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RECURSION RELATIONS AND THE ASYMPTOTIC BEHAVIOR OF THE EIGENVALUES OF THE LAPLACIAN

Peter B. Gilkey*

Introduction

There is a deep relation between the local geometry of a Riemannian manifold and the global geometry of the spectrum of the Laplacian. The heat equation gives rise to global invariants of the spectrum which describe the asymptotic behavior of the eigenvalues. These global invariants can be evaluated by integrating local invariants of the metric. McKean and Singer [20] and Berger [4] have used the techniques of classical differential geometry to compute the first 3 terms in the asymptotic expansion of the Laplacian on functions. Sakai [23] has computed the fourth term in this expansion and Patodi [22] has extended these results to the Laplacian acting on forms.

In an earlier paper [9] we computed the first four terms in the asymptotic expansion of an arbitrary second order differential operator with leading symbol given by the metric tensor. Our approach used the calculus of pseudo-differential operators to compute the parametrix and had the disadvantage of involving long and tedious combinatorial calculations. In this paper, we will give an independent derivation of the results of [9] using simpler methods which are based on the purely formal properties of the heat equation. This will yield some new results concerning the highest order part of the general term in the asymptotic expansion. We will also discuss the asymptotics which arise from a study of Hill's equation with suitable boundary conditions on the unit interval.

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The first part of this paper is devoted to the 1-dimensional case and the asymptotics which arise from a potential function. Some of the results of this section have been previously obtained by McKean and van Moerbeke [19]. We will then discuss the general case by using similar techniques.

We first review briefly the asymptotics that we shall be studying. Let M be a smooth compact Riemannian manifold without boundary of dimension m . Let G be the metric. In local coordinates $X = (x_1, \dots, x_m)$, let $G = g_{ij} dx^i dx^j$. We adopt the convention of summing over repeated indices. Indices i, j, k, \dots run from 1 thru m and index a frame for the tangent space $T(M)$. Let (g^{ij}) be the inverse matrix of the matrix (g_{ij}) and let $g = \det(g_{ij})^{1/2}$. The Riemannian measure on M is given by $|d\text{vol}| = g dx^1 \wedge \dots \wedge dx^m$.

Rather than restricting attention to the geometric Laplacian, we shall consider more general differential operators. Let V be a smooth vector bundle over M and let $D: C^\infty(V) \rightarrow C^\infty(V)$ be a second order differential operator with leading symbol given by the metric tensor. (The reals are embedded in the algebra of endomorphisms of V to act by scalar multiplication). If we choose a local frame for V , we may express

$$D = -(g^{ij} \partial/\partial x_i \partial x_j + P^k \partial/\partial x_k + Q).$$

The square matrices P^k and Q are not invariantly defined. They depend on the choice of the frame and of the coordinate system.

Let V_x be the fibre of V over a point x . Choose a smooth fibre metric on V . Let $L^2(V)$ be the completion of $C^\infty(V)$ with respect to the global integrated inner product. As a Banach space, $L^2(V)$ is independent of the Riemannian metric on M and of the fibre metric on V . For $t > 0$, $\exp(-tD): L^2(V) \rightarrow C^\infty(V)$ is an infinitely smoothing operator of trace class. Let $K(t, x, y, D): V_y \rightarrow V_x$ be the kernel of $\exp(-tD)$. K is a smooth endomorphism-valued function of (t, x, y) . One can use the calculus of pseudo-differential operators depending on a complex parameter as developed by Seeley in [24] to show that $K(t, x, y, D)$ vanishes to infinite order for $x \neq y$. If $x = y$, K has an asymptotic expansion as $t \rightarrow 0^+$ of the form:

$$K(t, x, x, D) \sim (4\pi t)^{-m/2} \sum_{n=0}^{\infty} t^n E_n(x, D).$$

The normalizing constant of $(4\pi)^{-m/2}$ was chosen so the first term $E_0(x, D) = I$. The surprising fact is that the endomorphisms $E_n(x, D)$

are *local* invariants of the differential operator D . Relative to a local frame for V and in a local system of coordinates, we can express $E_n(x, D)$ functorially as a non-commutative polynomial in the derivatives of the metric tensor and in the derivatives of the matrices P^k and Q with coefficients which are smooth functions of the metric. This polynomial is universal in the sense that the coefficients depend only on the dimension m and are independent of the vector bundle V and the operator D .

If D is self-adjoint, let $\{\lambda_i, \theta_i\}_{i=1}^\infty$ be a spectral resolution of D into a complete orthonormal basis of eigensections θ_i and eigenvalues λ_i . For such an operator, the kernel function is given by

$$K(t, x, y, D) = \sum_i \exp(-t\lambda_i) \theta_i(x) \otimes \theta_i^*(y).$$

Therefore

$$\begin{aligned} \text{Tr}(\exp(-tD)) &= \sum_i \exp(-t\lambda_i) = \int_M \text{Tr}(K(t, x, x, D)) |d\text{vol}(x)| \\ &\sim (4\pi t)^{-m/2} \sum_{n=0}^\infty t^n \int_M \text{Tr}(E_n(x, D)) |d\text{vol}(x)|. \end{aligned}$$

Let $a_n(x, D) = \text{Tr}(E_n(x, D))$ and $a_n(D) = \int_M a_n(x, D) |d\text{vol}(x)|$. $a_n(D)$ is a spectral invariant of the operator D .

In the first section, we will express the invariants $E_n(x, D)$ in terms of tensorial expressions. We will develop some of the formal properties of these local invariants. In the second section, we will consider the case $m = 1$ and compute E_n for $n = 0, 1, 2, 3, 4$ by using various recursive relations. We will also obtain information on the leading terms for general dimension m . We will apply this to generalize a theorem of McKean and van Moerbeke to arbitrary dimension m . In the third section, we will consider the asymptotic invariants which arise from suitable boundary problems on an interval. In the fourth section, we will generalize these methods to compute E_n for $n = 0, 1, 2, 3$ and general dimension m . We shall also obtain some interesting elliptic complexes.

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Section 1

Let $\alpha = (a_1, \dots, a_m)$ be a multi-index. Let $d_X^\alpha = (\partial/\partial x_1)^{a_1} \dots (\partial/\partial x_m)^{a_m}$ and let $|\alpha| = a_1 + \dots + a_m$. If R is a scalar or matrix valued function, let $R_{|\alpha} = d_X^\alpha R$. We adopt the convention that $R = R_{|\alpha}$ if $|\alpha| = 0$. Let α, β, γ be multi-indices. Let Φ denote the set of formal variables $\Phi = \{g_{ij\alpha}, P^k_{|\beta}, Q_{|\gamma}\}$ for $|\alpha| > 0$; these variables represent the formal derivatives of the symbol of D . Let $\bar{\mathcal{P}}_m$ be the non-commutative polynomial algebra in the variables of Φ with coefficients which are smooth functions of the $\{g_{ij}\}$ variables. Let X be a system of local coordinates on M and let s be a local frame for V . If $P \in \bar{\mathcal{P}}_m$, we can evaluate $P(X, s, x, D)$ to define an endomorphism of V_x . If this endomorphism is independent of the choice of X and s , we say that P is invariant. We will denote the common value by $P(x, D)$. Let \mathcal{P}_m be the sub-algebra of $\bar{\mathcal{P}}_m$ consisting of all invariant polynomials. For each (m, n) , $E_n(x, D)$ defines an element of \mathcal{P}_m .

We define a grading on the algebra $\bar{\mathcal{P}}_m$ as follows: let $\text{ord}(g_{ij\alpha}) = |\alpha|$, $\text{ord}(P^k_{|\beta}) = |\beta| + 1$, and $\text{ord}(Q_{|\gamma}) = |\gamma| + 2$. This defines the notion of order on the generators Φ ; we extend this in the natural way to define a grading on \mathcal{P}_m .

LEMMA 1.1: *Let $P \in \bar{\mathcal{P}}_m$. P is homogeneous of order n if and only if*

$$P(c^{-1}X, s, x, c^2D) = c^n P(X, s, x, D)$$

for every (X, s, D) and for every positive constant c .

PROOF OF LEMMA 1.1: Let $Y = c^{-1}X$ be a new coordinate system on M . $d_Y^\alpha = c^{|\alpha|} d_X^\alpha$. If G is the metric defined by the leading symbol of D , then $c^{-2}G$ is the metric defined by the leading symbol of c^2D . In the old coordinate system,

$$c^2D = -(c^2g^{ij}\partial^2/\partial x_i\partial x_j + c^2P^k\partial/\partial x_k + c^2Q).$$

Consequently, in the new coordinate system,

$$\begin{aligned} c^2D &= -(g^{ij}\partial^2/\partial y_i\partial y_j + cP^k\partial/\partial y_k + c^2Q) \\ g_{ij\alpha}(c^{-1}X, s, x, c^2D) &= c^{|\alpha|}g_{ij\alpha}(X, s, x, D) \\ P^k_{|\beta}(c^{-1}X, s, x, c^2D) &= c^{|\beta|+1}P^k_{|\beta}(X, s, x, D) \\ Q_{|\gamma}(c^{-1}X, s, x, c^2D) &= c^{|\gamma|+2}Q_{|\gamma}(X, s, x, D) \end{aligned}$$

This proves $P(c^{-1}X, s, x, c^2D) = c^n P(X, s, x, D)$ if and only if P is homogeneous of order n and completes the proof of the lemma. ■

If $P \in \mathcal{P}_m$, decompose $P = P_0 + \dots + P_n$ as a sum of homogeneous polynomials. By Lemma 1.1,

$$P_n(X, s, x, D) = \frac{1}{n!} (d/dc)^n P(c^{-1}X, s, x, c^2D) = \frac{1}{n!} (d/dc)^n P(x, c^2D).$$

This implies P_n is invariant. Similarly by induction we show each of the P_k is invariant separately. If $\mathcal{P}_{n,m}$ is the subspace of invariant polynomials of order n , this gives a direct sum decomposition of $\mathcal{P}_m = \bigoplus \mathcal{P}_{n,m}$ as a graded algebra.

LEMMA 1.2: For each m , E_n defines an invariant $E_{n,m} \in \mathcal{P}_m$. $E_{n,m}$ is homogeneous of order $2n$.

PROOF: Let $c \neq 0$. Since $\exp(-t(c^2D)) = \exp(-c^2t(D))$, the kernel functions satisfy

$$K(t, x, y, c^2D)|\text{dvol}|(c^{-2}G) = K(c^2t, x, y, D)|\text{dvol}(G)|$$

where G denotes the metric induced by the leading symbol of D . Since $|\text{dvol}(G)| = c^m |\text{dvol}(c^{-2}G)|$, this implies that

$$(4\pi t)^{-m/2} \sum_{n=0}^{\infty} t^n E_n(x, c^2D) c^{-m} \sim (4\pi c^2 t)^{-m/2} \sum_{n=0}^{\infty} (c^2 t)^n E_n(x, D).$$

Therefore $E_n(x, c^2D) = c^{2n} E_n(x, D)$ so $E_{n,m} \in \mathcal{P}_{n,m}$ for any m by Lemma 1.1. ■

We will now express the invariants $E_{n,m}$ in terms of tensorial expressions. We will show that these expressions are universal; this will enable us to drop the dependence on the dimension m and simply talk of the invariants E_n . Let ∇ be a connection on V and let ∇^G be the Levi-Civita connection on $T(M)$. These two connections induce a connection we will also denote by ∇ on $T^*M \otimes V$. The Bochner or reduced Laplacian D is defined by the diagram:

$$\begin{array}{ccc} D_{\nabla}: C^{\infty}(V) & \xrightarrow{\nabla} & C^{\infty}(T^*M \otimes V) \\ & & \xrightarrow{\nabla} C^{\infty}(T^*M \otimes T^*M \otimes V) \xrightarrow{-G \otimes 1} C^{\infty}(V). \end{array}$$

Let ω be the connection matrix of ∇ on V . Decompose $\omega = dx^k \otimes \omega_k$ where the ω_k are endomorphisms of V . Let $\Gamma_{ij}^k = \frac{1}{2}g^{kh}(g_{ihj} + g_{jhi} - g_{ijh})$ be the Christoffel symbols. Then

$$D_{\nabla} = -(g^{ij}\partial^2/\partial x_i\partial x_j + (2g^{ij}\omega_j - g^{jk}\Gamma_{jk}^i)\partial/\partial x_i + (g^{ij}\omega_{ij} + g^{ij}\omega_i\omega_j - g^{jk}\omega_i\Gamma_{jk}^i))$$

There is a unique connection ∇ on V so that $D_{\nabla} - D$ is a 0th order operator. The connection matrix of ∇ is given by $w_k = \frac{1}{2}(g_{ki}P^i + g_{ki}g^{jh}\Gamma_{jh}^i) \in \bar{\mathcal{P}}_m$.

Fix this connection henceforth. Let $\mathcal{E} = D_{\nabla} - D$. \mathcal{E} is an invariantly defined endomorphism of V given in local coordinates by the formula:

$$\mathcal{E} = Q - g^{ij}\omega_{ij} - g^{ij}\omega_i\omega_j + g^{jk}\omega_i\Gamma_{jk}^i \in \bar{\mathcal{P}}_m.$$

The triple (G, ∇, \mathcal{E}) is determined by the operator D . Conversely, we may express the derivatives of P^k and of Q in terms of the $\{g_{ij\alpha}, \omega_{ij\beta}, \mathcal{E}_{i\gamma}\}$ variables with coefficients which are smooth functions of the metric. This implies that D is determined by the triple (G, ∇, \mathcal{E}) and that we may regard $\bar{\mathcal{P}}_m$ as a free algebra in the $\{g_{ij\alpha}, \omega_{ij\beta}, \mathcal{E}_{i\gamma}\}$ variables by making an appropriate change of basis.

Let $\{e_1, \dots, e_m\}$ be a local orthonormal frame for $T(M)$. Let R_{ijkl} be the components of the curvature tensor of the Levi-Civita connection. Let W_{ij} be the components of the curvature of the connection ∇ on V .

$$R_{ijkl} = G(\{\nabla_{e_i}^G \nabla_{e_j}^G - \nabla_{e_j}^G \nabla_{e_i}^G - \nabla_{[e_i, e_j]}^G\} e_k, e_l)$$

$$W_{ij} = \nabla_{e_i} \nabla_{e_j} - \nabla_{e_j} \nabla_{e_i} - \nabla_{[e_i, e_j]}: V \rightarrow V$$

W_{ij} is an endomorphism valued tensor. R_{ijkl} is a scalar tensor which we let act on V by scalar multiplication. If θ is any tensor field on M , let $\theta_{:j_1 \dots j_s}$ be the tensor field obtained by multiple covariant differentiation. Let Ψ be the set of polynomials $\{R_{i_1 i_2 i_3 i_4 j_1 \dots j_s}, W_{i_1 i_2 j_1 \dots j_s}, \mathcal{E}_{:j_1 \dots j_s}\}$ for $s = 0, 1, \dots$ and for all possible values of $\{i_1, i_2, i_3, i_4, j_1, \dots, j_s\}$. For a fixed value of s , these polynomials are homogeneous of order $2 + s$. They are tensorial in the sense that their value only depends on the value of the local frame at a given point.

LEMMA 1.3: *We can construct invariant polynomials by taking various non-commutative products of elements of $\Psi = \{R_{i_1 i_2 i_3 i_4 j_1 \dots j_s}, W_{i_1 i_2 j_1 \dots j_s}, \mathcal{E}_{:j_1 \dots j_s}\}$ and by contracting pairs of indices. The set of all invariant polynomials \mathcal{P}_m is spanned by the collection of invariants constructed in this way.*

PROOF: Let X be a system of geodesic coordinates centred at some point x_0 of M . Then $g_{ij}(x_0) = \delta_{ij}$ is the Kronecker index. Using Cartan's lemma (which can be found in [1]), we can express all the partial derivatives of the metric tensor at x_0 in terms of the covariant derivatives of the components of the curvature tensor R_{ijkl} . Similarly, let $s(x_0)$ be a basis for the fibre of V at x_0 . Extend s to a local frame for V by parallel transport along the geodesic rays from x_0 . The coordinate system is unique up to the action of the orthogonal group $O(m)$. The frame s is unique up to the action of the general linear group $GL(\dim(V))$. Relative to such a parallel frame, we may express all the partial derivatives of the connection matrix in terms of the derivatives of the metric and the $W_{i_1 i_2; j_1 \dots j_s}$ variables. Finally, we may express all the partial derivatives of the endomorphism \mathcal{E} in terms of the covariant derivatives of \mathcal{E} and in terms of the derivatives of the metric and of the connection matrix. We may therefore express any $P \in \mathcal{P}_m$ as a polynomial in the Ψ variables.

Since P is invariant, it is independent of the choice of frame and coordinate system. If we express $P = P(\Psi)$ in terms of the Ψ variables, then this is invariant under the action of $O(m) \times GL(\dim(V))$. Invariance under the action of $GL(\dim(V))$ is automatic for such an expression since the Ψ variables are tensorial elements of the bundle of endomorphisms of V which are independent of the choice of frame. If $\dim(V) = 1$, we could apply H. Weyl's theorem [27] on the invariants of the orthogonal group to conclude that any invariant can be expressed in terms of contractions of indices. It is an easy exercise (see for example [1]) to extend H. Weyl's theorem to cover endomorphism-valued invariants as well. This shows that any invariant polynomial $P \in \mathcal{P}_m$ can be expressed in terms of contractions of non-commutative expressions of elements of Ψ . This completes the proof of Lemma 1.3 ■

We can write a basis for $\mathcal{P}_{n,m}$ by first writing all possible contractions of indices and by then reducing using the generalized Bianchi identities.

LEMMA 1.4:

- (a) $\{1\}$ is a basis for $\mathcal{P}_{0,m}$.
- (b) $\{\mathcal{E}, R_{ijij}\}$ spans $\mathcal{P}_{2,m}$ and is a basis if $m \geq 2$.
- (c) $\{\mathcal{E}^2, \mathcal{E}_{;kk}, \mathcal{E}R_{ijij}, W_{ij}W_{ij}, R_{ijij}R_{klkl}, R_{ijik}R_{ljlj}, R_{ijkl}R_{ijkl}, R_{ijij;kk}\}$ spans $\mathcal{P}_{4,m}$ and is a basis if $m \geq 4$.

There is a similar basis for $\mathcal{P}_{6,m}$ given in [9]. We sum over repeated

indices in these expressions. The elements are universal in the sense that the only dependence on the dimension m is the range over which the indices are summed. If the dimension m is low, these elements are not linearly independent. For example, if $m = 2$ then $R_{ijij}R_{klkl} = 2R_{ijik}R_{ijlk} = R_{ijkl}R_{ijkl}$.

There is a natural restriction map $r: \bar{\mathcal{P}}_m \rightarrow \bar{\mathcal{P}}_{m-1}$ defined as follows: let Φ be the collection of variables $\{g_{ij\alpha}, P^k_{\beta}, Q_{\gamma}\}$ for $|\alpha| > 0$. Let $\alpha = (a_1, \dots, a_m)$, let $\beta = (b_1, \dots, b_m)$, and let $\gamma = (c_1, \dots, c_m)$. If $\phi \in \Phi$, define $\text{deg}_l(\phi)$ by:

$$\text{deg}_l(g_{ij\alpha}) = \delta_{i,l} + \delta_{j,l} + a_l, \quad \text{deg}_l(P^k_{\beta}) = \delta_{k,l} + b_l, \quad \text{deg}_l(Q_{\gamma}) = c_l.$$

$\text{deg}_l(\phi)$ is the total number of times that the index l appears in the variable ϕ . If the index m does not appear in ϕ , then $\phi \in \bar{\mathcal{P}}_{m-1}$. Define

$$r(\phi) = \begin{cases} 0 & \text{if } \text{deg}_m(\phi) \neq 0 \\ \phi & \text{if } \text{deg}_m(\phi) = 0 \end{cases} \in \bar{\mathcal{P}}_{m-1}$$

and extend r to an algebra homomorphism from $\bar{\mathcal{P}}_m \rightarrow \bar{\mathcal{P}}_{m-1}$. Since r preserves the order of a polynomial, r induces maps $r: \bar{\mathcal{P}}_{n,m} \rightarrow \bar{\mathcal{P}}_{n,m-1}$.

We can describe the action of r geometrically as follows: let M_1 be a manifold of dimension $m - 1$. Let (V_1, D_1) be given over M_1 . Let $M = M_1 \times S^1$ with the product metric where S^1 is the unit circle. Let $\pi_1: M \rightarrow M_1$ be the projection onto the first factor and let $V = \pi_1^* V_1$ be the pull-back bundle over M . If s_1 is a local frame for V_1 , let $s = \pi_1^* s_1$ be the induced local frame for V . Let $X_1 = (x_1, \dots, x_{m-1})$ be local coordinates on M_1 and let x_m be arc length on S^1 . Let $X = (x_1, \dots, x_{m-1}, x_m)$ be the product coordinates on M . Define $D = D_1 - (\partial^2 / \partial x_m^2): C^\infty(V) \rightarrow C^\infty(V)$. The symbol of D is independent of the last coordinate so $\phi(X, s, x, D) = 0$ if $\text{deg}_m(\phi) \neq 0$. Conversely, if $\text{deg}_m(\phi) = 0$, then $\phi(X, s, x, D) = r(\phi)(X_1, s_1, \pi_1 x, D_1)$. Therefore, if $P \in \bar{\mathcal{P}}_m$, then

$$r(P)(X_1, s_1, \pi_1 x, D_1) = P(X, s, x, D) \quad \text{for } D = D_1 - (\partial^2 / \partial x_m^2).$$

This shows that if P is invariant, then $r(P)$ is invariant. Therefore r induces an algebra morphism $r: \mathcal{P}_m \rightarrow \mathcal{P}_{m-1}$ and $r: \mathcal{P}_{n,m} \rightarrow \mathcal{P}_{n,m-1}$ as well.

The restriction map r is defined by setting to zero any variable involving the last index. By Lemma 1.3, we can express any invariant polynomial in terms of contractions of indices of products of elements of Ψ . The restriction is defined on such an element by restrict-

ing the limits of summation to range from 1 thru $m - 1$ instead of from 1 to m . This shows that r is always surjective.

THEOREM 1.5: *For any integers (n, m)*

- (a) $r: \mathcal{P}_{n,m} \rightarrow \mathcal{P}_{n,m-1}$ is always surjective.
- (b) $r: \mathcal{P}_{n,m} \rightarrow \mathcal{P}_{n,m-1}$ is injective if $n < m$.
- (c) The kernel of $r: \mathcal{P}_{n,m} \rightarrow \mathcal{P}_{n,m-1}$ is the integrand of the Chern-Gauss-Bonnet theorem which is defined in [5].

PROOF: Let $\mathcal{P}_{n,m}^G$ be the subset of $\mathcal{P}_{n,m}$ consisting of polynomials in the $\{g_{ij\alpha}\}$ variables alone. Then r restricts to a map $r: \mathcal{P}_{n,m}^G \rightarrow \mathcal{P}_{n,m-1}^G$. Theorem 1.5 is proved in [8] and [12] if we replace $\mathcal{P}_{n,m}$ by $\mathcal{P}_{n,m}^G$. The proof given in these references for $\mathcal{P}_{n,m}^G$ immediately generalizes to $\mathcal{P}_{n,m}$. We shall omit the details of the proof since we shall not need this result in our development. ■

We shall prove that $r(E_{n,m}) = E_{n,m-1}$. Therefore, if we express $E_{n,m}$ in terms of a universal basis such as that given in Lemma 1.4, the coefficients will be independent of the dimension m . Consequently, we shall drop the dependence on the dimension in our notation in the following sections. Before proving this result, we must develop some other functorial properties of the invariants E_n .

Let M_i be Riemannian manifolds of dimension m_i with vector bundles V_i and operators D_i for $i = 1, 2$. Let $M = M_1 \times M_2$ with the product metric and let $\pi_i: M \rightarrow M_i$ be projection on the i^{th} factor. Let $V = \pi_1^* V_1 \otimes \pi_2^* V_2$ and let $D = D_1 \otimes 1 + 1 \otimes D_2: C^\infty(V) \rightarrow C^\infty(V)$. The connection ∇ defined by D on V is $\pi_1^*(\nabla_1) \otimes 1 + 1 \otimes \pi_2^*(\nabla_2)$ and the endomorphism \mathcal{E} defined by D on V is $\pi_1^*(\mathcal{E}_1) \otimes 1 + 1 \otimes \pi_2^*(\mathcal{E}_2)$. Since the operator splits, $\exp(-tD) = \exp(-tD_1) \otimes \exp(-tD_2)$. Therefore $K(t, (x_1, x_2), (y_1, y_2), D) = K(t, x_1, y_1, D_1) \otimes K(t, x_2, y_2, D_2)$. By comparing the coefficient of t in the asymptotic expansions, this proves:

LEMMA 1.6:

$$E_{n,m}((x_1, x_2), D_1 \otimes 1 + 1 \otimes D_2) = \sum_{i+j=n} E_{i,m_1}(x_1, D_1) \otimes E_{j,m_2}(x_2, D_2).$$

Let $\dim(M_1) = m_1 = m - 1$ and let M_2 be the unit circle. Let $D_2 = -d^2/dx_m^2$ and let V_2 be the trivial line bundle over S^1 . Since all the derivatives of the symbol of D_2 vanish identically, $E_{n,1}(x_2, D_2) = 0$ for $n > 0$. Since $E_0 = I$,

$$E_{n,m}((x_1, x_2), D_1 \otimes 1 - (\partial^2/\partial x_m^2)) = E_{n,m-1}(x_1, D_1).$$

This implies that $r(E_{n,m}) = E_{n,m-1}$ and completes the proof of the Lemma.

LEMMA 1.7: $r(E_{n,m}) = E_{n,m-1}$ for all (n, m) . In other words, if we express E_n in terms of a universal basis such as that given in Lemma 1.3, the coefficients are independent of the dimension m of the manifold.

We will use Lemma 1.7 to apply the results of section 2 for 1-dimensional manifolds in section 4 to give information about E_n for arbitrary dimension.

There is one final functorial property of the invariants E_n that it will be convenient to have.

LEMMA 1.8: Let ϵ be a real constant. Then

$$E_n(x, D - \epsilon I) = \sum \epsilon^k / k! E_{n-k}(x, D).$$

In particular, this implies, that the coefficient of \mathcal{E}^n in $E_n(x, D)$ is $1/n!$

PROOF: We have chosen the normalization so $E_0(x, D) = I$. The endomorphism defined by $D - \epsilon I$ is $\mathcal{E} + \epsilon I$. Since $\exp(-t(D - \epsilon I)) = \exp(t\epsilon) \exp(-tD)$, the kernel functions satisfy the relation:

$$K(t, x, y, D - \epsilon I) = \exp(t\epsilon) K(t, x, y, D).$$

By comparing powers in the asymptotic expansion, this proves the desired relationship. If $E_n(x, D) = c(n)\mathcal{E}^n + \text{other terms}$, then $E_n(x, D - \epsilon I) = c(n)\epsilon^n + \text{lower powers of } \epsilon$. This proves that $c(n) = 1/n!$ and completes the proof of the lemma. ■

We apply Lemma 1.8 to the following situation. Let $\Delta_p^m = (d\delta + \delta)$ be the ordinary Laplacian acting on the space of smooth p -forms on a manifold of dimension m . Let $\Delta\wp$ be the reduced or Bochner Laplacian defined by the Levi-Civita connection. Both these operators are natural in the sense of Epstein and Stredder [7, 26]. Let $\Delta_p^m(s) = s\Delta_p^m + (1-s)\Delta\wp$ for $s \in [0, 1]$ be a family of non-negative self-adjoint elliptic second order differential operators. Let $\text{spec}^p(M, s) = \{0 \leq \lambda_1^p(s) \leq \lambda_2^p(s) \dots\}$ be the spectrum of $\Delta_p^m(s)$ repeated according to multiplicity. The functions $\lambda_i^p(s)$ are continuous functions which describe the spectral flow from the operator Δ_p^m to $\Delta\wp$.

If M is Einstein, $\lambda_i^j(s)$ is linear. In general, however, these functions are not smooth.

If M is two-dimensional, let $K = -R_{1212}$ be the scalar curvature of M . $K_*|dvol|$ is a positive measure on the real line. If A is an open subset of R , then

$$K_*|dvol|(A) = \int_{K^{-1}(A)} |dvol|.$$

More generally, let M be of arbitrary dimension and let $\rho_{ij} = -R_{ikjk}$ be the Ricci tensor. Let $\rho_1(x) \leq \dots \leq \rho_n(x)$ be the eigenvalues of the Ricci tensor. Let $\rho_*|dvol|$ be the positive measure on the real line defined by:

$$\rho_*|dvol|(A) = \sum_{i=1}^n \int_{\rho_i^{-1}(A)} |dvol|.$$

This measure is a global invariant of the geometry of M .

THEOREM 1.9: *Let M and M' be Riemannian manifolds. Suppose $\text{spec}^1(M, s) = \text{spec}^1(M', s)$ for $0 \leq s \leq 1$. Then $\rho_*|dvol|_M = \rho_*|dvol|_{M'}$.*

PROOF: Let $D(s) = s\mathcal{E} + D_\nabla$ for some connection ∇ and some endomorphism \mathcal{E} . Then $E_n(D(s))$ is a polynomial of order n in s ; the coefficient of s^n is $\mathcal{E}^n/n!$ by Lemma 1.8. Given the value of $E_n(D(s))$ for at least $n + 1$ different values of s , we can recover the coefficients and consequently $\int_M |\text{Tr}(\mathcal{E}^n)| dvol$ is a spectral invariant of the operators $D(s)$ for $n + 1$ values of s . If Δ_1^n is the Laplacian acting on 1-forms, the induced connection is the Levi-Civita connection and the induced endomorphism is the Ricci tensor [11]. Consequently, $\int_M \text{Tr}(\rho^n)|dvol|$ is an invariant of $\text{spec}^1(M, s)$ given any $n + 1$ distinct values of s .

Let f be a continuous real valued function on the real line and let $f(\rho)(M) = \sum_{i=1}^n \int_M f(\rho_i(x))|dvol|$. We have just shown under the hypothesis of Theorem 1.8 that $x^k(\rho)(M) = x^k(\rho)(M')$ for $k = 0, 1, \dots$. Since the range of $\rho_i(x)$ is bounded and since the polynomials are dense in the ring of continuous functions on compact subsets, this implies $f(\rho)(M) = f(\rho)(M')$ for any f . This implies that $\rho_*|dvol|_M = \rho_*|dvol|_{M'}$ and completes the proof of the theorem. ■

The measure $\rho_*|dvol|$ contains a great deal of information about the Ricci tensor. We can deduce the following consequences from Theorem 1.8.

- COROLLARY 1.10: *Suppose $\text{spec}^1(M, s) = \text{spec}^1(M', s)$ for $0 \leq s \leq 1$.*
- (a) *if the eigenvalues of the Ricci tensor on M are constant, the same is true for M' .*
 - (b) *If there exist constants a and b so that the quadratic forms G and ρ satisfy $aG \leq \rho \leq bG$ on M , the same is true on M' .*

PROOF: The eigenvalues of the Ricci tensor on M are constant if and only if the corresponding measure $\rho_*|d\text{vol}|$ is pure point. This proves (a); (b) follows similarly. It is worth noting that to prove (a) it is not necessary to assume $\text{spec}^1(M, s) = \text{spec}^1(M', s)$ for all s but only for $m + 1$ distinct values. ■

We can restate Cor. 1.10(b) as follows: it is possible to hear the pinching of the Ricci tensor. In particular one can determine positive, zero, or negative Ricci curvature from the spectral geometry of the manifold. It is an interesting open question as to whether it is possible to hear the pinching of the full curvature operator by performing a similar analysis on $\text{spec}^2(M, s)$.

There are many other theorems of the type of Cor. 1.10 which we shall discuss in later sections which relate the global geometry of the manifold to the spectrum.

Section 2

In this section, we will discuss the invariants $E_n(x, D)$ for the circle. We will also discuss the corresponding invariants which arise from suitable boundary problems on the interval. Parametrize the circle by arc length. Since the metric is flat and since any connection on a vector bundle over the circle is locally flat, the $R_{ijkl, \dots}$ and $W_{ij, \dots}$ tensors vanish identically. We can always choose a flat frame locally. Relative to such a frame, $D = -d^2/dx^2 - \mathcal{E}$ and covariant differentiation agrees with ordinary differentiation. Let $\mathcal{E}^{(0)} = \mathcal{E}$ and $\mathcal{E}^{(n)} = d^n \mathcal{E}/dx^n$. $\mathcal{E}^{(n)}$ is of order $n + 2$ and $E_n(x, D)$ is a homogeneous polynomial of order $2n$ in the $\{\mathcal{E}^{(0)}, \dots, \mathcal{E}^{(2n-2)}\}$ variables.

THEOREM 2.1: *On the circle,*

$$(a) \quad E_n(x, D) = \frac{(n-1)!}{(2n-1)!} \left\{ \mathcal{E}^{(2n-2)} + \sum_{p+q=2n-4} \binom{2n-2}{p+1} + (-1)^p \mathcal{E}^{(p)} \mathcal{E}^{(q)} + \dots \right\}$$

we have omitted terms which are cubic or higher powers.

$$(b) \quad a_n(D) = \int_M \text{Tr}(E_n(x, D)) |d\text{vol}|$$

$$= (-1)^n \frac{(n-1)!}{(2n-2)!} \int_M \text{Tr}(\mathcal{E}^{(n-2)} \mathcal{E}^{(n-2)}) |d\text{vol}| \\ + \int_M Q_n(\mathcal{E}, \dots, \mathcal{E}^{(n-3)}) |d\text{vol}|.$$

Q_n is a polynomial which is homogeneous of order $2n$ and involves the derivatives of \mathcal{E} up to order $n-3$. We assume that $n \geq 2$.

- (c) Let \mathcal{E} be scalar and let $|\mathcal{E}|_n^2 = \sum_{k=0}^n \int_M \mathcal{E}^{(k)} \mathcal{E}^{(k)} |d\text{vol}|$ be the Sobolev norm of the function \mathcal{E} . Let $|\mathcal{E}|_\infty$ be the sup norm of \mathcal{E} . Then $|\mathcal{E}|_n^2 \leq (2n-2)!(n-1)!(-1)^n a_{n+2}(D) + q_n(a_0 D, \dots, a_{n+1} D, |\mathcal{E}|_\infty)$ for $n > 0$. q_n is a continuous function of its arguments.

We will prove (a) later in this section. Part (b) follows from (a) by integration by parts and part (c) follows from (b) by the usual Sobolev estimates. We omit the details of the proof of (c) since we will discuss similar estimates in more detail later in this section.

In addition to these general results, we will also prove

THEOREM 2.2: On the circle:

- (a) $E_0(D) = I$, $a_0(D) = \dim(V) \cdot \text{vol}(M)$
 (b) $E_1(D) = \mathcal{E}$, $a_1(D) = \int_M \text{Tr}(\mathcal{E}) |d\text{vol}|$
 (c) $E_2(D) = \frac{1}{6}(\mathcal{E}^{(2)} + 3\mathcal{E}^2)$, $a_2(D) = \frac{1}{6} \int_M 3 \text{Tr}(\mathcal{E}^2) |d\text{vol}|$
 (d) $E_3(D) = \frac{1}{60}(\mathcal{E}^{(4)} + 5\mathcal{E}^{(2)}\mathcal{E} + 5\mathcal{E}\mathcal{E}^{(2)} + 5\mathcal{E}^{(1)}\mathcal{E}^{(1)} + 10\mathcal{E}^3)$
 $a_3(D) = \frac{1}{60} \int_M \text{Tr}(-5\mathcal{E}^{(1)}\mathcal{E}^{(1)} + 10\mathcal{E}^3) |d\text{vol}|$
 (e) $E_4(D) = \frac{1}{840}(\mathcal{E}^{(6)} + 7\mathcal{E}^{(4)}\mathcal{E} + 7\mathcal{E}\mathcal{E}^{(4)} + 14\mathcal{E}^{(3)}\mathcal{E}^{(1)} + 14\mathcal{E}^{(1)}\mathcal{E}^{(3)} \\ + 21\mathcal{E}^{(2)}\mathcal{E}^{(2)} + a\mathcal{E}^{(2)}\mathcal{E}\mathcal{E} + a\mathcal{E}\mathcal{E}^{(2)} + (70-2a)\mathcal{E}\mathcal{E}^{(2)} \\ + b\mathcal{E}^{(1)}\mathcal{E}^{(1)}\mathcal{E} + b\mathcal{E}\mathcal{E}^{(1)}\mathcal{E}^{(1)} + (70-2b)\mathcal{E}^{(1)}\mathcal{E}\mathcal{E}^{(1)} + 35\mathcal{E}^4)$
 $a_4(D) = \frac{1}{840} \int_M \text{Tr}(7\mathcal{E}^{(2)}\mathcal{E}^{(2)} - 70\mathcal{E}\mathcal{E}^{(1)}\mathcal{E}^{(1)} + 35\mathcal{E}^4) |d\text{vol}|$

In (3) a and b are certain universal constants. Similar results have been obtained by McKean and van Moerbeke [19].

To prove these theorems, we derive some inductive relations among the coefficients. Let $V = S^1 \times R$ be the product line bundle with the flat metric. Let b be a real valued function on S^1 . Let $A = d/dx + b$ and $A^* = -d/dx + b$. Let

$$D_1 = A^*A = -(d^2/dx^2 + (b' - b^2))$$

$$D_2 = AA^* = -(d^2/dx^2 + (-b' - b^2)).$$

These are self-adjoint operators. The standard section of V is flat with respect to the connections induced by D_1 and D_2 . $\mathcal{E}(D_1) = b' - b^2$ and $\mathcal{E}(D_2) = -b' - b^2$.

LEMMA 2.3:

$$E_n(x, D_1) - E_n(x, D_2) = (1/(2n - 1)) d/dx(d/dx + 2b)E_{n-1}(x, D_1).$$

PROOF: Let $\{\theta, \lambda_i\}_{i=1}^{\infty}$ be a complete spectral resolution of D_1 . Then

$$d/dt K(t, x, x, D_1) = \sum -\lambda_i \exp(-t\lambda_i)\theta_i^2(x).$$

In this sum, we may assume that $\lambda_i \neq 0$. Since $\{A\theta_i/\sqrt{\lambda_i}, \lambda_i\}$ for $\lambda_i \neq 0$ is a complete spectral resolution of the restriction of D_2 to the orthogonal complement of the null space of D_2 ,

$$d/dt K(t, x, x, D_2) = - \sum \exp(-t\lambda_i)(A\theta_i)^2(x).$$

Since $A\theta_i = 0$ if $\lambda_i = 0$, we may sum over all values of i . Since $D_1\theta_i = \lambda_i\theta_i$,

$$\begin{aligned} & d/dt\{K(t, x, x, D_1) - K(t, x, x, D_2)\} \\ &= \sum \exp(-t\lambda_i)\{-D_1\theta_i \cdot \theta_i + (A\theta_i)^2\} \\ &= \sum \exp(-t\lambda_i)(\theta_i''\theta_i + (b' - b^2)\theta_i^2 + \theta_i'\theta_i' + 2b\theta_i'\theta_i + b^2\theta_i^2) \\ &= \sum \exp(-t\lambda_i)\frac{1}{2} d/dx(d/dx + 2b)(\theta_i^2) \\ &= \frac{1}{2} d/dx(d/dx + 2b)K(t, x, x, D). \end{aligned}$$

Therefore

$$\begin{aligned} & d/dt \left\{ \sum_{n=0}^{\infty} t^{n-1/2} (E_n(x, D_1) - E_n(x, D_2)) \right\} \\ & \sim \sum_{n=0}^{\infty} t^{n-1/2} \frac{1}{2} d/dx(d/dx + 2b)E_n(x, D_1). \end{aligned}$$

This implies that

$$(n - \frac{1}{2})(E_n(x, D_1) - E_n(x, D_2)) = \frac{1}{2} d/dx(d/dx + 2b)E_{n-1}(x, D_1)$$

which completes the proof of the lemma. ■

Let $b^{(k)} = d^k b/dx^k$ and let $E_n(x, D) = a(n)E^{(2n-2)} + \dots$. Then

$$E_n(x, D_1) - E_n(x, D_2) = 2a(n)b^{(2n-1)} + \dots$$

and

$$\frac{1}{2n-1} d/dx(d/dx + 2b)E_{n-1}(x, D_1) = \frac{a(n-1)}{2n-1} b^{(2n-1)} + \dots$$

This implies

$$a(n) = \frac{a(n-1)}{4n-2}.$$

By Lemma 1.8, $E_1 = \mathcal{E}$ so $a(1) = 1$. This implies that

$$a(n) = \frac{1}{2^{n-1} \cdot 1 \cdot 3 \cdots (2n-1)} = \frac{(n-1)!}{(2n-1)!}.$$

We may express

$$E_n(x, D) = a(n)(\mathcal{E}^{(2n-2)} + \sum_{p+q=2n-4} a(p, q)\mathcal{E}^{(p)}\mathcal{E}^{(q)} + \cdots).$$

If \mathcal{E} is self-adjoint, then $\exp(-tD)$ is self-adjoint so $K(t, x, x, D)$ is self-adjoint. This implies that $E_n(x, D)$ is a symmetric matrix in this case and therefore $a(p, q) = a(q, p)$.

Since $\mathcal{E}(D_1) = b' - b^2$ and $\mathcal{E}(D_2) = -b' - b^2$, $E_n(x, D_1) - E_n(x, D_2)$ is an odd function of the variables $\{b^{(k)}\}$. In particular, there are no quadratic terms in this difference. Since

$$E_n(x, D_1) - E_n(x, D_2) = \frac{1}{2n-1} d/dx(d/dx + 2b)E_{n-1}(x, D_1)$$

we conclude that $(d/dx + 2b)E_{n-1}(x, D_1)$ must also be an odd function of the variables $\{b^{(k)}\}$ and contains no quadratic terms. We omit cubic and higher terms to express

$$E_n(x, D_1) = a(n) \left\{ b^{(2n-1)} - \sum_p \binom{2n-2}{p} b^{(p)} b^{(2n-2-p)} \right. \\ \left. + \sum_p a(p, 2n-4-p) b^{(p+1)} b^{(2n-3-p)} + \cdots \right\},$$

$$(d/dx + 2b)E_n(x, D_1) = a(n) \left\{ b^{(2n)} + 2bb^{(2n-1)} - \sum_p \binom{2n-1}{p} b^{(p)} b^{(2n-1-p)} \right. \\ \left. + \sum_p a(p, 2n-4-p) (b^{(p+2)} b^{(2n-3-p)} \right. \\ \left. + b^{(p+1)} b^{(2n-2-p)}) + \cdots \right\}.$$

This implies

$$\sum_{p=1}^{2n-2} \left\{ -\binom{2n-1}{p} + a(p-2, 2n-2-p) + a(p-1, 2n-3-p) \right\} b^{(p)} b^{(2n-1-p)} = 0.$$

This is equivalent to the system of linear equations:

$$a(p-2, 2n-2-p) + a(p-1, 2n-3-p) = \binom{2n-1}{p} \quad p = 1, \dots, 2n-2$$

$$\therefore a(p, q) = a(q, p) \quad \text{and} \quad a(p, q) = 0 \quad \text{for } p < 0 \text{ or } q < 0.$$

This system of equations determines $a(p, q)$ uniquely. Since $a(p, q) = (-1)^p + \binom{2n-2}{p+1}$ is a solution of these equations, this completes the proof of Theorem. 2.1. ■

Theorem 2.2 is a consequence of Theorem 2.1 and Lemma 1.8. The linear and quadratic terms in E_n are given by Theorem 2.1. By Lemma 1.8, the coefficient of $(\mathcal{E})^n$ in E_n is $1/n!$ This proves assertions (a) thru (d) of Theorem 2.2. Write

$$E_4(x, D) = a_1 \mathcal{E}^{(2)} \mathcal{E} \mathcal{E} + a_2 \mathcal{E} \mathcal{E}^{(2)} \mathcal{E} + a_3 \mathcal{E} \mathcal{E} \mathcal{E}^{(2)} + b_1 \mathcal{E}^{(1)} \mathcal{E}^{(1)} \mathcal{E} + b_2 \mathcal{E}^{(1)} \mathcal{E} \mathcal{E}^{(1)} + b_3 \mathcal{E} \mathcal{E}^{(1)} \mathcal{E}^{(1)} + \dots$$

where we have already determined the other terms using Theorem 2.1 and Lemma 1.8. If \mathcal{E} is self-adjoint, then E_n is self-adjoint. This implies $a_1 = a_3 = a$ and $b_1 = b_3 = b$. Furthermore without the linear and quadratic terms,

$$E_4(x, D - \epsilon I) = \epsilon^2(2a + a_2) \mathcal{E}^{(2)} + \epsilon(2b + b_2) \mathcal{E}^{(1)} \mathcal{E}^{(1)} + \dots$$

By Lemma 1.8, this implies that $(2a + a_2)$ is $\frac{1}{2}$ the coefficient of $\mathcal{E}^{(2)}$ in E_2 and that $(2b + b_2)$ is the coefficient of $\mathcal{E}^{(1)} \mathcal{E}^{(1)}$ in E_3 . This implies that

$$2a + a_2 = \frac{1}{12} \quad 2b + b_2 = \frac{1}{12}$$

which completes the proof of Theorem 2.2. For vector-valued operators, the problem of non-commutativity makes it impossible to compute $\text{Tr}(E_n)$ for $n > 4$ using these methods. However, if we restrict to scalar operators, the relations given in Lemma 2.3 are sufficient to determine E_n for all n . ■

We can apply Theorem 2.1 to obtain some results concerning E_n for general dimension m . Let $\nabla^n \mathcal{E} \cdot \nabla^n \mathcal{E} = \mathcal{E}_{i_1 \dots i_n} \mathcal{E}_{i_1 \dots i_n}$. If \mathcal{E} is scalar or

if \mathcal{E} is self-adjoint, the Sobolev and sup norms can be defined by:

$$|\mathcal{E}|_n^2 = \sum_{i=0}^n \int_M \text{Tr}(\nabla^i \mathcal{E} \cdot \nabla^i \mathcal{E}) |d\text{vol}|,$$

$$|\mathcal{E}|_{\infty, n} = \sum_{i=0}^n \sup_{x \in M} \text{Tr}(\nabla^i \mathcal{E} \cdot \nabla^i \mathcal{E}).$$

The major result of the next theorem is that the invariants $\{a_k(x, D)\}$ give estimates on the Sobolev norms $\{|\mathcal{E}|_k^2\}$ and therefore by the Sobolev inequalities also give estimates on the sup norms $\{|\mathcal{E}|_{\infty, k}\}$.

THEOREM 2.4: *For a manifold of any dimension m and for $D = D_{\nabla} - \mathcal{E}$,*

(a) $E_n(x, D) = (n-1)/(2n-1)! \mathcal{E}_{;i_1 i_2 i_3 \dots i_{n-1} i_{n-1}} + \text{other terms which involve derivatives of order at most } 2n-4 \text{ of the endomorphism } \mathcal{E}$. Therefore $E_n(x, D) \neq 0$ for generic (x, D) .

$$\begin{aligned} \text{(b) } a_n(D) &= \int_M \text{Tr}(E_n(x, D)) |d\text{vol}| \\ &= (-1)^n \frac{(n-1)!}{(2n-1)!} |\mathcal{E}|_{n-2}^2 + \int_M \text{Tr}(Q_n) |d\text{vol}| \end{aligned}$$

where Q_n is a homogeneous polynomial of order $2n$ in the derivatives of the metric and of the connection form and the $\{\mathcal{E}_{;j_1 \dots j_s}\}$ variables for $j < n-2$. (We suppose that $n > 1$.)

(c) Let $m' = m$ if m is even and let $m' = m-1$ if m is odd. If $\dim(V) = 1$ (so D is scalar) or if D is self-adjoint with respect to some fibre metric, then $|\mathcal{E}|_n^2 \leq (-1)^n (2n-2)! / (n-1)! a_{n+2}(D) + Q_n(a_0(D), \dots, a_{n+1}(D), |\mathcal{E}|_{\infty, m'})$. Q_n is a continuous function of its arguments which depends on the sup norms of the $\{R_{i_1 i_2 i_3 i_4 j_1 \dots j_s}, W_{i_1 i_2 j_1 \dots j_s}\}$ tensors for $s \leq 2n$. In particular, if Δ_0^m is the Laplacian acting on functions and if $D = \Delta_0^m - \mathcal{E}$, then the connection induced by D is flat so Q_n depends only on the geometry of M .

PROOF: By using the Bianchi identities and a basis constructed as in Lemma 1.3, we may write $E_n(x, D) = c(n, m) \mathcal{E}_{;i_1 i_2 i_3 \dots i_{n-1} i_{n-1}} + R$ where the remainder term only involves at most $2n-4$ derivatives of \mathcal{E} . By Lemma 1.7, the coefficient $c(n, m)$ is independent of m . Therefore $c(n, m) = c(n, 1) = a(n) = (n-1)! / (2n-1)!$ by Theorem 2.1. This proves assertion (a).

We prove (b) and (c) in the scalar case; the general case follows by making proper allowance for the non-commutativity. Let $\bar{\mathcal{Q}}$ be the polynomial algebra in the $\{R_{i_1 i_2 i_3 i_4 i_5 \dots i_s}, W_{i_1 i_2 j_1 \dots j_s}\}$ variables. Let $D(s) = D_{\nabla} - s\mathcal{E}$. We expand $E_n(x, D(s)) = \sum_k s^k E_{n,k}(x, D)$. The polynomials $E_{n,k}(x, D)$ are invariant separately. In the decomposition of $E_n(x, D) = \sum_k E_{n,k}(x, D)$, $E_{n,0} \in \bar{\mathcal{Q}}$. $E_{n,1}$ is linear in the covariant derivatives of \mathcal{E} , $E_{n,2}$ is quadratic and so forth. Since $E_{n,0}$ is independent of \mathcal{E} , we may ignore $\int_M \text{Tr}(E_{n,0})(x, D)|\text{dvol}|$ in the proof of assertions (b) and (c).

We expand $E_{n,1} = \sum_q c(i_1 \dots i_q) \mathcal{E}_{:i_1 \dots i_q}$ for $c(i_1 \dots i_q) \in \bar{\mathcal{Q}}$. Integration by parts shows that

$$\int_M \text{Tr}(E_{n,1})(x, D)|\text{dvol}| = \int_M \text{Tr}(Q \cdot \mathcal{E})(x, D)|\text{dvol}|$$

for

$$Q = \sum_q (-1)^q c(i_1 \dots i_q)_{:i_1 \dots i_q} \in \bar{\mathcal{Q}}.$$

Since Q does not involve the covariant derivatives of \mathcal{E} , we may ignore this term as well in the proof of assertions (b) and (c).

Let $E(p, q, r) = \mathcal{E}_{:i_1 i_1 \dots i_p i_p k_1 \dots k_r} \mathcal{E}_{:j_1 j_1 \dots j_q j_q k_1 \dots k_r}$ and decompose

$$E_{n,2} = \sum_{p+q+r=n-2} c(p, q, r) \mathcal{E}(p, q, r) + R.$$

Since $p + q + r = n - 2$, the coefficients $c(p, q, r)$ have degree 0 and hence are real numbers. The remainder R is quadratic in the covariant derivatives of \mathcal{E} with coefficients which are of degree at least 2 in $\bar{\mathcal{Q}}$, R has the form:

$$R = \sum_{p+q \leq 2n-6} c(i_1 \dots i_p j_1 \dots j_q) \mathcal{E}_{:i_1 \dots i_p} \mathcal{E}_{:j_1 \dots j_q} \text{ for } c(*) \in \bar{\mathcal{Q}}.$$

By integrating by parts, we may express

$$\int_M \text{Tr}(R)|\text{dvol}| = \sum \int_M \text{Tr}(c(i_1 \dots i_p j_1 \dots j_q) \mathcal{E}_{:i_1 \dots i_p} \mathcal{E}_{:j_1 \dots j_q})|\text{dvol}|$$

where $c \in \bar{\mathcal{Q}}$ and we sum over $p + q \leq 2n - 6$, $p \leq q + 1$, $q \leq p + 1$. This implies both p and q are less than or equal to $n - 3$. Consequently, we can ignore this term in the proof of (b); we use Cauchy-Schwarz to control this term in the proof of (c).

By integrating by parts and using the curvature identities to rearrange the order in which indices are summed, we may express

$$\begin{aligned} &\sum c(p, q, r) \int_M \text{Tr}(\mathcal{E}(p, q, r)) |d\text{vol}| \\ &= \left(\sum (-1)^{p+q} c(p, q, r) \right) \int_M \text{Tr}(\nabla^{n-2} \mathcal{E} \cdot \nabla^{n-2} \mathcal{E}) |d\text{vol}| + \int_M \text{Tr}(R') |d\text{vol}| \end{aligned}$$

where R' is a quadratic error of the sort discussed above. If we restrict to $m = 1$, then $\mathcal{E}(p, q, r) = \mathcal{E}^{(2p+r)} \mathcal{E}^{(2q+r)}$. Consequently, by Lemma 1.7

$$a(\bar{p}, \bar{q}) = \sum_{2p+r=\bar{p}} c(p, q, r).$$

This implies that:

$$\begin{aligned} \sum (-1)^{p+q} c(p, q, r) &= \sum (-1)^{(\bar{p}-\bar{q})/2} a(\bar{p}, \bar{q}) \\ &= \sum (-1)^{(\bar{p}-\bar{q})/2} \{(-1)^{\bar{p}} + \binom{2n-2}{\bar{p}+1}\} a(n) \\ &= (-1)^n (n-1)! / (2n-2)!. \end{aligned}$$

We complete the proof of assertions (b) and (c) by estimating the integral of $\text{Tr}(E_{n,k})$ for $k \geq 3$. If $k = 3$, we may express

$$E_{n,3} = \sum c(i_1, \dots, i_p, j_1, \dots, j_q, k_1, \dots, k_r) \mathcal{E}_{:i_1 \dots i_p} \mathcal{E}_{:j_1 \dots j_q} \mathcal{E}_{:k_1 \dots k_r}$$

for $c(*) \in \bar{\mathcal{Q}}$. We suppose $p \geq q \geq r$ for notational convenience; the other terms are handled similarly. By integrating by parts, we may express $\int_M \text{Tr}(E_{n,3}) |d\text{vol}|$ in terms of expressions of this form for which $p \geq q \geq r$ and $q + 1 \geq p$. Since $2n - 6 = \text{ord}(c) + p + q + r$, this implies $p \leq n - 3$ which proves (b). There are similar estimates for $k > 3$ which complete the proof of (b).

Since $p \leq n - 3$, we may estimate this integrand by $|\mathcal{E}|_{n-3}^2$ and $|\mathcal{E}|_{\infty,r}$. If $n \leq m'$, estimate (c) is trivial so we may proceed by induction. Clearly (c) will follow if we can control $|\mathcal{E}|_{\infty,r}$. If $r \leq m'$, there is no difficulty. If $r < n - 3 - (m/2)$, we can estimate $|\mathcal{E}|_{\infty,r}$ by $|\mathcal{E}|_{n-3}$ using the Sobolev inequality. We therefore suppose that $r > m'$ and $r \geq n - 3 - \frac{1}{2}m$. Since $2n - 6 = \text{ord}(c) + p + q + r \geq 3r \geq 3(n - 3 - \frac{1}{2}m)$, we conclude $\frac{3}{2}m \geq n - 3$. Similarly $2n - 6 \geq 3r > 3m'$ which implies $3m > 3m'$. This is false if m is even. If m is odd, $r > m'$ implies $r \geq m$. This chain of inequalities shows $3m \geq 2n - 6 \geq 3r \geq 3m$ so $3m = 2n - 6$. This

implies m is even which is false. We estimate $\int_M \text{Tr}(E_{n,k})|d\text{vol}|$ for $k > 3$ by replacing “ $2n - 6$ ” by “ $2n - 8$ ” and by making a similar argument. We omit the details. This completes the proof of assertion (c) and of Theorem 2.4. ■

Theorem 2.4 has the following consequences for the isospectral deformation problem in the self-adjoint case. Let V be a vector bundle over M with a smooth fibre metric and let ∇ be a Riemannian connection on V . Let \mathcal{E} be a smooth self-adjoint endomorphism of V . Let $D(\mathcal{E}) = D_\nabla - \mathcal{E}$. Let $S(\mathcal{E}) = \{\tilde{\mathcal{E}} : \text{spec}(D(\mathcal{E})) = \text{spec}(D(\tilde{\mathcal{E}}))\}$ and let $S_\lambda(\mathcal{E}) = \{\tilde{\mathcal{E}} \in S(\mathcal{E}) : \|\tilde{\mathcal{E}}\|_{\infty, m'} \leq \lambda\}$. Since $a_n(D(\mathcal{E}))$ is constant on $S(\mathcal{E})$, by Theorem 2.4, $\|\cdot\|_k^2$ is bounded on $S_\lambda(\mathcal{E})$ for any λ . This proves

COROLLARY 2.5: *$S_\lambda(\mathcal{E})$ is a compact subset with respect to either the $\|\cdot\|_k^2$ or the $\|\cdot\|_{\infty, k}$ norms for any k and for any λ .*

It is clear that one could probably improve the estimate (c) of Theorem 2.4 by making better estimates on the troublesome cubic and higher order terms to eliminate the presence of the $\|\cdot\|_{\infty, m'}$ -norm. McKean and Van Moerbeke [19] have given such an argument for the case $m = 1$. Such an improvement would enable us to improve Cor. 2.5 by replacing $S_\lambda(\mathcal{E})$ by $S(\mathcal{E})$.

$S(\mathcal{E})$ need not consist of a single endomorphism. For example, let $M = S^m$ be the standard sphere and let $D = \Delta_0^m$ be the Laplacian acting on functions. The group of isometries $O(m + 1)$ acts naturally on $S(\mathcal{E})$ by composition. $S(\mathcal{E}) = \{\mathcal{E}\}$ if and only if \mathcal{E} is a constant function. If $m = 1$, it is known that there exist examples of functions \mathcal{E} for which $S(\mathcal{E})$ is very large (in particular the isometry group $O(2)$ does not act transitively). By taking sums of such operators on the product torus $T^m = S^1x \dots xS^1$ it is possible to find \mathcal{E} so that the isometry group of T^m does not act transitively on $S(\mathcal{E})$. It is an open question as to whether such examples are really generic or if it is possible to prove a rigidity theorem under the assumption that the isometry group of the manifold consists only of the identity map.

Section 3

In this section, we will generalize Theorem 2.2 to 1-dimensional manifolds with boundary. We first recall the general facts established by Seeley [25] and Greiner [16]. Let M be a compact manifold of dimension m with smooth boundary dM . Near the boundary, we choose coordinates so $M = \{x = (x_1, \dots, x_m) : x_m \geq 0\}$. We further

normalize the choice of coordinates so $\partial/\partial x_m$ is the extension to the normal vector field on dM by the geodesic flow. Let V and D be as before. A section f satisfies the Dirichlet boundary condition if the restriction of f to dM is zero. Let B be a first order differential operator on the restriction of $C^\infty V$ to dM . f satisfies Neuman boundary conditions if $(\nabla_{\partial/\partial x_m} + B)f = 0$.

Once suitable boundary conditions have been imposed, the operator $\exp(-tD): L^2(V) \rightarrow C^\infty(V)$ is well defined. Let $K(t, x, y, D)$ be the kernel of this operator. In the interior of the manifold, $K(t, x, y, D)$ is independent of the boundary conditions modulo an error term which dies to infinite order in t . Near the boundary, however, the kernel becomes highly dependent on the boundary conditions. If s is a local frame for V on dM , extend s to a local frame in a neighborhood of dM by parallel transport along the geodesic rays defined by the normal vector field $\partial/\partial x_m$. Then

$$\int_0^\epsilon K(t, x, x, D) dx_m \sim (4\pi t)^{-(m-1)/2} \sum_{n=0}^\infty t^n E_n^b(x, D) \quad (n = 0, \frac{1}{2}, 1, 1\frac{1}{2}, \dots)$$

This integrates K along the 1-dimensional geodesic fibres normal to the boundary; we express K relative to such a parallel transported frame. In this sum, n ranges not over integral values but rather over half-integers. We shall use the notation $E_n^b(x, D)_0$ for the invariants arising from Dirichlet boundary conditions and the notation $E_n^b(x, D)_\nu$ for the invariants arising from Neumann boundary conditions. In this expression, ϵ tends to zero but more slowly than t .

The invariants $E_n^b(x, D)$ are local invariants of the differential operator D which are polynomial in the derivatives of the symbol. The argument given in the proof of Lemma 1.2 shows that $E_n^b(x, D)_0$ is homogeneous of order $2n$ in the derivatives of the symbol of D . If we express

$$B = \sum_{i=1}^{m-1} R_i \nabla_{\partial/\partial x_i} + S$$

then $\{R_i\}$ and S are tensorial. We define $\text{ord}(R_{i|\alpha}) = |\alpha|$ and $\text{ord}(S_{|\alpha}) = |\alpha| + 1$ where α is only allowed to involve tangential derivatives. Then $E_n^b(x, D)_\nu$ is homogeneous of order $2n$ in the derivatives of the symbol of D and the $\{R_{i|\alpha}, S_{|\alpha}\}$ variables. We emphasize that $2n$ can be an *odd* integer since n is a half-integer for these boundary invariants.

We can choose normal coordinates for the boundary and parallel transport these coordinates along the geodesic rays of the normal

vector field. This normalizes the choice of coordinates up to the action of $O(m - 1)$. Let

$$L_{ij} = G(\nabla^G_{\partial/\partial x_i} \partial/\partial x_m, \partial/\partial x_j)$$

be the second fundamental form; this symmetric tensor field is defined for $1 \leq i, j \leq m - 1$. We can compute the derivatives of the metric in terms of the curvature tensor of M and the covariant derivatives of the second fundamental form. The normal direction plays a distinguished role. We argue as in the proof of Lemma 1.3 to show

LEMMA 3.1: *Let $x \in dM$ and let $E_n^b(x, D)_0$ and $E_n^b(x, D, B)_\nu$ be the invariants described above.*

- (a) $E_n^b(x, D)_0$ can be computed as a sum of non-commutative monomials in the $\{R_{i_1 i_2 i_3 i_4; j_1 \dots j_s}, W_{i_1 i_2; j_1 \dots j_s}, \mathcal{E}_{j_1 \dots j_s}, L_{k_1 k_2; k_3 \dots k_s}\}$ variables where the indices k range from 1 thru $m - 1$ and not from 1 thru m . We contract pairs of indices and sum from 1 thru $m - 1$. The index m is distinguished and we do not sum such indices.
- (b) $E_n^b(x, D, B)_\nu$ can be computed similarly where we also use the variables $\{R_{k_1; k_2 \dots k_s}, S_{; k_1 \dots k_s}\}$

The same arguments that were used in the first section show that:

LEMMA 3.2:

- (a) *If E_n^b is expressed in terms of the universal basis given in Lemma 3.1, then the coefficients are independent of the dimension m .*
- (b) $E_n^b(x, D - \epsilon I) = \sum \epsilon^k / k! E_{n-k}^b(x, D)$.
- (c) *If D is self-adjoint with respect to some fibre metric and the given boundary conditions, then E_n^b is a self-adjoint matrix.*

We may compute the index of an elliptic operator using these invariants. Let V_1 and V_2 be smooth vector bundles over M which are equipped with a smooth fibre metric. Let $A: C^\infty(V_1) \rightarrow C^\infty(V_2)$ be an elliptic first order differential operator. Let $D_1 = A^*A$ and $D_2 = AA^*$. We impose boundary conditions as follows. Let $\sigma_L(x, \zeta): V_1 \rightarrow V_2$ be the leading order symbol of A . This is an isomorphism for $x \in M$ and $\zeta \neq 0 \in T^*M_x$. Let ζ_m be the dual of the normal vector field near dM . We use $\sigma_L(x, \zeta_m): V_1 \rightarrow V_2$ to identify V_1 with V_2 near dM . Under this identification the differential operator A has the form:

$$A = \nabla_{\partial/\partial x_m} + B.$$

B is the tangential part of the Neuman boundary condition and acts on $C^\infty(V_1)$ over dM . We take Dirichlet boundary conditions on V_2 and Neuman boundary conditions on V_1 . Both D_1 and D_2 are self-adjoint with respect to these boundary conditions; A maps the domain of D_1 in $L^2(V_1)$ to the domain of D_2 in $L^2(V_2)$. Let $\text{index}(A) = \dim \ker A - \dim \text{coker } A = \dim \ker(D_1) - \dim \ker(D_2)$.

THEOREM 3.3: *Let $D_1 = A^*A$ and let $D_2 = AA^*$. Then $\int_M \{\text{Tr}(E_n(x, D_1)) - \text{Tr}(E_n(x, D_2))\} \text{dvol}_m + \int_{dM} \{\text{Tr}(E_{n-1/2}^b(x, D_1)_0) - \text{Tr}(E_{n-1/2}^b(x, D_2, B)_\nu)\} \text{dvol}_{m-1} = \text{index}(A)$ if $2n = m$ and 0 otherwise. We set $E_n(x, D) = 0$ if n is a half-integer.*

This gives a local formula for computing the index of a differential operator A with these boundary conditions. If M has no boundary, one can give a direct proof of the Gauss–Bonnet theorem, the Hirzebruch Signature formula and the Riemann–Roch formula using this formula [1, 8, 21]. In fact, the index theorem in general can be deduced from the special case of the twisted signature complex [1]. One can also derive the G -signature theorem and other generalized Lefschetz fixed point formulas using similar techniques [6, 12, 13, 17].

If the boundary is not empty, by using slightly different boundary conditions than those given above, one can derive the Chern–Gauss–Bonnet formulae for manifolds with boundary by heat equation methods [14]. For the signature and Dolbeault complexes, the corresponding boundary conditions are non-local. The analysis in this situation is more complicated and is discussed in [3].

We prove Theorem 3.3 as follows: Let $N(D_i, \lambda)$ be the eigenspace of D_i with eigenvalue λ . $N(D_i, \lambda)$ is a finite dimensional vector space of smooth sections to V_i which satisfy the appropriate boundary conditions. Since $AD_1 = D_2A$ and since A preserves the boundary conditions, A induces a map

$$A_\lambda : N(D_1, \lambda) \rightarrow N(D_2, \lambda).$$

If $\lambda \neq 0$, this map is an isomorphism so $\dim(N(D_1, \lambda)) = \dim(N(D_2, \lambda))$. Therefore

$$\begin{aligned} \text{Tr}(\exp(-tD_1)) - \text{Tr}(\exp(-tD_2)) &= \sum_\lambda \exp(-t\lambda) \\ &\quad \times (\dim(N(D_1, \lambda)) - \dim(N(D_2, \lambda))) \\ &= \dim(N(D_1, 0)) - \dim(N(D_2, 0)) \\ &= \text{index}(A). \end{aligned}$$

We can compute $\text{Tr}(\exp(-tD))$ by the formula:

$$\begin{aligned} \text{Tr}(\exp(-tD)) &= \int_M \text{Tr}(K(t, x, x, D))|d\text{vol}| \\ &\sim (4\pi t)^{-m/2} \sum_{n=0}^{\infty} t^n \int_M \text{Tr}(E_n(x, D))|d\text{vol}|_m \\ &\quad + (4\pi t)^{-(m-1)/2} \sum_{n=0}^{\infty} t^n \int_M \text{Tr}(E_n^b(x, D))|d\text{vol}|_{m-1}. \end{aligned}$$

By comparing powers of t in the expansion $\text{index}(A) = \text{Tr}(\exp(-tD_1)) - \text{Tr}(\exp(-tD_2))$, we complete the proof of Theorem 3.3. ■

We now specialize to the case $m = 1$. If x is a boundary point, parametrize M by arc length near x and choose the orientation so the normal d/dx_1 points inwards. This identifies M with a closed interval $[0, a]$ so that x corresponds to 0. Since M is 1-dimensional, the Neuman boundary conditions reduce to the equation $(d/dx_1 + S)f = 0$ where S is an endomorphism. If we express $D = -(d^2/dx^2 + \mathcal{E})$, then $E_n^b(x, D)_0$ is a polynomial in the $\{\mathcal{E}^{(k)}\}$ variables. We can only differentiate S tangentially. Since $m = 1$, this implies $E_n^b(x, D, B)_\nu$ is a polynomial in the $\{\mathcal{E}^{(k)}, S\}$ variables.

THEOREM 3.4: *If the dimension of M is 1 and if $x \in dM$, then*

- (a) $E_0^b(x, D)_0 = -\frac{1}{4}, \quad E_0^b(x, D, B)_\nu = \frac{1}{4},$
- (b) $E_{1/2}^b(x, D)_0 = 0, \quad E_{1/2}^b(x, D, B)_\nu = (4\pi)^{-1/2}2S,$
- (c) $E_1^b(x, D)_0 = -\frac{1}{4}\mathcal{E}, \quad E_1^b(x, D, B)_\nu = \frac{1}{4}(\mathcal{E} + 2S^2),$
- (d) $E_{1+1/2}^b(x, D)_0 = -\frac{1}{3}(4\pi)^{-1/2}\mathcal{E}^{(1)},$
 $E_{1+1/2}^b(x, D, B)_\nu = \frac{1}{3}(4\pi)^{-1/2}(2\mathcal{E}^{(1)} + 3S\mathcal{E} + 3\mathcal{E}S + 4S^3),$
- (e) $E_2^b(x, D)_0 = -\frac{1}{16}(\mathcal{E}^{(2)} + 2\mathcal{E}^2),$
 $E_2^b(x, D, B)_\nu = \frac{1}{16}(\mathcal{E}^{(2)} + 2\mathcal{E}^2 + 2S\mathcal{E}^{(1)} + 2\mathcal{E}^{(1)}S + cS^2\mathcal{E} + (8 - 2c)S\mathcal{E}S$
 $\quad + c\mathcal{E}S^2 + 4S^2),$
- (f) $E_{2+1/2}^b(x, D)_0 = -\frac{1}{30}(4\pi)^{-1/2}(2\mathcal{E}^{(3)} + 5\mathcal{E}^{(1)}\mathcal{E} + 5\mathcal{E}\mathcal{E}^{(1)}),$
 $E_{2+1/2}^b(x, D, B)_\nu = \frac{1}{30}(4\pi)^{-1/2}(3\mathcal{E}^{(3)} + 10\mathcal{E}^{(1)}\mathcal{E} + 10\mathcal{E}\mathcal{E}^{(1)} + 5\mathcal{E}^{(2)}S$
 $\quad + 5S\mathcal{E}^{(2)} + 16S^5 + \tau)$

where $\text{Tr}(\tau) = \text{Tr}(20\mathcal{E}^{(1)}S^2 + 30S\mathcal{E}^2 + 40S^3),$

- (g) $E_3^b(x, D)_0 = -\frac{1}{384}(3\mathcal{E}^{(4)} + 15\mathcal{E}^{(1)}\mathcal{E}^{(1)} + 12\mathcal{E}^{(2)}\mathcal{E} + 12\mathcal{E}\mathcal{E}^{(2)} + 16\mathcal{E}^3),$
 $E_3^b(x, D, B)_\nu = \frac{1}{384}(3\mathcal{E}^{(4)} + 21\mathcal{E}^{(1)}\mathcal{E}^{(1)} + 12\mathcal{E}^{(2)}\mathcal{E} + 12\mathcal{E}\mathcal{E}^{(2)} + 16\mathcal{E}^3$
 $\quad + 6S\mathcal{E}^{(3)} + 6\mathcal{E}^{(3)}S + 32S^6 + \nu)$

where $\text{Tr}(\nu) = \text{Tr}(24S^2\mathcal{E}^{(2)} + 48S^3\mathcal{E}^{(1)} + 96S^4\mathcal{E} + 48S\mathcal{E}^{(1)}$
 $\quad + 48S\mathcal{E}\mathcal{E}^{(1)} + 72S^2\mathcal{E}^2 + 24S\mathcal{E}S\mathcal{E}).$

PROOF: We will use the recursion relation of Lemma 2.3 in the proof of Theorem 3.4. Let $A = d/dx + b$, $D_1 = A^*A$, and $D_2 = AA^*$. We choose Dirichlet boundary conditions for D_2 and Neuman boundary conditions given by the operator b for D_1 . The argument given in the proof of Lemma 2.3 shows that:

$$d/dt(K(t, x, x, D_1) - K(t, x, x, D_2)) = \frac{1}{2}(d/dx)(d/dx + 2b)K(t, x, x, D_1).$$

If we integrate this relationship over $[0, \epsilon]$, the lefthand side becomes

$$\sum_{n=0}^{\infty} nt^{n-1}(E_n^b(x, D_1, b)_\nu - E_n^b(x, D_2)_0)$$

asymptotically. If we integrate the righthand side, we obtain

$$\frac{1}{2}(d/dx + 2b)K(t, x, x, D_1)|_0^\epsilon.$$

At 0 this is $\sum_i \exp(-t\lambda_i)\theta_i(d/dx + b)\theta_i$. Since we have chosen Neuman boundary conditions for D_1 , this expression vanishes at 0. The righthand side of this integral becomes $\frac{1}{2}(d/dx + 2b)K(t, \epsilon, \epsilon, D_1)$. In the interior, the kernel function is asymptotically equal to the free space kernel and has an asymptotic expansion of the form $(4\pi t)^{-1/2} \sum_{n=0}^{\infty} t^n E_n(\epsilon, D)$. We now let $\epsilon \rightarrow 0$ to prove that:

$$\begin{aligned} &\sum_{n=0}^{\infty} nt^{n-1}(E_n^b(x, D_1, b)_\nu - E_n(x, D_2)_0) \\ &\sim (4\pi t)^{-1/2} \sum_{n=0}^{\infty} t^{n\frac{1}{2}}(d/dx + 2b)E_n(x, D_1) \end{aligned}$$

We compare powers of t in the two expansions to show:

LEMMA 3.5: *Let n be an integer. Let $A = d/dx + b$, $D_1 = A^*A$, and $D_2 = AA^*$. Then there are recursion relations:*

- (a) $2(n + \frac{1}{2})\{E_{n+1/2}^b(x, D_1, b)_\nu - E_{n+1/2}^b(x, D_2)_0\} = (4\pi)^{-1/2}(d/dx + 2b)E_n(x, D_1);$
- (b) $2n\{E_n^b(x, D_1, b)_\nu - E_n^b(x, D_2)_0\} = 0.$

Let \mathcal{E} be an even matrix on $[-1, 1]$ which vanishes to infinite order at 1. Let $S^1 = [-1, 1]$ where we identify -1 with 1. The operator $D = -(d^2/dx^2 + \mathcal{E})$ extends smoothly to S^1 . Let $f(x) = -x$. Since D commutes with the involution f , we may decompose the spectrum of D on S^1 into the even and odd portion. The even portion will solve

the pure Neuman problem on $[0, 1]$; the odd portion will solve the Dirichlet problem on $[0, 1]$. Let $K(t, x, y, D)$ be the kernel of D on S^1 . Let $K(t, x, y, D)_0$ and $K(t, x, y, D)_\nu$ be the kernels of D on $[0, 1]$ with the Dirichlet and Neuman boundary conditions. Then $K(t, x, x, D)_0 + K(t, x, x, D)_\nu = 2K(t, x, x, D)$ for $x \in [0, 1]$. The factor of “2” is introduced by the normalization of the L^2 norms on $[0, 1]$ and on the double S^1 . Therefore:

$$\int_0^\epsilon K(t, x, x, D)_0 + \int_0^\epsilon K(t, x, x, D)_\nu = 2 \int_0^\epsilon K(t, x, x, D) \rightarrow 0 \text{ as } \epsilon \rightarrow 0.$$

This implies $E_n^b(x, D)_0 + E_n^b(x, D, 0)_\nu = 0$ if E is even. If n is a half-integer, this expression is odd and vanishes automatically. If n is a whole integer, however, this yields additional relationships:

LEMMA 3.6: *If \mathcal{E} is even and if $D = -(d^2/dx^2 + \mathcal{E})$, then $E_n^b(x, D)_0 + E_n^b(x, D, 0)_\nu = 0$.*

We begin the proof of Theorem 3.4 by computing E_0^b . Since E_0^b is homogeneous of order 0, there are universal constants so

$$E_0^b(x, D)_0 = c_1 \quad E_0^b(x, d, b)_\nu = c_2.$$

By Lemma 3.6, $c_1 + c_2 = 0$. If $b = 0$, then $D_1 = D_2 = -d^2/dx^2$. By Lemma 3.3, the index of the operators (D_1, D_2) with Neumann and Dirichlet boundary conditions is

$$E_0^b(0, D_1, 0)_\nu + E_0^b(1, D_1, 0)_\nu - E_0^b(0, D_2)_0 - E_0^b(1, D_2)_0 = 2(c_2 - c_2) = 4c_2.$$

If $f'' = 0$, then f is linear. There are no non-trivial linear functions with Dirichlet boundary conditions. The space of linear functions with Neuman boundary conditions is 1-dimensional and consists of the constants. Therefore the index of this problem is 1. This implies $c_2 = \frac{1}{4}$ and $c_1 = -\frac{1}{4}$ which proves (a). We shall prove (c), (e), and (g) before computing E_n^b where n is a half-integer.

Since E_1^b is homogeneous of order 2, there are universal constants so

$$E_1^b(x, D)_0 = c_3' \mathcal{E}$$

$$E_1^b(x, D, b)_\nu = c_3 \mathcal{E} + c_4 S^2.$$

By Lemma 3.2, $c_3' = E_0^b(x, D)_0 = -\frac{1}{4}$ and $c_3 = E_0^b(x, D, b)_\nu = \frac{1}{4}$. $\mathcal{E}(D_1) =$

$b' - b^2$ and $\mathcal{E}(D_2) = -b' - b^2$. By Lemma 3.5,

$$0 = E_1(x, D_1, b)_\nu - E_1(x, D_2)_0 = \frac{1}{4}((b' - b^2) + (-b' - b^2)) + c_4 b^2.$$

This implies that $c_4 = \frac{1}{2}$ which proves (c). By using Lemmas 3.2 and 3.6 together with the fact that E_2^b is homogeneous of order 4, we may express

$$\begin{aligned} E_2^b(x, D)_0 &= \frac{-1}{16}(c_5 \mathcal{E}^{(2)} + 2\mathcal{E}^2) \\ E_2^b(x, D, b)_\nu &= \frac{1}{16}(c_5 \mathcal{E}^{(2)} + 2\mathcal{E}^2 + c_6(S\mathcal{E}^{(1)} + \mathcal{E}^{(1)}S) + c(S^2\mathcal{E} + \mathcal{E}S^2) \\ &\quad + (8 - 2c)S\mathcal{E}S + c_7 S^4. \end{aligned}$$

By Lemma 3.5,

$$\begin{aligned} 0 &= c_5\{(b' - b^2)^{(2)} + (-b' - b^2)^{(2)}\} + 2(b' - b^2)^2 + 2(-b' - b^2)^2 \\ &\quad + 2c_6\{b(b' - b^2)^{(1)}\} + 8b^2(b' - b^2) + c_7 b^4. \end{aligned}$$

This yields the identities:

$$(4 - 8 + c_7)b^4 = 0, \quad (4 - 4c_5)b'b' = 0, \quad (-4c_5 + 2c_6)b''b = 0$$

This proves $c_7 = 4$, $c_5 = 1$, and $c_6 = 2$ which completes the proof of (e).

We compute E_3^b first for line bundles. By Lemmas 3.2 and 3.6,

$$\begin{aligned} E_3^b(x, D)_0 &= \frac{-1}{384}\{c_1 \mathcal{E}^{(4)} + c_2' \mathcal{E}^{(1)}\mathcal{E}^{(1)} + 24\mathcal{E}^{(2)}\mathcal{E} + 16\mathcal{E}^3\} \\ E_3^b(x, D, b)_\nu &= \frac{1}{384}\{c_1 \mathcal{E}^{(4)} + c_2 \mathcal{E}^{(1)}\mathcal{E}^{(1)} + 24\mathcal{E}^{(2)}\mathcal{E} + 16\mathcal{E}^3 + c_3 S\mathcal{E}^{(3)} \\ &\quad + 96S\mathcal{E}^{(1)}\mathcal{E} + c_4 S^2\mathcal{E}^{(2)} + 96S^2\mathcal{E}^2 + c_5 S^3\mathcal{E}^{(1)} + 96S^4\mathcal{E} + c_6 S^6\} \end{aligned}$$

where c_1, \dots, c_6 are new constants not related to the previous values. Lemma 3.6 does not imply $c_2 = c_2'$ since $\mathcal{E}^{(1)}\mathcal{E}^{(1)}$ vanishes if \mathcal{E} is even. By Lemma 3.5,

$$\begin{aligned} 0 &= c_{11}\{(b' - b^2)^{(4)} + (-b' - b^2)^{(4)}\} + c_2\{(b' - b^2)^{(1)}\}^2 + c_2'\{(-b' - b^2)^{(1)}\}^2 \\ &\quad + 24\{(b' - b^2)^{(2)}(b' - b^2) + (-b' - b^2)^{(2)}(b' - b^2)\} \\ &\quad + 16\{(b' - b^2)^3 + (-b' - b^2)^3\} + c_3 b(b' - b^2)^{(3)} + 96b(b' - b^2)^{(1)}(b' - b^2) \\ &\quad + c_4 b^2(b' - b^2)^{(2)} + 96b^2(b' - b^2)^2 + c_5 b^3(b' - b^2)^{(1)} \\ &\quad + 96b^4(b' - b^2) + c_6 b^6. \end{aligned}$$

This yields the relations:

$$\begin{aligned} (-32 + 96 - 96 + c_6)b^6 &= 0, & (192 - 192 - 2c_5 + 96)b'b^4 &= 0, \\ (96 - 96 - 2c_4 + c_5)b''b^3 &= 0, & (-2c_3 + c_4)b''''b^2 &= 0, \\ (-16c_1 + 48)b''''b' &= 0, & (c_2' + c_2 - 12c_1)b''''b'' &= 0, \\ (4c_2' - 4c_2 - 6c_3 + 96)b''''b'b &= 0. \end{aligned}$$

This implies that $c_6 = 32$, $c_5 = 48$, $c_4 = 24$, $c_3 = 12$, $c_1 = 3$ and that

$$c_2' + c_2 - 36 = 0, \quad c_2' - c_2 + 6 = 0$$

so $c_2 = 21$ and $c_2' = 15$. Consequently, when we express E_3^b as in (g), the error term ν has the form $\nu = \nu_0 + \nu_1 + \nu_2$ where

$$\begin{aligned} \text{Tr}(\nu_0) &= \frac{1}{384} \text{Tr}(24S^2\mathcal{E}^{(2)} + 48S^3\mathcal{E}^{(1)} + 96S^4\mathcal{E}) \\ \text{Tr}(\nu_1) &= \frac{1}{384} \text{Tr}(96S\mathcal{E}^{(1)}\mathcal{E}) \\ \text{Tr}(\nu_2) &= \frac{1}{384} \text{Tr}(96S^2\mathcal{E}^2) \end{aligned}$$

for line bundles. In the general case, we may expand $\nu_0 = c_4'S^2\mathcal{E}^{(2)} + c_4'\mathcal{E}^{(2)}S^2 + (24 - 2c_4')S\mathcal{E}^{(2)}S + \dots$. Since $\text{Tr}(S^2\mathcal{E}^{(2)}) = \text{Tr}(S\mathcal{E}^{(2)}S) = \text{Tr}(\mathcal{E}^{(2)}S^2)$, this lack of commutativity poses no problem and the formula for $\text{Tr}(\nu_0)$ holds true for vector bundles. Similarly we may expand $\nu_1 = d_1(S\mathcal{E}^{(1)}\mathcal{E} + \mathcal{E}\mathcal{E}^{(1)}S) + d_2(\mathcal{E}^{(1)}S\mathcal{E} + \mathcal{E}S\mathcal{E}^{(1)}) + d_3(S\mathcal{E}\mathcal{E}^{(1)} + \mathcal{E}^{(1)}\mathcal{E}S)$ where $2(d_1 + d_2 + d_3) = 96$. This implies $\text{Tr}(\nu_1) = 48 \text{Tr}(S\mathcal{E}\mathcal{E}^{(1)} + S\mathcal{E}^{(1)}\mathcal{E})$.

Similarly, $\text{Tr}(\nu_2) = d_4 \text{Tr}(\mathcal{E}^2S^2) + d_5 \text{Tr}(\mathcal{E}S\mathcal{E}S)$ for $d_4 + d_5 = 96$. Let \bar{b} be matrix valued and vanish to infinite order at the other endpoint of the interval. Let $\bar{A} = d/dx + \bar{b}$, $\bar{D}_1 = \bar{A}^*\bar{A}$, and $\bar{D}_2 = \bar{A}\bar{A}^*$. If n is integral, $E_{n+1/2}(x, \bar{D}) = 0$. Therefore, by Theorem 3.3:

$$\text{Tr}(E_n^b(0, \bar{D}_1, \bar{b})_\nu) - \text{Tr}(E_n^b(0, \bar{D}_2)_0) = 0.$$

We compute the coefficient of $\text{Tr}(\bar{b}^2\bar{b}'\bar{b}')$ and of $\text{Tr}(\bar{b}\bar{b}'\bar{b}\bar{b}')$ to show that

$$(-72 + d_4) \text{Tr}(\bar{b}\bar{b}\bar{b}'\bar{b}') = 0 \quad \text{and} \quad (-24 + d_5) \text{Tr}(\bar{b}\bar{b}'\bar{b}\bar{b}') = 0.$$

This implies $d_4 = 72$ and $d_5 = 24$ and completes the proof of (b). One could continue this process to compute $\text{Tr}(E_n^b)$ for $n > 3$ and n integral. The calculations become much longer but no new methods are required which were not used in the case $n = 3$.

We finish section three by determining $E_{n+1/2}^b$ for $n = 0, 1, 2$. Since $E_{1/2}^b$ is homogeneous of order 1, $E_{1/2}^b(x, D)_0 = 0$ and $E_{1/2}^b(x, D, b)_\nu = cS$ for some universal constant c . By Lemma 3.5 (a) with $n = 0$,

$$E_{1/2}^b(x, D_1, b)_\nu - E_{1/2}^b(x, D_2)_0 = (4\pi)^{-1/2}(d/dx + 2b)E_0(x, D_1).$$

This proves $cb = (4\pi)^{-1/2}(2b)$ so $c = 2(4\pi)^{-1/2}$ which completes the proof of (c). The proofs of (d) and (f) are along similar lines and are omitted since there are no new techniques required. ■

Section 4

In this section, we will determine $E_n(x, D)$ for $n = 0, 1, 2, 3$ and all m by generalizing the recursive formulas of the second section. Let V be a smooth vector bundle over M and let $D = -(g^{ij}\partial^2/\partial x_i\partial x_j + P^k\partial/\partial x_k + Q)$ be a second order elliptic operator on V with leading symbol given by the metric tensor. Let ∇^G be the Levi-Civita connection on $T(M)$ and let ∇ be the connection induced by D on V . Let $\omega = dx^k \otimes \omega^k$ be the connection matrix of ∇ on V ; ω_k is endomorphism-valued. We showed earlier that:

$$\begin{aligned} \nabla^G(\partial/\partial x_j) &= \Gamma_{ij}^k dx^i \otimes \partial/\partial x_k & \Gamma(dx^i) &= -\Gamma_{ik}^j dx^i \otimes dx^k \\ \Gamma_{ij}^k &= \frac{1}{2}g^{kl}(g_{ilj} + g_{jli} - g_{ijl}) & \omega_l &= \frac{1}{2}(g_{il}P^i + g_{il}g^{jk}\Gamma_{jk}^i) \\ \mathcal{E} &= Q - g^{ij}\omega_{ij} - g^{ij}\omega_i\omega_j + g^{jk}\omega_i\Gamma_{jk}^i & W_{ij} &= \omega_{ji} - \omega_{ij} + \omega_i\omega_j - \omega_j\omega_i. \end{aligned}$$

If $\Delta_p^m = (d\delta + \delta d)_p : C^\infty(\Lambda^p T^*M) \rightarrow C^\infty(\Lambda^p T^*M)$ is the Laplacian on p -forms, then the connection determined by Δ is the Levi-Civita connection. If $\text{ext}(dx^i) : \Lambda^p(T^*M) \rightarrow \Lambda^{p+1}(T^*M)$ is exterior multiplication, let $\text{int}(dx^i) : \Lambda^p(T^*M) \rightarrow \Lambda^{p-1}(T^*M)$ be the dual map, interior multiplication. If W_{ij} is the action of the curvature on $\Lambda^p(T^*M)$, then the endomorphism induced by Δ_p^m is given by the formula [10]

$$\mathcal{E}(\Delta_p^m) = \frac{1}{2}(\text{ext}(dx^i) - \text{int}(dx^i))(\text{ext}(dx^i) - \text{int}(dx^i))W_{ij}.$$

THEOREM 4.1: $E_n(x, \Delta_p^m) = (n - 1)!(2n - 1)!(\mathcal{E}_{i_1 i_1 \dots i_{n-1} i_{n-1}} - (n/4n + 2)R_{ijj;k_1 \dots k_{n-1} k_{n-1}}) + \text{other terms involving at most } 2n - 4 \text{ covariant derivatives of the } \{\mathcal{E}, R_{ijkl}, W_{ij}\} \text{ tensors.}$

If Δ_0^m is the scalar Laplacian acting on functions, the connection ∇ is flat so $\mathcal{E} = W_{ij} = 0$. This implies that

COROLLARY 4.2: $E_n(x, \Delta_0^m) = -[n(n-1)!(4n+2)(2n-1)!] \times R_{ijj;k_1k_1\dots k_{n-1}k_{n-1}} + \text{other terms involving at most } 2n-4 \text{ covariant derivatives of the curvature tensor. Thus generically, } E_n(x, \Delta_0^m) \neq 0.$

In addition to proving Theorem 3.1, we will also prove

THEOREM 4.3: Express $(R_{ijkl}, \dots, W_{ij}, \dots, E_i, \dots)$ in terms of the derivatives of the symbol of D using the equations above. Then we may compute E_0, E_1, E_2, E_3 for D :

$$\begin{aligned}
 \text{(a) } E_0 &= I \\
 \text{(b) } E_1 &= \frac{1}{6}(-R_{ijj} + 6\mathcal{E}) \\
 \text{(c) } E_2 &= \frac{1}{360}(-12R_{ijj;kk} + 5R_{ijj}R_{klkl} - 2R_{ijik}R_{ljlk} + 2R_{ijkl}R_{ijkl} \\
 &\quad - 60\mathcal{E}R_{ijj} + 180\mathcal{E}^2 + 60\mathcal{E}_{;kk} + 30W_{ij}W_{ij}) \\
 \text{(d) } E_3 &= \frac{1}{7!}(-18R_{ijj;kkll} + 17R_{ijj;k}R_{lnln;k} - 2R_{ijik;l}R_{njnk;l} - 4R_{ijik;l}R_{njnl;k} \\
 &\quad + 9R_{ijkl;n}R_{ijkl;n} + 28R_{ijj}R_{klkl;nn} - 8R_{ijik}R_{ljlk;nn} + 24R_{ijik}R_{ljln;k} \\
 &\quad + 12R_{ijkl}R_{ijkl;nn} - \frac{35}{9}R_{ijj}R_{klkl}R_{pqpq} + \frac{14}{3}R_{ijj}R_{klkn}R_{plpn} \\
 &\quad - \frac{14}{3}R_{ijj}R_{klnp}R_{klnp} + \frac{208}{9}R_{ijik}R_{jnln}R_{kplp} - \frac{64}{3}R_{ijik}R_{nplp}R_{jnkl} \\
 &\quad + \frac{16}{3}R_{ijik}R_{jnlp}R_{knlp} - \frac{44}{9}R_{ijkn}R_{ijlp}R_{knlp} - \frac{80}{9}R_{ijkn}R_{ilkp}R_{jlnp}) \\
 &\quad + \frac{1}{360}(8W_{ij;k}W_{ij;k} + 2W_{ij;j}W_{ik;k} + 6W_{ij;kk}W_{ij} + 6W_{ij}W_{ij;kk} \\
 &\quad - 12W_{ij}W_{jk}W_{ki} - 6R_{ijki}W_{ij}W_{kl} + 4R_{ijik}W_{jl}W_{kl} - 5R_{ijij}W_{kl}W_{kl} \\
 &\quad + 6\mathcal{E}_{;ijj} + 30\mathcal{E}_{;ii} + 30\mathcal{E}_{;ii}\mathcal{E} + 30\mathcal{E}_{;i}\mathcal{E}_{;i} + 60\mathcal{E}^3 + c\mathcal{E}W_{ij}W_{ij} \\
 &\quad + cW_{ij}W_{ij}\mathcal{E} + (30-2c)W_{ij}\mathcal{E}W_{ij} - 10R_{ijij}\mathcal{E}_{;kk} - 4R_{ijik}R_{;jk} \\
 &\quad - 12R_{ijj;k}\mathcal{E}_{;k} + d(\mathcal{E}_{;j}W_{ij;i} - W_{ij;i}\mathcal{E}_{;j}) - 30R_{ijij}\mathcal{E}\mathcal{E} - 12R_{ijj;kk}\mathcal{E} \\
 &\quad + 5R_{ijj}R_{klkl}\mathcal{E} - 2R_{ijik}R_{ljlk}\mathcal{E} + 2R_{ijkl}R_{ijkl}\mathcal{E}).
 \end{aligned}$$

for some universal constants c and d . ($c = 12$ and $d = -6$ from [9]).

If we set $W_{ij} = \mathcal{E} = 0$, we get the usual formula for $a_n(D_0)$ for $n = 0, 1, 2, 3$ which has been derived separately by McKean and Singer [19], Berger [4], and Sakai [23].

We prove Theorems 4.1 and 4.3 by defining an elliptic complex which is based on the Clifford algebra. This elliptic complex is not the spin complex. Let $\text{CLIF}(R^m)$ be the Clifford algebra of R^m with respect to the standard inner product. Let $*$ denote Clifford multiplication in this algebra. If $\{e_1, \dots, e_m\}$ is an orthonormal basis for R^m , then $\text{CLIF}(R^m)$ is the algebra generated by this basis subject to the relations: $e_i * e_j = -e_j * e_i$ if $i \neq j$ and $e_i * e_i = -1$. There is a natural innerproduct on the algebra $\text{CLIF}(R^m)$ which is invariant under

Clifford multiplication by a unit vector of R^m . If e^* denotes left multiplication by e , then $-e^*$ is the adjoint map. A more complete description of Clifford algebras may be found in Atiyah, Fott, Shapiro [2].

Let h be a real function on the m -dimensional torus T^m and let G be the conformally flat metric: $G = \exp(-h)(dx_1^2 + \dots + dx_m^2)$. The volume is given by $|\text{dvol}| = \exp(-\frac{1}{2}mh)|dx_1 \dots dx_m|$. For this metric, $\Gamma_{ij}^k = 0$ if all 3 indices are distinct. The non-zero Christoffel symbols are:

$$\Gamma_{ii}^i = \Gamma_{ji}^j = \Gamma_{ij}^j = -\frac{1}{2}h_{ji}, \quad \Gamma_{jj}^i = \frac{1}{2}h_{ji}.$$

(we do not sum over repeated indices in the above equations). Similarly $R_{ijkl} = 0$ if all 4 indices are distinct. The non-zero curvatures are:

$$R_{iji}^k = \frac{1}{4}(-2h_{ijk} - h_{ij}h_{ik}) \text{ if } j \neq k$$

$$R_{iji}^j = \frac{1}{4}\left(-2h_{jii} - 2h_{ijj} + \sum_{k \neq i, k \neq j} h_{ik}h_{jk}\right)$$

(we do not sum over repeated indices in the above equations).

Let $V = T^m \times \text{CLIF}(R^m)$ with the product metric on the fibre. We define a first order differential operator A and its adjoint $A' : C^\infty(V) \rightarrow C^\infty(V)$ by:

$$A = \exp\left(\frac{1}{4}mh\right) \left(\sum_j e_j^* \partial / \partial x_j \right) \exp\left(\frac{1}{4}(2-m)h\right)$$

$$A' = \exp\left(\frac{1}{4}(2+m)h\right) \left(\sum_j e_j^* \partial / \partial x_j \right) \exp\left(-\frac{1}{4}mh\right).$$

The associated Laplacians are given by

$$D_1 = A'A = -\exp\left(\frac{1}{4}(2+m)h\right) \left(\sum_i \partial^2 / \partial x_i^2 \right) \exp\left(\frac{1}{4}(2-m)h\right)$$

$$= -\exp(h) \left(\sum_i \partial^2 / \partial x_i^2 + \frac{1}{2}(2-m)h_{ii} \partial / \partial x_i \right.$$

$$\left. + \frac{1}{16}(4(2-m)h_{iii} + (2-m)^2 h_{ii}h_{ii}) \right)$$

$$D_2 = AA' = \exp\left(\frac{1}{4}mh\right) \left(\sum_j e_j^* \partial / \partial x_j \right) \exp(h) \left(\sum_k e_k^* \partial / \partial x_k \right) \exp\left(-\frac{1}{4}mh\right)$$

$$= -\exp(h) \left(\sum_i \partial^2 / \partial x_i^2 - \sum_{j \neq k} h_{ij}e_j^* e_k \partial / \partial x_k + \frac{1}{2}(2-m)h_{ik} \partial / \partial x_k \right.$$

$$\left. + \frac{1}{16}(-4mh_{iii} - m(4-m)h_{ii}h_{ii}) \right).$$

From these equations it follows immediately that:

$$\omega_i(D_1) = \frac{1}{2}e^{-h} \left(P^i(D_1) + \sum_j \Gamma_{jj}^i \right) = \frac{1}{4}((2-m)h_{ji} + (m-2)h_{ji}) = 0$$

$$\omega_i(D_2) = \frac{1}{2}e^{-h} \left(P^i(D_2) + \sum_j \Gamma_{jj}^i \right) = -\frac{1}{2} \sum_{j \neq i} h_{jj} e_j^* e_i$$

$$W_{ij}(D_1) = 0$$

$$W_{ij}(D_2) = \frac{1}{2} h_{jik} e_k^* e_i - \frac{1}{2} h_{jik} e_k^* e_j + \frac{1}{2} \sum_{k \neq i, k \neq j} h_{jk} h_{ik} e_i^* e_j \quad (i \neq j)$$

$$\mathcal{E}(D_1) = \frac{1}{16} e^h (-4(m-2)h_{jii} + (m-2)^2 h_{ji} h_{ji})$$

$$\mathcal{E}(D_2) = \frac{1}{16} e^h (-4m h_{jii} + (m^2 - 4) h_{ji} h_{ji}).$$

We generalize the recursion relations of Lemma 2.3 as follows:

LEMMA 4.4:

$$(2n - m)(\text{Tr } E_n(x, D_1) - \text{Tr } E_n(x, D_2)) \\ = \sum_k \exp(\frac{1}{2}hm) \partial^2 / \partial x_k^2 \exp(\frac{1}{2}h(2 - m)) \text{Tr } E_{n-1}(x, D_1).$$

Lemma 4.4 gives polynomial identities involving the derivatives of the function h which we will exploit to complete the proofs of Theorems 4.1 and 4.3.

PROOF: Let L be the line bundle $T^m \times R$ with the flat fibre metric and let $D_L = -\exp(\frac{1}{4}(2 + m)h) \sum_k \partial^2 / \partial x_k^2 \exp(\frac{1}{4}(2 - m)h) : C^\infty(L) \rightarrow C^\infty(L)$. Let $\{\lambda_i, \theta_i\}_{i=1}^\infty$ be a complete spectral resolution of D_L . Let $I = \{i_1, \dots, i_p\}$ with $1 \leq i_1 < \dots < i_p \leq m$. Let $e_I = e_{i_1}^* \dots e_{i_p}$. $e_I \in \text{CLIF}(R^m)$ and form an orthonormal basis. Relative to this frame, D_1 is a diagonal operator which is isomorphic to 2^m copies of D_L . A complete spectral resolution of D_1 is given by $\{\lambda_i, \theta_i e_I\}$. We argue as in the proof of Lemma 2.3 using the identity $(e_i^* e_I, e_j^* e_I) = \delta_{ij}$ to show:

$$d/dt \{ \text{Tr } K(t, x, x, D_1) - \text{Tr } K(t, x, x, D_2) \} \\ = \sum_{i,I} \exp(-t\lambda_i) \{ -(D_1 \theta_i e_I, \theta_i e_I) + (A \theta_i e_I, A \theta_i e_I) \} \\ = \sum_{i,I} \exp(-t\lambda_i) \exp(h) \sum_k \{ \theta_{ijk} \theta_i + \frac{1}{2}(2-m) h_{jk} \theta_{ijk} \theta_i + \frac{1}{4}(2-m) h_{kk} \theta_i \theta_i \\ + \frac{1}{16}(2-m)^2 h_{jk} h_{jk} \theta_i \theta_i + \theta_{ijk} \theta_{ijk} + \frac{1}{2}(2-m) h_{jk} \theta_{ijk} \theta_i + \frac{1}{16}(2-m)^2 h_{jk} h_{jk} \theta_i \theta_i \}$$

$$\begin{aligned}
 &= \sum_{i,l} \exp(-t\lambda_i) \exp(h) \sum_k \{ \theta_{i|kk} \theta_i + \theta_{i|k} \theta_{i|k} + (2-m) h_{i|k} \theta_{i|k} \theta_i \\
 &\quad + \frac{1}{8} (2-m)^2 h_{i|k} h_{i|k} \theta_i \theta_i + \frac{1}{4} (2-m) h_{i|kk} \theta_i \theta_i \} \\
 &= \sum_{i,l,k} \exp(-t\lambda_i) \exp(\frac{1}{2}hm) \frac{1}{2} \partial^2 / \partial x_k^2 \exp(\frac{1}{2}h(2-m)) (\theta_i e_l, \theta_i e_l) \\
 &\quad \frac{1}{2} \exp(\frac{1}{2}hm) \sum_k \partial^2 / \partial x_k^2 \exp(\frac{1}{2}h(2-m)) \text{Tr } K(t, x, x, D_1).
 \end{aligned}$$

Comparing coefficients of t in the two asymptotic expansions completes the proof of the lemma. ■

We use Lemma 4.4 to complete the proof of Theorem 4.1: by Lemma 1.4, we may express $E_n(x, D) = a(n, m) \mathcal{E}_{:i_1 i_1 \dots i_{n-1} i_{n-1}} + b(n, m) R_{ijj;k_1 k_1 \dots k_{n-1} k_{n-1}} +$ other terms involving at most $n - 4$ covariant derivatives of the $\{\mathcal{E}, W_{ij}, R_{ijkl}\}$ tensors. By Lemma 1.7, the coefficients $a(n, m)$ and $b(n, m)$ are independent of the dimension m . By Theorem 2.1, $a(n, m) = a(n, 1) = (n - 1)! / (2n - 1)!$. Let $2n + 2 = m$. By Lemma 4.4,

$$\sum_k \exp(\frac{1}{2}hm) \partial^2 / \partial x_k^2 \exp(\frac{1}{2}h(2-m)) \text{Tr } E_n(x, D_1) = 0.$$

This implies $\text{Tr } E_n(x, D_1)$ is zero. We compute:

$$\begin{aligned}
 \text{Tr } \mathcal{E}_{:i_1 i_1 \dots i_{n-1} i_{n-1}} &= 2^m \exp(nh) \frac{1}{4} (2-m) \partial^{2n} / \partial x_1^{2n}(h) + \dots \\
 \text{Tr } R_{ijj;k_1 k_1 \dots k_{n-1} k_{n-1}} &= 2^m \exp(nh) (1-m) \partial^{2n} / \partial x_1^{2n}(h) + \dots
 \end{aligned}$$

This implies $\frac{1}{4}(2-m)a(n) + (1-m)b(m) = 0$ for $m = 2n + 2$ and consequently

$$b(n) = -\frac{1}{4} a(n)(2n) / (2n + 1) = -\frac{n(n-1)!}{(4n+2)(2n-1)!}$$

which completes the proof of Theorem 4.1. ■

We prove Theorem 4.3 in a similar fashion. (a) and (b) follow from Theorem 4.1. By applying Theorem 4.1 and Lemma 1.4, we may express

$$\begin{aligned}
 E_2(x, D) &= \frac{1}{360} (-12R_{ijj;kk} + c_1 R_{ijj} R_{kkkl} + c_2 R_{ijk} R_{ijlk} + c_3 R_{ijkl} R_{ijkl} + c_4 \mathcal{E} R_{ijj} \\
 &\quad + 60 \mathcal{E}_{:kk} + c_5 \mathcal{E}^2 + c_6 W_{ij} W_{ij}).
 \end{aligned}$$

By Lemma 1.8, the coefficient of \mathcal{E}^2 in $E_2(x, D)$ is $\frac{1}{2}$ so $c_5 = 180$. The

coefficient of $\mathcal{E}R_{ijj}$ in $E_2(x, D)$ is the same as the coefficient of R_{ijj} in $E_1(x, D)$ so by 4.3(b) $c_4 = -60$. Finally, if we consider a product manifold $M = M_1 \times M_2$ and $D = D_1 \otimes 1 + 1 \otimes D_2$, then $E_2((x_1, x_2), D) = E_2(x_1, D_1) + E_2(x_2, D_2) + (2c_1/360)R_{ijj}(M_1)R_{ijj}(M_2) + \text{other terms}$. By Lemma 1.6, $E_2((x_1, x_2), D) = E_2(x_1, D_1) + E_2(x_2, D_2) + E_1(x_1, D_1)E_1(x_2, D_2)$. This implies that $2c_1/360 = (-\frac{1}{6})^2$ and therefore $c_1 = 5$. Thus $E_2(x, D)$ has the form:

$$E_2(x, D) = \frac{1}{360}(-12R_{ijj;kk} + 5R_{ijj}R_{klk} + c_2R_{ijk}R_{ljk} + c_3R_{ikl}R_{ijl} - 60R_{ijj}\mathcal{E} + 60\mathcal{E}_{;kk} + 180\mathcal{E}^2 + c_6W_{ij}W_{ij}).$$

We will determine c_6 by applying Lemma 4.4 to the case $m = 2$ and $n = 2$.

$$\begin{aligned} \mathcal{E}(D_1) &= 0 & \mathcal{E}(D_2) &= \exp(h)(-\frac{1}{2}h_{11} - \frac{1}{2}h_{22}) \\ W_{ij}(D_1) &= 0 & W_{12}(D_2) &= -\frac{1}{2}h_{11}e_1^*e_2 - \frac{1}{2}h_{22}e_2^*e_1 \\ R_{121}^2 &= -\frac{1}{2}(h_{11} + h_{22}). \end{aligned}$$

Since the dimension of V is 4, $\text{Tr } E_1(x, D_1) = \frac{4}{6}\exp(h)(h_{11} + h_{22})$. The right hand side of Lemma 4.4 becomes $\frac{4}{6}\exp(2h)(h_{11}h_{11}) + \text{other terms}$. Similarly

$$\begin{aligned} \text{Tr } E_2(x, D_1) - \text{Tr } E_2(x, D_2) &= \frac{-1}{360}(-60R_{ijj}\mathcal{E} + 60\mathcal{E}_{;kk} + 180\mathcal{E}^2 \\ &\quad + c_6W_{ij}W_{ij})(x, D_2) \\ &= \frac{4}{360}\exp(2h)(30 + 30 - 45 + \frac{1}{2}c_6)(h_{11}h_{11}) \\ &\quad + \text{other terms}. \end{aligned}$$

Therefore by Lemma 4.4

$$\frac{8}{360}(15 + \frac{1}{2}c_6) = \frac{4}{6} \text{ so } c_6 = 30.$$

We determine c_2 and c_3 by applying Lemma 4.4 to the case $m = 6$, $n = 3$. Since $2n - m = 0$, $\text{Tr } E_2(x, D_1) = 0$. We compute

$$\begin{aligned} \mathcal{E}(D_1) &= \exp(h) \sum_k (-h_{kk} + h_{ik}h_{ik}) & W_{ij}(D_1) &= 0 \\ R_{ijj} &= \exp(h) \sum_k (-5h_{kk} + 5h_{ik}h_{ik}) = 5\mathcal{E}(D_1). \end{aligned}$$

Consequently, $-12R_{ijj;kk} + 60\mathcal{E}_{;kk} = 0$ and $6R_{ijj}R_{klk} - 60R_{ijj}\mathcal{E} +$

$$180\mathcal{E}^2(D_1) = 5\mathcal{E}^2(x, D_1)$$

$$5\mathcal{E}^2(D_1) = \exp(2h)(5h_{/11}h_{/11} + 0h_{/12}h_{/12}) + \text{other terms},$$

$$c_2R_{ijk}R_{ijk} = \exp(2h)\left(\frac{15c_2}{2}h_{/11}h_{/11} + 8c_2h_{/12}h_{/12}\right) + \text{other terms},$$

$$c_3R_{ijkl}R_{ijkl} = \exp(2h)(5c_3h_{/11}h_{/11} + 8c_3h_{/12}h_{/12}) + \text{other terms}.$$

This implies $c_2 + c_3 = 0$ and $10 + 15c_2 + 10c_3 = 0$. Consequently $c_2 = -2$ and $c_3 = 2$ which completes the proof of Theorem 4.3(c). The proof of part (d) is along similar lines and we will omit the details of the computation. ■

If $m = 1$, let x be parametrization by arc length; $d/dx = \exp(\frac{1}{2}h) d/dx_1$. With this change of variable, $A = \exp(\frac{1}{2}h)e_1^*(d/dx_1 + \frac{1}{4}h_{/1}) = e_1^*(d/dx + \frac{1}{4}dh/dx)$. If we set $b = \frac{1}{4}dh/dx$, then $A = e_1^*(d/dx + b)$ and $A' = e_1^*(d/dx - b)$. Therefore $D_1 = -(d^2/dx^2 + (b' - b^2))$ and $D_2 = -(d^2/dx^2 + (-b' - b^2))$. This shows that Lemma 4.4 reduces to Lemma 2.3 if $m = 1$.

If $m = 2$, $G = \exp(-h)(dx_1^2 + dx_2^2)$. If M is oriented, the metric induces a holomorphic structure on M . An orthonormal frame for $\Lambda^{0,1}(M)$ is given by $\exp(\frac{1}{2}h)(dx_1 - idx_2)$. Relative to this frame and the constant frame for $\Lambda^{0,0}(M)$, the operator of the Dolbeault complex is given by:

$$\sqrt{2}\bar{\partial} = \exp(\frac{1}{2}h)(\partial/\partial x_1 + i\partial/\partial x_2)$$

$$A = \exp(\frac{1}{2}h)(e_1\partial/\partial x_1 + e_2\partial/\partial x_2) = e_1 \exp(\frac{1}{2}h)(\partial/\partial x_1 + (e_2^*e_1)\partial/\partial x_2).$$

Since $(e_2^*e_1) = -1$, A is isomorphic to the direct sum of 4 copies of $\sqrt{2}\bar{\partial}$. Let Δ_p^2 be the Laplacian on p -forms and let δ'' be the adjoint of $\bar{\partial}$. $\Delta_0^2 = 2\delta''\bar{\partial}$. Δ_1^2 is isomorphic to the direct sum of 2 copies of the operator $2\bar{\partial}\delta''$. Therefore

$$\text{Tr } E_n(x, D_1) = 4 \text{Tr } E_n(x, \Delta_0^2)$$

$$\text{Tr } E_n(x, D_2) = 2 \text{Tr } E_n(x, \Delta_1^2).$$

Since $\Delta_0^2 = -\exp(h)(\partial^2/\partial x_1^2 + \partial^2/\partial x_2^2)$, we may rewrite Lemma 4.4 in the form:

$$(2n - 2)(4 \text{Tr } E_n(x, \Delta_0^2) - 2 \text{Tr } E_n(x, \Delta_1^2)) = 4\Delta_0^2 \text{Tr } E_{n-1}(x, \Delta_0^2).$$

Δ_0^2 is locally isomorphic to Δ_2^2 so we can rewrite this equation in the

form:

$$(n-1)(\text{Tr } E_n(x, \Delta_0^2) - \text{Tr } E_n(x, \Delta_1^2) + \text{Tr } E_n(x, \Delta_2^2)) = \Delta_0^2 \text{Tr } E_{n-1}(x, \Delta_0^2).$$

We have proved this for any metric of the form $G = \exp(-h)(dx_1^2 + dx_2^2)$ on T^2 . Since these are local invariants, this identity holds for any metric on a 2-dimensional manifold which locally has this form for some choice of coordinate system. Any 2-dimensional manifold admits a local holomorphic structure relative to which the metric is conformally flat and hence this identity holds for any 2-dimensional manifold. This proves the identity which was first proved by McKean and Singer [19]:

COROLLARY 4.5: *Let $m = 2$, then $(n-1)(\text{Tr } E_n(x, \Delta_0^2) - \text{Tr } E_n(x, \Delta_1^2) + \text{Tr } E_n(x, \Delta_2^2)) = \Delta_0^2 \text{Tr } E_{n-1}(x, \Delta_0^2)$.*

If $m = 2$, $\mathcal{E}(\Delta_0^2) = 0$ so $E_{n-1}(x, \Delta_0^2) = b(n)R_{ij;k_1k_1\dots k_{n-2}k_{n-2}}$ + other terms where $b(n) \neq 0$ is given in Theorem 4.1. This proves that generically $\Delta_0^2 \text{Tr } E_{n-1}(x, \Delta_0^2) \neq 0$.

THEOREM 4.6: *Let $a_n^m(x, d + \delta) = \sum_p (-1)^p \text{Tr } E_n(x, \Delta_p^m)$. Then,*

- (a) $a_n^m(x, d + \delta) = 0$ if m is odd.
- (b) $a_n^m(x, d + \delta) = 0$ if $2n < m$.
- (c) $a_n^m(x, d + \delta)$ is the integrand of the Chern–Gauss–Bonnet theorem [5] if $2n = m$.
- (d) $a_n^m(x, d + \delta) \neq 0$ if m is even and if $2n > m$ generically.

PROOF: Δ_p^m is locally isomorphic to Δ_{m-p}^m by Poincaré duality so the alternating sum vanishes if m is odd. This proves (a). By Theorem 3.3

$$\begin{aligned} & \int_M a_n^m(x, d + \delta) |d\text{vol}| \\ &= \begin{cases} 0 & \text{if } 2n \neq m \\ \text{the Euler–Poincaré characteristic of } m & \text{if } 2n = m. \end{cases} \end{aligned}$$

Theorem 4.6(b) shows the alternating sum vanishes locally if $2n < m$ while (c) implies that the integrand is the formula given by Chern [5] for $2n = m$. We have proved [15] that $a_n^m(x, d + \delta) = \text{divergence}(Q_n^m)$ for some functorially defined 1-form Q_n^m . If $m = 2$, then Cor. 4.5 shows $a_n^2(x, d + \delta) = \Delta_0^2(P_n^2)$ for some scalar P_n^2 if $n > 1$. The formula of [15a] gives Q_n^m for $2n = m + 2$ and shows P_n^m is not Δ_0^m of a scalar function if $m > 2$ is even.

Parts (b) and (c) are well known [1, 8, 21]. We proved (d) if $m = 2$ using Cor. 4.5. We complete the proof of (d) by induction. Let $m = m_1 + 2$ where m_1 is even. Let N be a manifold of dimension m_1 and let R be a Riemann-surface. Let $M = N \times R$ with the product metric. There exist natural isomorphisms

$$\Delta_p^M = \bigoplus_{p=r+s} (\Delta_r^N \otimes 1 + 1 \otimes \Delta_s^R).$$

By Lemma 1.5, this implies that

$$E_n((x_1, x_2)\Delta_p^M) = \sum_{r+s=p} \sum_{u+v=n} E_u(x_1, \Delta_r^N) \otimes E_v(x_2, \Delta_s^R).$$

Consequently, when we perform the alternating sum,

$$a_n^m((x_1, x_2), (d + \delta)_M) = \sum_{u+v=n} a_u^{m_1}(x_1, (d + \delta)_N) a_v^2(x_2, (d + \delta)_R).$$

By (b), $a_u^{m_1}(x_1, (d + \delta)_N) = 0$ for $2u < m_1$ and $a_{m_1/2}^{m_1}(x_1, (d + \delta)_N)$ is the integrand of the Chern–Guass–Bonnet theorem and is non-zero generically. Since $a_v^2(x_2, (d + \delta)_R) = c(v)R_{ijj;k_1 k_1 \dots k_{v-1} k_{v-1}} +$ lower order terms where $c(v)$ is non-zero if $v \neq 0$, this implies that the coefficient of $R_{ijj;k_1 \dots k_r}$ in $a_n^m(x, (d + \delta))$ is non-zero if $r = n - \frac{1}{2}m_1 - 1 \geq 0$. Therefore $a_n^m(x, (d + \delta))$ is generically non-zero provided that $n \geq 2m$ which completes the proof of (d). ■

For any operator D , $a_0(D) = \dim(V)$ is the dimension of the vector bundle on which the operator acts. This implies $a_0(A^*A) - a_0(AA^*) = 0$ for any operator A such that the leading symbol of A^*A and AA^* is given by the metric tensor. If A is the operator of the DeRham, signature, or spin complex, then $a_n(x, A^*A) - a_n(x, AA^*) = 0$ for $2n < m$ [1, 8]. It is natural to conjecture that some such vanishing takes place for any elliptic complex. This is not, however, the case.

THEOREM 4.7: *Let $n \neq 0$ and $m \neq 0$. There is a manifold of dimension m and an elliptic complex $A: C^\infty(V_1) \rightarrow C^\infty(V_2)$ over M such that $a_n(x, A^*A) - a_n(x, AA^*) \neq 0$ for some point x of the manifold.*

PROOF: If $2n = m$, we can take the DeRham complex for any manifold of dimension m and non-zero Euler characteristic. If $2n \neq m$, we take the elliptic complex described in Lemma 4.4. Since

$2n \neq m$, it suffices to show $\text{Tr } E_{n-1}(x, D_1) = 0$. We compute:

$$\begin{aligned} \mathcal{E}_{:i_1 i_1 \dots i_{n-2} i_{n-2}} &= \exp(2(n-1)h) \left(-\frac{1}{4}(m-2)h_{/11 \dots 11} + \text{other terms}\right) \\ R_{ijj:k_1 k_1 \dots k_{n-2} k_{n-2}} &= \exp(2(n-1)h) \left(- (m-1)h_{/11 \dots 11} + \text{other terms}\right). \\ \text{Tr } E_{n-1}(x, D_1) &= \frac{2^m(n-2)!}{(2n-3)!} \exp(2(n-1)h) \left(-\frac{1}{4}(m-2) \right. \\ &\quad \left. + \frac{(m-1)(n-1)}{(4n-2)} h_{/11 \dots 11} \right) + \text{other terms.} \end{aligned}$$

If $\text{Tr } E_{n-1}(x, D_1) = 0$, then $(m-2)(4n-2) = 4(n-1)(m-1)$ which implies $2n = m$. This completes the proof of Theorem 4.7. ■

We can use Theorem 4.3 together with the formulas given at the beginning of this section to compute $a_n(x, \Delta_p^m)$ for $n = 0, 1, 2, 3$. We shall only list the results of this computation for $n = 0, 1, 2$ to avoid lengthy expressions. These results can also be found in [9, 22].

THEOREM 4.8: *If Δ_p^m is the Laplacian on p -forms on an m -dimensional manifold, then:*

- (a) $a_0(x, \Delta_p^m) = \binom{m}{p}$,
- (b) $a_1(x, \Delta_p^m) = \left\{ \frac{-1}{6} a_p^m + \binom{m-2}{p-1} \right\} R_{ijj}$,
- (c) $a_2(x, \Delta_0^m) = \frac{1}{360} (-12R_{ijj,kk} + 5R_{ijkl}R_{ijkl} - 2R_{ijik}R_{ijlk} + 2R_{ijkl}R_{ijkl})$,
 $a_2(x, \Delta_1^m) = \frac{1}{360} (60R_{ijj,kk} - 60R_{ijj}R_{kkk} + 180R_{ijik}R_{ijlk} - 30R_{ijkl}R_{ijkl})$
 $+ ma_2(x, \Delta_0^m)$,
 $a_2(x, \Delta_2^m) = \frac{1}{360} \{ 60(m-2)R_{ijj,kk} + (300-60m)R_{ijj}R_{kkk}$
 $+ 180(m-6)R_{ijik}R_{ijlk} + (240-30m)R_{ijkl}R_{ijkl}$
 $+ \binom{m}{2} a_0(x, \Delta_0^m) \}.$

If $m \geq 4$, $a_2(x, \Delta_p^m) = \{ \binom{m-4}{p} + \binom{m-4}{p-4} \} a_2(x, \Delta_0^m) + \{ \binom{m-4}{p-1} + \binom{m-4}{p-3} \} a_2(x, \Delta_1^m) + \binom{m-4}{p-2} a_2(x, \Delta_2^m)$.

If $m \leq 4$, we use the relation $a_2(x, \Delta_p^m) = a_2(x, \Delta_{m-p}^m)$ for $p = 3, 4$.

PROOF: We shall omit the derivation of these formulas since they are well known; a derivation is contained in [11] which uses Theorem 4.3. ■

If P is a scalar invariant, let $P(M) = \int_M P(G)|d\text{vol}|$. For example $1(M)$ is the volume of M . The following theorem is due to Patodi and is a direct consequence of the formulas of Theorem 4.8:

THEOREM 4.9: *Let M and N be Riemannian manifolds and suppose that the spectrum of Δ_p^m on M is equal to the spectrum of Δ_p^m on N for $p = 0, 1, 2$. Then*

- (a) $1(M) = 1(N)$, $R_{ijj}(M) = R_{ijj}(N)$, $R_{ijj}R_{klkl}(M) = R_{ijj}R_{klkl}(N)$,
 $R_{ijk}R_{ljk}(M) = R_{ijk}R_{ljk}(N)$, $R_{ijkl}R_{ijkl}(M) = R_{ijkl}R_{ijkl}(N)$;
 (b) if M has constant scalar curvature c then so does N ;
 (c) if M is Einstein, so is N ;
 (d) if M has constant sectional curvature c then so does N ;
 (e) if M is isometric to the standard sphere of radius 1, so is N .

PROOF: Since $a_n(x, \Delta_p^m)$ depends only on the spectrum of Δ_p^m , $a_n(\Delta_p^m)(M) = a_n(\Delta_p^m)(N)$ for all n and $p = 0, 1, 2$. We use these equations for $n = 0, 1, 2$ to prove (a); the matrix of coefficients of $R_{ijj}R_{klkl}$, $R_{ijk}R_{ljk}$, and $R_{ijkl}R_{ijkl}$ in a_2 for $p = 0, 1, 2$ has rank 3 so we can solve for these invariants. M has constant scalar curvature c if and only if $(R_{ijj}R_{klkl} + 4cR_{ijj} + 4c^2)(M) = 0$. Since the same identity must hold true for N , N has constant scalar curvature. (c) and (d) follow similarly; (e) follows from (d) and $\text{vol}(M) = \text{vol}(N)$. ■

There are many other similar results which follow from the formulas of Theorem 4.3. We mention just one which is proved in [11] which utilizes the computation of a_3 : let $\Delta_1^m(s) = s(\Delta_1^m) + (1-s)(\Delta_1^1)$ be the operator of Theorem 1.8.

THEOREM 4.10: *Let N and M be two Riemannian manifolds. Suppose that Δ_0^m and Δ_2^m have the same spectrum on N and M and that $\Delta_1^m(s)$ has the same spectrum on N and M for all $s \in [0, 1]$. Then if N is a local symmetric space, so is M . The Ricci tensor is parallel transported in a local symmetric space so the eigenvalues of the Ricci tensor are constant. Under these assumptions, the eigenvalues of the Ricci tensor on N are equal to the eigenvalues of the Ricci tensor on M .*

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