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ON THE GELFOND–FELDMAN MEASURE OF ALGEBRAIC INDEPENDENCE

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In 1950, A.O. Gelfond and N.I. Feldman established their well-known measure of algebraic independence of α^β and α^{β^2}, where $\alpha \neq 0$ is algebraic, $\log \alpha \neq 0$ and β is a cubic irrationality [8]:

For $\epsilon > 0$, there is a $t(\epsilon) > 0$ such that for every non-zero polynomial $P(x, y) \in \mathbf{Z}[x, y]$ with $t = d_P + h_P > t(\epsilon)$, we have

$$\log |P(\alpha^\beta, \alpha^{\beta^2})| > -\exp(t^{4+\epsilon}).$$

The numbers α^β and α^{β^2} had been proven algebraically independent by Gelfond [7]. The total degree of a polynomial P will be abbreviated as d_P or $\deg P$ and the logarithm of its height as h_P or $\log ht\, P$.

In Theorem 1 of this paper we improve the lower bound $-\exp(t^{4+\epsilon})$ to $-\exp(Cd_P^3(d_P + h_P))$. We obtain this result from the proof of Theorem 2. But in contrast with the theorems of Gelfond and Feldman and even the recent generalization of Gelfond's result on $\alpha^\beta, \alpha^{\beta^2}$ by M. Waldschmidt and the author [4], where α is taken to be well-approximated by algebraic numbers, the arithmetic nature of α by itself is not specified in Theorem 2. However, as in [4], [2], both lower and upper bounds on expressions in these numbers are used, even when no transcendence measure for any of them is known. This basic idea comes from G.V. Choodnovsky's proof of the algebraic independence of three numbers [5] based on Gelfond's method. The result of [2] applied to more instances than did the earlier lower bound of Gelfond and Feldman, but was weaker than it in the case they considered.

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Since then G.V. Choodnovsky has established the algebraic independence of $n+1$ algebraically independent numbers by a remarkable iterative application of a variant of Gelfond's method. There it was necessary for him to develop a useful extension of the concept of a resultant to polynomials having a common factor (see Lemma 5 in Section 3 below). We employ these so-called semi-resultants to prove a theorem on a sort of simultaneous approximation problem which will generalize and sharpen somewhat the result of Gelfond and Feldman. (Actually the sharpening is obtainable from the original structure of proof as well.) Choodnovsky himself has announced a strengthening of the Gelfond–Feldman result to a lower bound of $-\exp(C(h_P+d_P)^3)$, but the details are yet to appear. We will obtain our general result by making use of the following corollary of a theorem of M. Mignotte and M. Waldschmidt [10]:

Let $a \in \mathbf{C}$, $a \neq 0$, $\log a \neq 0$ with b algebraic but irrational. There are absolute constants $C_0 > 0$ and $t_0 > 0$ such that for non-zero $P(x), Q(x) \in \mathbf{Z}[x]$ with

$$\deg P + \deg Q + \log ht\, P + \log ht\, Q = t \geq t_0,$$

we have

$$\log \max\{|P(a)|, |Q(a^b)|\} > -t^{C_0}.$$

Earlier work in this direction would have sufficed, except that it applied only when $\deg P + \deg Q$ was below some fixed bound.

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1. Statement of Results

The first result applies to the situation of Gelfond and Feldman:

THEOREM 1. *Let $\alpha \in \mathbf{C}$ be algebraic, $\alpha \neq 0$, $\log \alpha \neq 0$ and $\beta \in \mathbf{C}$ be cubic irrational. Then there is a positive, effectively computable constant C such that for every non-zero polynomial $Q(x, y) \in \mathbf{Z}[x, y]$,*

$$\log |Q(\alpha^{\beta}, \alpha^{\beta^2})| > -e^{Cd_Q^2(d_Q+h_Q)\delta},$$

where $\delta = \min\{\deg_x^* Q, \deg_y^* Q\}$.

We use an asertisk to denote the maximum of 1 and the function value. Theorem 1 follows from the proof of the more general result:

THEOREM 2. *Let $\alpha \in \mathbf{C}$, $\alpha \neq 0$, $\log \alpha \neq 0$. Let $\beta \in \mathbf{C}$ be a cubic irrational. Then there is a positive, effectively computable constant D such that for all relatively prime polynomials $P(x, y)$, $Q(x, y, z) \in \mathbf{Z}[x, y, z]$, we have that*

$$\log \max\{|P(\alpha, \alpha^{\beta})|, |Q(\alpha, \alpha^{\beta}, \alpha^{\beta^2})|\} > -r^D,$$

where

$$\log r = d_P^3 d_Q^2 (d_P h_Q + d_Q h_P + d_P d_Q) \deg_z^* Q.$$

Theorem 2 actually holds in a somewhat stronger form involving the partial degrees of P and Q.

Note 1. Any two of the three numbers α, α^{β}, α^{β^2} could appear in the left hand polynomial in Theorem 2 as long as the corresponding polynomials P, Q remain relatively prime.

Note 2. If $Q(\alpha, \alpha^{\beta}, \alpha^{\beta^2}) = 0$, then we have as good a measure of algebraic independence for any two of these numbers which are independent as Gelfond and Feldman did for α^{β} and α^{β^2} when α is algebraic. Thus we can conclude that when α is not algebraic, but is very well approximated by algebraic numbers, then the three numbers α, α^{β}, α^{β^2} are algebraically independent. An example of such an α is

$$\alpha = \sum_{n=4}^{\infty} (-1)^n 2^{-3^{\cdot^{\cdot^{\cdot^{n}}}}},$$

where ... stands for the intermediate consecutive integers. For when α is irrational, but $Q(\alpha, \alpha^{\beta}, \alpha^{\beta^2}) = 0$, then there is an effectively computable C such that for rational approximations p/q to α, we have

$$\log|\alpha - p/q| > -|q|^C.$$

Note 3. If only two of the numbers α, α^{β} and α^{β^2} are involved in P and Q, we can eliminate any common occurrence and then use the result of Mignotte and Waldschmidt to produce our conclusion. If only one of the numbers is involved, forming the resultant gives the conclusion immediately.

Note 4. It is possible to deduce a non-trivial lower bound for arbitrary pairs of relatively prime polynomials $P(x, y, z)$, $Q(x, y, z)$ in $\mathbf{Z}[x, y, z]$: If z occurs in P and Q, then form R, the resultant eliminating z between the smallest prime factors P_1, Q_1 of P and Q, respectively. Apply Theorem 2 to R and Q_1, using Lemma 5 below. (R and Q_1 are relatively prime by the standard representation of R as a polynomial linear combination of P_1 and Q_1.)

Finally, consider the analytic curve

$$t \mapsto (\alpha^t, \alpha^{\beta t}, \alpha^{\beta^2 t}),$$

where α^z denotes as usual $\exp(z \log \alpha)$, for some fixed $\log \alpha$. It follows from Theorem 2 that the coordinates exhibit no surprising algebraic degeneracy. Since any one of the coordinates may even be an algebraic number γ (e.g. when $t = \log \gamma / \log \alpha$), we see that at times the analytic curve intersects algebraic surfaces over $\bar{\mathbf{Q}}$, the algebraic closure of $\bar{\mathbf{Q}}$. But by Remark 4, it does not even come close (in an appropriate sense) to algebraic curves over $\bar{\mathbf{Q}}$, except of course at $t = 0$.

What we actually will prove is the following more exact result, where C_0 still denotes the constant in the result of Mignotte and Waldschmidt:

THEOREM 3. *Let $\alpha \in \mathbf{C}$, $\alpha \neq 0$, $\log \alpha \neq 0$. Let $\beta \in \mathbf{C}$ be a cubic irrational. Then for a given $C_1 > C_0$, there is an effectively computable constant C, depending only on C_1, β and $\log \alpha$, such that if there are two relatively prime polynomials $P(x, y)$, $Q(x, y, z) \in \mathbf{Z}[x, y, z]$ satisfying*

$$\log \max\{|P(\alpha, \alpha^\beta)|, |Q(\alpha, \alpha^\beta, \alpha^{\beta^2})|\} < -r^C$$

where

$$\log r = A^2 B(\deg_y^* P)(\deg_z^* Q),$$

$$A = (\deg_y^* P)(\deg_x Q + \deg_z^* Q) + (\deg_x^* P)(\deg_y Q + \deg_z^* Q),$$

$$\begin{aligned} B = {} & (\deg_z^* Q) \deg P + (\deg_y^* P) \deg Q + (\deg_x P)(\deg_y^* Q) \\ & + (\deg_y Q) \log(1 + \deg P) \\ & + (\deg_y^* P) \log htQ + (\deg_y Q + \deg_z^* Q) \log htP, \end{aligned}$$

then there is a polynomial $U(x) \in \mathbf{Z}[x]$ with $U(\alpha) \neq 0$ and

$$\deg U + \log htU < r^{C/(12C_1)},$$

(2)

$$\log |U(\alpha)| < -r^{C/4}.$$

The proof of the corresponding inequality for α^β is obtained from the following proof merely by interchanging α and α^β everywhere below. For large enough C, in terms of C_0, the inequalities (2) for α and α^{β} are contradicted by the result of Mignotte and Waldschmidt, thus establishing Theorem 2. Theorem 1 follows on letting P be the minimal polynomial for α over $\mathbf{Z}$ and, if $\deg_y Q < \deg_z Q$, interchanging α^{β} and α^{β^2} in the argument.

2. Proof of Theorem 3

We assume that Theorem 3 is false for large enough C.

Step 0. Preliminary Remarks. Without loss of generality we may assume that both P and Q are irreducible over $\mathbf{Z}$ (see Lemma 2). By our denial of (2), we have $\deg_y P > 0$ and a non-trivial lower bound on the leading coefficient $p(\alpha)$ of $P(\alpha, y)$:

$$\log |p(\alpha)| \geq -r^{C/4}.$$

Since there are at most $\deg_y P$ distinct roots, we are forced to admit that a root ξ_1 of $P(\alpha, y) = 0$ nearest α^{β} satisfies

$$\log |\xi_1 - \alpha^{\beta}| < -r^{7C/8}. \tag{3}$$

Moreover since $\xi_1 - \alpha^{\beta}$ formally divides $Q(\alpha, \xi_1, z) - Q(\alpha, \alpha^{\beta}, z)$, we see easily that

$$\log |Q(\alpha, \xi_1, \alpha^{\beta^2})| < -r^{7C/8}/2.$$

The polynomial $Q(\alpha, \xi_1, z)$ is not identically zero since P and Q are relatively prime. If z did not occur after the substitution of ξ_1 for α^{β}, then we would simply form the resultant (see Lemma 5 below) of $P(\alpha, y)$ and $Q(\alpha, y, z)$ with respect to y to fulfill (2). Similarly we even obtain a lower bound on the leading coefficient $q(\alpha, \xi_1)$ of z in $Q(\alpha, \xi_1, z)$. Arguing as before for P, we see that a root ξ_2 of $Q(\alpha, \xi_1, z) = 0$ nearest to α^{β^2} satisfies

$$\log |\xi_2 - \alpha^{\beta^2}| < -r^{3C/4}. \tag{4}$$

In the proof, the Gothic lower case letters $\mathfrak{l}, \mathfrak{n}$ will be used to denote triples of integers given by the corresponding Greek letters, and absolute value signs will denote the sup norm. E.g. $\mathfrak{l} = (\lambda_0, \lambda_1, \lambda_2) \in \mathbf{Z}^{(3)}$ and $|\mathfrak{l}| = \max\limits_i |\lambda_i|$. The coordinates will be non-negative except in Lemma 5. In addition we set $\mathfrak{b}=(1,\beta,\beta^2)$, $\mathfrak{l}\cdot\mathfrak{b}=\lambda_0+\lambda_1\beta+\lambda_2\beta^2$ and similarly for $\mathfrak{n}\cdot\mathfrak{b}$. We also let $B_{\mathfrak{l}}=(\mathfrak{l}\cdot\mathfrak{b})b^2\log\alpha$, where b is the positive leading coefficient of the minimal polynomial for β over $\mathbf{Z}$. The letters $c_1, c_2, c_3, \ldots$ will denote positive constants depending only on β and $\log\alpha$. For simplicity of notation in the proof, we will assume that all degrees in the proof are at least 1 and all heights at least 2.

For $u\in\mathbf{Z}[\alpha,\xi_1,\xi_2]$, we mean by

$$\deg_{\mathbf{Z}} u\leq d,\quad ht_{\mathbf{Z}} u\leq H$$

that there is a non-zero polynomial $p(x,y,z)\in\mathbf{Z}[x,y,z]$ with

$$\deg p\leq d,\quad ht\, p\leq H,$$
$$u=p(\alpha,\xi_1,\xi_2).$$

The partial degrees of p will be referred to as $\deg_\alpha u$ and $\deg_{\xi_i} u$. The fact that u might be represented by several such p will present no problem.

Let

$$N_0=[r^{C/(24C_1)}],\quad N_1=[r^{C/6}]$$

It is easy to verify that when C is large enough, then

$$r^{C/3}<N_1^3\log N_1<r^{5C/8},$$
$$N_0^{3/2}\log N_0<r^{C/12C_1}.$$

For $N_0\leq N\leq N_1$, define

$$L_N=[(C/C_1)^{1/4}(AN(\deg_z Q)(\deg_y P))^{1/2}],$$
$$D_N=[N^{3/2}(C/C_1)^{-3/4}(A(\deg_z Q)(\deg_y P))^{-3/2}].$$

We note for use later in Steps 4 and 5 that
(i) since, from the definition of N_0 and of L_N,

$$\log N\geq\log N_0\geq(C/24C_1)A^2B(\deg_y P)\deg_z Q$$
$$>C^{3/4}A^2B(\deg_y P)\deg_z Q\geq C^{1/4}ABL_N^2/N,$$

we know that

$$C^{1/4}ABN^2L_N^2<N^3\log N,\text{ and}$$

(ii) since

$$C^{1/4}AN(\deg_z Q)\deg_y P < L_N^2,$$

we know that

$$C^{1/4}AN(\deg_z Q)\deg_y P < L_N^2,$$

we know that

$$C^{1/4}ANL_N(\deg_z Q)(\deg_y P)D_N \log N < N^3 \log N.$$

Step 1. We show that there exist $\Phi_{\mathfrak{n}}(x, y, z) \in \mathbb{Z}[x, y, z]$, $|\mathfrak{n}| < N$, without a common nonconstant factor such that

$$\deg \Phi_{\mathfrak{n}} \leq c_1 NL_N, \quad \log ht\, \Phi_{\mathfrak{n}} \leq c_2 D_N \log N$$

and such that when we set $\varphi(\mathfrak{n}) = \Phi_{\mathfrak{n}}(\alpha, \alpha^\beta, \alpha^{\beta 2})$, the function

$$F_N(z) = \sum_{|\mathfrak{n}|<N} \varphi(\mathfrak{n}) \exp((\mathfrak{n} \cdot \mathfrak{b})z)$$

satisfies

$$F_N^{(p)}(B_{\mathfrak{l}}) = 0$$

for all $0 \leq p < D_N$ and $|\mathfrak{l}| < L_N/4$. For after multiplying all these expressions by $b^{2D_N}(\alpha^{1+\beta+\beta 2})^{c_3 NL_N}$, we obtain a linear homogeneous system of at most $3N^3/64$ equations in N^3 unknowns having coefficients polynomials over $\mathbf{Z}$ in α, α^β, $\alpha^{\beta 2}$. We apply Lemma 1 below to solve them *formally*, i.e. with x, y, z substituted for $\alpha, \alpha^\beta, \alpha^{\beta 2}$. We obtain the desired $\Phi_{\mathfrak{n}}(x, y, z)$ on dividing the formal solutions in $\mathbf{Z}[x, y, z]$ by their greatest common divisor, noting that $NL_N \leq 24 D_N \log N$, and keeping in mind Gelfond's result on the height of a factor (Lemma 2 below).

Step 2. An application of the Cauchy integral formula on the circle, say, $|z| = N^{3/2}$, which is standard in transcendence proofs (see, e.g. [7, pp. 157–158]) shows that for all $0 \leq p < D_N$ and $|\mathfrak{l}| < L_N \log N$,

$$\begin{aligned}\log |F_N^{(p)}(B_{\mathfrak{l}})| &\leq c_4(N^2 + D_N \log N) - N^3(\log (N/2))/64 \\ &< -N^3 \log N/65.\end{aligned}$$

Step 3. A result of R. Tijdeman's (Lemma 4 below) shows that either

Case (i) there exist $0 \leq p_0 < D_N$ and $|\mathfrak{l}| < c_5 L_N$ with

$$\log |F_N^{(p_0)}(B_{\mathfrak{l}})| > -c_6 N^3 \log N \tag{5}$$

or else

Case (ii) for all $|\mathfrak{n}| < N$,

$$\log|\varphi(\mathfrak{n})| < -N^3 \log N. \tag{6}$$

We now consider the two cases separately.

Case (i). When we substitute ξ_1 for α^β and ξ_2 for α^{β^2} in

$$b^{2p_0}(q(\alpha, \alpha^\beta))^{c_7NL_N} F_N^{(p_0)}(B_{\mathfrak{l}}), \tag{7}$$

we obtain an element of $\mathbf{Z}[\alpha, b\beta, \xi_1, \xi_2]$ which is non-zero by inequalities (3), (4), (5). On taking the relative norm to $\mathbf{Z}[\alpha, \xi_1, \xi_2]$ we get a non-zero q there with

$$\deg_Z q \le c_9 NL_N, \quad \log ht_Z q \le c_{10} D_N \log N$$

and

$$\log|q(\alpha, \xi_1, \xi_2)| < -N^3 \log N/66.$$

Either the leading coefficient $w(\alpha, \xi_1)$ of $q(\alpha, \xi_1, z)$ or else $q(\alpha, \xi_1, \xi_2)/w(\alpha, \xi_1)$ has absolute value at most

$$e^{-N^3(\log N)/132}.$$

When $w(\alpha, \xi_1)$ is larger, we see from Lemma 5 and the fact that $Q(\alpha, \xi_1, \xi_2) = 0$ that the semi-resultant $r(\alpha, \xi_1) \in \mathbf{Z}[\alpha, \xi_1]$ of $q(\alpha, \xi_1, z)$ and $Q(\alpha, \xi_1, z)$ satisfies the inequalities

$$\begin{aligned} \deg_\alpha r &\le c_{11} NL_N(\deg_x Q + \deg_z Q), \\ \deg_{\xi_1} r &\le c_{11} NL_N(\deg_y Q + \deg_z Q), \\ \log ht_Z r &\le c_{12}((\deg_z Q) D_N \log N \\ &\qquad + NL_N(\log ht\, Q + \deg_z Q + \log(1 + \deg Q))), \\ \log|r(\alpha, \xi_1)| &< -N^3 \log N/389, \end{aligned} \tag{8}$$

which are also satisfied by $w(\alpha, \xi_1)$ when it is smaller.

We finally take the resultant of $r(\alpha, y)$ or $w(\alpha, y)$ and $P(\alpha, y)$ to obtain a non-zero element $s(\alpha) \in \mathbf{Z}[\alpha]$ with

$$\deg_\alpha s \le c_{13} ANL_N,$$

$$
\begin{aligned}
\log ht_{\mathbf{Z}} s \le\ & c_{14}(NL_N(\deg_z Q(\deg P + \log ht\, P) + \deg_y P(\deg Q + \log ht\, Q) \\
(9) \qquad & + \deg_y Q(\deg_x P + \log ht\, P + \log(1+\deg P))) + \\
& + D_N(\log N)\deg_z Q(\deg_y P)) \\
\le\ & c_{15}(NL_N B + D_N \log N(\deg_z Q)\deg_y P),
\end{aligned}
$$

$$\log|s(\alpha)| < -N^3 \log N/390.$$

Case (ii). By our denial of (2), we can assume α to be transcendental. Since the $\Phi_{\mathfrak{n}}(x, y, z)$ are without a common factor, not all the $\Phi_{\mathfrak{n}}(\alpha, \xi_1, z)$ are zero. It is conceivable that the leading coefficient of some non-zero $\Phi_{\mathfrak{n}}(\alpha, \xi_1, z)$ has absolute value at most $\exp(-N^3 \log N/389)$ and thus itself satisfies (8).

When the leading coefficients are at least $\exp(-N^3 \log N/389)$, we will be interested in the monic polynomials $\Psi_{\mathfrak{n}}(z)$ obtained from the $\Phi_{\mathfrak{n}}(\alpha, \xi_1, z)$ on dividing by the leading coefficient. In fact, we apply Lemma 5 to alter the definition of $F_N(z)$ somewhat. For this, let $Q_1(z)$ be the minimal monic polynomial for ξ_2 over $\mathbf{Q}(\alpha, \xi_1)$. Let σ_1 be the maximal non-negative integer such that for all $\mathfrak{n}$ we can write

$$\Phi_{\mathfrak{n}}(\alpha, \xi_1, z) = (Q_1(z))^{\sigma_1} R_{\mathfrak{n}}(z),$$

for some polynomial $R_{\mathfrak{n}}(z)$, whose coefficients naturally lie in $\mathbf{Q}(\alpha, \xi_1)$.

If $\sigma_1 = 0$, then we obtain a non-zero $r(\alpha, \xi_1) \in \mathbf{Z}[\alpha, \xi_1]$, satisfying (8), from Lemma 5 by forming the semi-resultant of $Q(\alpha, \xi_1, z)$ and some $\Phi_{\mathfrak{n}}(\alpha, \xi_1, z)$ not divisible by $Q_1(z)$. To check the bound on the absolute value, evaluate $Q_1(z)$ and $\Psi_{\mathfrak{n}}(z)$ at $z = \xi_2$.

If $\sigma_1 > 0$, we define a new function $G_N(z)$ by replacing $\varphi(\mathfrak{n})$ in the definition of $F_N(z)$ with $R_{\mathfrak{n}}(\alpha^{\beta^2})$. Then for $0 \le p < D_N$ and $|\mathfrak{l}| < L_N \log N$ the expressions

$$b^{2D_N}(\alpha^{1+\beta+\beta^2})^{c_3 NL_N} G_N^{(p)}(B_{\mathfrak{l}}) \tag{10}$$

are polynomials, say $S_{p,N} = \Sigma\, a_j \alpha^{\beta j}$, of degree at most $c_1 NL_N$ with coefficients from $\mathbf{Q}(\beta, \alpha, \xi_1, \alpha^{\beta^2})$ having height at most $\exp(c_{16} D_N \log N)$.

Recall that we solved for $\Phi(x, y, z)$ formally in Step 1, on replacing α by x, α^β by y and α^{β^2} by z in the equation for $F_N(z)$ corresponding to (10). Therefore when we resubstitute ξ_1 for α^β and z for α^{β^2} in the original equations, the expressions remain zero even after division by $Q_1(z)^{\sigma_1}$ and substitution anew of α^{β^2} for z. Thus $S_{p,N}(\xi_1) = 0$, and consequently

$$
\begin{aligned}
|S_{p,N}(\alpha^\beta)| &\le \sum |a_j|\, |\alpha^{\beta j} - \xi_1^j| \\
&\le N^{c_{17} D_N} |\alpha^\beta - \xi_1| \le e^{-r^{7C/8}}/3.
\end{aligned}
$$

We now apply the Hermite interpolation formula to find that for $N^{3/2} \le |z| < N^2$,

$$G_N(z) = \frac{1}{2\pi i} \int\limits_{|\rho|=N^2} \frac{G_N(\rho)}{\rho - z} \prod_{\mathfrak{l}} \left(\frac{z - B_{\mathfrak{l}}}{\rho - B_{\mathfrak{l}}}\right)^{D_N} d\rho$$

$$- \frac{1}{2\pi i} \sum_{\mathfrak{l}} \sum_{p=0}^{D_N - 1} \frac{G_N^{(p)}(B_{\mathfrak{l}})}{p!} \int\limits_{|\rho - B_{\mathfrak{l}}| = m/2} \left(\frac{z - B_{\mathfrak{l}}}{\rho - z}\right)^p \prod_{\mathfrak{l}' \neq \mathfrak{l}} \left(\frac{z - B_{\mathfrak{l}}}{\rho - B_{\mathfrak{l}'}}\right)^{D_N} d\rho,$$

where $m = \min\limits_{\mathfrak{l} \neq \mathfrak{l}'} |B_{\mathfrak{l}} - B_{\mathfrak{l}'}|$; $|\mathfrak{l}|, |\mathfrak{l}'| < L_N$. Using standard estimates (see [4, p. 69] for details), we see that

$$\begin{aligned} \log |G_N|_{N^{3/2}} &\le \max \{-D_N L_N^3 (\log N)/3, -r^{3C/4}/4\} \\ &< -N^3 \log N/193. \end{aligned}$$

Applying the Cauchy integral formula on $|z| = N^{3/2}$ shows that for all $0 \le p < D_N$ and $|\mathfrak{l}| \le N \log N$,

$$\log |G_N^{(p)}(B_{\mathfrak{l}})| \le -N^3 \log N/194.$$

Now from the definition of σ_1, we find that some $R_{\mathfrak{n}}(\alpha^{\beta^2})$ is non-zero. However there are still two possibilities for G_N, as in Step 3 for F_N.

If G_N falls into Case (i), then for the relevant $p_0, \mathfrak{l}_0$, we can substitute ξ_1 for α^β and z for α^{β^2} in (7) and

$$(11) \qquad b^{2p_0} q(\alpha, \alpha^\beta)^{c_7 N L_N} p(\alpha)^{c_8 N L_N} G_N^{(p_0)}(B_{\mathfrak{l}})$$

and form the semi-resultant of the polynomial $q(\alpha, \xi_1, z)$ obtained from (7) and $Q(\alpha, \xi_1, z)$ to obtain a non-zero $r(\alpha, \xi_1) \in \mathbf{Z}[\alpha, \xi_1]$. If the leading coefficient w of $q(\alpha, \xi_1, z)$ has absolute value at least

$$e^{-N^3}(\log N)/388,$$

we consider $Q_1(z)$ and the monic polynomial $q(\alpha, \xi_1, z)/w$ evaluated at $z = \xi_2$ to see that $r(\alpha, \xi_1)$ satisfies (8). If, on the other hand, $|w|$ is less than the bound cited, it is less than the bound in (8). Recall that (11) was obtained formally from (7) on replacing some of the α^β (those coming from $\varphi(\mathfrak{n})$) with ξ_1 and α^{β^2} with z, dividing by $Q_1(z)^{\sigma_1}$, and then replacing the remaining α^β with ξ_1 and z with α^{β^2}. Actually the first step may as well have been to replace all α^β with ξ_1. Now $Q_1(z)$ is monic. So w is obtained from some coefficient of α^{β^2} in (7) by replacing α^β with ξ_1, and hence $w \in \mathbf{Z}[\alpha, \xi_1]$, satisfying the remaining inequalities in (8). At any rate, we have seen that if G_N falls into Case (i), we can find an $r(\alpha, \xi_1)$ satisfying the inequalities (8).

If G_N falls into Case (ii), then we simply form the semiresultant of $Q(\alpha, \xi_1, z)$ and of a $\Phi_{\mathrm{n}}(\alpha, \xi_1, z)$ for which σ_1 is the exact power of divisibility by $Q_1(z)$. We then use $Q_1(z)$ and (the monic polynomial associated with) the corresponding $R_{\mathrm{n}}(z)$ evaluated at α^{β^2} to show that the semi-resultant satisfies (8), from our previous agreement on the lower bound for the absolute value of the leading coefficient of $\Phi_{\mathrm{n}}(\alpha, \xi_1, z)$ and thus of $R_{\mathrm{n}}(z)$.

Thus we have shown that even in Case (ii) we can obtain a non-zero $r(\alpha, \xi_1)$ satisfying (8). Taking the resultant with $P(\alpha, y)$ gives us an $s(\alpha) \in \mathbf{Z}[\alpha]$ satisfying (9) just as in Case (i).

Step 4. There is a factor $t_N(\alpha) = u(\alpha)^{v_N}$ of $s(\alpha)$ with $u(\alpha) \in \mathbf{Z}[\alpha]$ irreducible over $\mathbf{Z}$, $v_N \geq 1$, satisfying

$$\begin{gathered}\deg t_N \leq c_{13} A N L_N,\\ \log ht\, t_N \leq c_{18}(BNL_N + D_N \log N(\deg_y P)\deg_z Q),\\ \log |t_N(\alpha)| < -N^3 \log N/391.\end{gathered}$$

To verify this, refer to Lemma 3 below and the inequalities following the definition of L_N and D_N. By the same token we can apply Lemma 5 to t_N and t_{N+1}, $N_0 \leq N < N_1$ to see that the underlying irreducible polynomial $u(\alpha)$ is the same for N and $N+1$ and hence for all N, $N_0 \leq N \leq N_1$.

Step 5. Thus $t_{N_1}(\alpha) = u(\alpha)^{v_{N_1}}$, and

$$\begin{aligned}-N_1^3 \log N_1/391 > \log |t_{N_1}(\alpha)| &= v_{N_1} \log |u(\alpha)|\\ &\geq c_{13} A N_1 L_{N_1} \log |u(\alpha)|.\end{aligned}$$

Now since $C^{1/4} L_N^2 \leq N \log N/AB$, we see that

$$\begin{aligned}\log |u(\alpha)| &\leq -C^{1/9} N_1^{3/2} (\log N_1)^{1/2}\\ &\leq -r^{C/4},\end{aligned}$$

as desired.

By the definition of D_N,

$$\begin{aligned}D_N \log N(\deg_z Q) \deg_y P &\leq N^{3/2}(C/C_1)^{-3/4}(\deg_z Q)^{-1/2}\\ &\quad \times (\deg_y P)^{-1/2} A^{-3/2} \log N\\ &\leq N^{3/2} \log N/2,\end{aligned}$$

and by inequality (i) after the definition of L_N and D_N, we see that

$$c_{18}BNL_N = c_{18}(ABN^2L_N^2)^{1/2}(B/A)^{1/2} \leq N^{3/2}\log N/2.$$

Therefore we can conclude from Lemma 2 below that

$$\begin{aligned}\log ht\, u + \deg u &\leq c_{16}(N_0L_{N_0}(A+B) + D_{N_0}\log N_0(\deg_z Q)\deg_y P)\\ &\leq N_0^{3/2}\log N_0\\ &\leq r^{C/(12C_1)}.\end{aligned}$$

3. Lemmas used in the proof

LEMMA 1: *Let R and S be positive integers with $16R \leq S$. Let $a_{ij} \in \mathbf{Z}[x, y, z]$ satisfy*

$$\deg a_{ij} \leq d, \quad ht\, a_{ij} \leq H,$$

where $H \geq 1$. Then the system of equations

$$\begin{gathered} a_{11}\varphi_1 + \cdots + a_{1S}\varphi_S = 0\\ \vdots\\ a_{R1}\varphi_1 + \cdots + a_{RS}\varphi_S = 0\end{gathered}$$

has a non-trivial solution $\varphi_1, \ldots, \varphi_S \in \mathbf{Z}[x, y, z]$ with

$$\deg \varphi_k \leq 3d, \quad ht\, \varphi_k \leq (1+d)^6 SH.$$

For a proof, see [1, pp. 16–18].

LEMMA 2. *Suppose $P(x, y, z)$, $Q(x, y, z) \in \mathbf{C}[x, y, z]$. Then*

$$(ht\, P)(ht\, Q) \leq (ht\, PQ)\exp(3\deg PQ).$$

For a proof, see [7, Lemma 2, p. 135] or, for the fundamental one variable case, [11, Lemma 3, p. 149] or [9].

LEMMA 3: *Suppose $\omega \in \mathbf{C}$ and $P(x) \in \mathbf{Z}[x]$, $P(x) \not\equiv 0$, satisfy $|P(\omega)| < e^{-\lambda d(h+d)}$ where $\lambda > 3$, $d \geq \deg P$, $e^h \geq ht\, P$. Then there is a factor $Q(x)$ of $P(x)$ which is a power of an irreducible polynomial in $\mathbf{Z}[x]$ such that*

$$\log |Q(\omega)| < -(\lambda - 1)d(h + d).$$

For a proof, see [7, Lemma VI, p. 147].

LEMMA 4: *Suppose* $F(z) = \Sigma_{|\mathfrak{n}|<N}\varphi(\mathfrak{n}) \exp((\mathfrak{n} \cdot \mathfrak{b})z)$, *and set*

$$B_1 = \max_{|\mathfrak{n}|<N} (|\mathfrak{n} \cdot \mathfrak{b}|^*, |B_{\mathfrak{n}}|^*)$$

$$B_0 = \min_{0<|\mathfrak{n}|<N} (1, |\mathfrak{n} \cdot \mathfrak{b}|, |B_{\mathfrak{n}}|),$$

where in the definition of B_0, *the coordinates of* $\mathfrak{n}$ *may be negative, but not all zero, and*

$$E = \max_{\substack{|\mathfrak{l}|<L \\ 0\le p<P}} |F^{(p)}(B_{\mathfrak{l}})|.$$

Then, if $L \le N$ *and* $PL^3 \ge 2N^3 + 13B_1^2$, *we have*

$$\max |\varphi(\mathfrak{n})| \le L^3\sqrt{(N^3)!}\, e^{7B_1^2}\left(\frac{1}{2B_0B_1}\right)^{N^3}\left(\frac{72B_1}{B_0L^{3/2}}\right)^{PL^3} E.$$

This is a special case of what Tijdeman proved in [12, Theorem 3, pp. 87–88].

DEFINITION: Let $P(x) = p_0\Pi_{i=1}^m(x - t_i)$, $Q(x) = q_0\Pi_{j=1}^n(x - u_j)$. Then the *semi-resultant* r of P and Q is defined by

$$r = p_0^n q_0^m \prod{}' (t_i - u_j),$$

where $'$ means that the product is taken over all $(i, j) \notin M = \{(i, j) \mid t_i = u_j\}$. Clearly r is non-zero.

LEMMA 5 (Choodnovsky): *Let* $P(x) = p_0x^m + \cdots + p_m$, $Q(x) = q_0x^n + \cdots + q_n \in \mathbf{C}[x]$. (i) *Their semi-resultant* r *can be written as a polynomial over* $\mathbf{Z}$ *in the* p_i *and* q_j *with*

$$\deg_{p_i} \le n,$$
$$\deg_{q_j} \le m,$$
$$ht\, r \le 8^{mn}.$$

(ii) *Let* $\theta \in \mathbf{C}$, *and let* $P_1(x), Q_1(x) \in \mathbf{C}[x]$ *be monic, relatively prime, and satisfy*

$$\max\{|P_1(\theta)|, |Q_1(\theta)|\} < 1.$$

Then when $P_1(x)$ and $Q_1(x)$ divide $P(x)$ and $Q(x)$, respectively,

$$|r| \leq \max\{|P_1(\theta)|, |Q_1(\theta)|\} \max\{1, |\theta|\}^{\deg P_1(\deg Q_1)} \times (ht^*P)^n (ht^*Q)^m e^{mnk}$$

where k is an absolute constant.

For a proof, see [6, Lemma 4.12] or [3].

REFERENCES

[1] W.D. BROWNAWELL: Gelfond's method for algebraic independence. *Trans. A.M.S. 210* (1975) 1–26.

[2] W.D. BROWNAWELL: Pairs of polynomials small at a number to certain algebraic powers, Séminaire DELANGE PISOT POITOU, 17e année (1975/76) n° 11.

[3] W.D. BROWNAWELL: Some remarks on semi-resultants, pp. 205–210, in *Advances in Transcendence Theory*, A. Baker and D.W. Masser, eds., Academic Press, 1977.

[4] W.D. BROWNAWELL and M. WALDSCHMIDT: The algebraic independence of certain numbers to algebraic powers. *Acta Arith. 32* (1977), 63–71.

[5] G.V. CHOODNOVSKY: Algebraic independence of some values of the exponential function, *Mat. Zametki 5* (1974) 661–672 [English translation: *Math Notes 15* (1974) 391–398].

[6] G.V. CHOODNOVSKY: *Analytic methods in diophantine approximations* (in Russian), Institute of Mathematics, Ukrainian SSR Academy of Sciences, Preprint IM-74-9, Kiev (1974).

[7] A.O. GELFOND: *Transcendental and Algebraic Numbers*, GITTL, Moscow, 1952 [English translation: Dover, New York, 1960].

[8] A.O. GELFOND and N.I. FELDMAN: On the measure of relative transcendence of certain numbers (in Russian), *Izv. Akad. Nauk SSSR, Ser. mat. 14* (1950) 493–500.

[9] K. MAHLER: An application of Jensen's formula to polynomials, *Mathematika 7* (1960) 98–100.

[10] M. MIGNOTTE and M. WALDSCHMIDT: Linear forms in logarithms and Schneider's Method. *Math. Ann. 231* (1978) 241–267.

[11] R. TIJDEMAN: On the algebraic independence of certain numbers, *Nederl. Akad. Wet. Proc. Ser. A 74 = Indag. Math. 33* (1971), 146–162.

[12] R. TIJDEMAN: An auxiliary result in the theory of transcendental numbers, *J. Number Theory 5* (1973), 80–94.

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