ell^n} \equiv 1 - \sigma_n^{\ell^n} \equiv 0 \bmod(\ell \mathbf{Z}_\ell[\langle\sigma_n\rangle]).$$

Then from linear algebra

$$V_n = \prod_{i=1}^{z_n} M_{n,i}$$

where $\ell^n \geq \operatorname{rank} M_{n,1} \geq \operatorname{rank} M_{n,2} \geq \cdots \geq \operatorname{rank} M_{n,z_n} \geq 1$;

$z_n = \operatorname{rank} V_n/V_n^{1-\sigma_n}$; and

$M_{n,i}$ is an elementary abelian ℓ-group which

is also a cyclic $\mathbf{Z}_\ell[\langle\sigma_n\rangle]$ module.

If we let $\alpha_{n,i} = \operatorname{rank} M_{n,i}$, then $M_{n,i}^{(1-\sigma_n)^{\alpha_{n,i}}} = \{1\}$ but $M_{n,i}^{(1-\sigma_n)^{\alpha_{n,i}-1}} \neq \{1\}$. Also the integers $\alpha_{n,1}, \ldots, \alpha_{n,z_n}$ are uniquely determined by V_n.

Now we let $\bar{N}_{i,j}: V_j \to V_i$ be the map induced by $N_{i,j}$, and we let $\bar{J}_{j,i}: V_i \to V_j$ be the map induced by $J_{j,i}$. Since $N_{i,j}$ is surjective for $j > i \geq n_0$ by Lemma 3, then $\bar{N}_{i,j}$ is surjective for $j > i \geq n_0$.

Unless otherwise noted, we shall assume $n \geq n_2$, where $n_2 = \max(n_0, n_1)$. (See Theorem 1 for the definition of n_1.) Let $y_{0,n}$ denote the number of $\alpha_{n,i}$ with $\alpha_{n,i} = \ell^n$. (Note: If $\alpha_{n,1} < \ell^n$, then $y_{0,n} = 0$.) We claim that $y_{0,n+1} \leq y_{0,n}$. In fact, if $M_{n+1,i}$ is a cyclic $\mathbf{Z}_\ell[\langle\sigma_{n+1}\rangle]$ module factor of V_{n+1} with rank $= \ell^{n+1}$, then letting $M = \bar{N}_{n,n+1}(M_{n+1,i})$, we see that $\bar{J}_{n+1,n}(M) = \bar{J}_{n+1,n}(\bar{N}_{n,n+1}(M_{n+1,i})) = M_{n+1,i}^{T_{n+1}}$. Since

$$T_{n+1} \equiv (1-\sigma_{n+1})^{(\ell-1)\ell^n} \bmod(\ell \mathbf{Z}_\ell[\langle\sigma_{n+1}\rangle])$$

by Lemma 1, then $\operatorname{rank} \bar{J}_{n+1,n}(M) = \ell^{n+1} - (\ell-1)\ell^n = \ell^n$. Then it is easy to see that M is a cyclic $\mathbf{Z}_\ell[\langle\sigma_n\rangle]$ module factor of V_n with rank $= \ell^n$. So for every cyclic module factor of V_{n+1} with rank $= \ell^{n+1}$, there is a corresponding cyclic module factor of V_n with rank $= \ell^n$. Then we see that $y_{0,n+1} \leq y_{0,n}$. Since each $y_{0,n} \geq 0$, there are integers $\gamma_0 \geq 0$ and $f_0 \geq n_2$ such that $y_{0,n} = \gamma_0$ for all $n \geq f_0$.

Next let $y_{1,n}$ denote the number of $\alpha_{n,i}$ with $\alpha_{n,i} = \ell^n - 1$. By arguments similar to those above, $y_{1,n+1} \leq y_{1,n}$ for all $n \geq f_0$, and there are integers $\gamma_1 \geq 0$ and $f_1 \geq f_0$ such that $y_{1,n} = \gamma_1$ for all $n \geq f_1$. Similarly if $y_{j,n}$ (for $1 < j \leq s-1$) denotes the number of $\alpha_{n,i}$ with $\alpha_{n,i} = \ell^n - j$, then there are nonnegative integers $\gamma_2, \ldots, \gamma_{s-1}$ and $g = f_{s-1} \geq \cdots \geq f_2 \geq f_1$ such that $y_{j,n} = \gamma_j$ for all $n \geq f_j$. We observe that $V_g = \prod_{i=1}^{z_g} M_{g,i}$ with

$$\alpha_{g,i} = \begin{cases} \ell^g & \text{for } i \leq \gamma_0 \\ \ell^g - 1 & \text{for } \gamma_0 < i \leq \gamma_0 + \gamma_1 \\ \vdots & \\ \ell^g - s + 1 & \text{for } \sum_{j=1}^{s-2} \gamma_j < i \leq \sum_{j=1}^{s-1} \gamma_j \end{cases}$$

and $\alpha_{g,i} < \ell^g - s + 1$ for $i > \sum_{j=1}^{s-1} \gamma_j$. We let y denote $\sum_{j=1}^{s-1} \gamma_j$, and we let M_g denote the product of all $M_{g,i}$ with $\alpha_{g,i} < \ell^g - s + 1$. So $V_g = (\prod_{i=1}^{y} M_{g,i}) \times M_g$. For each $n > g$, we define inductively $M_{n,i}$ $(1 \leq i \leq y)$ and M_n such that $\bar{N}_{n-1,n}(M_{n,i}) = M_{n-1,i}$; $\bar{N}_{n-1,n}(M_n) = M_{n-1}$; $V_n = (\prod_{i=1}^{y} M_{n,i}) \times M_n$; and $\alpha_{n,i}$ satisfies the same equation as $\alpha_{g,i}$ (for $1 \leq i \leq y$) except with n replacing g. For $n \geq g$, we then define inductively $\mathbf{Z}_\ell[\langle\sigma_n\rangle]$ modules $B_n \supseteq A_n^\ell$ so that the image of B_n in V_n is M_n and $N_{n,n+1}(B_{n+1}) = B_n$. We note that $|A_n/B_n| = |V_n/M_n| = \ell^{\beta_n}$ for $n \geq g$, where

$$\beta_n = \gamma_0\ell^n + \gamma_1(\ell^n - 1) + \cdots + \gamma_{s-1}(\ell^n - s + 1).$$

Our next step is to estimate $B_n^{\langle\tau_n\rangle} = \{a \in B_n \mid a^{\tau_n} = a\} = B_n \cap A_n^{\langle\tau_n\rangle}$. First we note that for $n > g$, $J_{n,n-1}(A_{n-1}) \subseteq A_n^{\langle\tau_n\rangle}$, and

$$((J_{n,n-1}(A_{n-1})) \cdot B_n)/B_n = ((J_{n,n-1}(N_{n-1,n}(A_n))) \cdot B_n)/B_n = (A_n^{T_n} \cdot B_n)/B_n.$$

Since $T_n \equiv (1 - \sigma_n)^{(\ell-1)\ell^{n-1}} \bmod(\ell\mathbf{Z}_\ell[\langle\sigma_n\rangle])$ by Lemma 1, then

$$\begin{aligned} |(A_n^{T_n} \cdot B_n)/B_n| &= |A_n/B_n| \cdot [|A_n/(A_n^{T_n} \cdot B_n)|]^{-1} \\ &= \ell^{\beta_n} \cdot \ell^{-(\gamma_0 + \cdots + \gamma_{s-1})(\ell-1)\ell^{n-1}} \\ &= \ell^{\beta_{n-1}}. \end{aligned}$$

Since $A_n^{\langle\tau_n\rangle}/(B_n \cap A_n^{\langle\tau_n\rangle}) \cong (A_n^{\langle\tau_n\rangle} \cdot B_n)/B_n$, then $|A_n^{\langle\tau_n\rangle}/(B_n \cap A_n^{\langle\tau_n\rangle})| = |(A_n^{\langle\tau_n\rangle} \cdot B_n)/B_n| \geq \ell^{\beta_{n-1}}$. Then

$$|B_n^{\langle\tau_n\rangle}| = |B_n \cap A_n^{\langle\tau_n\rangle}| \leq |A_n^{\langle\tau_n\rangle}| \cdot \ell^{-\beta_{n-1}} \leq |A_{n-1}| \cdot \ell^{s-1} \cdot \ell^{-\beta_{n-1}}$$

by Lemma 4. Since $|A_{n-1}/B_{n-1}| = \ell^{\beta_{n-1}}$, then $|A_{n-1}| \cdot \ell^{-\beta_{n-1}} = |B_{n-1}|$. So $|B_n^{\langle\tau_n\rangle}| \leq |B_{n-1}| \cdot \ell^{s-1}$ for $n > g$.

Now we let $W_n = B_n/B_n^{\ell}$. We can apply the same procedures to the W_n that we applied to the V_n. So we can find submodules C_n of the B_n containing B_n^{ℓ}, nonnegative integers $\delta_0, \delta_1, \ldots, \delta_{s-1}$, and an integer $r \geq g$ such that $N_{n,n+1}(C_{n+1}) = C_n$ for $n \geq r$, and $|B_n/C_n| = \ell^{\epsilon_n}$ for $n \geq r$, where

$$\epsilon_n = \delta_0\ell^n + \delta_1(\ell^n - 1) + \cdots + \delta_{s-1}(\ell^n - s + 1).$$

Also $|C_n^{\langle\tau_n\rangle}| \leq |C_{n-1}| \cdot \ell^{s-1}$ for $n > r$. We note that for $n \geq r$, $|A_n/C_n| = \ell^{\theta_n}$ with

$$\theta_n = (\gamma_0 + \delta_0)\ell^n + (\gamma_1 + \delta_1)(\ell^n - 1) + \cdots + (\gamma_{s-1} + \delta_{s-1})(\ell^n - s + 1).$$

We can then consider C_n/C_n^{ℓ} and repeat the above procedures. Eventually we obtain submodules H_n of the A_n and an integer $n_3 \geq r$ such that $N_{n,n+1}(H_{n+1}) = H_n$ for $n \geq n_3$, and H_n/H_n^{ℓ} has no cyclic $\mathbf{Z}_\ell[\langle\sigma_n\rangle]$ module factor with rank $\geq \ell^n - s + 1$ for $n \geq n_3$. (Remark: Our procedure terminates after a finite number of steps because $|A_{n_1}|$ is finite and $|\bar{N}_{n_1,n}(M)| \geq \ell^{\ell^{n_1} - s + 1} \geq \ell$ for each cyclic $\mathbf{Z}_\ell[\langle\sigma_n\rangle]$ module factor M with rank $\geq \ell^n - s + 1$.) Also $|A_n/H_n| = \ell^{\omega_n}$ for $n \geq n_3$, where

$$\omega_n = \psi_0\ell^n + \psi_1(\ell^n - 1) + \cdots + \psi_{s-1}(\ell^n - s + 1)$$

for some nonnegative integers $\psi_0, \psi_1, \ldots, \psi_{s-1}$. Furthermore $|H_n^{\langle\tau_n\rangle}| \leq |H_{n-1}| \cdot \ell^{s-1}$ for $n > n_3$. We observe that

$$\omega_n = (\psi_0 + \psi_1 + \cdots + \psi_{s-1})\ell^n - (\psi_1 + 2\psi_2 + \cdots + (s-1)\psi_{s-1}).$$

We let $\omega = \psi_0 + \psi_1 + \cdots + \psi_{s-1}$ and $\psi = \psi_1 + 2\psi_2 + \cdots + (s-1)\psi_{s-1}$. So $\omega_n = \omega\ell^n - \psi$. Now by equation 1, $|A_n| = \ell^{e_n}$ with $e_n = \mu\ell^n + \lambda n + \nu$ for n sufficiently large. Since $|A_n/H_n| = \ell^{\omega_n}$, then $|H_n| = \ell^{e'_n}$ for n sufficiently large, where

$$e'_n = \mu'\ell^n + \lambda' n + \nu' \tag{3}$$

with $\mu' = \mu - \omega$, $\lambda' = \lambda$, and $\nu' = \nu + \psi$. Also, from the structure of A_n/H_n, routine calculations show that

$$|A_{n_1}| \geq |N_{n_1,n_3}(A_{n_3})/N_{n_1,n_3}(H_{n_3})| = \ell^{\omega\ell^{n_1}-\psi} \geq \ell^{\omega(\ell^{n_1}-s+1)}.$$

So if $|A_{n_1}| = \ell^{e_{n_1}}$, then $\omega \leq e_{n_1}/(\ell^{n_1} - s + 1)$. To complete the proof of Theorem 1, it suffices to show that $\mu' = 0$. Then $\mu = \omega \leq e_{n_1}/(\ell^{n_1} - s + 1)$.

So we consider H_n with the following properties: H_n/H_n^ℓ has no cyclic $\mathbf{Z}_\ell[\langle\sigma_n\rangle]$ module factor with rank $\geq \ell^n - s + 1$ for $n \geq n_3$; $N_{n,n+1}(H_{n+1}) = H_n$ for $n \geq n_3$; and $|H_n^{\langle\tau_n\rangle}| \leq |H_{n-1}| \cdot \ell^{s-1}$ for $n > n_3$. We let $P_n = H_n/H_n^\ell$. Now

$$P_n/P_n^{1-\tau_n} = (H_n/H_n^\ell)/(H_n/H_n^\ell)^{1-\tau_n} \cong H_n/(H_n^\ell \cdot H_n^{1-\tau_n})$$

$$\cong (H_n/H_n^{1-\tau_n})/(H_n/H_n^{1-\tau_n})^\ell.$$

Since $N_{n-1,n}(H_n) = H_{n-1}$ for $n > n_3$, then $H_n/\ker N_{n-1,n} \cong H_{n-1}$. Also it is easy to see that $\ker N_{n-1,n} \supseteq H_n^{1-\tau_n}$ and $|H_n/H_n^{1-\tau_n}| = |H_n^{\langle\tau_n\rangle}|$. It then follows that $|\ker N_{n-1,n}/H_n^{1-\tau_n}| \leq \ell^{s-1}$. Since H_{n-1}/H_{n-1}^ℓ is the direct product of cyclic $\mathbf{Z}_\ell[\langle\sigma_{n-1}\rangle]$ modules with ranks $< \ell^{n-1} - s + 1$ for $n > n_3$, then $(H_n/\ker N_{n-1,n})/(H_n/\ker N_{n-1,n})^\ell$ is the direct product of cyclic $\mathbf{Z}_\ell[\langle\sigma_n\rangle]$ modules with ranks $< \ell^{n-1} - s + 1$ for $n > n_3$. Then $(H_n/H_n^{1-\tau_n})/(H_n/H_n^{1-\tau_n})^\ell$ and hence $P_n/P_n^{1-\tau_n}$ are direct products of cyclic $\mathbf{Z}_\ell[\langle\sigma_n\rangle]$ modules with ranks $< \ell^{n-1}$ for $n > n_3$. Since $1 - \tau_n = 1 - \sigma_n^{\ell^{n-1}} \equiv (1 - \sigma_n)^{\ell^{n-1}} \bmod(\ell\mathbf{Z}_\ell[\langle\sigma_n\rangle])$, we must have $P_n^{1-\tau_n} = \{1\}$. So

$$|P_n| = |P_n/P_n^{1-\tau_n}| = |(H_n/H_n^{1-\tau_n})/(H_n/H_n^{1-\tau_n})^\ell|$$

$$= \frac{|H_n/\ker N_{n-1,n}| \cdot |\ker N_{n-1,n}/H_n^{1-\tau_n}|}{|(H_n/H_n^{1-\tau_n})^\ell|}$$

$$\leq \frac{|H_{n-1}| \cdot |\ker N_{n-1,n}/H_n^{1-\tau_n}|}{|H_{n-1}^\ell|}$$

$$\leq |P_{n-1}| \cdot \ell^{s-1}.$$

Next we note that $H_n^{T_n} \subset H_n^\ell$ since

$$T_n \equiv (1 - \sigma_n)^{(\ell-1)\ell^{n-1}} \equiv (1 - \tau_n)^{\ell-1} \bmod(\ell\mathbf{Z}_\ell[\langle\sigma_n\rangle]), \quad \text{and } P_n^{1-\tau_n} = \{1\}.$$

On the other hand, $H_n/H_n^{T_n}$ is a finitely generated module over the principal ideal domain $\mathbf{Z}_\ell[\langle\sigma_n\rangle]/(T_n)$. Since

$$\begin{aligned}(H_n/H_n^{T_n})/(H_n/H_n^{T_n})^\ell &\cong H_n/(H_n^\ell \cdot H_n^{T_n})\\ &\cong (H_n/H_n^\ell)/(H_n/H_n^\ell)^{T_n}\\ &= P_n/P_n^{T_n} = P_n,\end{aligned}$$

and P_n is the direct product of cyclic modules with ranks $< \ell^{n-1} \leq (\ell-1)\ell^{n-1}$, then $H_n/H_n^{T_n}$ is the direct product of cyclic modules over $\mathbf{Z}_\ell[\langle\sigma_n\rangle]/(T_n)$ with ranks $< (\ell-1)\ell^{n-1}$. So $H_n/H_n^{T_n}$ is an elementary abelian ℓ-group by Lemma 2. Hence $H_n^{T_n} \supseteq H_n^\ell$. Thus $H_n^\ell = H_n^{T_n} = J_{n,n-1}(N_{n-1,n}(H_n)) = J_{n,n-1}(H_{n-1})$. So $|H_n^\ell| \leq |H_{n-1}|$. Then

$$|H_n| = |H_n/H_n^\ell| \cdot |H_n^\ell| = |P_n| \cdot |H_n^\ell| \leq |P_{n-1}| \cdot \ell^{s-1} \cdot |H_{n-1}|.$$

Let $t_{n-1} = \operatorname{rank} P_{n-1}$. Then $|H_n| \leq |H_{n-1}| \cdot \ell^{t_{n-1}+s-1}$ for $n > n_3$. So $|H_{n_3+1}| \leq |H_{n_3}| \cdot \ell^{t_{n_3}+s-1}$;

$$\begin{aligned}|H_{n_3+2}| \leq |H_{n_3+1}| \cdot \ell^{t_{n_3+1}+s-1} &\leq (|H_{n_3}| \cdot \ell^{t_{n_3}+s-1}) \cdot \ell^{(t_{n_3}+s-1)+s-1}\\ &= |H_{n_3}| \cdot \ell^{2t_{n_3}+3(s-1)};\end{aligned}$$

and by induction, $|H_{n_3+i}| \leq |H_{n_3}| \cdot \ell^v$, where $v = it_{n_3} + (s-1)(i+1)(i)/2$ for $i \geq 1$. Now by equation 3, $|H_{n_3+i}| = \ell^{e'_{n_3+i}}$ with $e'_{n_3+i} = \mu'\ell^{n_3+i} + \lambda'(n_3+i) + \nu'$ for i sufficiently large. We note that $\lim_{i\to\infty}(e'_{n_3+i}/\ell^{n_3+i}) = \mu'$. From our above calculations, $e'_{n_3+i} \leq e'_{n_3} + it_{n_3} + (s-1)(i+1)(i)/2$. Then it is easy to see that $\lim_{i\to\infty}(e'_{n_3+i}/\ell^{n_3+i}) = 0$. So $\mu' = 0$. This is what we wanted to show, and hence the proof of Theorem 1 is complete.

4. Some special cases

The following corollaries are immediate consequences of Theorem 1.

COROLLARY 1: *Let K/k_0 be a $\mathbf{Z}_\ell$-extension of k_0, where k_0 is a finite extension of $\mathbf{Q}$. Assume that there is only one ramified prime in K/k_0, and it is totally ramified. If $|A_0| = \ell^{e_0}$, where A_0 is the ℓ-class group of k_0, then the Iwasawa invariant μ of K/k_0 satisfies $\mu \leq e_0$.*

REMARK: Corollary 1 is also proved in [2].

COROLLARY 2: *Let K/k_0 be a $\mathbf{Z}_\ell$-extension of k_0, where $[k_0:\mathbf{Q}] = 2$. If ℓ splits in k_0, and both primes above ℓ are totally ramified in K/k_0, then $\mu \leq e_1/(\ell-1)$.*

REMARK: Let K be the $\mathbf{Z}_\ell$-extension of k_0 contained in the field obtained by adjoining to k_0 all ℓ-power roots of unity. Assume that k_0 is an imaginary quadratic extension of $\mathbf{Q}$. In [1] Ferrero proves that $\mu \leq e_0$ for this special type of $\mathbf{Z}_\ell$-extension. Also if k_0 is any abelian extension of $\mathbf{Q}$ and $\ell = 2$ or 3, Ferrero shows that $\mu = 0$ for this special type of $\mathbf{Z}_\ell$-extension.

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