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TORSION ALGEBRAIC CYCLES AND A THEOREM OF ROITMAN

S. Bloch

Let X be a smooth projective algebraic variety defined over an algebraically closed field k. Let $CH^n(X)$ denote the chow group of codimension n algebraic cycles on X modulo rational equivalence [8]. In this note I study the l-torsion subgroup $CH^n(X)(l) \subset CH^n(X)$ for l prime to char k. Two sorts of results are proved.

(i) $CH^n(X)(l)$ is related to the l-adic étale cohomology via a map

$$\lambda^n : CH^n(X)(l) \longrightarrow H^{2n-1}_{\acute{e}t}(X, \mathbb{Q}_l/\mathbb{Z}_l(n)).$$

(ii) λ^n is shown to be an isomorphism for $n = \dim X$, i.e., for 0-cycles.

The map λ^1 is also an isomorphism, arising from the identification $CH^1(X) \cong H^1_{\acute{e}t}(X, \mathbb{G}_m)$ together with the Kummer sequence

$$0 \longrightarrow \mu_{l^\nu} \longrightarrow \mathbb{G}_m \xrightarrow{l^\nu} \mathbb{G}_m \longrightarrow 0.$$

When $k = \mathbb{C}, H^{2n-1}_{\acute{e}t}(X, \mathbb{Q}_l/\mathbb{Z}_l(n)) \cong H^{2n-1}_{cl}(X, \mathbb{Q}_l/\mathbb{Z}_l)$ (cl = complex topology) is identified up to a finite group with the torsion in the Griffiths intermediate jacobian $J^n(X)$, and λ^n coincides with the Abel-Jacoby map of Griffiths, restricted to torsion cycles. The existence of λ is used in [1] to identify chow groups of curves on Fano 3-folds with certain generalized Prym varieties.

The fact that λ^n is an isomorphism for $n = \dim X$ is a very beautiful theorem of the Soviet mathematician A.A. Roitman. Unfortunately (and perhaps for reasons which have nothing to do with mathematics), details of his proof have been long delayed in appearing. I think the importance of the result and its relevance to the subject matter of this paper justify the inclusion here of my own somewhat awkward proof.

The construction of the λ^n is surprisingly deep. I am forced to use

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the full strength of some theorems in [2] (which I like to think are non-trivial) together with the Weil conjectures as proved by Deligne [3] (no question of triviality there.) These prerequisites are discussed in §1. The construction of λ^n is given in §2. In §3 the various functoriality properties of λ are discussed. §4 contains the proof of Roitman's theorem, and §5 is a brief discussion of relations between algebraic K-theory and torsion in étale cohomology; in particular, relations between K_1 of a surface and torsion in $H^3_{ét}$.

I am indebted to J. Murre for suggesting that λ should exist and showing me its value in calculations for chow groups of Fano 3-folds.

1. Prerequisites

We fix once for all a smooth projective variety X defined over an algebraically closed field k, and a prime $l \neq \operatorname{char} k$. In this section we recall briefly results from [2] which will be used in the sequel.

Let $\mathbf{H}^q(\mu_{l^\nu}^{\otimes n})$ denote the Zariski sheaf on X associated to the presheaf

$$U \longrightarrow H^q_{ét}(U, \mu_{l^\nu}^{\otimes n})$$

where $U \subset X$ is Zariski open, q, ν, n are given integers, and $\mu_{l^\nu}^{\otimes n}$ denotes the étale sheaf of l-th roots of 1 on X, tensored with itself n times. The Leray spectral sequence associated to the morphisms of sites $X_{ét} \to X_{\text{zariski}} \to \text{point}$ is

$$(1.1) \qquad E_2^{p,q} = H^p(X, \mathbf{H}^q(\mu_{l^\nu}^{\otimes n})) \Rightarrow H^{p+q}_{ét}(X, \mu_{l^\nu}^{\otimes n}).$$

Let X^r denote the set of points of X of codimension r. The filtration by codimension of support on $R\Gamma(X_{ét}, \mu_{l^\nu}^{\otimes n})$ gives rise to another spectral sequence

$$(1.2) \qquad E_1^{p,q} = \coprod_{x \in X^p} H^{q-p}_{\text{galois}}(k(x), \mu_{l^\nu}^{\otimes n-p}).$$

The basic fact is that the spectral sequences (1.1) and (1.2) coincide from E_2 onward. In fact, we can localize (1.2) for the Zariski topology to obtain a complex of sheaves

$$(1.3) \quad 0 \longrightarrow \mathbf{H}^q(\mu_{l^\nu}^{\otimes n}) \longrightarrow \coprod_{x \in X^0} i_x H^q(k(x), \mu_{l^\nu}^{\otimes n}) \longrightarrow \cdots \longrightarrow$$
$$\coprod_{x \in X^{q-1}} i_x H^1(k(x), \mu_{l^\nu}^{\otimes n-q+1}) \longrightarrow \coprod_{x \in X^q} i_x \mu_{l^\nu}^{\otimes n-q}$$

where $i_x(A)$ for an abelian group A denotes the constant sheaf A supported on the Zariski closure of the point x. The main theorem of [2] says that (1.3) is an acyclic resolution of $\mathbf{H}^q$. Taking global sections, the cohomology of $E_1^{\cdot q}$ is seen to be $H^{\cdot}(X, \mathbf{H}^q)$.

Two corollaries will be particularly useful.

Corollary 1.4:

$$H^p(X, \mathbf{H}^q(\mu_{l^\nu}^{\otimes n})) = (0) \quad \text{for } p > q. \tag{1.4}$$

In particular, for $n \geqslant 1$, $E_2^{n+i,n-i+1} = (0)$, $i \geq 1$, *so there is a boundary map from* (1.1)

$$H^{n-1}(X, \mathbf{H}^n(\mu_{l^\nu}^{\otimes n})) \longrightarrow H_{ét}^{2n-1}(X, \mu_{l^\nu}^{\otimes n}).$$

Corollary 1.5: $H^{n-1}(X, \mathbf{H}^n(\mu_{l^\nu}^{\otimes n}))$ *is the cohomology of the complex*

$$\coprod_{x \in X^{n-2}} H^2(k(x), \mu_{l^\nu}^{\otimes 2})) \longrightarrow \coprod_{x \in X^{n-1}} k(x)^*/k(x)^{*l^\nu} \xrightarrow{\partial_{l^\nu}} \coprod_{x \in X^n} Z/l^\nu Z. \tag{1.5}$$

One further fact of this sort which will be used is the existence of a resolution (*cf.* [6]).

$$\coprod_{x \in X^{n-1}} k(x)^* \xrightarrow{\partial} \coprod_{x \in X^n} Z \longrightarrow CH^n(X) \longrightarrow 0. \tag{1.6}$$

The map ∂_{l^ν} *in* (1.5) *is obtained by reduction modulo* l^ν *from* ∂ *in* (1.6).

2. Construction of λ

We consider the diagram which exact rows

$$\begin{tikzcd}
0 \arrow[r] & \coprod_{X^{n-1}} k(x)^*/k^* \arrow[r, "l^\nu"] \arrow[d, "\partial"] & \coprod k(x)^*/k^* \arrow[r] \arrow[d, "\partial"] & \coprod k(x)^*/k(x)^{*l^\nu} \arrow[r] \arrow[d, "\partial_{l^\nu}"] & 0 \\
0 \arrow[r] & \coprod_{X^n} Z \arrow[r, "l^\nu"] & \coprod Z \arrow[r] & \coprod Z/l^\nu Z \arrow[r] & 0
\end{tikzcd} \tag{2.1}$$

The serpent lemma applied to (2.1) says the horizontal row in (2.2) below is exact

(2.2)
$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}\,\partial/l^\nu\,\mathrm{Ker}\,\partial \arrow[r] \arrow[ddr, "\rho_{l^\nu}"'] & \mathrm{Ker}\,\partial_{l^\nu} \arrow[r] \arrow[d, "(1.5)"] & \mathrm{CH}^n(X)_{l^\nu} \arrow[r] & 0 \\
& & H^{n-1}(X, \mathbf{H}^n(\mu_{l^\nu}^{\otimes n})) \arrow[d, "(1.4)"] & & \\
& & H^{2n-1}_{\acute{e}t}(X, \mu_{l^\nu}^{\otimes n}). & &
\end{tikzcd}$$

Passing to the limit over ν we get

$$\rho : \mathrm{Ker}\,\partial \longrightarrow H^{2n-1}_{\acute{e}t}(X, Z_l(n)) \tag{2.3}$$

The key fact necessary to construct λ is

Lemma 2.4: *The image of ρ is torsion.*

Proof: Suppose for a moment that $k = \bar{\mathbb{F}}_p$ is the algebraic closure of a finite field. Fix $x \in \mathrm{Ker}\,\partial$ and assume x and X are defined over $\mathbb{F}_{p^a}$, $X = X_0 \underset{\mathrm{Sp}\mathbb{F}_{p^a}}{\times} \mathrm{Sp}\,\bar{\mathbb{F}}_p$. Take

$$f = (p^a\text{-th power frobenius on } X_0) \times (\text{Identity on } \bar{\mathbb{F}}_p).$$

Then f acts compatibly with ρ

$$f\rho(x) = \rho(f(x)) = p^{an}\rho(x).$$

By the Weil conjectures (proved in [3]), p^{an} is not a proper value of f on $H^{2n-1}_{\acute{e}t}(X, Q_l(n))$, so $\rho(x)$ is torsion.

In general, we may spread X out to a smooth scheme $\mathscr{X}$ over Sp R, where R is a valuation ring with quotient field k and residue field $k_0 = \bar{\mathbb{F}}_p$. We have a specialization isomorphism

$$s : H^{2n-1}(X_k, Z_l(n)) \xrightarrow{\sim} H^{2n-1}(X_0, Z_l(n)).$$

In fact, specialization (actually cospecialization, cf. [4], Arcata exposé, section V) induces a map of complexes of E_1 terms (1.2)

$$E_1^{\cdot,q}(k) \longrightarrow E_1^{\cdot,q}(k_0). \tag{2.5}$$

To see this, let $Z_k \subset X_k$ be closed, and let $\mathscr{Z} \subset \mathscr{X}$ denote the closure of Z_k in $\mathscr{X}$. Let $U_k = X_k - Z_k$, $\mathscr{U} = \mathscr{X} - \mathscr{Z}$. Consider the long exact sequences of local cohomology

$$\begin{tikzcd}
\cdots \arrow[r] & H^r_{\mathscr{Z}}(\mathscr{X}, \mathbb{Z}/l^\nu\mathbb{Z}) \arrow[r] \arrow[d, "\gamma"] & H^r(\mathscr{X}, \mathbb{Z}/l^\nu\mathbb{Z}) \arrow[r] \arrow[d, "\alpha"] & H^r(\mathscr{U}, \mathbb{Z}/l^\nu\mathbb{Z}) \arrow[r] \arrow[d, "\beta"] & \cdots \\
& H^r_{Z_k}(X_k, \mathbb{Z}/l^\nu\mathbb{Z}) \arrow[r] & H^r(X_k, \mathbb{Z}/l^\nu\mathbb{Z}) \arrow[r] & H^r(U_k, \mathbb{Z}/l^\nu\mathbb{Z}). &
\end{tikzcd}$$

$\mathscr{X}$ is smooth over Sp R and hence *locally acyclic* (op. cit.) so α and β are isomorphisms. Hence γ is an isomorphism. We obtain a specialization map

$$\begin{tikzcd}
H^r_{Z_k}(X_k, \mathbb{Z}/l^\nu\mathbb{Z}) \arrow[r, "\gamma^{-1}"'] \arrow[rr, bend right, "\tau"'] & H^r_{\mathscr{Z}}(\mathfrak{X}, \mathbb{Z}/l^\nu\mathbb{Z}) \arrow[r, dash] & H^r_{Z_0}(X_0, \mathbb{Z}/l^\nu\mathbb{Z}).
\end{tikzcd}$$

Such a τ can be thought of as a specialization map for étale *homology* (cf. [2]). Examining the construction of the spectral sequence (1.2) in [2], one easily constructs the arrow in (2.5).

Notice now that the subgroups $\coprod_{x \in X_k^{n-1}} k^* \subset \coprod_{x \in X_k^{n-1}} k(x)^*$ (resp. $\coprod k_0^* \subset \coprod_{x \in X_0^{n-1}} k_0(x)^*$) are l-divisible and map to zero in $\coprod_{x_k^n} \mathbb{Z}$ (resp. $\coprod_{X_0^n} \mathbb{Z}$). A standard specialization argument yields homomorphisms t fitting into a commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker} \tilde{\partial}_k \arrow[r] \arrow[d, "t"] & \coprod_{X_k^{n-1}} k(x)^*/k^* \arrow[r] \arrow[d, "t"] & \coprod_{X_k^n} \mathbb{Z} \arrow[r] \arrow[d, "t"] & Ch^n(X_k) \arrow[r] \arrow[d, "t"] & 0 \\
0 \arrow[r] & \operatorname{Ker} \tilde{\partial}_0 \arrow[r] & \coprod_{X_0^{n-1}} k_0(x)^*/k_0^* \arrow[r] & \coprod_{X_0^n} \mathbb{Z} \arrow[r] & CH^n(X_0) \arrow[r] & 0
\end{tikzcd}$$

Using divisibility we see easily that $\tilde{\rho}$ in (2.3) factors through a $\tilde{\rho}_k$ with domain Ker $\tilde{\partial}_k$ (resp. $\tilde{\rho}_0$ with domain Ker $\tilde{\partial}_0$). Moreover $\tilde{\rho}_0 \circ t = s \circ \tilde{\rho}_k$. Since Image $\tilde{\rho}_0$ is torsion and s is an isomorphism, Image $\tilde{\rho}_k$ = Image ρ_k is necessarily torsion as well. Q.E.D.

Construction of λ: We take the direct limit of exact sequences (2.2)

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker} \partial / l^{\nu} \operatorname{Ker} \partial \arrow[r] \arrow[d, "\times l"] & \operatorname{Ker} \partial_{l^{\nu}} \arrow[r] \arrow[d] & CH^{n}(X)_{l^{\nu}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{Ker} \partial / l^{\nu+1} \operatorname{Ker} \partial \arrow[r] & \operatorname{Ker} \partial_{l^{\nu+1}} \arrow[r] & CH^{n}(X)_{l^{\nu+1}} \arrow[r] & 0
\end{tikzcd}$$

and maps

$$\begin{tikzcd}
\operatorname{Ker} \partial_{l^{\nu}} \arrow[r] \arrow[d] & H^{2n-1}_{\acute{e}t}(X, \mu_{l^{\nu}}^{\otimes n}) \arrow[d] \\
\operatorname{Ker} \partial_{l^{\nu+1}} \arrow[r] & H^{2n-1}_{\acute{e}t}(X, \mu_{l^{\nu+1}}^{\otimes n})
\end{tikzcd}$$

The image in $H^{2n-1}_{\acute{e}t}(X, \mu_{l^{\nu}}^{\otimes n})$ of the torsion in $H^{2n-1}_{\acute{e}t}(X, Z_l(n))$ dies in the limit group $H^{2n-1}_{\acute{e}t}(X, Q_l/Z_l(n))$, so we obtain a map

$$CH^{n}(X)(l) \longrightarrow H^{2n-1}_{\acute{e}t}(X, Q_l/Z_l(n)). \tag{2.7}$$

We define λ_l^n to be the negative of (2.7). (The reason for the change of sign is to get (3.6).)

3. Functoriality of λ_l^n

LEMMA 3.1: *The presentation*

$$\coprod_{x \in X^{n-1}} k(x)^* \xrightarrow{\partial} \coprod_{x \in X^{n}} Z \longrightarrow CH^{n}(X) \longrightarrow 0$$

is functorial under pullback by a flat morphism $f: W \to X$ *and under direct image by a proper morphism* $g: X \to Y$.

PROOF: The maps

$$\coprod_{w \in W^{n}} Z \xleftarrow{f^*} \coprod_{x \in X^{n}} Z \xrightarrow{g_*} \coprod_{y \in Y^{n-m}} Z \quad (m = \text{fibre dim } X/Y)$$

are defined as usual in cycle theory [8]. Note that flatness insures that $f^{-1}(\{\bar{x}\})$ has codimension n for any $x \in X^{n}$, so f^* is everywhere defined. Let $x \in X^{n-1}$, $y \in W^{n-1}$ and suppose the cycle $f^{-1}(\{\bar{x}\})$ contains $\{\bar{y}\}$ with multiplicity m. Then $f^*: k(x)^* \to k(y)^*$ is the m-th power of the map induced by the morphism of schemes $\{\bar{y}\} \to \{\bar{x}\}$. Similarly, $g_*: k(x)^* \to k(g(x))^*$ is the norm map if $[k(g(x)):k(x)] < \infty$ and zero otherwise. The fact that these maps are compatible is classical cycle theory, and is left for the reader. Q.E.D.

LEMMA 3.2: *The spectral sequence* (1.2) *is functorial under pullback by flat morphisms and direct image by proper morphisms.*

PROOF: Let $w \xrightarrow{f} X \xrightarrow{g} Y$ be as above. For $q \geq 0$ let $H_q^*(X, \mathbf{Z}/l^\nu\mathbf{Z}) = \varinjlim_{\substack{Z \subset X \\ \operatorname{codim} Z = q}} H_Z^*(X, \mathbf{Z}/l^\nu\mathbf{Z})$. Since f is flat, we get $f^*: H_q^*(X, \mathbf{Z}/l^\nu\mathbf{Z}) \to H_q^*(W, \mathbf{Z}/l^\nu\mathbf{Z})$. This suffices to show f^*-functoriality for (1.2) (cf. [2]). The existence of compatible maps

$$g_*: H_q^*(X, \mu_{l^\nu}^{\otimes n}) \longrightarrow H_{q-m}^*(Y, \mathbf{Z}/l^\nu\mathbf{Z})$$

is the duality theory for étale cohomology (It can be interpreted as saying that étale homology is covariant for proper maps, cf. [2].) and suffices to establish covariant g_* functoriality for the spectral sequence (1.2). Notice, however, that g_* shifts indices

$$g_*: E_1^{p,q} = \coprod_{x \in X^p} H^{q-p}(k(x), \mu_{l^\nu}^{\otimes m}) \longrightarrow \coprod_{y \in Y^{p-m}} H^{q-p}(k(y), \mathbf{Z}/l^\nu\mathbf{Z})$$
$$= E_1^{p-m,q-m}.$$

Combining these two lemmas (again compatibilities will not be pursued in detail) we obtain

PROPOSITION 3.3: *With notation as above, there is a commutative diagram*

$$\begin{tikzcd}
CH^n(W)(l) \arrow[r, "f^*"] \arrow[d, "\lambda_l^n"] & CH^n(X)(l) \arrow[r, "g_*"] \arrow[d, "\lambda^n{}_l"] & CH^{n-m}(Y)(l) \arrow[d, "\lambda_l^{n-m}"] \\
H_{\text{ét}}^{2n-1}(W, \mathbf{Q}_l/\mathbf{Z}_l(n)) & H_{\text{ét}}^{2n-1}(X, \mathbf{Q}_l/\mathbf{Z}_l(n)) \arrow[l, "f^*"'] \arrow[r, "g_*"] & H_{\text{ét}}^{2n-2m-1}(Y, \mathbf{Q}_l/\mathbf{Z}_l(n-m)).
\end{tikzcd}$$

Now let $z \in CH^q(X)$ be a cycle class, and let $[z] \in H_{\text{ét}}^{2q}(X, \mathbf{Z}_l(q))$ be the class of z in cohomology.

PROPOSITION 3.4: *The diagram*

$$\begin{tikzcd}
CH^n(x)(l) \arrow[r, "\cdot z"] \arrow[d, "\lambda_l^n"] & CH^{n+q}(X)(l) \arrow[d, "\lambda_l^{n+q}"] \\
H^{2n-1}(X, \mathbb{Q}_l/\mathbb{Z}_l(n)) \arrow[r, "\cdot{[z]}"] & H^{2n+2q-1}(X, \mathbb{Q}_l/\mathbb{Z}_l(n+q))
\end{tikzcd}$$

commutes, where $\cdot z$ *denotes multiplication in the Chow ring and* $\cdot[z]$ *the cup product induced by the obvious bilinear map*

$$\mathbb{Q}_l/\mathbb{Z}_l(n) \times \mathbb{Z}_l(q) \longrightarrow \mathbb{Q}_l/\mathbb{Z}_l(n+q).$$

PROOF: We may assume z is the class of an irreducible subvariety $Z \subset X$. Let S denote the set of all pairs $(x, R^*(x))$ where $x \in X^{n-1}$ and $R(x) \subset k(x)$ is a finitely generated k-subalgebra. For $A \subset S$ a subset, let $A' \subset X^n$ denote the set of all generic points of $\{\bar{x}\} - \operatorname{Sp} R(x)$ as x runs through A. Given $A \subset S$ a finite subset, we can arrange, by moving Z, for Z to "meet A properly", i.e., for the cycle intersection $Z \cdot \{\bar{x}\}$ to be defined for all x in A or A'. We obtain in this case a diagram

$$\begin{tikzcd}
\coprod_A R(x)^*/R(x)^{*l} \arrow[r, "\partial^{n-1}_{l^\nu,A}"] \arrow[d, "\cdot Z"] & \coprod_{A'} \mathbb{Z}/l^\nu\mathbb{Z} \arrow[d, "\cdot Z"] \\
\coprod_{y \in X^{n+q-1}} k(y)^*/k(y)^{*l^\nu} \arrow[r, "\partial^{n+q-1}_{l^\nu}"] & \coprod_{y \in X^{n+q}} \mathbb{Z}/l^\nu\mathbb{Z}
\end{tikzcd}$$

and hence a map $\cdot Z : \operatorname{Ker} \partial^{n-1}_{l^\nu,A} \to \operatorname{Ker} \partial^{n+q-1}_{l^\nu}$.

The diagrams (cf. (2.2))

$$\begin{tikzcd}
\operatorname{Ker} \partial^{n-1}_{l^\nu,A} \arrow[r, "\cdot Z"] \arrow[d] & \operatorname{Ker} \partial^{n+q-1}_{l^\nu} \arrow[d] \\
CH^n(X)_{l^\nu} \arrow[r, "\cdot Z"] & CH^{n+q}(X)_{l^\nu}
\end{tikzcd}
\qquad
\begin{tikzcd}
\operatorname{Ker} \partial^{n-1}_{l^\nu,A} \arrow[r, "\cdot Z"] \arrow[d] & \operatorname{Ker} \partial^{n+q-1}_{l^\nu} \arrow[d] \\
H^{2n-1}(X, \mathbb{Q}_l/\mathbb{Z}_l(n)) \arrow[r, "\cdot{[Z]}"] & H^{2n+2q-1}(X, \mathbb{Q}_l/\mathbb{Z}_l(n+q))
\end{tikzcd}$$

Commutes. Given $u \in CH^n(X)_{l^\nu}$ we can choose A finite but sufficiently large so $u \in \operatorname{Image}(\operatorname{Ker} \partial_{l^\nu,A} \to CH^n(X)_{l^\nu})$. It is now straightforward to verify $[Z] \cdot \lambda_{l^\nu}(x) = \lambda_{l^\nu}(u \cdot z)$. Q.E.D.

Proposition 3.5: *Let X, Y be smooth projective varieties, m and n integers, and Γ a cycle on $Y \times X$ of dimension* $= \dim X + m - n$. *Then Γ induces correspondences*

$$\Gamma_*: CH^m(Y) \longrightarrow CH^n(X)$$
$$\Gamma_*: H^{2m-1}(Y, Q_l/\mathbb{Z}_l(m)) \to H^{2n-1}(X, Q_l/\mathbb{Z}_l(n))$$

and the diagram

$$\begin{tikzcd} CH^m(Y)(l) \arrow[r, maps to, "\Gamma*"] \arrow[d, "\lambda"] & CH^n(X)(l) \arrow[d, "\lambda"'] \\ H^{2m-1}(Y, Q_l/\mathbb{Z}_l(m)) \arrow[r, "\Gamma*"] & H^{2n-1}(X, Q_l/\mathbb{Z}_l(n)) \end{tikzcd}$$

commutes.

Proof: $\Gamma_* = p_{2*} \circ (\cdot \Gamma) \circ p_1^*$ where p_1, p_2 are the two projections on $Y \times X$. The assertions follow from (3.3)–(3.4). Q.E.D.

Proposition 3.6: *The map λ^1: $CH^1(X)(l) \to H^1(X, Q_l/\mathbb{Z}_l(1))$ is the natural isomorphism arising from the Kummer sequence*

$$0 \longrightarrow \mu_{l^\nu} \longrightarrow \mathbb{G}_m \xrightarrow{l^\nu} \mathbb{G}_m \longrightarrow 0$$

and the identification

$$CH^1(X) \cong H^1(X, \mathbb{G}_m).$$

Proof: We have in the Zariski topology an exact sequence

$$0 \longrightarrow O_X^*/O_X^{*l^\nu} \longrightarrow (k(X)^*/k(X)^{*l^\nu})_X \xrightarrow{\partial_{l^\nu}} \coprod_{x \in X^1} i_{x*}\mathbb{Z}/l^\nu\mathbb{Z} \longrightarrow 0$$

and a commutative diagram of cohomology

$$\begin{tikzcd} \operatorname{Ker} \partial_{l^\nu} \cong H^1_{\acute{e}t}(X, \mu_{l^\nu}) \arrow[r] \arrow[d, phantom, "\wr\Vert"] & H^1_{\acute{e}t}(X, G_m) \arrow[d, phantom, "\wr\Vert"] \\ \Gamma(X, O^*/O^{*l^\nu}) \arrow[r, maps to] & H^1(X, O^*). \end{tikzcd}$$

For $\bar{f} \in \operatorname{Ker} \partial_{l^\nu} \subset k(X)^*/k(X)^{*l^\nu}$, we lift to $f \in k(X)^*$ and write

$$f_\alpha = fg_\alpha^{l^\nu} \quad f_\alpha \in \Gamma(U_\alpha, O_X^*)$$

for some open cover U_α of X. The image of $\bar{f}$ in $H^1(X, O_X^*)$ is represented by the cocycle $g_\alpha/g_\beta = (f_\alpha/f_\beta)^{l^{-\nu}}$. This cocycle is associated to the divisor $-l^{-\nu}(f)$. The assertion of the proposition is now straightforward. (Note that we changed the sign to define λ in (2.7).) Q.E.D.

Suppose now that the ground field is C, the complex numbers. Let $CH^n(X)(l)_0 \subset CH^n(X)(l)$ denote the subgroup of cycles homologous to zero (in $H^{2n}(X, Z)$). Griffiths has defined [5] a complex structure on the torus

$$J^n(X) = H^{2n-1}(X, R)/H^{2n-1}(X, Z)$$

and a cycle map $CH^n(X)_0 \to J^n(X)$. He obtains in this way a map

$$\psi_l^n : CH^n(X)(l)_0 \to H^{2n-1}(X, Q_l/Z_l).$$

Proposition 3.7: *Identify $Q_l/Z_l \cong Q_l/Z_l(n)$ by taking $e^{2\pi i l^{-\nu}}$ as the generator of the l^ν-th roots of* 1. *Then $\psi_l^n = \lambda_l^n$.*

Proof: For $x \in X^{n-1}$ and $f \in k(x)^*$, let $|f|$ denote the support of the divisor (f). Such an f gives a map $f : \{\bar{x}\} - |f| \to P^1 - \{0, \infty\}$. We fix a simple path l on P^1 with $\partial l = (0) - (\infty)$. $f^{-1}(l)$ is a chain on $\{\bar{x}\}$ representing a class $f^{-1}(l) \in H_{2e-1}(\{\bar{x}\}, |f|; Z)$, where $e = \dim\{\bar{x}\}$.

Suppose now that $F = (\ldots f_i \ldots) \in \operatorname{Ker} \partial$. Since $|f_i|$ has complex dimension $e - 1$, we get an exact sequence

$$0 \longrightarrow H_{2e-1}(X; Z) \longrightarrow H_{2e-1}\Big(X, \bigcup |f_i|; Z\Big) \longrightarrow H_{2e-2}\Big(\bigcup |f_i|; Z\Big)$$

and the assignment $F \mapsto F^{-1}(l) = \Sigma f_i^{-1}(l)$ clearly defines a map $-\lambda' : \operatorname{Ker} \partial \to H_{2e-1}(X, Z)$. Similarly, one defines $-\lambda'_{l^\nu} : \operatorname{Ker} \partial_{l^\nu} \to H_{2e-1}(X, Z/l^\nu Z)$. In fact, the image of Ker ∂ is torsion and λ'_{l^ν} coincides with λ_{l^ν} (2.2) up to the identification

$$H_{2e-1}(X, Z/l^\nu Z) \cong H^{2n-1}(X, Z/l^\nu Z), \; n = \dim X - e + 1.$$

The fact that $\lambda'(\operatorname{Ker} \partial)$ is torsion is precisely (2.4), but one can also argue as follows, using intermediate jacobians. Fix $x \in X_e$, $f \in k(x)^*$ non-constant. Recall the Griffiths jacobian is defined by

$$J_{e-1}(X) = (H^{2e-1,0}(X) + \cdots + H^{e,e-1}(X))^*/H_{2e-1}(X, Z).$$

The cycle map $CH_{e-1}(X)_0 \xrightarrow{\psi} J_{e-1}(X)$ is defined (roughly speaking) by $\psi(\gamma) = \int_\Gamma$, where $\partial\Gamma = \gamma$. If we take $\gamma = f^{-1}(\alpha) - f^{-1}(\beta)$ for $\alpha, \beta \in \mathbb{P}^1$, we can take $\Gamma = f^{-1}(l_{\alpha\beta})$ for a path $l_{\alpha\beta}$ from α to β. Since γ is trivial in $CH_{e-1}(X)$, we conclude that $\int_{f^{-1}(l_{\alpha\beta})} \omega$ is constant independent of α, β for $\omega \in H^{2e-1,0} + \cdots + H^{e,e-1}$. Allowing $\alpha \to \beta$, we see that this integral is trivial. Since any cycle in $\lambda'(\text{Ker }\partial)$ can be represented as a sum of chains $f^{-1}(l)$, we conclude

$$\int_{\lambda'(\text{Ker }\partial)} (H^{2e-1,0} + \cdots + H^{e,e-1}) = (0),$$

so $\lambda'(\text{Ker }\partial)$ is torsion.

Suppose now $\gamma \in CH_{e-1}(X)_0$ and $l^\nu\gamma = 0$. We compare the two prescriptions for obtaining a torsion homology class:

Griffiths prescription: Write $\gamma = \partial\Gamma$ and note $\int_\Gamma = l^{-\nu}$. period.

λ'-prescription: Write $l^\nu\gamma = \partial F$, $F = (\ldots f_i \ldots)$. Then $F^{-1}(l)$ is a cycle mod l^ν representing $-\lambda'_{l^\nu}(\gamma)$. Since $l^\nu\Gamma - \partial F =$ period and $\int_{\partial F}$ is trivial on $H^{2e-1,0} + \cdots + H^{e,e-1}$, it follows that these two prescriptions coincide. Q.E.D.

PROPOSITION 3.8: *λ is compatible with specialization. Given $\mathscr{X}$ smooth and projective over* Sp R *(R local) with general fibre X and special fibre X_0, there exist for l prime to* char X_0 *specialization maps*

$$\sigma : CH^n(X)(l) \longrightarrow CH^n(X_0)(l)$$

$$\tau : H^{2n-1}(X, Q_l/Z_l(n)) \xrightarrow{\cong} H^{2n-1}(X_0, Q_0/Z_l(n)),$$

and $\lambda\sigma = \tau\lambda$.

PROOF: The existence of σ is classical [8]. For τ, cf. the discussion following (2.6) and also [4]. To verify $\lambda\sigma = \tau\lambda$, we may assume after base change that R is a valuation ring with algebraically closed quotient and residue fields. For $x \in X^{n-1}$ ($X \subset \mathscr{X}$, is the generic fibre) let $R(x) \subset k(x)$ denote the ring of all functions on the normalization of $\{\bar{x}\}$ (closure in $\mathscr{X}$) which have no pole along components of the special fibre. Let $\mathscr{Y}_x$ denote the same open set of smooth points on this normalization, and let $Y_x \subset \mathscr{Y}_x$ be the general fibre. The morphism $Y_X \hookrightarrow \mathscr{Y}_x$ is locally acyclic, so $H^1(Y_x, \mu_{l^\nu}) \cong H^1(\mathscr{Y}_x, \mu_{l^\nu})$ [4]. Removing all horizontal divisors on $\mathscr{Y}_x$ (i.e. divisors whose support contains no component of the special fibre) we obtain in the limit

$$k(x)^*/k(x)^{*l^\nu} \cong R(x)^*/R(x)^{*l^\nu}.$$

There is a commutative diagram

$$\begin{tikzcd}
\coprod\limits_{x\in X^{n-1}} k(x)^*/k(x)^{*l^\nu} \arrow[rr, "\partial_{l^\nu}(X)"] \arrow[d, phantom, "\wr\|"] & & \coprod\limits_{x\in X^n} \mathbb{Z}/l^\nu\mathbb{Z} \arrow[dd, "\begin{array}{l}\text{take closure}\\ \text{and intersect}\\ \text{with } X_0\end{array}"] \\
\coprod R(x)^*/R(x)^{*l^\nu} \arrow[d] & & \\
\coprod\limits_{x\in X_0^{n-1}} k_0(x_0)^*/k_0(x_0)^{*l^\nu} \arrow[rr, "\partial_{l^\nu}(X_0)"] & & \coprod\limits_{x_0\in X_0^n} \mathbb{Z}/l^\nu\mathbb{Z}
\end{tikzcd}$$

The two squares

$$\begin{tikzcd}
H^{2n-1}(X, \mathbb{Q}_l/\mathbb{Z}_l(n)) \arrow[d, "\tau"'] & \operatorname{Ker} \partial_{l^\nu}(X) \arrow[l] \arrow[r, two heads] \arrow[d] & CH^n(X)_{l^\nu} \arrow[d, "\sigma"] \\
H^{2n-1}(X_0, \mathbb{Q}_l/\mathbb{Z}_l(n)) & \operatorname{Ker} \partial_{l^\nu}(X_0) \arrow[l] \arrow[r, two heads] & CH^n(X_0)_{l^\nu}
\end{tikzcd}$$

also commute, so one deduces $\lambda\sigma = \tau\lambda$. Q.E.D.

One final compatibility. Suppose $n = \dim X$ and let $\mathrm{Alb}(X)$ denote the albanese of X (the dual of the Picard variety). The e_m-pairing gives an isomorphism

$$\mathrm{Alb}(X)(l) \cong H^{2n-1}(X, \mathbb{Q}_l/\mathbb{Z}_l(n)).$$

There is also a map $CH^n(X)(l) \to \mathrm{Alb}(X)(l)$ obtained by mapping a zero cycle on X to the sum of the corresponding points on the albanese.

Proposition 3.9: *With notation as above, the diagram*

$$\begin{tikzcd}
CH^n(X)(l) \arrow[r] \arrow[dr, "\lambda"'] & \mathrm{Alb}(X)(l) \arrow[d, phantom, "\wr\|"] \\
& H^{2n-1}(X, \mathbb{Q}_l/\mathbb{Z}_l(n))
\end{tikzcd}$$

commutes.

PROOF: We will see in the next section that $CH^n(X)(l) \cong \mathrm{Alb}(X)(l)$. Let $C \overset{i}{\hookrightarrow} X$ be a general linear space section of dimension 1. Then it is known that $CH^1(C)(l) \twoheadrightarrow \mathrm{Alb}(X)(l)$. Using this and (3.3) we reduce to verifying commutativity for

$$\begin{tikzcd}
CH^1(C)(l) \arrow[r] \arrow[d, phantom, "\wr\|"] & \mathrm{Alb}(X)(l) \arrow[d] \\
H^1(C, Q_l/Z_l(1)) \arrow[r, "i_*"] & H^{2n-1}(X, Q_l/Z_l(n))
\end{tikzcd}$$

This compatibility is known. Q.E.D.

4. On a theorem of Roitman

Roitman has announced a proof of the following important

THEOREM 4.1: *Let X be a smooth projective variety over an algebraically closed field k. Let $CH_0(X)_{\mathrm{tors}}$ denote the torsion subgroup of the chow group $CH_0(X)$ of zero cycles on X modulo rational equivalence, and let $\mathrm{Alb}(X)_{\mathrm{tors}}$ be the torsion subgroup of the Albanese of X. Then the natural map*

$$\psi : CH_0(X)_{\mathrm{tors}} \longrightarrow \mathrm{Alb}(X)_{\mathrm{tors}}$$

is an isomorphism.

I will present in detail the proof of a slightly weaker result.

THEOREM 4.2: *The above map ψ is surjective and is an isomorphism prime to the characteristic of k.*

PROOF: *Surjectivity.* The subgroup $CH_0(X)_{\deg 0}$ of zero cycles of degree 0 is known to be divisible, so it suffices to show $CH_0(X)_l \twoheadrightarrow \mathrm{Alb}(X)_l$, when the subscript indicates the kernel of multiplication by a prime l. We use induction on dimension X.

When $\dim X = 1$, $\mathrm{CH}_0(X)$ is the Picard group, and the assertion is well-known. Assume $\dim X > 1$ and let $Y \hookrightarrow X$ be a smooth hyperplane section. The top horizontal arrow in the diagram

$$\begin{tikzcd}
CH_0(Y)_l \arrow[r] \arrow[d] & \mathrm{Alb}(Y)_l \arrow[d] \\
CH_0(X)_l \arrow[r] & \mathrm{Alb}(X)_l
\end{tikzcd}$$

may be assumed surjective, so it suffices to show $\mathrm{Alb}(Y)_l \twoheadrightarrow \mathrm{Alb}(X)_l$. Ignoring twists by roots of 1 we have

$$\mathrm{Hom}(H^1_{\acute{e}t}(X, Z/lZ), Z/lZ) \twoheadrightarrow \mathrm{Alb}(X)_l.$$

(This is true even for $l = \mathrm{char}\, k$) so it suffices to show $H^1_{\acute{e}t}(X, Z/lZ) \hookrightarrow H^1_{\acute{e}t}(Y, Z/lZ)$. This follows from Zariski's connectedness theorem.

Injectivity. We use a presentation of the chow group

$$\coprod_{x\in X_1} k(x)^* \xrightarrow{\partial} \coprod_{x\in X_0} Z \longrightarrow CH_0(X) \longrightarrow 0$$

where X_i denotes the set of points on X whose Zariski closure has dimension i. Fix a cycle z on X and an integer $l \geq 1$ such that $lz \underset{\mathrm{rat}}{\sim} 0$. Assume further that $\psi(z) = 0$. We must show $z \underset{\mathrm{rat}}{\sim} 0$. We may assume l prime.

Let $x_0, \ldots, x_n \in X_1, f_i \in k(x_i)$, $C_i = \{\bar{x}_i\}$ so that $lz = \partial(f_1, \ldots, f_n)$.

First reduction: We may assume the C_i are smooth, and no more than two C_i pass through any point.

Indeed, blowing up a point on X changes neither $CH_0(X)$ nor $\mathrm{Alb}(X)$. Blow up a succession of points so that the strict transform $\tilde{C}$ of C becomes a disjoint union of smooth curves

$$\tilde{C} = \coprod \tilde{C}_i.$$

Let $\pi : \tilde{X} \to X$ denote the blowing down map. Let $\tilde{z}$ be a cycle on $\tilde{C}$ with $\pi\tilde{z} = z$, and let

$$w = l\tilde{z} - \tilde{\partial}(f_0, \ldots, f_n)$$

where f_i is viewed as a function on $\tilde{C}_i$, and

$$\tilde{\partial} : \coprod_{\tilde{x}\in \tilde{X}_1} k(\tilde{x})^* \longrightarrow \coprod_{\tilde{x}\in\tilde{X}_0} Z.$$

Note $\pi(w) = 0$. Let $E_1, \ldots, E_r$ denote the connected components of the exceptional locus of π. w can be written $w = w_1 + \cdots + w_r$ where each w_i has degree 0 and is supported on E_i. Since E_i is a union of projective spaces we can find lines $l_{ij} \subset E_i$ with w_i supported on $\bigcup_j l_{ij}$ and functions g_{ij} on l_{ij} such that $\partial(g_{i1}, \ldots, g_{ij}, \ldots) = w_i$. We can further arrange that $C' = \tilde{C} \cup \bigcup_{ij} l_{ij}$ has at most two components through any point as desired.

Second reduction: We may assume X is a surface.

Indeed, let notation be as above, and let Y be a general linear space section of large degree and dimension 2 containing C. Our hypothesis about C will guarantee that Y is smooth. Moreover, $\mathrm{Alb}(X) \cong \mathrm{Alb}(Y)$ and $lz \underset{\mathrm{rat}}{\sim} 0$ on Y. Hence we may replace X by Y.

Third reduction: We may assume $|z|$ consists of smooth points of $C = \bigcup C_i$.

It suffices to write $z = z_0 + \cdots + z_n$ with $|z_i| \subset C_i$, and then move z_i by a rational equivalence on C_i so that $|z_i| \cap C_j = \emptyset$, all $i \neq j$.

Fourth reduction: We may assume z represents an l-torsion point in the (generalized) jacobian $J(C)$.

This is trickier. We proceed by induction on N = number of pairs of irreducible components C_i, C_j such that some point of $C_i \cap C_j$ is a zero or pole of f_i or f_j. If $N = 0$, then the class of lz lies in $\mathrm{Ker}(J(C) \to J(\tilde{C})) = K$. Since K is divisible and lies in $\mathrm{Ker}(J(C) \to CH_0(X))$ we can replace z by $z + k$ for some $k \in K$ and suppose $lz = 0$ in $J(C)$.

Suppose now $N > 0$. Write $C_\infty = \bigcup_{i=1}^{n} C_i$. Renumbering if necessary, we may assume some point of $C_0 \cap C_\infty$ is a zero or pole of f_0. Then adding irreducible curves in general position to C_0 and C_∞, we may assume C_0 linearly equivalent to C_∞ and very ample. (C_0 may no longer be irreducible.) We may further suppose that a general element of the corresponding pencil $\{C_t\}$ is smooth.

We next write

$$z = z_0 - z_\infty$$

with $|z_0| \subset C_0$, $|z_\infty| \subset C_\infty$ and degree z_0 = degree $z_\infty = 0$. This can be done in such a way that $lz_0 = \delta$ in $J(C_0)$, with $|\delta| \subset C_0 \cap C_\infty$. Let $\mathscr{C} \to \mathbf{P}^1$ be obtained by blowing up the base locus of the pencil $\{C_t\}$. δ gives rise to a divisor Δ on $\mathscr{C}$ with $\delta = \Delta \cdot C_0$.

Write Δ also for the corresponding section of the relative Picard scheme $\Delta : \mathbf{P}^1 \to \mathbf{Pic}(\mathscr{C}/\mathbf{P}^1)$. Multiplication by l induces a map $l_* : \mathbf{Pic} \to \mathbf{Pic}$ and we consider the scheme $l_*^{-1}(\Delta) \subset \mathbf{Pic}$. This is a principal homogeneous space under $\mathbf{Pic}_l = l_*^{-1}(0)$ and is non-empty. Let Z be the completion of the normalization of an irreducible component of $l_*^{-1}(\Delta)$ containing the point of $J(C_0) \subset \mathbf{Pic}(\mathscr{C}/\mathbf{P}^1)$ corresponding to z_0. It follows from Lemma 1 (below) that Z maps onto $\mathbf{P}^1$. Let $w_0, w_\infty \in Z$ be points of Z lying over 0 and ∞, with w_0 coinciding in **Pic** with the class of z_0.

With reference to the diagram

(1)
$$\begin{tikzcd} & X \\ Z\times\mathscr{C} \arrow[ur, "\pi"] \arrow[r, dash, "\pi'"] \arrow[d, "p"', "\mathbf{P}^1"] & \mathscr{C} \arrow[u, "\sigma"'] \arrow[d, "f"] \\ Z \arrow[r, "q"'] & \mathbf{P}^1 \end{tikzcd}$$

there exists a divisor D on $Z\times_{\mathbf{P}^1}\mathscr{C}$ such that

$$lD \sim \pi'^{*}\Delta + p^{*}(t)$$

for some divisor t on Z. If we fix a divisor w on Z with $lw \sim (w_0) - (w_\infty)$ we find

$$\begin{aligned}\pi_*(p^*((w_0)-(w_\infty))\cdot D) = \pi_*(p^*w\cdot\pi'^*\Delta) &= \sigma_*(\pi'_*p^*w\cdot\Delta)\\ &= \sigma_*(f^*q_*w\cdot\Delta)\sim 0\end{aligned}$$

because $q_*w = 0$.

Let $z'_\infty = \pi_*(p^*w_\infty\cdot D)$. We have $|z'_\infty|\subset C_\infty$, $z'_\infty \sim z_0$, and

$$lz'_\infty \sim \pi_*(p^*w_\infty\cdot\pi'^*\Delta)\sim\sigma_*(f^*(\infty)\cdot\Delta) = \delta.$$

In fact, $[lz'_\infty] = [\delta]$ in $J(C_\infty)$. In particular, there exist functions g_i on C_i, $i = 1,\dots,n$ such that the g_i have no zeros or poles on $C_i\cap C_j$ $(i, j = 1,\dots,n)$ and such that

$$\sum (g_i) = lz'_\infty - \delta.$$

Then taking $z' = z'_\infty - z_\infty$ we find $z'\sim z$ and

$$\partial(f_1g_1, f_2g_2,\dots,f_ng_n) = lz'_\infty - \delta - (lz_\infty - \delta) = lz'.$$

Since $N(z,(f_0,\dots,f_n)) > N(z',(f_1g_1,\dots,f_ng_n))$ the induction is complete.

For simplicity, I assume henceforth l prime to char k.

LEMMA 1: *With notation as above, all irreducible components of the scheme* $\mathrm{Pic}(\mathscr{C}/\mathbf{P}^1)_l$ *map onto non-empty open sets of* $\mathbf{P}^1$.

PROOF: The fibre of $\mathbf{Pic}_l$ over a point $0 \in \mathbb{P}^1$ is $H^1(C_0, \mathbb{Z}/l\mathbb{Z})$ (up to twisting). Since an étale cover of C_0 lifts to an étale cover of $\mathscr{C} \times_{\mathbb{P}^1} \mathrm{Sp}(\hat{\mathcal{O}}_{\mathbb{P}^1,0})$ we see that a closed point in the fibre of $\mathbf{Pic}_l$ over 0 necessarily spreads out to cover some open set in $\mathbb{P}^1$. Q.E.D.

Returning now to the proof that

$$\psi : CH_0(X)_{\mathrm{tors}} \longrightarrow \mathrm{Alb}(X)_{\mathrm{tors}}$$

is injective prime to the characteristic, we have a cycle z, an integer l prime to char k, and a very ample curve C such that $|z| \subset$ smooth points of C, $[lz] = 0$ in $J(C)$, and $\psi(z) = 0$. We must show $z \sim 0$ in $CH_0(X)$. For this we have

Fifth reduction: We may assume C smooth.

Indeed, we fix a general pencil $\mathscr{C}$ as before with $C = C_0$ and denote by Z the completion of the normalization of an irreducible component of Pic_l containing a point w_0 lying over $0 \in \mathbb{P}^1$ and mapping to $[z] \in J(C_0)$. We pick some general, smooth fibre $C_1 \subset \mathscr{C}$ and a point $w_1 \in Z$ lying over the same point $1 \in \mathbb{P}^1$. As before there is a divisor D on $Z \times_{\mathbb{P}^1} \mathscr{C}$ well-defined up to rational equivalence and vertical fibres such that (cf. diagram (1))

$$\pi_*(p^* w_0 \cdot D) \sim z$$

$$lD \sim p^*(t) \quad \text{some divisor } t \text{ on } Z.$$

Let $z' = \pi_*(p^* w_1 \cdot D)$, and let w be a divisor on Z such that $lw \sim (w_1) - (w_0)$. Then

$$z' \sim z + \pi_*(p^*(lw) \cdot D) \sim z$$

and $|z'| \subset C_1$. Also $[lz'] = 0$ in $J(C_1)$, completing the reduction.

And now (the moment you've all been waiting for) comes the point.

KEY LEMMA: *Let X be a smooth projective surface, $C \subset X$ a smooth hyperplane section. Let l be an integer prime to the characteristic. Then the two maps*

$$\begin{tikzcd} & \mathrm{Alb}(X)_l \\ J(C)_l \arrow[ur] \arrow[dr] & \\ & CH_0(X)_l \end{tikzcd}$$

have the same kernel.

Notice that this lemma will prove the theorem.

Proof of lemma: As before we take a general (lefschetz) pencil $\mathscr{C}$ with $C = \mathscr{C}_0$. We have a morphism of schemes

$$\operatorname{Pic}(\mathscr{C}/\mathbb{P}^1) \xrightarrow{h} \operatorname{Alb}(X) \underset{\operatorname{Sp} k}{\times} \mathbb{P}^1.$$

Let $V = \operatorname{Ker} h$, $V_{l^\nu} \subset V$ the l^ν-torsion. There are diagrams

$$\begin{tikzcd}
V'_{l^\nu} \underset{\text{def.}}{=} \operatorname{Ker}(H^1_{\acute{e}t}(C, Z/l^\nu) \arrow[r] \arrow[d, hook, "``l^\nu\text{''}"'] & H^3_{\acute{e}t}(X, Z/l^\nu)) \arrow[r, hook] & V_{l^\nu,0} \arrow[d, hook] \\
V'_{l^{\nu+\mu},0} = \operatorname{Ker}(H^1_{\acute{e}t}(C, Z/l^{\nu+\mu}) \arrow[r] & H^3_{\acute{e}t}(X, Z/l^{\nu+\mu}) \arrow[r, hook] & V_{l^{\nu+\mu},0}
\end{tikzcd}$$

which induce an isomorphism in the limit

$$\operatorname{Ker}(H^1_{\acute{e}t}(C, Q_l/Z_l) \longrightarrow H^3_{\acute{e}t}(X, Q_l/Z_l)) \xrightarrow{\sim} V_{l\text{-tors},0}.$$

Thus, given $\gamma \in J(C)_l \cong H^1_{\acute{e}t}(C, Z/lZ)$ with $\gamma \mapsto 0$ in $\operatorname{Alb}(X)$, $\gamma' = "l^\nu"\gamma \in V'_{l^\nu,0}$ for $\nu \gg 0$, i.e., γ' is a *vanishing cycle*. I claim V_{l^ν} contains an irreducible component Z such that any $\gamma' \in V'_{l^\nu,0}$ can be written

$$\gamma' = \sum n_i \gamma_i$$

with $\gamma_i \in Z_0$ and $\Sigma\, n_i = 0$. Assume this claim for a moment. Given $\gamma \in V_{l,0}$, write $\gamma' = "l^\nu"\gamma = \Sigma\, n_i\gamma_i$. Just as before, there is a divisor D on $Z \times_{\mathbb{P}^1} \mathscr{C}$ and (notation as in diagram 1)

$$l^\nu D \sim p^*(t)$$

$$\gamma' \sim \pi_*\Big(p^* \sum n_i(\gamma_i) \cdot D\Big) \text{ in } CH_0(X).$$

Since V_{l^ν} is étale over 0 the γ_i are smooth points of Z, so $\Sigma\, n_i(\gamma_i) \in J(Z)$, a divisible group. It follows that $\gamma' \sim 0$ in $CH_0(X)$ whence $\gamma \sim 0$ in $CH_0(X)$ as desired. (Note $\gamma' = \gamma$ viewed as a point of order l^ν.)

Proof of claim: The group $V'_{l^\nu,0}$ of vanishing cycles is known to be generated by certain cycles $\delta_i \pmod{l^\nu}$ (*the* vanishing cycles) which are all conjugate under the monodromy group of the pencil.

Let Z be the component of V_{l^ν} containing one (and hence all) the δ_i. Associated with each δ_i is a $\sigma_i \in \pi_1(\mathbb{P}^1 - \Sigma, 0)$ where Σ is the finite set of points where the fibres of f are singular. The σ_i generate π_1, and for $\chi \in V'_{l^\nu,0}$, the Picard-Lefschetz formula says

$$\sigma_i(\chi) = \chi \pm \langle \chi \cdot \delta_i \rangle \delta_i$$

where $\langle \ \rangle$ denotes the intersection pairing. Since the δ_i generate $V'_{l^\nu,0}$ we can write

$$\gamma' = \sum m_i \delta_i.$$

The trick is to get $\Sigma\, m_i \equiv 0 \pmod{l^\nu}$. Suppose $\Sigma\, m_i = m$ and that $m_1 | m_i$, all i. Let q be such that $\langle \delta_1 \cdot \delta_1 \rangle$ is invertible in $\mathbb{Z}/l^\nu\mathbb{Z}$. Such a q must exist, otherwise by Picard-Lefschetz, writing $V' = \mathrm{Ker}(H^1(C, \mathbb{Z}_l) \to H^3(X, \mathbb{Z}_l))$, we would have $\sigma_q(\delta_1) - \delta_1 \in lV'$. Since the $\sigma_q(\delta_1)$ generate V' and rank V' is even, this is not possible. Let r be such that

$$\sigma_q^r(\delta_1) = \delta_1 \pm r < \delta_1 \cdot \delta_q \rangle \delta_q = \delta_1 + \frac{m}{m_1}\, \delta_q.$$

Then $\sigma_q^r(\delta_1) \in Z_0$, and

$$\gamma' = m_1 \sigma_q^r(\delta_1) + \sum_{i>1} m_i \delta_i - m\delta_q.$$

This verifies the claim and completes the proof of the theorem.

5. Relations with algebraic K-theory

In this final section I want to reconsider the map ρ in (2.3) from the point of view of algebraic K-theory. K_i will denote the Zariski sheaf on X associated to the i-th Quillen K-group, [6]. We will take $n = \dim X$. $K_i(X)$ will denote the i-th global K-group of (the category of vector bundles on) the variety, and $SK_1(X) = \mathrm{Ker}(K_1(X) \to k^* = \Gamma(X, O_X^*))$. Finally $T(l) \subset H^{2n-1}_{ét}(X, \mathbb{Z}_l(n))$ will denote the torsion subgroup (Pontryagin dual to the torsion subgroup of the Neron-Severi group of X).

THEOREM 5.1: *There is a surjective map* $H^{n-1}(X, K_n) \twoheadrightarrow T(l)$. *When* $\dim X = 2$, *we obtain* $SK_1(K) \twoheadrightarrow T(l)$.

PROOF: A construction of Tate gives a map

$$K_2(F) \longrightarrow H^2(F, \mu_l^{\otimes 2}) \quad l \text{ prime to char } F$$

for any field F. We obtain a commutative diagram

$$\begin{tikzcd}
\coprod_{x\in X^{n-2}} K_2(k(x)) \arrow[r] \arrow[d] & \coprod_{x\in X^{n-1}} k(x)^* \arrow[r] \arrow[d] & \coprod_{x\in X^n} \mathbb{Z} \arrow[d] \\
\coprod_{x\in X^{n-2}} H^2(k(x), \mu_{l^\nu}^{\otimes 2}) \arrow[r] & \coprod_{x\in X^{n-1}} k(x)^*/k(x)^{*l^\nu} \arrow[r] & \coprod_{x\in X^n} \mathbb{Z}/l^\nu\mathbb{Z}
\end{tikzcd}$$

where the top line comes via a spectral sequence

$$\coprod_{x\in X^p} K_{q-p}(k(x)) \Rightarrow K_{q-p}(X)$$

due to Quillen. Quillen also shows that the E_1 complexes compute the cohomology of the sheaves K_p just as in (1.3). We deduce from the above diagram (cf. (2.3)) a map

$$\bar{\rho}: H^{n-1}(X, K_n) \longrightarrow T(l).$$

For $\nu \gg 0$ we have $T(l) \subset H^{2n-1}(X, \mathbb{Z}(n))/l^\nu \subset H^{2n-1}(X, \mathbb{Z}/l^\nu\mathbb{Z}(n))$ and a diagram

$$\begin{tikzcd}
0 \arrow[r] & H^{n-1}(X, K_n)/l^\nu \arrow[r] \arrow[d, "\bar{\rho}"] & H^{n-1}(X, K_n/l^\nu K_n) \arrow[r, tail] \arrow[d, "\alpha"] & CH^n(X)_{l^\nu} \arrow[r] \arrow[d, "\lambda"] & 0 \\
0 \arrow[r] & T(l) \arrow[r, tail] & H^{2n-1}(X, \mathbb{Z}/l^\nu\mathbb{Z}(n)) \arrow[r] \arrow[d] & H^{2n-1}(X, \mathbb{Z}/l^\nu\mathbb{Z}(n))/T(l) \arrow[r] & 0 \\
 & & 0 & &
\end{tikzcd}$$

where α comes as in (2.2) and is surjective because for $n = \dim X$, $H^{n-1}(X, H^n) \simeq H_{\acute{e}t}^{2n-1}(X)$, and λ is an isomorphism by §4. We conclude that $\bar{\rho}$ is surjective. When $n = 2$, an examination of (5.2) shows $SK_1(X) \twoheadrightarrow H^1(X, K_2)$, so we obtain $SK_1(X) \twoheadrightarrow T(l)$. Q.E.D.

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