COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 38, nº 2 (1979), p. 241-252

<http://www.numdam.org/item?id=CM_1979__38_2_241_0>

COMPOSITIO MATHEMATICA, Vol. 38, Fasc. 2, 1979, pag. 241–252

THE DEGREES OF CERTAIN STRATA OF THE DUAL VARIETY

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1. Introduction

Let X be a nonsingular, projective variety over an algebraically closed field k. For a given imbedding $X \subset \mathbb{P}^m$, the dual variety $\check{X}$ is the subvariety of $\check{\mathbb{P}}^m$ (the dual projective space) of hyperplanes tangent to X. One expects that a general tangent hyperplane H be as nice as possible, i.e., that it be tangent to X at a single point and that the corresponding hyperplane section have an ordinary quadratic singularity at the point x of tangency (cf. Katz [6] p. 227). This last condition means that the affine tangent cone of $H \cap X$ at x is quadratic and its associated projective quadric is nonsingular. J. Roberts [9] introduced a filtration of $\check{X}$ by the rank of the abovementioned quadric and raised the question of computing the degree of each stratum. He observed that said stratification is the image of the Thom–Boardman second order singularities of the map $Y \to \check{\mathbb{P}}^m$, where Y is the variety of pairs (x, H) in $X \times \check{\mathbb{P}}^m$ such that x lies in H. He used a method outlined by Porteous ([8] p. 286–307) to compute the cohomology class of the second order singularity corresponding to the first stratum.

We offer in this note a derivation of the formula for the cohomology class of each of the 2nd. order singularities envisaged in Roberts' stratification of $\check{X}$. As a matter of fact, we compute the classes of certain cycles which map onto those 2nd. order singularities provided char. $k \neq 2$. Our starting point is the observation that the 1st. order singularity $\{(x, H) : H$ is tangent to X at $x\}$ parametrizes a family of quadrics in the projective bundle of tangent directions of X. Thus, we reduce the question to that of calculating the "generic" cohomology class of the subfamily of quadrics the vertices of which are required to have specified dimension. Our discussion yields as

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special cases formulas for the degrees of the varieties parametrizing the quadrics of $\mathbb{P}^n$ with vertices of specified dimensions.

2. Quadrics

All schemes are of finite type / k.

Let V be an $(n+1)$-dimensional k-vector space and let $\mathbb{P}^n = \mathbb{P}(V)$ be the associated projective space (of 1-quotients of V). Let $Q \subset \mathbb{P}^n$ be a quadric hypersurface with homogeneous equation s in $S_2 V$ (= 2nd. symmetric power). We recall that the singular locus of Q (if nonempty) is a linear subspace, the vertex of Q. If L is a linear subspace of $\mathbb{P}^n$, say $L = \mathbb{P}(W)$, where

$$0 \to W' \to V \to W \to 0$$

is an exact sequence of vector spaces, then one can easily show that L lies in the vertex of Q iff s belongs to $S_2 W' \subset S_2 V$.

(2.1) This discussion naturally globalizes to families of quadrics. Let Z be a scheme and let F (resp. L) be a locally free (resp. invertible) 0_Z-Module. Let $p : \mathbb{P} = \mathbb{P}(F) \to Z$ be the corresponding projective bundle. Fix a section

$$s : \mathcal{O}_{\mathbb{P}} \to L \otimes \mathcal{O}_{\mathbb{P}}(2).$$

We denote by Q or Q_s the scheme of zeros of s in $\mathbb{P}$. We will think of Q as a family of quadrics parametrized by Z, even though Q may not be flat / Z. In fact, we do allow s to vanish identically on fibres of p.

(2.2) Given a map $t : T \to Z$, form the fibre diagram:

$$\begin{tikzcd} \mathbb{P}_T \arrow[r, "t_{\mathbb{P}}"] \arrow[d, "p_T"'] & \mathbb{P} \arrow[d, "p"] \\ T \arrow[r, "t"] & Z \end{tikzcd}$$

Let W be a locally free quotient of F_T and set

$$W' = \ker(F_T \twoheadrightarrow W).$$

We say that $\mathbb{P}(W)$ lies in the vertex of $Q_T = Q \times_Z T$ if the section

$$(p_T)_* t_{\mathbb{P}}^* s = t^* p_* s$$

of $(L\otimes S_2F)_T$ factors through $(L\otimes S_2W')_T$.

(2.3) Fix a nonnegative integer r. We wish to get the equations for the subscheme of Z over which the fibres of Q have vertices of dimension $\geq$r. For this, we look at the Grassmann bundle $g:\underline{G}\to Z$ parametrizing the locally free quotients of rank $r+1$ of F over Z. Let

$$0\to R'\to F_{\underset{\sim}{G}}\to R\to 0$$

be the universal sequence. Set

$$\mathbb{H}=\operatorname{coker}(S_2R'\subset S_2F_{\underset{\sim}{G}}).$$

We construct the diagram,

(2.3.1)
$$\begin{tikzcd}
 & & \mathcal{O}_G \arrow[d, "(p_*s)_G"'] \arrow[dr, "h"] & & \\
0 \arrow[r] & L\otimes S_2R' \arrow[r] & L\otimes S_2F_{\underset{\sim}{G}} \arrow[r] & L\otimes\mathbb{H} \arrow[r] & 0,
\end{tikzcd}$$

and denote by $\tilde{Z}_r$ the scheme of zeros of h in $\underset{\sim}{G}$ (cf. Altman and Kleiman [1], 2.2).

The Proposition below shows that $\tilde{Z}_r$ parametrizes the r-subspaces of the fibres of $\mathbb{P}\to Z$ which lie in the vertices of the corresponding fibres of $Q\to Z$.

(2.4) PROPOSITION: *A map $t:T\to \underset{\sim}{G}$ factors through $\tilde{Z}_r$ iff $\mathbb{P}(t^*R)$ lies in the vertex of $Q(gt)=Q\times_Z T$.*

PROOF: By the definition of a scheme of zeros, t factors through $\tilde{Z}_r$ iff $t^*h=0$ holds. By the construction of h, the last condition holds iff $t^*g^*p_*s$ factors through $t^*(L\otimes S_2R')$, thus proving the assertion.

(2.5) PROPOSITION: *Suppose Z is quasi-projective and Cohen–Macaulay. If $\tilde{Z}_r$ is empty or of the correct codimension $z=\operatorname{rank}\mathbb{H}(=\binom{n+2}{2}-\binom{n-r+1}{2})$ in $\underset{\sim}{G}$ (where $n+1=\operatorname{rank}F$), then we have the formula,*

$$[\tilde{Z}_r]=c_z(L\otimes\mathbb{H})\cap[\underset{\sim}{G}]$$

in the Chow–Fulton homology group $A.(\underset{\sim}{G})$.

PROOF: The 1st. hypothesis implies $\underset{\sim}{G}$ is Cohen–Macaulay. Hence, by the 2nd. one, the Koszul complex

$$\Lambda^{\cdot}(\mathsf{H}\otimes L)\to \mathcal{O}_{\tilde{Z}_r}$$

is exact. The assertion follows from general properties of the cohomology–homology theory of Fulton [3].

(2.6) LEMMA: *The following formula holds in the Grothendieck ring* $K^{\cdot}(\underset{\sim}{G})$:

$$\mathsf{H} = S_2R + R(F-R).$$

PROOF: The assertion follows from the natural exact sequence,

$$0 \longrightarrow R'\otimes R \longrightarrow \mathsf{H} \longrightarrow S_2R \longrightarrow 0.$$

The 2nd. map exists and is surjective because $S_2R' \to S_2R$ is the zero map. The 1st. one comes from the bilinear map

$$b: R'\times R \longrightarrow \mathsf{H}$$

locally defined by the formula $b(x',\hat{x}) = (x'x)\tilde{}$, where x' denotes a local section of R', $\hat{x}$ the image in R of a local section x of F, and $\tilde{}$ the class in H of a local section of S_2F. The details are routine and will be omitted.

(2.7) PROPOSITION: *Set* $Z_r = g(\tilde{Z}_r)$. The map

$$g^{-1}(Z_r - Z_{r+1}) \cap \tilde{Z}_r \longrightarrow Z_r - Z_{r+1}$$

induced by $g: \underset{\sim}{G} \to Z$ is bijective.

PROOF: The fibre of Q over a point in $Z_R - Z_{r+1}$ is a quadric with vertex of dimension exactly r. Now, if two distinct r-spaces are contained in the vertex of a quadric, so is their join, which has dimension $\geq r+1$.

3. Universal flat family of quadrics

Let S be a scheme and let V be a locally free $\mathcal{O}_S$-Module of rank $n+1$. Then $Z = \mathsf{P}(S_2V)\check{}$ parametrizes the universal flat family $Q \to Z$

of quadrics of $\mathbb{P}(V)$ (cf. Altman and Kleiman [1]). Set $F = V_Z$ and set $L = \mathcal{O}_{\mathbb{P}(S_2V)^\vee}(1)$. Then $Q \subset \mathbb{P} = \mathbb{P}(F)$ is the scheme of zeros of the section $s = v \otimes L$ of $O_{\mathbb{P}}(2) \otimes L$, where v is defined in the diagram below:

$$\begin{tikzcd} \check{L}_{\mathbb{P}} \arrow[d] \arrow[dr] & \\ S_2 V_{\mathbb{P}} \arrow[r, two heads] & \mathcal{O}_{\mathbb{P}}(2). \end{tikzcd}$$

For the universal family we can prove the following.

(3.1) Proposition: (i) *$\tilde{Z}_r$ has the correct codimension in $\underset{\sim}{G} = G_{r+1}(V) \times_S \mathbb{P}(S_2V)^\vee$ (see* (2.5).

(ii) *$\tilde{Z}_r \to G_{r+1}(V)$ is a projective subbundle of $\underset{\sim}{G} \to G_{r+1}(V)$ with fibre dimension $\binom{n-r+1}{2} - 1$. In particular, $\tilde{Z}_r$ is integral if S is.*

Proof: The 1st. assertion is a consequence of the 2nd. Now we work with the diagram (2.3.1). Given a map $t : T \to \underset{\sim}{G}$, the following assertions are equivalent: (a) t factors through $\tilde{Z}_r$; (b) $t^*h = 0$; (c) $t^*(h \otimes L) = 0$; (d) the quotient t^*L_G of $t^*(S_2F)_G$ factors through $(S_2F)_G \twoheadrightarrow (S_2R')^\vee$; (e) t factors through the linear embedding $\mathbb{P}(S_2R')^\vee \subset \mathbb{P}(S_2V)_{\underset{\sim}{G}}^\vee$. Thus we have that $\tilde{Z}_r$ is equal to $\mathbb{P}(S_2R')^\vee$. The assertion follows since the rank of S_2R' is $\binom{n-r+1}{2}$.

When $S = \operatorname{Spec}(k)$ and $\operatorname{char}(k) = 0$, (2.5) and (2.7) yield the following result.

(3.2) Proposition: *The degree of the subvariety Z_r of $\check{\mathbb{P}}^N$ ($N = \binom{n+2}{2} - 1$) of quadrics of $\mathbb{P}^n$ with vertices of dimension at least r is the degree of the Chern class $c_{(r+1)(n-r)}(-S_2R')$ (see 2.3).*

Proof: Let e denote the class of a hyperplane of $\check{\mathbb{P}}^N$ and set $d = \dim Z_r$. We have,

$$\deg Z_R = \deg([Z_r]e^d) = \deg(g_*([\tilde{Z}_r]g^*e^d)),$$

by the projection formula and because $\tilde{Z}_r \to Z_r$ is purely inseparable (by 2.7) hence birational since we've assumed $\operatorname{char}(k) = 0$. Now, denoting by q the 2nd. projection from $\check{\mathbb{P}}^N \times G_{r+1}(V)$, we have,

$$q_*([\tilde{Z}_r]g^*e^d) = q_*(g^*e^d c_z(\mathsf{H} \otimes L)) = c_{(r+1)(n-r)}(\mathsf{H})$$

by well-known properties of Chern classes and of the Gysin map of a projective bundle. The assertion now follows from the definition of H and the fact that $F = V_Z$ has trivial Chern classes.

(3.3) REMARKS: (i) For $r = 0$, one gets $\deg Z_0 = n + 1$, which is otherwise clear: the locus of singular quadrics of $\mathbb{P}^n$ is given by the vanishing of the determinant of an $(n+1)$-symmetric matrix in the homogeneous coordinates of $\check{\mathbb{P}}^N$. It would be interesting to show directly for arbitrary r that the degree of the Chern class in (3.2) is the one prescribed by the formula of Giambelli: $\Pi_0^r \binom{n+r+1}{2i+1}/\binom{n+r+1}{i}$ (cf. Baker [2] p. 111).

(ii) It can easily be shown that $\tilde{Z}_0 \to Z_0$ is just the dual map of $\mathbb{P}^n = \mathbb{P}(V)$ for the Veronese embedding in $\mathbb{P}(S_2 V)$. According to Katz ([6] p. 227), this map is an isomorphism off Z_1 if $\mathrm{char}(k) \neq 2$ or n is even (and has nontrivial inseparable degree otherwise). More generally, it can be shown each $\tilde{Z}_r \to Z_r$ is an isomorphism off Z_{r+1}, provided char. $k \neq 2$. Indeed, there is a natural embedding $\mathbb{P}(S_2 V)^\vee \subset \mathbb{P}(V \otimes V)^\vee = \mathbb{P}(\underline{\mathrm{Hom}}(V^\vee, V))^\vee$ under which a quadric is identified with a symmetric linear map $V^\vee \to V$ (up to scalar). Each $\tilde{Z}_r \to Z_r$ turns out to be just the restriction to $\mathbb{P}(S_2 V)^\vee$ of the usual desingularization of the determinantal locus of maps of rank $\leq n - r$. I ignore whether Z_r is always Cohen–Macaulay, but it seems that Kempf's methods [7] should yield a proof.

4. Stratification of $\check{X}$

We focus now on a nonsingular, projective variety X of dimension $n + 1$ embedded in some $\mathbb{P}^m$. We assume, for simplicity, that X is not contained in a hyperplane.

Put $S = \check{\mathbb{P}}^m$ and put $\underline{X} = X \times S$. Set $M = \mathcal{O}_X(1) \otimes \mathcal{O}_S(1)$. There is a natural section w of M such that the scheme of zeros Y of w in $\underline{X}$ consists of the pairs (x, H) with x in $X \cap H$. Thus Y is the total space of the family of hyperplane sections of X.

(4.1) Set $Z = \{(x, H) : H$ is tangent to X at $x\}$. Thus, $\check{X}$ is the image of Z in $\check{\mathbb{P}}^m$. We will show that Z is naturally the scheme of zeros in $\underline{X}$ of a certain map of Modules. We begin by recalling that a hyperplane H is tangent to X at a point x iff $X \cap H$ is singular at x. On the other hand, one knows that $X \cap H$ is singular at x iff its total transform $(H \cap X)^*$ in the blowup $B(x)$ of X at x contains (at least) twice the exceptional divisor.

Let $b : B \to \underline{X} \times_S \underline{X}$ be the blowingup of the diagonal. Let E denote the exceptional divisor. Let b_i denote the composition of b with the ith. projection. Regard B as a scheme / $\underline{X}$ via b_1. The restriction of b_2 to a fibre $b_1^{-1}(x, H)$ is just the blowup $B(x) \to X$ of X at x. Consider the diagram of maps of $\mathcal{O}_B$-Modules,

$$\begin{tikzcd}
 & & \mathcal{O}_B \arrow[d, "b_2^*w"'] \arrow[dr, "u"] & & \\
0 \arrow[r] & b_2^*M \otimes \mathcal{O}_B(2) \arrow[r] & b_2^*M \arrow[r] & b_2^*M \otimes \mathcal{O}_{2E} \arrow[r] & 0.
\end{tikzcd}$$

Here $\mathcal{O}_B(2)$ denotes the square of the Ideal of E. By construction, the map u (see the diagram) vanishes on a fibre $b_1^{-1}(x, H)$ iff $(H \cap X)^*$ contains twice $E(x)$.

We give Z the structure of scheme of zeros of u in $\underline{X}$.

(4.1.1) LEMMA: *The class of Z in the Chow ring $A(\underline{X})$ is equal to the top Chern class of $(\Omega_X^1 + \mathcal{O})M$.*

PROOF: By Altman & Kleiman ([1], Prop. 2.3), Z is also the scheme of zeros of a section of the locally free sheaf $F = b_{1*}(b_2^* M \otimes \mathcal{O}_{2E})$. Using the exact sequence,

$$0 \longrightarrow \mathcal{O}_E(1) \longrightarrow \mathcal{O}_{2E} \longrightarrow \mathcal{O}_E \longrightarrow 0,$$

and the formulae

$$R^i b_* \mathcal{O}_B(j) = \begin{cases} I^j & \text{if } i = 0 \leq j; \\ 0 & \text{if } i \geq 1, j \geq 0 \end{cases}$$

where I stands for the Ideal of the diagonal, one gets $F = M(\Omega_X^1 + \mathcal{O})$ in $K(\underline{X})$. Because the fibre of Z over a point x in X consists of the tangent hyperplanes to X at x, we see that Z has codimension rank(F) in $\underline{X}$. The assertion now follows from ([4], Cor. p. 153).

(4.2) Now we show that Z naturally parametrizes a family of quadrics in the bundle of tangent directions of X. For this, restrict the above diagram over Z. Since u_Z is zero, therefore b_2^*w factors through a section $\underline{w}$ of $(b_2^*M \otimes \mathcal{O}_B(2))_Z$. One needs here the fact that $\mathcal{O}_{2E}$ is flat / $\underline{X}$. Recall that E is equal to the projectivized relative tangent bundle of $\underline{X}/S$. Put $F = \Omega_{\underline{X}/S}^1$ restricted to Z, and set $\mathbf{P} = \mathbf{P}(F)$. Thus, $\mathbf{P}$ is the restriction of E over Z. Finally, further restricting $\underline{w}$ to $\mathbf{P}$, we get a section s of $\mathcal{O}_{\mathbf{P}}(2) \otimes L$ (where $L = M_Z$), thereby defining a (possibly nonflat) family of quadrics $Q \to Z$.

By construction, the fibre of Q over $z=(x,H)$ in Z is just the intersection of the effective divisor $(X\cap H)^*-2E(x)$ with $E(x)$. That intersection is precisely the projectivized tangent quadric cone of $X\cap H$ at x, provided the multiplicity m_z of $X\cap H$ at x is 2. Of course, if $m_z>2$ then $Q(z)=E(x)$, that is, s vanishes on the whole fibre of $\mathbb{P}$ over z.

(4.3) Having constructed the family $Q\to Z$, we now interpret the corresponding schemes $\tilde{Z}_r$, or rather their images Z_r in Z. Namely, Z_r consists of the pairs (x,H) such that the projectivized tangent cone of $X\cap H$ at x is a quadric in the projectivized tangent space of X at x with vertex of dimension $\geq r$.

Let $f:\underset{\sim}{G}\to\check{\mathbb{P}}^m$ be the composition $\underset{\sim}{G}\to Z\to\check{\mathbb{P}}^m$. Set $G=G_{r+1}(\Omega^1_X)$ and set

$$A=S_2R+R(\Omega^1_X-R)+\Omega^1_X+\mathcal{O}_G \text{ in } K(G) \tag{4.3.1}$$

(4.4) THEOREM: *If $\tilde{Z}_r$ is empty or of pure dimension $d=m-1-\binom{r+2}{2}$, then*

$$f_*[\tilde{Z}_r]=n_r\check{h}^{m-d}$$

holds in $A(\check{\mathbb{P}}^m)$, where $\check{h}$ denotes the class of a hyperplane and n_r is the degree of the $((n-r)(r+1)+n+1)$th. Chern class of $A\otimes\mathcal{O}_X(1)$.

PROOF: n_r is the degree of $[\tilde{Z}_r]f^*\check{h}^d$. The formula will follow upon pushing down this zero cycle to G.

$$\begin{array}{ccccc} \tilde{Z}_r\subset \underset{\sim}{G} & \hookrightarrow & G\times\check{\mathbb{P}}^m & \to & G\\ \downarrow & & \downarrow & & \\ Z & \hookrightarrow & X\times\check{\mathbb{P}}^m. & & \end{array}$$

Indeed, using the formula for $\tilde{Z}_r$ in $A.(\underset{\sim}{G})$ (2.5) and applying the projection formula, we get, in $A(G\times\check{\mathbb{P}}^m)$,

$$[\tilde{Z}_r]=c_z(\mathsf{H}\otimes M)\cap[\underset{\sim}{G}].$$

Since the vertical maps in the above cartesian square are smooth, the class of $\underset{\sim}{G}$ is the pullback of that of Z in $X\times\check{\mathbb{P}}^m$. By (4.1.1), $[Z]$ is equal to the top Chern class of $K\otimes M$, where we put $K=\Omega^1_X+\mathcal{O}_X$. Thus, we get,

$$[\tilde{Z}_r]=c_z(\mathsf{H}\otimes M)c_{n+2}(K\otimes M)=c_{z+n+2}(A\otimes M).$$

The last equality follows from the definition of A (4.3.1) and the formula (2.6) for H, plus standard properties of Chern classes. Finally, multiplying the last expression by $\check{h}^d$ and pushing down to G, the assertion follows.

The case $r = 0$ is particularly easy to handle explicitly:

(4.5) COROLLARY (Roberts [9]): *We have the formula,*

$$n_0 = \sum_0^{n+1} \binom{n+3-i}{2} \deg((n+1)c_i(\Omega^1_X) + 2c_1(\Omega^1_X)c_{i-1}(\Omega^1_X)).$$

PROOF: We compute $q_*(y)$, where we put $y = c_{2n+1}(A \otimes 0_X(1))$ and denote by q the structure map of $G = G_{r+1}(\Omega^1_X) \to X$. Since we have rank $R = 1$ (as $r = 0$), we may write,

$$A = R\Omega^1_X + \Omega^1_X + \mathcal{O}_G.$$

Putting $x = c_1(0_X(1))$, we get,

$$y = \sum_0^{n+1} \binom{i+2}{2} c_{2n+1-i}(A)x^i$$

(by a general formula for the Chern class of the tensor product by a line bundle and the fact that x^i vanishes for $i > n+1 = \dim X$). Similarly, there are formulas,

$$q_*(c_s(A)) = \sum_0^s c_{s-j}(\Omega^1_X) q_*(c_j(R\Omega^1_X)),$$

and

$$q_*(c_j(R\Omega^1_X)) = \sum_0^j \binom{n+1-j+t}{t} c_{j-t}(\Omega^1_X) q_*(v^t),$$

where we put $v = c_1(R)$. The last expression is equal to zero for $j < n$ (because $q_*(v^t)$ vanishes in that range) or $j > n+1$; it is equal to $(n+1)[X]$ for $j = n$, and to $2c_1(\Omega^1_X)$ for $j = n+1$. Thus, we get

$$q*(y) = \sum_0^{n+1} \binom{i+2}{2} x^i((n+1)c_{n+1-i}(\Omega^1_X) + 2c_1(\Omega^1_X)c_{n-i}(\Omega^1_X)).$$

The assertion follows upon computing degrees of these zero cycles.

(4.6) REMARKS: A local computation reveals that the rank of the jacobian map J_z of $Z \to \check{\mathbf{P}}^m$ at a point $z = (x, H)$ is $m - 1 - (n + 1)$ plus the rank of the hessian matrix of a local equation of $H \cap X$ at x (Katz [6] p. 225). The hessian in turn depends only on the quadratic term q of that local equation. Now, if char $k \neq 2$, the rank of the hessian of q is equal to $n - r$, where r denotes the dimension of the vertex of the projective quadric cut out by q in the space of tangent directions of X at x (Hodge and Pedoe [5] p. 207). Therefore, Z_r is equal to the 1st. order singularity $\{z : \text{Rank}\, J_z \leq m - 1 - (r + 1)\}$, at least set-theoretically. Since Z is itself the 1st. order singularity of $Y \to \check{\mathbf{P}}^m$ (Katz [6], Remarque 3.1.5, p. 218), we see that Z_r is in fact a 2nd. order Thom–Boardman singularity.

(ii) The constructions (4.1–4.3) go through for an arbitrary smooth map $\underline{X} \to S$ together with the scheme of zeros Y of a section of an invertible $\mathcal{O}_{\underline{X}}$-Module M. In particular, (4.4) applies to an arbitrary linear system, under the additional hypothesis that Z be of the right dimension. This condition is of course fulfilled for a very ample linear system.

(4.7) A CLASSICAL EXAMPLE (S. Roberts [10]): We apply the formula (4.5) to get the degree of the "condition" that a surface of degree m in $\mathbf{P}^3$ have a biplanar double point. We take $X = \mathbf{P}^3$ embedded in $\mathbf{P}^N$ by the complete system of surfaces of degree m. Here we have $\deg X = m^3$ and the Chern polynomial $c(\Omega^1_X) = (1 - x)^4$. Thus the sought for number is

$$n_0 = 30m^3 - 120m^2 + 150m - 60.$$

For the sake of completeness, we sketch a proof that the hypothesis of the theorem (4.4) is indeed fulfilled for "nearly all" embeddings $X \subset \mathbf{P}^m$.

Let I_x denote the Ideal sheaf of a point x in X.

(4.8) PROPOSITION: *Suppose $H^1(X, I_x^3\mathcal{O}_X(1)) = 0$ holds for all x in X. Then each $\tilde{Z}_r$ is integral and of the correct dimension.*

PROOF: Set $W = \{(x, H) : x$ is a triple point of $H \cap X\}$. The hypothesis implies $W \to X$ is a smooth projective subbundle of $X \times \check{\mathbf{P}}^m$ of the correct dimension $m - 1 - \binom{n+2}{2}$ ([11] Theorem (4.2.3), (2), p. 39). Furthermore, there is a smooth, surjective map of bundles / X,

$$\begin{tikzcd} U = Z - W \arrow[r] \arrow[dr] & \mathbb{P}(S_2\Omega^1_X)^{\vee} \arrow[d] \\ & X \end{tikzcd}$$

sending (x, H) to the (point representing the) projectivized tangent quadric cone of $H \cap X$ at x. Let $\underline{\tilde{Z}}_r$ denote the subscheme of

$$G \times_X \mathbb{P}(S_2\Omega^1_X)^{\vee}$$

(where $G = G_{r+1}(\Omega^1_X)$) defined in (2.3) for the universal family of quadrics of $\mathbb{P}(\Omega^1_X)$. It is clear that (at least set theoretically), $(\tilde{Z}_r)_U$ is the smooth pullback of $\underline{\tilde{Z}}_r$ via the map just defined. Thus, $(\tilde{Z}_r)_U$ has the correct dimension (in view of (3.1)). To finish the proof, it suffices to show that the dimensions of the fibres of $\tilde{Z}_r$ over W are appropriately bounded. This follows from the observation that $(\tilde{Z}_r)_W$ is equal to the whole of $G \times_X W$, at least set theoretically. The dimension of the latter is $(n-r)(r+1) + \dim W$, which is strictly less than the correct dimension of $\tilde{Z}_r$. Since the dimension of each component of $\tilde{Z}_r$ must be bigger than or equal to the correct one (as $\tilde{Z}_r$ is locally cut out by rank H equations), it follows that no such component lies over W, whence $\tilde{Z}_r$ is the closure of $(\tilde{Z}_r)_U$. Thus, $\tilde{Z}_r$ is integral and of the correct dimension as asserted.

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(Oblatum 13–IX–1977)

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