COMPOSITIO MATHEMATICA

LORENZO ROBBIANO

A theorem on normal flatness

Compositio Mathematica, tome 38, n° 3 (1979), p. 293-298

<http://www.numdam.org/item?id=CM_1979__38_3_293_0>

COMPOSITIO MATHEMATICA, Vol. 38, Fasc. 3, 1979, pag. 293–298

A THEOREM ON NORMAL FLATNESS

Lorenzo Robbiano*

Introduction

The beginning of the story is in [4] where the first theorem of transitivity of normal flatness is given.

More or less the problem is the following. Let R be a local ring, $I, J, \Lambda = I + J$ ideals of R and assume that J is generated by a regular sequence mod I; then how is it possible to relate the properties that $G(I)$ (the graded ring associated to I) is a free R/I-module and $G(\Lambda)$ is a free R/Λ-module?

As I said, the first answer was in [4] and it was the starting point for successive improvements (see for instance [1], [2], [5], [8]); a theory of normal flatness was constructed ([3]) and also some questions on normal torsionfreeness were solved (see for instance [1], [6], [7], [8]). In this kind of problems it always happened that the authors assumed J to be in particular position with respect to I (essentially, as I said, J is generated by a regular sequence mod I). The purpose of the present paper is to produce a new theorem (Theorem 10) of transitivity of normal flatness in the case that I, J are in "symmetric position"; in order to prove it, I firstly achieve some results which give a connection between the graded rings $G(I)$, $G(J)$, $G(\Lambda)$; in this way it is also possible to give a new insight in the proofs of some known theorems.

In this paper all rings are supposed to be commutative, Noetherian and with identity.

Let R be a ring, $I, J, \Lambda = I + J$ ideals of R; we denote by $R(I; J)$ the "Rees algebra" associated with the pair $(I; J)$ i.e. the graded ring $\bigoplus_n R_n(I; J)$ where $R_n(I, J) = \bigoplus_{r+s=n} I^r J^s$ and the multiplication is the obvious one.

*This work was supported by C.N.R. (Consiglio Nazionale delle Ricerche).

0010-437X/79/03/0293-06 $00.20/0

Of course $R(I;R) = R(I)$ the usual Rees algebra.

We shall denote by $G(I;J)$ the graded ring associated with the pair $(I;J)$ i.e. the graded ring $R(I;J)\otimes_R R/\Lambda$.

It is clear that $G(I;J) = \bigoplus G_n(I;J)$ where $G_n(I,J) = \bigoplus_{r+s=n} G_{r,s}(I;J)$ and $G_{r,s}(I;J) = I^rJ^s \otimes_R R/\Lambda \simeq I^rJ^s/I^rJ^s\Lambda$.

It is also clear that we have canonical epimorphisms

$$G(I)\otimes_R G(J) \twoheadrightarrow G(I;J) \twoheadrightarrow G(\Lambda)$$

LEMMA 1: *Let r, n be integers such that $0 \le r < n$ and call $s = n - r$; let R be a ring, $I, J, \Lambda = I + J$ ideals of R and assume that $I^{r+1} \cap J^{s-1} = I^{r+1}J^{s-1}$. Then the canonical epimorphism*

$$I^{r+1}\Lambda^{s-2}/I^{r+1}\Lambda^{s-1} \oplus I^rJ^{s-1}/I^rJ^{s-1}\Lambda \twoheadrightarrow I^r\Lambda^{s-1}/I^r\Lambda^s$$

is an isomorphism.

PROOF: An element of the direct sum is of the form $(\bar{\alpha}, \bar{\beta})$ where $\alpha \in I^{r+1}\Lambda^{s-2}$, $\beta \in I^rJ^{s-1}$.

With the obvious meaning of the symbols its image is $\overline{\alpha + \beta}$.

Suppose now that $\alpha + \beta \in I^r\Lambda^s$; then

$$\begin{aligned}
\alpha \in\ & I^{r+1}\Lambda^{s-2} \cap (I^rJ^{s-1} + I^r\Lambda^s) \\
&= I^{r+1}\Lambda^{s-2} \cap (I^rJ^{s-1} + I^rJ^s + I^{r+1}J^{s-1} + I^{r+2}J^{s-2} + \cdots + I^{r+s}) \\
&= I^{r+1}\Lambda^{s-2} \cap (I^rJ^{s-1} + I^{r+2}\Lambda^{s-2}) = I^{r+2}\Lambda^{s-2} \\
&\quad + (I^{r+1}\Lambda^{s-2} \cap I^rJ^{s-1}) \subseteq I^{r+2}\Lambda^{s-2} + I^{r+1} \cap J^{s-1} \\
&= I^{r+2}\Lambda^{s-2} + I^{r+1}J^{s-1} = I^{r+1}\Lambda^{s-1} \text{ which means } \bar{\alpha} = 0.
\end{aligned}$$

Now we have

$$\begin{aligned}
\beta \in\ & I^rJ^{s-1} \cap I^r\Lambda^s \\
&= I^rJ^{s-1} \cap (I^{r+s} + \cdots + I^{r+2}J^{s-2} + I^{r+1}J^{s-1} + I^rJ^s) \\
&= I^{r+1}J^{s-1} + I^rJ^s + I^rJ^{s-1} \cap I^{r+2}\Lambda^{s-2} \\
&\subseteq I^rJ^{s-1}\Lambda + J^{s-1} \cap I^{r+1} = I^rJ^{s-1}\Lambda \text{ which means } \bar{\beta} = 0.
\end{aligned}$$

PROPOSITION 2. *Let n be a positive integer; let R be a ring, $I, J, \Lambda = I + J$ ideals of R. Assume that $I^r \cap J^s = I^rJ^s$ for every pair of positive integers r, s such that $r + s = n$. Then the canonical epimorphism $G_{n-1}(I;J) \twoheadrightarrow G_{n-1}(\Lambda)$ is an isomorphism.*

PROOF. It is an easy consequence of Lemma 1.

LEMMA 3. *Let r, s be positive integers; with the usual notations assume that $I^r \cap J = I^rJ$ and $G_s(J)$ is a free R/J-module. Then the canonical epimorphism $G_r(I)\otimes_R G_s(J) \xrightarrow{\pi} G_{r,s}(I;J)$ is an isomorphism.*

PROOF. Let $\alpha = rk_{R/J}(G_s(J)) = rk_{R/\Lambda}(J^s/J^s\Lambda)$ and let us choose $\theta_1, \ldots, \theta_\alpha \in J^s$ so that $\bar{\theta}_1, \ldots, \bar{\theta}_\alpha$ is a free basis of $G_s(J)$. Then the homomorphism $\rho\colon G_r(I) \otimes G_s(J) \longrightarrow (I^r/I^r\Lambda)^\alpha$ defined by $\rho(\bar{a} \otimes \bar{\theta}_i) = (0, \ldots, 0, \bar{a}, 0, \ldots, 0)$ where $\bar{a}$ is in the ith place, is clearly an isomorphism.

Now we can consider the homomorphism $\tau\colon (I^r/I^r\Lambda)^\alpha \longrightarrow I^rJ^s/I^rJ^s\Lambda$ defined by $\tau(\bar{a}_1, \ldots, \bar{a}_\alpha) = \overline{\Sigma\, a_i\theta_i}$.

It is clear that $\pi = \tau \circ \rho$, hence it is enough to prove that τ is injective. Let $\bar{0} = \tau(\bar{a}_1, \ldots, \bar{a}_\alpha) = \overline{\Sigma\, a_i\theta_i}$; this means that $\Sigma\, a_i\theta_i \in I^rJ^s\Lambda$ hence $\Sigma\, a_i\theta_i = \Sigma\, b_i\theta_i \bmod J^{s+1}$ where $b_i \in I^r\Lambda$. Therefore $\Sigma\,(a_i - b_i)\theta_i = 0 \bmod J^{s+1}$ which implies $a_i - b_i \in J$ for every i. We get $a_i \in I^r \cap (I^r\Lambda + J) = I^r\Lambda + I^r \cap J = I^r\Lambda + I^rJ = I^r\Lambda$ for every i, and this concludes the proof.

LEMMA 4. *With the usual notations assume that* $I \cap J = IJ$ *and* $G_r(J)$ *is a free* R/J*-module for* $r = 1, \ldots, n-1$. *Then* $I \cap J^n = IJ^n$.

PROOF. It is an easy consequence of the fact that, given I_1, I_2 ideals of a ring R, $I_1 \cap I_2 = I_1 \cdot I_2$ is equivalent to $\mathrm{Tor}_1^R(R/I_1, R/I_2) = 0$.

THEOREM 5. *Let* R *be a ring,* $I, J, \Lambda = I + J$ *ideals of* R *such that* $I \cap J = IJ$. *Let* n *be a positive integer and assume that* $G_i(I)$ *is a free* R/I*-module for* $i = 1, \ldots, n-1$ *and* $G_j(J)$ *is a free* R/J*-module for* $j = 1, \ldots, n-1$.

Then the canonical epimorphism $\bigoplus_{\lambda+\mu=\nu} G_\lambda(I) \otimes_R G_\mu(J) \twoheadrightarrow G_\nu(\Lambda)$ *is an isomorphism, hence* $G_\nu(\Lambda)$ *is a free* R/Λ*-module for* $\nu = 1, \ldots, n-1$.

PROOF. Using Lemma 4 we get $I^r \cap J^s = I^rJ^s$ for $r = 1, \ldots, n-1$, $s = 1, \ldots, n-1$; hence $\bigoplus_{\lambda+\mu=\nu} G_{\lambda,\mu}(I; J) \simeq G_\nu(\Lambda)$ for $\nu = 1, \ldots, n-1$ by Proposition 2.

Using Lemma 3 we are done.

REMARK. It is possible to use theorem 5 to prove the following (known) fact. Let R be a local ring, let $a_1, \ldots, a_n$; $b_1, \ldots, b_m$ be R-sequences such that if we call $I = (a_1, \ldots, a_n)$, $J = (b_1, \ldots, b_m)$, $I \cap J = IJ$.

Then $a_1, \ldots, a_n, b_1, \ldots, b_m$ is a regular R-sequence.

PROOF. (Hint). Use Theorem 5 to prove that $G(I + J)$ is a polynomial ring in $n + m$ indeterminates.

Lemma 6. *With the usual notations denote by* $\bar{\ }$ *the reduction modulo* I. *Then the canonical sequence of* R/Λ*-modules*

$$0 \longrightarrow I \cap J^n/IJ^n + I \cap J^{n+1} \longrightarrow J^n/J^{n+1} \xrightarrow{\pi_n} \bar{J}^n/\bar{J}^{n+1} \longrightarrow 0$$

is exact for every positive integer n.

Proof. $\bar{J}^n/\bar{J}^{n+1} \simeq (J^n + I)/(J^{n+1} + I) \simeq J^n/(J^{n+1} + J^n \cap I)$.
Hence $\operatorname{Ker} \pi_n \simeq (J^{n+1} + J^n \cap I)/(J^{n+1} + J^n I)$

$$\simeq (I \cap J^n)/(I \cap J^n \cap (IJ^n + J^{n+1})) \simeq (I \cap J^n)/(IJ^n + I \cap J^{n+1}).$$

Henceforth we assume that R *is local.*

Corollary 7. *With the notations of Lemma 6, assume that* R *is local and let* n *be a positive integer. Then the following conditions are equivalent*

(i) π_r *is an isomorphism for every* $r \geq n$.
(ii) $I \cap J^r = IJ^r$ *for every* $r \geq n$.

Proof. The only thing to be proved is that (i) implies $I \cap J^n = IJ^n$. Now, if π_r is an isomorphism for every $r \geq n$, then $I \cap J^n = IJ^n + I \cap J^{n+1} = IJ^n + IJ^{n+1} + I \cap J^{n+2} = IJ^n + I \cap J^{n+2} = \cdots = IJ^n + I \cap J^{n+k}$. Hence $I \cap J^n = \bigcap_k (IJ^n + I \cap J^{n+k}) = IJ^n$.

Corollary 8. *With the usual notations assume that* R *is local* $I \cap J = IJ$ *and* $G(J)$ *is a free* R/J*-module. Then* $G(\bar{J}) \simeq G(J) \otimes_R R/\Lambda$, *hence is a free* R/Λ*-module.*

Proof. Using Lemma 4 we get that $I \cap J^n = IJ^n$ for every n, hence we can apply Corollary 7.

Lemma 9. *Let* R *be a local ring,* I, J, Λ *as above,* $B = R/I$, $A = B/\bar{\Lambda} \simeq R/\Lambda$ ($\bar{\ }$ *denotes* "modulo I").

Let L *be a finite free* B*- module and* T *a submodule of* L. *Suppose*

(i) $T_\wp = 0$ *for every* $\wp \in Ass(A)$.
(ii) $J^n/J^n\Lambda$ *is a free* A*-module for every* n.
(iii) $(I \cap J^n)_\wp = (IJ^n)_\wp$ *for every* $\wp \in Ass(A)$.

Then $T = 0$.

Proof. If we make the identification of L with $(R/I)^\alpha$ for a suitable α, then T can be considered as the reduction modulo I of a submodule K of R^α. Because of the type of the argument, we may assume that $\alpha = 1$, so it is enough to prove that $K \subset I$.

Using (i) we easily get that $T \subset \bar{\Lambda}$, hence $K \subset \Lambda$. Therefore if $x \in K$ there exists an element $a \in I$ such that $x - a \in J$. Let $\vartheta_1, \ldots, \vartheta_r$ be a minimal system of generators of J, such that $\bar{\vartheta}_1, \ldots, \bar{\vartheta}_r$ is a free basis of $J/J\Lambda$. Then $x - a = \Sigma\, \lambda_i \vartheta_i$ and using (i) we get an element $\lambda \notin \bigcup \wp$, $\wp \in Ass(A)$ such that $\lambda(x-a) = \Sigma\, \lambda\lambda_i\vartheta_i \in I$. Hence $\lambda(x-a) = \Sigma\, \lambda\lambda_i\vartheta_i \in I \cap J$.

From (iii) we get an element $\lambda' \notin \bigcup \wp$, $\wp \in Ass(A)$ such that

$$\lambda\lambda'(x-a) = \Sigma\, (\lambda\lambda'\lambda_i)\vartheta_i \in IJ \subset J\Lambda.$$

From (ii) we deduce $\lambda\lambda'\lambda_i \in \Lambda$ for every i, hence $\lambda_i \in \Lambda$ for every i, hence $x \in I + \Lambda J = I + J^2$. Going on with the same type of argument we get $x \in I + J^r$ for every r, hence $x \in I$.

Let us state a notation: if $\mathfrak{a}$ is an ideal of a ring R, we denote by $\mathscr{Z}_{\mathfrak{a}}(R)$ the set $\{x \in R / \bar{x} \in \mathscr{Z}(R/\mathfrak{a})\}$ where $^-$ denotes the reduction modulo $\mathfrak{a}$ and $\mathscr{Z}(\cdots)$ means "zerodivisors of ..."

We are ready to prove the following

THEOREM 10. *Let R be a local ring, $I, J, \Lambda = I + J$ ideals of R such that $I \cap J = IJ$. Then the following conditions are equivalent*:

(1) *$G(I)$ and $G(J)$ are free.*

(2) *$G(I) \otimes_R R/\Lambda$ and $G(J) \otimes_R R/\Lambda$ are free and $G(IR_\wp)$ and $G(JR_\wp)$ are free for every $\wp \in Ass(R/\Lambda)$.*

(3) *$G(\Lambda)$ is free, $G(IR_\wp)$ and $G(JR_\wp)$ are free for every $\wp \in Ass(R/\Lambda)$ and $\mathscr{Z}_I(R)$, $\mathscr{Z}_J(R)$ are contained in $\mathscr{Z}_\Lambda(R)$.*

PROOF. (1)$\Rightarrow$(3) $G(\Lambda)$ free follows from Theorem 5.

Hence we have to prove that $\mathscr{Z}_I(R) \subseteq \mathscr{Z}_\Lambda(R)$ (the same argument works for $\mathscr{Z}_J(R)$).

If we call $B = R/I$, $\mathscr{Z}_I(R) \subseteq \mathscr{Z}_\Lambda(R)$ is equivalent to $\mathscr{Z}_{(0)}(B) \subseteq \mathscr{Z}_{\bar{\Lambda}}(B)$ where $^-$ denotes the reduction modulo I. Using Corollary 8 we get that $G(\Lambda)$ is a free $R/\Lambda \simeq B/\bar{\Lambda}$-module. Let now x, y be elements of B such that $xy = 0$ and $x \in \wp \in E\wp$ where $E = Ass(B/\bar{\Lambda})$. Being $G(\Lambda)$ free, $Ass(B/\bar{\Lambda}^n) \subseteq Ass(B/\bar{\Lambda})$, hence $y \in \bigcap_n \bar{\Lambda}^n = (0)$.

(2)$\Rightarrow$(1) Let $\vartheta_1, \ldots, \vartheta_\alpha$ be a minimal system of generators of I^n, such that $\bar{\vartheta}_1, \ldots, \bar{\vartheta}_\alpha$ is a free basis of $I^n/I^n\Lambda$.

Let $0 \longrightarrow T \longrightarrow (R/I)^\alpha \xrightarrow{\pi} I^n/I^{n+1} \longrightarrow 0$ be the exact sequence of R/I-modules defined by $\pi(e_i) = \bar{\vartheta}_i \in I^n/I^{n+1}$.

Then we can apply Lemma 9, because (i) and (ii) are clearly satisfied and (iii) follows from Lemma 4.

In conclusion $T=0$ and $G_n(I)$ is free. This works for every n and, in the same way, for J.

Since (1)$\Rightarrow$(2) is obvious, we only have to prove (3)$\Rightarrow$(1). Since $I\cap J=IJ$, by Proposition 2 we get

$$\Lambda/\Lambda^2 \simeq I/I\Lambda \oplus J/J\Lambda.$$

Let $\vartheta_1,\ldots,\vartheta_\alpha$ be a minimal system of generators of I such that $\bar{\vartheta}_1,\ldots,\bar{\vartheta}_\alpha$ is a free basis of $I/I\Lambda$.

Let $0\longrightarrow T\longrightarrow (R/I)^\alpha\longrightarrow I/I^2\longrightarrow 0$ be the exact sequence of R/I-modules defined by $\pi(e_i)=\bar{\vartheta}_i\in I/I^2$.

By hypothesis $T_\wp=0$ for every $\wp\in Ass(R/\Lambda)$, hence, for every $x\in T$ there exists $\lambda\not\in\mathscr{X}_\Lambda(R)$ such that $\lambda x=0$. Since $\mathscr{X}_I(R)\subseteq\mathscr{X}_\Lambda(R)$, we get $x=0$. Hence I/I^2 and J/J^2 (using the same argument) are free. Let us assume, by induction, that $G_i(I), G_i(J)$ are free for $i=0,\ldots,\nu-1$. Using Lemma 4 we get $I^\lambda\cap J^\mu=I^\mu$ for $\lambda+\mu\leqslant\nu+1$ hence $G_\nu(\Lambda)\simeq\bigoplus_{\lambda+\mu=\nu}G_{\lambda,\mu}(I;J)$ by Proposition 2.

In particular $I^\nu/I^\nu\Lambda$ and $J^\nu/J^\nu\Lambda$ are free R/Λ-modules.

Using the same argument as before we get that $G_\nu(I)$ and $G_\nu(J)$ are free.

REFERENCES

[1] ACHILLES, SCHENZEL, VOGEL: Bemerkungen über normale Flachheit und normale Torsionsfreiheit und Anwendungen. (Preprint).

[2] HERRMANN, SCHMIDT: Zur Transitivität der normalen Flachheit. *Invent. Math.* 28 (1975).

[3] HERRMANN, SCHMIDT, VOGEL: *Theorie der normalen Flachheit.* Teuber-Texte zur Math. (1977).

[4] HIRONAKA: Resolution of singularities of an algebraic variety over a field of characteristic zero. *Ann. of Math.* 79 (1964).

[5] GROTHENDIECK: Éléments de géométrie algébrique I.H.E.S. Publ. Math. Paris 32 (1967).

[6] ROBBIANO: An algebraic property of $\mathbf{P}^1\times\mathbf{P}^r$. (In preparation).

[7] ROBBIANO, VALLA: Primary powers of a prime ideal. *Pacific J. Math.* 63 (1976).

[8] ROBBIANO, VALLA: On normal flatness and normal torsionfreeness. *J. of Algebra* 43 (1976).

(Oblatum 15-XI-1977)

Istituto Matematico
dell'Università di Genova
Via L.B. Alberti, n° 4 –
16132 GENOVA (Italia)