COMPOSITIO MATHEMATICA

KNUD LØNSTED
STEVEN L. KLEIMAN

Basics on families of hyperelliptic curves

Compositio Mathematica, tome 38, nº 1 (1979), p. 83-111
<http://www.numdam.org/item?id=CM_1979__38_1_83_0>

COMPOSITIO MATHEMATICA, Vol. 38, Fasc. 1, 1979, pag. 83–111
Sijthoff & Noordhoff International Publishers – Alphen aan den Rijn
Printed in the Netherlands

BASICS ON FAMILIES OF HYPERELLIPTIC CURVES

Knud Lønsted and Steven L. Kleiman*

0. Introduction

The central theme of this paper is the basic study of smooth families of projective hyperelliptic curves parametrized by locally noetherian base schemes. However, a certain number of general features of smooth families of curves have been included. Among these is a fairly thorough study of the formation of quotients with respect to finite groups. One purpose of this paper has been to prepare the way for the construction of various kinds of moduli spaces for hyperelliptic curves to be carried out in a forthcoming paper by O.A. Laudal and the first author.

The main problem concerning families of hyperelliptic curves is to single out those families that deserve to be named *hyperelliptic families.* A priori this may be done in several ways, each of which is a generalization of one of the classical characterizations of hyperelliptic curves over algebraically closed field. The point is to prove that the ways are equivalent.

There is no restriction on the characteristics of the residue fields. In particular, characteristic 2 has been included at all stages. As a consequence we have been obliged to renounce the use of (hyperelliptic) Weierstrass points as a basic technical tool, so they do not emerge until the last section.

The material is organized as follows:

1. Conventions.
2. Image of a finite morphism.
3. Generalities on families of curves.
4. Quotients by finite groups.
5. Hyperelliptic families of curves.

* Partially supported by the Danish National Research Council under grant no. MPS 7407275. Guest at Copenhagen University, som skal have mange tak.

In section 2 we introduce the notion of *co-flat* morphisms of schemes. This is particularly useful when the morphism is finite, in which case it is closely related to the formation of a scheme-theoretical image (Prop. 2.6). Section 3 starts with the study of the structure and cohomology of families of curves of genus zero. Then we give a cohomological characterization of the dualizing sheaf on an arbitrary family of smooth projective curves (Prop. 3.8).

In section 4 we first consider a (noetherian) ring R and a finite group G that acts on an R-algebra A. We introduce a subring $\Sigma_R^G(A)$ of the ring of invariants A^G, called the ring *generated by the G-symmetric functions* over R. In the case where $\Sigma_R^G(A) = A^G$, it turns out that the formation of A^G commutes with base change. On the scheme level one has the corresponding notion, and it is proved that for a smooth family of quasi-projective curves the corresponding equality *always* holds, provided the group acts faithfully in the fibers (Theorem 4.12).

The study of hyperelliptic curves begins in section 5. The starting point is the characterization of a hyperelliptic curve defined over a field by means of the canonical map, i.e., the map defined by the canonical divisor. It is proved that a suitable generalization of this to a family is equivalent to the family being a double covering of a family D of curves of genus zero, and to the existence of a global *canonical involution* (Theorem 5.5). A family of curves that satisfies these equivalent conditions is called a *hyperelliptic family*. It is proved that the family D and the double covering of D are unique up to automorphism of D (Cor. 5.10).

In section 6 we define the *Weierstrass subscheme* of a hyperelliptic family of curves as the branch locus of the canonical morphism, with the usual scheme structure. It also equals the branch locus of the double covering of D, and (as a special case of the Riemann-Hurwitz formula) we show that it has the expected properties (Prop. 6.3).

In section 7 we finally introduce the notion of *hyperelliptic Weierstrass sections* that generalizes the classical concept of hyperelliptic Weierstrass points. We prove that a family of curves is hyperelliptic if and only if it acquires a hyperelliptic Wierstrass section after a faithfully flat base change. Furthermore, all sections of the Weierstrass subscheme are hyperelliptic Weierstrass sections (Theorem 7.3).

The paper closes with an appendix due to the second author, who wishes to express his gratitude to Daniel Grayson for pointing out some oversights in a preliminary version. The appendix contains part of a general base change theory presented in a series of lectures given at the University of Copenhagen in May, 1977.

1. Conventions

We freely use the terminology and the results of Grothendieck ([6]). For simplicity all schemes are assumed locally noetherian and all morphisms of schemes are of finite type, unless the contrary is explicitly mentioned. Recall that a *geometric point* s of a scheme S is a morphism $\mathrm{Spec}(k) \to S$ (not necessarily of finite type), where k is an *algebraically closed field.* The inverse image of a sheaf F of $\mathcal{O}_S$-Modules by this morphism will be denoted by $F(s)$, whereas pull-backs to an S-scheme T of a relative situation over S will be denoted by lower subscripts T.

A morphism $p: C \to S$ is called a *smooth family of projective curves of genus* g if p is smooth and projective and the geometric fibers of p are connected curves of genus g. It is well-known that it makes no difference to replace the word "projective" by "proper" in this connection when $g \geqq 2$, (see e.g. [3]).

In section 4 we shall call a ring extension $A \subset B$ *quasi-finite* provided all the fibers of $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$ are finite. Thus, we drop the usual extra condition that B be essentially of finite type over A (cf. [6, II. 6.2.3]).

2. Image of a finite morphism

Let S be a locally noetherian scheme, and let X and Y be S-schemes of finite type. Associated to an S-morphism $f: X \to Y$ we have the co-morphism ${}^cf: \mathcal{O}_Y \to f_*\mathcal{O}_X$. Put $L = \mathrm{Coker}({}^cf)$. We then have an exact sequence of quasi-coherent $\mathcal{O}_Y$-Modules,

(2.1)
$$\mathcal{O}_Y \xrightarrow[{}^cf]{} f_*\mathcal{O}_X \to L \to 0.$$

LEMMA 2.2: *If the morphism f above is affine, then the formation of the exact sequence* (2.1) *commutes with base change on S.*

PROOF: The condition on f ensures that f_*F commutes with base change for any quasi-coherent sheaf F of $\mathcal{O}_X$-Modules, so the lemma follows from the right-exactness of pull-back. ■

Consider now a *finite* S-morphism $f: X \to Y$. The scheme-theoretical image of f, $\mathrm{Im}(f)$, is defined to be the closed subscheme of Y with Ideal $\mathrm{Ker}({}^c f)$. From (2.1) we may form two short exact sequences:

$$0 \to \mathrm{Ker}({}^c f) \to \mathcal{O}_Y \to \mathcal{O}_{\mathrm{Im}(f)} \to 0; \tag{2.3}$$

$$0 \to \mathcal{O}_{\mathrm{Im}(f)} \to f_* \mathcal{O}_X \to L \to 0. \tag{2.4}$$

Since f is proper these sequences consist of coherent $\mathcal{O}_Y$-Modules.

DEFINITION 2.5: An S-morphism of finite type $f: X \to Y$ will be called *co-flat* if $L = \mathrm{Coker}({}^c f)$ is S-flat.

PROPOSITION 2.6: *Let $f: X \to Y$ be a finite S-morphism, and assume that X is S-flat. Then f is co-flat if and only if the image* $\mathrm{Im}(f)$ *is S-flat and its formation commutes with base change on S.*

Furthermore, when f is co-flat, then the canonical morphism $X \to$ $\mathrm{Im}(f)$ *is flat if and only if the corresponding morphism is flat for every geometric fiber over S.*

PROOF: Let $T \to S$ be a morphism, and let $\varphi: Y_T \to Y$ be the induced morphism. Then the exact sequence $\varphi^* \mathcal{O}_Y \to \varphi^* f_* \mathcal{O}_X \to \varphi^* L \to 0$ is canonically isomorphic to

$$\mathcal{O}_{Y_T} \to (f_T)_* \mathcal{O}_{X_T} \to L_T \to 0, \tag{2.7}$$

by Lemma 2.2. We again form two short exact sequences:

$$0 \to \mathrm{Ker}({}^c f_T) \to \mathcal{O}_{Y_T} \to \mathcal{O}_{\mathrm{Im}(f_T)} \to 0, \tag{2.8}$$

$$0 \to \mathcal{O}_{\mathrm{Im}(f_T)} \to (f_T)_* \mathcal{O}_{X_T} \to L_T \to 0. \tag{2.9}$$

The image $\mathrm{Im}(f_T)$ is the pull-back of $\mathrm{Im}(f)$ to T precisely when (2.8) is obtained by applying φ^* to (2.3).

Assume first that f is co-flat. Since f is affine, $f_* \mathcal{O}_X$ is S-flat and (2.4) shows that $\mathcal{O}_{\mathrm{Im}(f)}$ is S-flat. Therefore, φ^* applied to (2.3) and (2.4) yields the short exact sequences (2.8) and (2.9). Hence $\mathrm{Im}(f)$ is S-flat and commutes with base change.

Suppose next that $\mathrm{Im}(f)$ is S-flat and that its formation commutes with base change. This implies that the homomorphism $\mathcal{O}_{\mathrm{Im}(f)} \to f_* \mathcal{O}_X$ is universally injective, cf. Cor. A.2, so the quotient L is S-flat, i.e., f is co-flat.

The last assertion is a special case of the following.

LEMMA 2.10: *Let $f: X \to Y$ be an S-morphism of flat S-schemes. Then f is flat if and only if $f_s: X_s \to Y_s$ is flat for all geometric points s of S.*

PROOF: This is a special case of [6, Cor. IV. 11.3.11]. ■

REMARKS 2.11: (a) If in addition we assume Y to be S-flat in Prop. 2.6, the proof of the first assertion may be reduced to general base change theory as follows: Consider the complex

$$0 \to \mathcal{O}_Y \to f_*\mathcal{O}_X \to 0$$

of coherent $\mathcal{O}_Y$-Modules, flat over S. The first cohomology group is L, which commutes with base change. The zero'th cohomology group is the Ideal J of Im(f). So, by Prop. A.8.(ii), J commutes with base change if and only if L is S-flat.

(b) With the same notation as in Def. 2.5 we observe that f is co-flat if and only if f has a co-flat pull-back to some faithfully flat Y-scheme, not necessarily of finite type.

EXAMPLES 2.12: (1) A closed S-embedding $X \hookrightarrow Y$ is co-flat.

(2) Assume that X is S-flat. Then a finite, flat S-morphism $f: X \to Y$ is co-flat. In order to prove this, one may assume that f is surjective, since it maps X onto an open and closed subset of Y. The question is then local on Y, and it follows from this: Let $A \to B$ be a faithfully flat, finite ring homomorphism of noetherian rings. Then A is a direct summand in B, i.e. B/A is A-flat, (see [6, Prop. IV.2.2.17]).

For the sake of completeness we include the following result, which will not be used in the sequel.

PROPOSITION 2.13: *Let $f: X \to Y$ be a finite S-morphism, and assume that X is S-flat. Then there exists a stratification $\amalg S_i$ of S with the following property: For any S-scheme $T \to S$ the image of $f_T: X_T \to Y_T$ is flat and commutes with base-change on T if and only if $T \to S$ factors through $\amalg S_i \to S$.*

PROOF: Take for $\amalg S_i$ the flattening stratification for L, (see [8, Lecture 8]), and apply Prop. 2.6. ■

This proposition is an essential step in the construction of the moduli spaces for hyperelliptic curves that was mentioned in the introduction.

3. Generalities on families of curves

In this section $p: C \to S$ denotes a smooth, projective family of curves of genus $g \geqq 0$. Assume first that p has a section $e: S \to C$.

Then e is a closed embedding. The restriction of the Ideal of Im(e) to a fiber over S is locally generated by a non-zerodivisor. It follows that the Ideal of Im(e) is invertible and that Im(e) is the support of an effective Cartier divisor E on C relative to S, where the invertible sheaf L_E associated to E is given by the exact sequence,

$$0 \to L_E^{-1} \to \mathcal{O}_C \to e_*\mathcal{O}_S \to 0. \tag{3.1}$$

(See [6, IV.21] or [8, Lecture 9-10] for details about relative Cartier divisors.)

We now examine families of curves $p: C \to S$ of genus 0.

DEFINITION 3.2: A smooth, projective family of curves $p: C \to S$ of genus 0 is called a *twisted* $\mathbb{P}^1_S$.

One simple class of twisted projective lines over S consists of the morphisms $\mathbb{P}(V) \to S$, where V is a locally free sheaf of rank 2 on S. We have the following criterion for a twisted $\mathbb{P}^1_S$ to be of this form.

PROPOSITION 3.3: *Let $p: C \to S$ be a twisted $\mathbb{P}^1_S$. Assume that there exists an invertible sheaf L on C such that* deg $L_s = 1$ *for all geometric points s of S. (This holds with $L = L_E$ as above, when p admits a section e.)*

Then C is S-isomorphic to $\mathbb{P}(V)$ for some locally free sheaf V on S of rank 2.

PROOF: Since the geometric fibers of p are projective lines, we have $R^1p_*L = 0$ in the fibers; so $V = p_*L$ is locally free and commutes with base change ([16, III.7.8] or [9, section 5]). The Riemann–Roch formula applied to a geometric fiber shows that $rkV = 2$. The natural map $p^*V \to L$ is surjective since its restriction to every geometric fiber is surjective. Let $\Phi: C \to \mathbb{P}(V)$ denote the associated S-morphism. The restriction of Φ to a fiber over S is an isomorphism, so Φ is quasi-finite. Since Φ is proper, it is finite by Chevalley's theorem. The restriction to the fibers of the co-morphism ${}^c\Phi: \mathcal{O}_{\mathbb{P}(V)} \to \Phi_*\mathcal{O}_C$ is bijective. We conclude that ${}^c\Phi$ is bijective, hence that Φ is an isomorphism, by the following local assertion: Let $R \to A$ be a local homomorphism of local noetherian rings, and let $M \to N$ be an A-linear map between finite A-modules, where N is assumed flat over R. Suppose that the induced map $M \otimes_R k \to N \otimes_R k$ is bijective, where k denotes the residue field of R. Then $M \to N$ is bijective. This assertion follows immediately from Nakayama's lemma and the vanishing of $\mathrm{Tor}_1^R(N, k)$). ■

Since a smooth morphism always has a section locally in the étale topology ([6, IV.17.6.3]), we have

COROLLARY 3.4: *For every twisted* $\mathbb{P}^1_S$, $p: C \to S$, *there exists an étale surjective morphism* $T \to S$ *such that* $C_T \simeq \mathbb{P}^1_T$.

REMARK 3.5: Every twisted $\mathbb{P}^1_S$ gives rise to an element in the Brauer group of S, $\mathrm{Br}(S)$. Therefore, if $\mathrm{Br}(S) = 0$, then every twisted $\mathbb{P}^1_S$ is of the form $\mathbb{P}(V)$ described above. As an example we can mention that $\mathbb{P}^1_{\mathbb{Z}}$ itself is the only twisted $\mathbb{P}^1_{\mathbb{Z}}$.

Now let $p: C \to S$ be an arbitrary smooth family of projective curves, and let F be a coherent sheaf on C which is flat over S. Then $R^i p_* F = 0$ in the fibers for $i \geqq 2$. Hence $R^i p_* F = 0$ for $i \geqq 2$, and $R^1 p_* F$ commutes with base change ([6, III.7.8]). Consequently, $p_* F$ is locally free and commutes with base change if and only if $R^1 p_* F$ is locally free. Furthermore, the property that $R^i p_* F$ be locally free and commute with base change is local on S (for the Zariski topology), and it may be checked after a faithfully flat base change (i.e., it is local in the *fpqc*-topology).

PROPOSITION 3.6: *Let* $p: C \to S$ *be a twisted* $\mathbb{P}^1_S$, *and let* L *be an invertible sheaf on* C. *Then* $p_* L$ *and* $R^1 p_* L$ *are locally free and commute with base change.*

PROOF: According to Cor. 3.4 and the remarks above, the proposition need only be proved for $C = \mathbb{P}^1_S$ and under the additional assumption that S be connected. Since L is flat over S, the integer $n = \deg L_s$ is independent of the point s of S. If $n \geqq 0$, then $R^1 p_* L = 0$ in the fibers, so $R^1 p_* L = 0$, $p_* L$ is locally free, and both commute with base change. If $n < 0$, then $p_* L = 0$ in the fibers, so $p_* L = 0$, $R^1 p_* L$ is locally free, and both commute with base change. ■

REMARK 3.7: When S is connected and $C = \mathbb{P}(V)$ in the proposition above, it is well-known (cf. [6, II.4.2.7]) that there is a unique expression $L = \mathcal{O}_{\mathbb{P}(V)}(n) \otimes p^* M$, with $n \in \mathbb{Z}$ and M an invertible sheaf on S. This gives a more explicit description of the sheaf $p^* L$ which, however, will not be needed here.

The last topic in this section is a characterization of the dualizing sheaf on a smooth, projective family of curves.

PROPOSITION 3.8: *Let* $p: C \to S$ *be a smooth, projective family of curves of genus* $g \geqq 0$, *and let* L *be an invertible sheaf on* C. *Then* L *is* *isomorphic to the sheaf of relative differentials* $\Omega^1_{C/S}$ *if and only if it satisfies the following two conditions*:

(i) *There exists an isomorphism* $\varphi : \mathcal{O}_S \xrightarrow{\sim} R^1p_*L$;

(iiv) *For every geometric point s of S one has* $\deg L_s = 2g-2$.

Moreover, the choice of an isomorphism φ *determines in a canonical fashion a specific isomorphism* $L \xrightarrow{\sim} \Omega^1_{C/S}$.

PROOF: It is easy to see that $\Omega^1_{C/S}$ satisfies the two conditions, so assume that L satisfies (i) and (ii). Fix an isomorphism φ. According to Prop. 3.6 the sheaf R^1p_*L commutes with base change; so, by Grothendieck duality, we get an isomorphism $\mathcal{O}_S \xrightarrow{\sim} p_*(L^{-1} \otimes \Omega^1_{C/S})$ that commutes with base change. The adjoint homomorphism $\mathcal{O}_C \to L^{-1} \otimes \Omega^1_{C/S}$ is injective on the geometric fibers of p, hence universally injective by Cor. A.2. Consequently the cokernel Q is flat over S. Consider the exact sequence of S-flat coherent $\mathcal{O}_C$-Modules,

$$0 \to \mathcal{O}_C \to L^{-1} \otimes \Omega^1_{C/S} \to Q \to 0.$$

Its restriction to the fiber over a geometric point s of S remains exact, and taking Euler-characteristics shows that $\chi(Q_s) = 0$. Since the support of Q_s is finite, one has $Q_s = 0$. By Nakayama's lemma $Q = 0$, so $\mathcal{O}_C \xrightarrow{\sim} L^{-1} \otimes \Omega^1_{C/S}$, or, equivalently, $L \xrightarrow{\sim} \Omega^1_{C/S}$. ∎

4. Quotients by finite groups

Let R be a commutative ring and let A denote an R-algebra. For any finite group G of R-automorphisms of A, we let A^G denote the G-invariant R-subalgebra of A. Every $a \in A$ satisfies an integral equation over A^G of the form,

$$\prod_{g \in G} (x - g(x)) = 0. \tag{4.1}$$

Put $n = |G|$ (the order of G). Denote the coefficient of x^{n-j} in this equation by $(-1)^j \sigma_j(x)$. In this way we define mappings $\sigma_j : A \to A^G$, $j = 0, 1, \ldots, n$, that we call the *G-symmetric functions* on A.

Let $s_1, \ldots, s_n$ denote the usual elementary symmetric functions in n variables $x_1, \ldots, x_n$, i.e.,

$$\begin{aligned}
s_1(x_1, \ldots, x_n) &= x_1 + x_2 + \cdots + x_n,\\
s_2(x_1, \ldots, x_n) &= x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n,\\
&\vdots\\
s_n(x_1, \ldots, x_n) &= x_1x_2 \ldots x_n.
\end{aligned}$$

If $G = \{g_1, g_2, \ldots, g_n\}$ is an ordering of the elements of G, where g_1 is the identity, then we have for all $a \in A$

$$\sigma_j(a) = s_j(a, g_2(a), \ldots, g_n(a)), \quad j = 1, \ldots, n.$$

The function σ_0 is the constant map 1, $\sigma_1 = \mathrm{Tr}$ is the *trace map*, and $\sigma_n = N$ is the *norm map*. Note that Tr is A^G-linear, whereas N is multiplicative.

DEFINITION 4.2: Let $\Sigma_R^G(A) = R[\bigcup_j \sigma_j(A)]$. We say that A^G is *generated over R by the G-symmetric functions* if $\Sigma_R^G(A) = A^G$.

EXAMPLE 4.3: If $n = |G|$ is invertible in R, e.g., when R contains a field of characteristic zero, then $\Sigma_R^G(A) = A^G$ for all finite groups G. In fact, $A^G = \mathrm{Tr}(A)$.

We shall occasionally write $\Sigma_R(A)$ or $\Sigma(A)$ for $\Sigma_R^G(A)$, when no confusion is possible.

The fundamental property of $\Sigma(A)$ is that the base change map on $\Sigma(A)$ is surjective for *any* base change $R \to R'$. More precisely,

PROPOSITION 4.4: *Fix $R \to A$ and G as above, and let $R \to R'$ be an R-algebra. Put $M' = M \otimes_R R'$ for any R-module M. Then one has a commutative diagram of R'-modules,*

$$\begin{array}{ccccccc} R' & \to & \Sigma_R^G(A)' & \to & (A^G)' & \to & A' \\ \| & & \downarrow \varphi & & \downarrow \psi & & \| \\ R' & \to & \Sigma_{R'}^G(A') & \to & (A')^G & \to & A' \end{array}$$

where φ is surjective.

If R' is a flat R-algebra, then φ and ψ are both bijective. Furthermore, if R' is a faithfully flat R-algebra, then $\Sigma_R(A) = A^G$ if and only if $\Sigma_{R'}(A') = A'^G$.

PROOF: The first row is obtained by tensoring

$$R \to \Sigma(A) \to A^G \to A$$

with R' over R. An element $g \in G$ acts on $a \otimes r' \in A'$ by $g(a \otimes r') = g(a) \otimes r'$. It follows that the natural map $A^G \to A'$ factors through A'^G, thus inducing the map $\psi : (A^G)' \to (A')^G$. In order to see that the image of $\Sigma(A)'$ in $(A^G)'$ is mapped by ψ into $\Sigma(A')$, it suffices to show that $\sigma_j(a) \otimes 1 \in \Sigma(A')$ for all $a \in A$, $j = 1, \ldots, n$, where σ_j denotes the

G-symmetric functions on A. This is clear, since $\sigma_j(a)\otimes 1 = \sigma'_j(a\otimes 1)\in\Sigma(A')$, where σ'_j denote the G-symmetric functions on A'. This provides us with a map $\varphi:\Sigma(A')'\to\Sigma(A')$. To prove the surjectivity of φ we need the following general lemma, whose proof is left to the reader.

LEMMA 4.5: *Let $s_j(x)$ denote the j'th elementary symmetric functio.. in n variables $\mathfrak{x}=(x_1,\ldots,x_n)$, and let $\mathfrak{x},\mathfrak{y},\ldots,\mathfrak{z}$ denote a finite set of n-tuples of variables. Then $s_j(\mathfrak{x}+\mathfrak{y}+\cdots+\mathfrak{z})$ may be expressed as a polynomial with integer coefficients in the symmetric functions s_k, $k\leqq j$, where the arguments in the various s_k are monimials in $\mathfrak{x},\mathfrak{y},\ldots,\mathfrak{z}$, i.e., of the form*

$$\mathfrak{x}^p\cdot\mathfrak{y}^q\cdot\ldots\cdot\mathfrak{z}^r=(x_1^p\cdot y_1^q\cdot\ldots\cdot z_1^r,\ldots,x_n^p\cdot y_n^q\cdot\ldots\cdot z_n^r).$$

We continue the proof of Prop. 4.4. The image of φ is an R'-subalgebra of $\Sigma(A')$; so in order to prove that it is all of $\Sigma(A')$, it suffices to prove that it contains the generators of $\Sigma(A')$, i.e., the elements $\sigma'_j(a')$ where $a'\in A'$. When $a'=a\otimes r'$ one has $\sigma'_j(a')=\sigma_j(a)\otimes r'^j$, which certainly belongs to $\mathrm{Im}(\varphi)$. For an arbitrary $a'=\Sigma_{i=1}^p a_i\otimes r'_i$ one has $\sigma'_j(a')=\sigma'_j(a_1\otimes r'_1+\cdots+a_p\otimes r'_p)$, and it follows immediately from Lemma 4.5 and the previous arguments that this element belongs to $\Sigma(A')$.

When R' is flat over R, it is well-known that ψ is bijective (see [5]), and it follows that φ is injective, so bijective by the first part of the proposition. If R' is faithfully flat, then $\Sigma(A)\to A^G$ is a bijection if and only if $\Sigma(A)'\to(A^G)'$ is a bijection, so if and only if $\Sigma(A')\to(A')^G$ is bijective. ■

Assume for a moment that R is noetherian and that $R\to A$ is of finite type. By (4.1), A is integral over $\Sigma_R^G(A)$, so $\Sigma_R^G(A)$ is of finite type over R by Noether's lemma (see e.g. [13, III.12. Lemma 10]). Moreover, A and A^G are then finite $\Sigma(A)$-modules. In this case the difference between $\Sigma(A)$ and A^G is relatively small.

PROPOSITION 4.6: *Let $R\to A$ and G be as described at the beginning of the section. Then one has*

(i) *If the extension $\Sigma_R^G(A)\subset A^G$ is quasi-finite,*[1] *then the natural map $\varphi:\mathrm{Spec}\,A^G\to\mathrm{Spec}\,\Sigma_R^G(A)$ is a homeomorphism.*

(ii) *If A is a domain and G is a subgroup of $\mathrm{Aut}_R(A)$, then φ is birational.*

[1] Cf. section 1.

PROOF: Assume first that $\Sigma(A) \subset A^G$ is a quasi-finite extension. The map φ is surjective and closed since the extension is integral. So we only need to prove that φ is injective. Recall that the extension $A^G \subset A$ is always quasi-finite, so in our case the extension $\Sigma(A) \subset A$ will be quasi-finite. Let p be a prime ideal of $\Sigma(A)$ and let $r_1, \ldots, r_m$ be the prime ideals of A that lie over p. Put $q = r_1 \cap A^G$. We may assume that the r_i's have been numbered in such a way that $r_1, r_2, \ldots, r_s$ all contract to q and $r_{s+1}, \ldots, r_m$ contract to different prime ideals of A^G if $s < m$. (This means that $\{r_1, \ldots, r_s\}$ is a set of conjugate prime ideals of A with respect to G and that $\{r_{s+1}, \ldots, r_m\}$ is a union of sets of conjugate prime ideals.) We claim that $s = m$. Assume that $s < m$. Choose an element $u \in A$ such that $u \in r_1$, $u \notin r_i$ for $i > 1$, and set $v = N(u) = \Pi_{g \in G}\, g(u)$. We have $v \in r_1 \cap \Sigma(A) = p$. Let $q' = r_{s+1} \cap A^G$. Since $g(u) \notin r_{s+1}$ for any $g \in G$, we have $v \notin q'$. This contradicts $p\,.\,A^G \subset q'$. Consequently $s = m$ and hence q is the only prime ideal of A^G that lies over p. Thus φ is injective.

Suppose next that A is a domain with fraction field L, and set $K = L^G$. Then K is the fraction field of A^G. Denote the fraction field of $\Sigma(A)$ by K'. Clearly $K' \subset K$. Let $\mathrm{Tr}: L \to K$, resp. $N: L \to K$, denote the classical trace and norm maps. As $G = \mathrm{Aut}_K L$, these maps coincide with the ones we have defined. Since $K \subset L$ is finite separable, there exists an $x \in L$ with $\mathrm{Tr}(x) \neq 0$. Write $x = y/z$ with $y, z \in A$, and set $a = x \cdot N(z)$. Then $a \in A$ and $\mathrm{Tr}(a) = N(z) \cdot \mathrm{Tr}(x) \neq 0$. Any element in K is of the form b/c with $b, c \in A^G$. We have

$$\frac{b}{c} = \frac{b \cdot \mathrm{Tr}(a)}{c \cdot \mathrm{Tr}(a)} = \frac{\mathrm{Tr}(ba)}{\mathrm{Tr}(ca)},$$

which shows that $b/c \in K'$. Thus $K' = K$. ■

THEOREM 4.7: *Let R be a commutative ring, A an R-algebra, and G a subgroup of $\mathrm{Aut}_R(A)$. Assume A is Dedekind domain with perfect residue fields at the maximal ideals. Then A^G is a Dedekind domain, the extension $A^G \subset A$ is finite and flat of degree $|G|$, and $\Sigma_R^G(A) = A^G$.*

PROOF: The only nonstandard assertion is the equality $\Sigma(A) = A^G$. The classical proof of the finiteness of A over A^G (see e.g. [12]) actually shows that A is finite as a $\Sigma(A)$-module. Therefore $\Sigma(A)$ is noetherian by the Eakin–Nagata theorem ([4] and [10]). Consequently A^G is a finite $\Sigma(A)$-module, being a submodule of A. To prove $\Sigma(A) = A^G$ it suffices to prove that $\Sigma(A)_p = A_p^G$ for all maximal ideals p of $\Sigma(A)$. Since we may replace R by $\Sigma(A)$ itslef in the definition of $\Sigma(A)$, this and Prop. 4.4 show that we may assume that $B = \Sigma(A)$ is local. Then $C = A^G$ is a local by Prop. 4.6, hence a discrete valuation ring, and A is a semi-local Dedekind domain, in particular, a principal ideal domain.

Let p and q denote the maximal ideals of B and C respectively, and let $r_1, \ldots, r_s$ denote the maximal ideals of A. Let w (resp. $v_1, \ldots, v_s$ denote the valuations corresponding to q (resp. $r_1, \ldots, r_s$). We first prove that p contains a uniformizing parameter for C, that is, a generator of q. Pick a generator u of r_1 such that $v_i(u) = 0$ for $i \geqq 2$. Set $\pi = N(u)$, then $\pi \in p$. The ramification index for q at any r_i is independent of i and equals $e = \text{card}\{g \in G \mid g(r_i) = r_i\}$. We have $v_1(g(u)) = 1$ if $g(r_1) = r_1$, and $v_1(g(u)) = 0$ otherwise. Hence by standard valuation theory we get

$$e \cdot w(\pi) = v_1(\pi) = \sum_{g \in G} v_1(g(u)) = e.$$

So $w(\pi) = 1$, that is, π generates q.

Let k and k' be the residue fields of B and C. The last obstacle to proving $B = C$ is that the extension $k \subset k'$ may a priori be nontrivial. Only at this point does the assumption on the residue fields of A come in. Because of it, k' is a subfield of finite index of a perfect field. Hence k' is perfect. By the same argument k is perfect, so the extension $k \subset k'$ is separable. Choose a finite galois extension $k \subset k''$ that contains k', and choose a finite flat local extension $B \subset B'$, where B' is a local ring with residue field k''. (Set $B' = B[X]/(f(X))$, where $k'' = k[X]/(\bar{f}(X))$; see [6, 0.10.3] for details.) We have a commutative diagram

$$\begin{array}{ccccc} B & \hookrightarrow & C & \hookrightarrow & A \\ \downarrow & & \downarrow & & \downarrow \\ B' & \to & C' & \to & A' \end{array}$$

where $C' = C \otimes_B B'$ and $A' = A \otimes_B B'$ and where the vertical maps are finite and étale. Since $B' = \Sigma_{B'}^G(A)$, C' is local by Prop. 4.6, thus a discrete valuation ring. (A finite étale extension of a normal domain is locally normal.) Since $k' \otimes_k k''$ is a finite product of copies of k'', it follows that the residue field of C' equals k''. Let p' and q' denote the maximal ideals of B' and C'. Then $p' \cdot C' = q'$. This implies that p' contains a uniformizing parameter t for C'. (It may be chosen among a set of generators of p'.)

The extension $B' \subset C'$ is finite; so the conductor of B' in C' is a nonzero ideal, say, $t^d \cdot C'$ for some $d \in \mathbb{N} \cup \{0\}$. We identify the residue fields of B' and C' and denote the residue class of an element z in B' or C' by $\bar{z}$. For any $c \in C'$ we may find a $b \in B'$ so that $\bar{b} = \bar{c}$. Let $x \in C'$ be arbitrary. It follows that we may find elements

$x_0, x_1, \ldots, x_{d-1} \in B'$ so that

$$x - (x_0 + x_1 t + \cdots + x_{d-1} t^{d-1}) = x_d,$$

where x_d belongs to the conductor. The right hand side of this equality belongs to B', thus $x \in B'$. Consequently $B' = C'$, and since the extension $B \subset B'$ is faithfully flat, we have $B = C$. ■

Let $p: X \to S$ be a morphism of finite type of locally noetherian schemes, and let G be a finite group of S-automorphisms of X that acts admissibly in the sense of Grothendieck ([5]). Then one may cover X by G-invariant affine open subschemes U_i with rings A_i such that the quotient X/G is an S-scheme obtained by glueing together the $\mathrm{Spec}(A_i^G)$. The quotient X/G is of finite type over S and the natural morphism $X \to X/G$ is finite and surjective. Moreover, it is obvious that X/G is proper over S if and only if X is proper over S.

DEFINITION 4.8: With the same notation as above we say that X/G is *co-generated by the G-symmetric functions over S*, if the covering $\{U_i\}$ of X may be chosen so that for each i, $p(U_i)$ is contained in an open affine subscheme V_i of S with ring R_i such that $A_i^G = \Sigma_{R_i}^G(A_i)$.

THEOREM 4.9: *Let $p: X \to S$ be a flat morphism of schemes and let G be a finite group of S-automorphisms of X that acts admissibly. Assume that for every geometric point s of S the quotient X_s/G of the fibers X_s is co-generated by the G-symmetric functions over s.*

Then X/G is co-generated by the G-symmetric functions over S, and the formation of the natural projection $X \to X/G$ commutes with base change on S. In particular, for every geometric point s of S one has

$$(X/G)_s = X_s/G.$$

Furthermore, then X/G is flat over S, and $X \to X/G$ is flat if and only if $X_s \to X_s/G$ is flat for every geometric point s of S. Moreover, when the latter holds, then X/G is quasi-projective (resp. locally projective) over S if and only if p is quasi-projective (resp. locally projective).

PROOF: We may assume that S is affine with (noetherian) ring R, and we first consider an open affine G-invariant subscheme U of X with ring A. Then $U/G = \mathrm{Spec}(A^G)$ and $U \to U/G$ corresponds to the inclusion $A^G \subset A$. Here A^G is noetherian (of finite type over R) and A is a finite A^G-module. Consider the following complex of finite A^G-

modules, flat over R:

(4.10) $$0 \longrightarrow A \xrightarrow{\alpha} A^{\oplus|G|} \longrightarrow 0,$$

where $\alpha(a) = (a - g(a))_{g \in G}$, the summands of $A^{\oplus|G|}$ being indexed by the elements of G. Clearly $\mathrm{Ker}(\alpha) = A^G$. Let $R \to \Omega$ be a homomorphism of R into an algebraically closed field Ω. The base change map $\varphi : \Sigma_R^G(A) \otimes_R \Omega \to \Sigma_\Omega^G(A \otimes \Omega)$ is surjective by Prop. 4.3. By hypothesis, $\Sigma^G(A \otimes \Omega)$ is equal to $(A \otimes \Omega)^G$. Hence $\psi : A^G \otimes_R \Omega \to (A \otimes_R \Omega)^G$ is surjective; that is, the natural base change homomorphism of the zero'th cohomology of the complex (4.10), $U^0(\Omega) : \mathrm{Ker}(\alpha) \otimes_R 1_\Omega \to \mathrm{Ker}(\alpha \otimes_R 1_\Omega)$, is surjective (see the Appendix). This holds for all Ω, so $\mathrm{Ker}(\alpha) = A^G$ commutes with arbitrary base change on R by Thm. A.5 (i); in particular, ψ is bijective. It follows that $X \to X/G$ commutes with base change on S. We have $(A^G/\Sigma(A)) \otimes_R \Omega = 0$ for all Ω. Here we may replace Ω by an arbitrary residue field of A, so $\Sigma(A) = A^G$, by Nakayama's lemma, i.e. X/G is co-generated by the G-symmetric functions over S.

The calculations above show that $A^G \otimes_R \Omega \to A \otimes_R \Omega$ is injective for all Ω; hence with Ω replaced by any residue field of R. Therefore $A^G \to A$ is universally injective and A/A^G and A^G are flat over R by Cor. A.2. Thus, X/G is flat over S. The assertion about the flatness of $X \to X/G$ follows form Lemma 2.10, and the statement about the (quasi)-projectivity is an application of [6, II.6.6.4]. ■

REMARK 4.11: It follows from Rem. A.9 that if the morphism p in Thm. 4.9 is *projective* and if we just assume the co-generation of X_s/G for a *single* geometric point s of S, then the conclusions of the theorem still hold with S replaced by an open neighbourhood of s.

THEOREM 4.12: *Let $p : C \to S$ be a smooth, (quasi-) projective family of curves, and let G be a finite group of S-automorphisms of C that acts faithfully in the geometric fibers of p. Then the quotient C/G is a smooth, (quasi-) projective family of curves over S, the formation of which commutes with base change on S. Furthermore, the natural projection $C \to C/G$ is finite and faithfully flat of degree equal to* $|G|$. ■

PROOF: By a faithful action of G in the fiber over a point s of S we mean that the induced map $G \to \mathrm{Aut}_s(C_S)$ should be injective. The proof of the theorem is an obvious combination of Thm. 4.7 and Thm. 4.9. ■

EXAMPLE 4.13: The following example, proposed to us by Andy Magid, shows that the faithfullness of the action of G in the fibers is a necessary condition. Let $R = \mathbb{Z}$, $A = \mathbb{Z}[X]$, and let $G = \{1, \sigma\}$, where $\sigma : A \to A$ is given by $\sigma(X) = -X$. Then $A^G = \mathbb{Z}[X^2]$. However, G acts trivially on $\mathbb{F}_2[X]$, so $(A \otimes \mathbb{F}_2)^G \to A \otimes \mathbb{F}_2$ is the identity, whereas $(A^G \to A) \otimes \mathbb{F}_2$ is the inclusion $\mathbb{F}_2[X^2] \subset \mathbb{F}_2[X]$.

EXAMPLE 4.14: The condition that A be one-dimensional in Thm. 4.7 seems to be essential in the case of unequal characteristics. Here is an example, mainly due to A. Thorup, of a 2-dimensional regular ring A for which the conclusion in the theorem fails to hold. Let $A = \mathbb{Z}[i][X]$ be the polynomial ring over the Gaussian integers, and let the nontrivial element of $G = \mathbb{Z}/(2)$ act on A by complex conjugation of the coefficients of a polynomial. It is easy to see that $A^G = \mathbb{Z}[X]$, whereas $\Sigma_{\mathbb{Z}}^G(A) = \mathbb{Z}[2X, X^2]$.

5. Hyperelliptic families of curves

Let C_0 denote a smooth projective curve of genus $g \geqq 2$ defined over an algebraically closed field k. We recall the following:

CLASSICAL FACT 5.0: *Let $f : C_0 \to \mathbb{P}_k^{g-1}$ denote the canonical morphism, that is, the morphism defined by the canonical divisor on C_0. Then either f is an embedding, or* Im(*f*) *is isomorphic to $\mathbb{P}_k^1$ and the induced morphism $h : C_0 \to \mathbb{P}_k^1$ has degree* 2.

PROOF: See e.g. [1]. ■

The curve C_0 is called *hyperelliptic* in the latter case. It is well-known that the following conditions for an arbitrary C_0 are equivalent:

(5.1) C_0 is hyperelliptic;
(5.2) There exists a k-morphism $h : C_0 \to \mathbb{P}_k^1$ of degree 2;
(5.3) $\mathrm{Aut}_k(C_0)$ contains a involution σ so that the quotient C_0/σ is k-isomorphic to $\mathbb{P}_k^1$.

The involution σ is uniquely determined by (5.3) and it is called the *canonical involution* on C_0.

We shall reprove the equivalence of these conditions and their generalizations to families of curves, thus only taking for granted the fact (5.0).

Let $p: C \to S$ be a smooth, projective family of curves of genus $g \geqq 2$.

DEFINITION 5.4: An S-morphism $\sigma: C \to C$ is called a *(global) canonical involution* if σ induces an involution in each geometric fiber C_0 of p that satisfies condition (5.3).

The existence of σ implies that all geometric fibers of p are hyperelliptic. We note that σ is uniquely determined since the automorphism scheme $Aut_S(C)$ is unramified over S (see [3]). Furthermore σ *is* an involution since σ^2 induces the identity in every geometric fiber and $Aut_S(C)$ is unramified over S.

THEOREM 5.5: *Let $p: C \to S$ be a smooth, projective family of curves of genus $g \geqq 2$. Then the following conditions are equivalent:*

(i) *There exists a finite, surjective S-morphism $h: C \to D$ of degree* 2, *where $q: D \to S$ is some twisted $\mathbf{P}^1_S$.*

(ii) *There exist a faithfully flat morphism $T \to S$ and a finite faithfully flat T-morphism $h: C_T \to \mathbf{P}^1_T$ of degree* 2.

(iiiv) *C admits a global canonical involution σ over S.*

(iv) *The image of the canonical morphism $f: C \to \mathbf{P}(p_*\Omega^1_{C/S})$ is a twisted $\mathbf{P}^1_S$, and its formation commutes with base change* (*that is, f is co-flat*).

Note that the morphism h in (i) above is automatically flat by Prop. 2.10, thus the degree of h is well-defined.

Before the proof of the theorem we insert two lemmas that give somewhat more information than the corresponding implications in the theorem:

LEMMA 5.6: *Let $h: X \to Y$ be a morphism of locally noetherian schemes that satisfies the following two conditions:*

(i) *h is finite and faithfully flat of degree* 2;

(ii) *h is separable at the fibers over the associated points of Y.*

Then there is a natural action of the group $G = \mathbf{Z}/(2)$ on X such that the quotient X/G exists and h is isomorphic to the natural projection $X \to X/G$.

PROOF: The question is local on Y, so assume for a moment that X and Y are affine with rings B and A and that h corresponds to a ring homomorphism $A \to B$. Then A is a direct summand in B, and we may assume that B and B/A are free A-modules of rank 2 and 1 by shrinking Y. Let $\{1, y\}$ be an A-basis for B. Then y satisfies an equation of the form $y^2 + ay + b = 0$, with uniquely determined $a, b \in A$. Define an A-linear map $\sigma: B \to B$ by $\sigma(1) = 1$, $\sigma(y) = -y - a$. One readily checks that σ is independent of the choice of y, that $\sigma^2 = id$, and that σ is an A-algebra automorphism of B. We claim that $\sigma \neq id$ and that $B^\sigma = A$. If 2 is a nonzerodivisor in A, this is obvious. In general, the second condition on h implies that the element a does not belong to any associated prime of A; that is, a is a nonzerodivisor. The claim is an immediate consequence of this.

It is straightforward to verify that σ is the only non-trivial A-automorphism of B that leaves A fixed. Furthermore, our construction of σ commutes with localization. This shows that it globalizes to the original X and yields an Y-automorphism of X, also denoted by σ, with the desired properties. ■

LEMMA 5.7: *Assume that $p: C \to S$ satisfies condition* (i) *in Thm.* 5.5. *Then there exists a closed embedding $j: D \to \mathbf{P}(p_*\Omega^1_{C/S})$ such that $f = j \circ h$, where f is the canonical morphism.*

PROOF: We have an exact sequence on D,

$$0 \to \mathcal{O}_D \to h_*\mathcal{O}_C \to L \to 0, \tag{5.8}$$

where L is invertible, according to example (2.12) (2). Put $M = L^{-1} \otimes \Omega^1_{D/S}$. We shall prove that h^*M is naturally isomorphic to $\Omega^1_{C/S}$ by means of Prop. 3.8. Note that we have $R^ip_* = (R^iq_*) \circ h_*$, for $i \geqq 0$, since h is affine. This implies first that $R^1p_*(h^*M) = R^1q_*(M \otimes h_*\mathcal{O}_C)$. The latter sheaf is dual to $q_*((L^{-1} \otimes h_*\mathcal{O}_C)^\vee)$ by Grothendieck duality. We tensor (5.8) with L^{-1} and take duals, getting

$$0 \to \mathcal{O}_D \to (L^{-1} \otimes h_*\mathcal{O}_C)^\vee \to L \to 0.$$

Then we apply q_* and get a new exact sequence,

$$0 \to q_*\mathcal{O}_D \to q_*((L^{-1} \otimes h_*\mathcal{O}_C)^\vee) \to q_*L. \tag{5.9}$$

Here $q_*\mathcal{O}_D = \mathcal{O}_S$ by Serre's computation of chomology of projective n-space ([6, III.2.1.12]). We shall show in a moment that $\deg(L_s) = -g - 1$ for all geometric points s of S, hence $q_*L = 0$. This proves that $\mathcal{O}_S = q_*((L^{-1} \otimes h_*\mathcal{O}_C)^\vee)$. Thus $R^1p_*(h^*M) = \mathcal{O}_S$, so the first condition in Prop. 3.8 is satisfied. To compute $\deg(h^*M_s)$ we remark that h^* multiplies the degree of an invertible sheaf by 2 and $(\Omega^1_{D/S})_s$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-2)$; so, one gets $\deg(h^*M_s) = 2 \cdot \deg M_s = 2g-2$.

The natural map $q^*q_*M \to M$ is surjective and induces a proper S-morphism $j: D \to \mathbb{P}(q_*M)$ which is a closed embedding in the fibers, hence a closed embedding. Using (5.8) and the fact that $q_*\Omega^1_{D/S} = 0$ it is easy to check that the isomorphism $h^*M \to \Omega^1_{C/S}$ induced by the isomorphism in (5.9) gives rise to an identification of $p_*\Omega^1_{C/S}$ with q_*M, and that h^* applied to $q^*q_*M \to M$ yields the natural surjection $p^*p_*\Omega^1_{C/S} \to \Omega^1_{C/S}$. This proves that $f = j \circ h$.

The only thing left is the computation of $\deg L_s$. We consider the restriction of h to the fibers over s, $h_s: C_s \to \mathbb{P}^1_k$, where $k = k(s)$. Let $\mathbb{P}^1_k = U_0 \cup U_1$ be a standard covering by two copies of A^1_k such that h_s is unramified at the points $\mathbb{P}^1_k \setminus U_0$ and $\mathbb{P}^1_k \setminus U_1$. Let x and x_1 denote the coordinates on U_0 and U_1. It is not difficult to see that $(h_s)_*\mathcal{O}_{C_s}$ is given by rank-two free modules over $k[x]$ and $k[x_1]$ with bases $\{1, y\}$ and $\{1, y_1\}$, where y and y_1 satisfy equations of the form

$$y^2 + a(x) \cdot y + b(x) = 0,$$

$$y_1^2 + a_1(x_1) \cdot y_1 + b_1(x_1) = 0.$$

Here $a, b \in k[x]$, $\deg(a) \leqq g+1$, $\deg(b) = 2g+2$, and $a_1(x_1) = x_1^{g+1}a(1/x_1)$, $b_1(x_1) = x_1^{2g+2}b(1/x_1)$, $y_1 = x_1^{g+1}y$. This shows that (5.8) splits in the fiber and that $L_s = \mathcal{O}_{\mathbb{P}^1}(-g-1)$. ■

COROLLARY 5.10: *Assume that $p: C \to S$ satisfies condition* (i) *of Theorem 5.5. Then D is uniquely determined up to S-isomorphism and h is unique up to S-automorphism of D.*

REMARK 5.11: The sheaves $h_*\mathcal{O}_C$ and $(L^{-1} \otimes h_*\mathcal{O}_C)^\vee$ in the preceeding proof are locally isomorphic, and in the case where (5.8) splits, they are globally isomorphic. This happens when S is the spectrum of a field and $D = \mathbb{P}^1_S$ as was seen above, and it happens in the general case provided 2 is invertible on S because then $\mathrm{Tr}: h_*\mathcal{O}_C \to \mathcal{O}_D$ is surjective.

For all geometric points s of S one has $M_s \simeq \mathcal{O}(g-1)$ in the fiber D_s. It follows that after a faithfully flat base change $T \to S$, the embedding j is isomorphic to the Veronese morphism $\mathbb{P}^1_T \hookrightarrow \mathbb{P}^{g-1}_T$.

PROOF OF THEOREM 5.5: (i) ⇒ (iv): $\mathrm{Im}(f)$ is a twisted $\mathbb{P}^1_S$ by Lemma 5.7. In order to prove that $\mathrm{Im}(f)$ commutes with base change we note that $h: C \to D$ is co-flat and $\mathrm{Coker}({}^c f) = j_*(\mathrm{Coker}({}^c h)$ is S-flat since $\mathrm{Coker}({}^c h)$ is S-flat. We then conclude by Prop. 2.6.

(iv) $\Rightarrow$ (i): The morphism $C \to \mathrm{Im}(f)$ has degree 2 in every geometric fiber by (5.0).

(i) $\Rightarrow$ (ii): This follows from Cor. 3.4.

(ii) $\Rightarrow$ (iv): This is a consequence of (i) $\Rightarrow$ (iv) since co-flatness may be checked after a faithfully flat base change as noted in (2.11)(b).

(i) $\Rightarrow$ (iii): This is a special case of Lemma 5.6. In order to verify condition (ii) of the lemma we notice that, by the flatness of q, an associated point y of D is an associated point of the fiber over $q(y)$; i.e., y is the generic point of the fiber (see [6, IV.3.3.1]). Since the fibers of p have genus $g \geqq 2$, they are not rational, so the induced morphism $C_{q(y)} \to D_{q(y)}$ is generically separable; that is, h is separable in the fiber over y.

(iii) $\Rightarrow$ (i): Apply Thm. 4.11. ■

DEFINITION 5.12: A smooth, projective family of curves $p: C \to S$ is called *hyperelliptic* if it satisfies the equivalent conditions of Theorem 5.5.

REMARK 5.13: Let $p: C \to S$ denote a smooth, projective family of curves, and let T be a locally noetherian S-scheme, not necessarily of finite type over S. If p is hyperelliptic, then the pull-back p_T is hyperelliptic by condition (i) in Thm. 5.5. On the other hand, if p_T is hyperelliptic and T is faithfully flat over S, then p is hyperelliptic by condition (iv) in the theorem and Remark 2.11(b).

In particular, if S is the spectrum of a field k, then C is hyperelliptic if and only if the extension of C to an algebraic closure of k is hyperelliptic.

PROPOSITION 5.14: *Let $p: C \to S$ be a smooth, projective family of curves, where S is a connected regular 1-dimensional scheme. Denote the generic point of S by η. Then p is a hyperelliptic family if and only if C_η is a hyperelliptic curve.*

PROOF: "If": The canonical involution σ_η on C_η gives rise to a rational morphism $S \dashrightarrow Aut_S(C)$ which extends to a morphism, by the valuative criterion of properness. Hence σ_η extends to an S-automorphism σ of C. Since $Aut_S(C)$ is unramified over S, the restrictions of σ to the fibers of p are nontrivial. It follows from Thm. 4.11 that the quotient $C/\{1, \sigma\} = D$ is a smooth, projective family of curves, whose generic fiber has genus 0 by the assumption on σ_η.

Hence all fibers of D have genus 0; i.e., D is a twisted $\mathbb{P}^1_S$. Consequently, our family p is hyperelliptic by, say, condition (i) in Thm. 5.5.

"Only if": Obvious. ■

6. The Weierstrass subscheme

Let $p: C \to S$ denote a hyperelliptic family of curves of genus $g(\geqq 2)$ as defined in the previous section. By D we denote the image of the canonical morphism $f: C \to \mathbb{P} = \mathbb{P}(p_* \Omega^1_{C/S})$, and we let $h: C \to D$ denote the induced morphism. Recall that the restriction of h to a fiber over S is generically separable, as we noticed in the proof of Thm. 5.5.

DEFINITION 6.1: The branch locus of the canonical morphism f, endowed with the scheme structure defined by the zero'th Fitting Ideal of $\Omega^1_{C/\mathbb{P}}$, is called the *Weierstrass subscheme* of C and is denoted by $W_{C/S}$.

Note that $\Omega^1_{C/\mathbb{P}}$ and $\Omega^1_{C/D}$ are canonically isomorphic, so $W_{C/S}$ is also the branch locus of h. We have a sequence of $\mathcal{O}_C$-Modules,

$$0 \to h^* \Omega^1_{D/S} \to \Omega^1_{C/S} \to \Omega^1_{C/D} \to 0, \tag{6.2}$$

which commutes with base change on S. It is exact on the right and the restriction of the map $h^* \Omega^1_{D/S} \to \Omega^1_{C/S}$ to a geometric fiber is injective since $h^* \Omega^1_{D/S}$ is torsionfree here and the restriction is injective on the open dense set where h is unramified. As $\Omega^1_{C/S}$ is S-flat, Cor. A.2. applies. It shows that the sequence (6.2) is *universally exact* and that $\Omega^1_{C/D}$ is S-flat.

The formation of the zero'th Fitting Ideal $F^0(\Omega^1_{C/D})$ commutes with base change, hence so does the formation of $W_{C/S}$. The sequence (6.2) defines locally a presentation of $\Omega^1_{C/D}$. After tensoring (6.2) with $(\Omega^1_{C/S})^{-1}$ it becomes apparent that the Ideal of $W_{C/S}$ is isomorphic to $h^* \Omega^1_{D/S} \otimes (\Omega^1_{C/S})^{-1}$, thus invertible. Furthermore, $\Omega^1_{C/D}$ and the structure sheaf of $W_{C/S}$ are locally isomorphic; i.e., $W_{C/S}$ is flat over S. Consequently, $W_{C/S}$ is the closed subscheme of C associated to a Cartier divisor on C relative to S, also denoted by $W_{C/S}$. The restriction of the subscheme $W_{C/S}$ to a fiber over a point s of S is thus finite, so $W_{C/S}$ is quasi-finite over S, and since $W_{C/S}$ is closed in a projective S-scheme, $W_{C/S}$ is thus finite over S. The degree of $W_{C/S}$ over S may be computed in a fiber: There it is equal to $-\deg(h^* \Omega^1_{C/S} \otimes (\Omega^1_{C/S})^{-1}) = -(2(-2) - (2g-2)) = 2g+2$.

These results may be summarized as follows:

PROPOSITION 6.3: *The Weierstrass subscheme $W_{C/S}$ of C is the subscheme associated to an effective Cartier divisor on C relative to S. It is finite and flat over S of degree $2g+2$, and its formation commutes with base change.*

REMARK 6.4: Our computations for $W_{C/S}$ are actually a special case of the Riemann–Hurwitz formula for a finite flat S-morphism between two smooth families of projective curves. In this connection we might note that $F^0(\Omega^1_{C/D}) = \mathrm{Ann}(\Omega^1_{C/D})$ and that it is usually called the *Kähler different* of h.

In the classical case, where S is the spectrum of an algebraically closed field of characteristic different from 2, it is well-known that $h: C \to D$ has precisely $2g+2$ distinct ramification points, and that they are equal to the fixed points of C under the action of the canonical involution. The first fact follows from a local computation of the length of $W_{C/S}$ in terms of the ramification indices (all equal to 1). The second fact will be generalized below to an arbitrary hyperelliptic family of curves, $p: C \to S$, with no restriction on the characteristics.

PROPOSITION 6.5: *Let σ be the canonical involution of a hyperelliptic family of curves $p: C \to S$. Then the Weierstrass subscheme $W_{C/S}$ of C is equal to the fixed point subscheme of C under the action of the group $\{1, \sigma\}$.*

PROOF: Let $h: C \to D$ denote the morphism in Thm. 5.5.(i), and let $U \subset D$ be an affine open subscheme with ring A such that $B = \Gamma(h^{-1}(U), \mathcal{O}_C)$ is a free A-module of rank two. Let $\{1, y\}$ be an A-basis for B. Then y satisfies an equation of the form

$$y^2 + ay + b = 0, \tag{6.6}$$

where $a, b \in A$. It follows that $\Omega^1_{B/A}$ is isomorphic to $B/(2y+a)$, hence so is $\Gamma(h^{-1}(U), W_{C/S})$.

On the other hand, the fixed point subscheme of an affine scheme with ring B under a finite group G is defined by the ideal I_G of B generated by the set $\{x - g(x) \mid x \in B, g \in G\}$. In our case the set of generators becomes $\{x - \sigma(x) \mid x \in B\}$, so we see that I_G is generated by $y - \sigma(y) = 2y + a$. ■

The equation (6.6) above yields an easy definition of a scheme structure on the discriminant locus $\Delta_{C/D} \subset D$; we let its Ideal be the one locally generated by $a^2 - 4b$. It is easy to verify that this definition is independent of the choice of y, and that the formation of $\Delta_{C/D}$ commutes with base change on S. Furthermore, $2y + a$ is a nonzerodivisor in B, since it defines $W_{C/S}$ on $h^{-1}(U)$. Consequently, $a^2 - 4b = (2y + a)^2$ is a nonzerodivisor in A. This shows that $\Delta_{C/D}$ is the subscheme of D associated to a Cartier divisor relative to S, also denoted by $\Delta_{C/D}$, and that we have the formula

$$(6.7) \qquad 2W_{C/S} = h^*\Delta_{C/D}.$$

Here h^* is well-defined since h is flat (see [6, IV.21]). The *norm* of $2y + a$ is equal to $-a^2 + 4b$, which shows that the divisor $\Delta_{C/D}$ is the norm of the divisor $W_{C/S}$. We shall generalize the formula (.67) in section 7.

In the case that S is the spectrum of a field of characteristic 2 (or, more generally, if S is equicharacteristic of this characteristic), then $\Delta_{C/D}$ is the square (or the double) of the Cartier divisor locally defined by the element $a \in A$, which we might denote by $\frac{1}{2}\Delta_{C/D}$. In this case $2y + a = a$, so $W_{C/S} = h^*(\frac{1}{2}\Delta_{C/D})$. In particular, *$h: C \to D$ ramifies in at most $g + 1$ distinct points* (cf. the proof of Lemma 5.7.).

COROLLARY 6.8: *The Weierstrass subscheme $W_{C/S}$ is étale over S at the points over a point s of S if and only if* $\mathrm{char}(\kappa(s)) \neq 2$.

PROOF: Since $W_{C/S}$ is finite and flat over S of degree $2g + 2$, it is étale over a point s of S if and only if the geometric fiber over s contains $2g + 2$ distincts points (see [6, IV.16.7.2]). Hence, the corollary follows from Remark 6.4 and the calculations above. ■

REMARK 6.9: The corollary above is a generalization of F. Oort's analogous statement in the case where S is the spectrum of a perfect field, see [11].

7. Hyperelliptic Weierstrass sections

Let $p: C \to S$ be a smooth, projective family of curves of genus $g \geqq 2$. Assume for a moment that S is the spectrum of an algebraically closed field k. Recall that a k-point P on C is called a *hyperelliptic Weierstrass point* if $\dim_k H^0(C, \mathcal{O}_C(2P)) = 2$. It is well-known that C is a hyperelliptic curve if and only if it possesses at least one hyperelliptic Weierstrass point.

Let again S be an arbitrary locally noetherian scheme and let $e: S \to C$ be a section of p. Assume that $e(s)$ is a hyperelliptic Weierstrass point on the fiber C_s for every geometric point s of S. In the case where S is *reduced* this is sufficient to call the section a *hyperelliptic Weierstrass* section. However, in the general case we must take the infinitesimal behavior of e into account. Let L_e be the invertible sheaf on C associated to the Cartier divisor determined by $e(S)$. Then L_e is defined by the exact sequence

$$0 \to L_e^{-1} \to \mathcal{O}_C \to e_* \mathcal{O}_S \to 0. \tag{7.1}$$

It follows that one has $\mathcal{O}_C \subset L_e \subset L_e^{\otimes 2} \subset \cdots$, and consequently, $\mathcal{O}_S \subset p_* L_e \subset p_* L_e^{\otimes 2} \subset \cdots$. On the fibers of p we have $\mathcal{O}_S = p_* L_e$, therefore $\mathcal{O}_S = p_* L_e$ on S.

DEFINITION 7.2: Let $p: C \to S$ be a smooth, projective family of curves of genus $g \geqq 2$. Let $e: S \to C$ be a section of p and denote by L_e the invertible sheaf on C that is defined by (7.1). Then e is called a *hyperelliptic Weierstrass section* of p provided

(i) The sheaf $p_* L_e^{\otimes 2}$ is locally free and commutes with base change,

(ii) $rk p_* L_e^{\otimes 2} = 2$.

The following theorem provides an excuse for the name of the Weierstrass subscheme $W_{C/S}$ of a hyperelliptic family of curves.

THEOREM 7.3: *A smooth, projective family of curves $p: C \to S$ is hyperelliptic if and only if it acquires a hyperelliptic Weierstrass section after a faithfully flat base change $T \to S$. In fact, if C is hyperelliptic, one can take $T = W_{C/S}$.*

Furthermore, any section of p that factors through $W_{C/S}$ is a hyperelliptic Weierstrass section.

PROOF: Assume first that $e: S \to C$ is a hyperelliptic Weierstrass section of a smooth, projective family of curves, $p: C \to S$. With the same notation as above, the sheaf $p_* L_e^{\otimes 2}$ is locally free of rank 2 and it commutes with base change. The natural map $p^* p_* L_e^{\otimes 2} \to L_e^{\otimes 2}$ is surjective on the fibers, hence surjective; so we get an S-morphism $h: C \to D$, where $D = \mathbb{P}(p_* L_e^{\otimes 2})$ is a twisted $\mathbb{P}_S^1$. The morphism h is finite and faithfully flat of degree 2 by Prop. 2.10. Hence C is hyperelliptic according to Thm. 5.5. (i). By Remark 5.13 this proves one direction in the first assertion of the theorem.

Assume now that $p: C \to S$ is a hyperelliptic family of projective curves. After a base change to $T = W_{C/S}$, which is finite and faithfully flat by Prop. 6.3, we obtain the diagonal section $W_{C/S} \to W_{C/S} \times_S W_{C/S}$. This provides us with a section $e: T \to C_T$ that factors through $W_{C_T/T}$. In order to prove the remaining parts of the theorem it will therefore suffice to show that any section $e: S \to W_{C/S}$ is a hyperelliptic Weierstrass section of C. For this we shall need two technical lemmas.

LEMMA 7.5: *Let $p: C \to S$ be a hyperelliptic family of curves, and let $h: C \to D$ be the morphism in Thm.* 5.5(i). *For any invertible sheaf M on D the sheaf $p_*(h^*M)$ is locally free and commutes with base change.*

PROOF: We have $p_*(h^*M) = q_*((h_*\mathcal{O}_C) \otimes M)$ by the projection formula. From the exact sequence (5.8) we get an exact sequence,

$$0 \to M \to h_*\mathcal{O}_C \otimes M \to L \otimes M \to 0. \tag{7.6}$$

The sheaves in (7.6) are S-flat, so (7.6) remains exact after restriction to the fiber of q over any geometric point s of S. it follows easily from the weak 4-lemma and Prop. 3.6 applied to M and $L \otimes M$ that the base change map $q_*(h_*\mathcal{O}_C \otimes M)(s) \to q_{s,*}(h_{s,*}\mathcal{O}_{C_s} \otimes M_s)$ is surjective. Thus, $q_*(h_*\mathcal{O}_C \otimes M) = p_*(h^*M)$ is locally free and commutes with base change by [6, III.7.8]. ■

LEMMA 7.7: *Same notation as in Lemma 7.5. Let σ denote the global canonical involution on C, and let E be an effective Cartier divisor on C relative to S with associated invertible sheaf L_E. By $N(E)$ and $N(L_E)$ we denote the norms of E and L_E.*

Then σ^*L_E is the sheaf associated with σ^*E, and we have $h^*(N(E)) = E + \sigma^*E$, and $h^*(N(L_E)) = L_E \otimes \sigma^*L_E$. Moreover, $p_*(L_E \otimes \sigma^*L_E)$ is locally free and commutes with base change.

PROOF: The last assertion is an application of Lemma 7.5, and the first assertions follow immediately from an easy local computation. ■

END OF PROOF OF THEOREM 7.3: A section $e: S \to C$ that factors through $W_{C/S}$ determines an effective Cartier E on C with support $e(S)$ for which $\sigma^*E = E$ and $\sigma^*L_E = L_E$ with the same notation as in

Lemma 7.7. Consequently, $p_*(L_E^{\otimes 2}) = p_*(L_E \otimes \sigma^* L_E)$ is locally free and commutes with base change. Finally $rk(p_* L_E^{\otimes 2}) = 2$, since $e(s)$ is a hyperelliptic Weierstrass point on C_s for any geometric point s of S. This proves that e is a hyperelliptic Weierstrass section. ■

REMARK 7.8: (1) Note that Lemma 7.7 is a generalization of the formula (6.7) since for $E = W_{C/S}$ we have $\sigma^* W_{C/S} = W_{C/S}$ and $N(W_{C/S}) = \Delta_{C/D}$.

(2) One may actually prove a little more than stated in Thm. 7.3, at least in the case where S is reduced, namely, a section of p is a hyperelliptic Weierstrass section if and *only if* it factors through $W_{C/S}$.

A. Appendix: Rudiments of a general base change theory

In this section we fix the following notation: A and B are noetherian rings, and $A \to B$ is a homomorphism of finite type. By p we denote a prime ideal of A, and κ is the residue field of A at p. Let $S = \{f \in B \mid f \otimes 1_\kappa \in (B \otimes_A \kappa)^*\}$. Note that S is multiplicatively closed. Finally, $v: G \to F$ denotes a B-linear map between two B-modules.

We call $v: G \to F$ *universally injective over* A if $v \otimes 1_M : G \otimes_A M \to F \otimes_A M$ is injective for all A-modules M.

PROPOSITION A.1: *Assume that G and F are finite B-modules and that F is flat over A. Then the following three conditions are equivalent*:

(i) *There exists an $f \in S$ such that v_f is universally injective*;

(ii) *$v \otimes 1_\kappa$ is injective*;

(iii) *v_q is injective and $(F/v(g))_q$ is flat over A for all q in* Spec(B) *that lie over p.*

PROOF: The implication (i)$\Rightarrow$(ii) is trivial, and the implication (ii)$\Rightarrow$(iii) follows immediately from the case that A and B are local, treated in [5, Exp. IV, Cor. 5.7]. Finally, assume (iii) holds. Consider the set of all primes q of B such that v_q is injective and $(F/v(G))_q$ is flat over A. This set is open in Spec(B); say its complement is $V(I)$. By hypothesis, the set contains the closed fiber $V(p \cdot B_p)$ of Spec(B_p). Hence $I \cdot B_p + p \cdot B_P = B_p$ holds. Therefore there are elements $f \in I$ and $g \in B - p$ such that f/g is congruent to 1 modulo $p \cdot B_p$. Then f is an element of S such that v_f is injective and $(F/v(G))_f$ is flat. Therefore (i) holds. ■

We remark that when the conditions above hold, then the module G_f is flat over A.

COROLLARY A.2: *Same notation and hypotheses as above. The map $v: G \to F$ is universally injective over A if and only if $v \otimes 1_\kappa$ is injective for all residue fields κ of A. Moreover, when this holds, then $F/(v(G))$ and G are flat over A.*

Some more notation: by $K^{\cdot}$ we denote a complex of finite B-modules that are flat over A

$$\cdots \to K^{i-1} \xrightarrow{d^{i-1}} K^i \xrightarrow{d^i} K^{i+1} \to \cdots$$

The i'th coboundary of $K^{\cdot}$ is denoted by $B^i = B^i(K^{\cdot})$ and $b^i: B^i \to K^i$ is the inclusion map. By $Z^i = Z^i(K^{\cdot})$ we denote the ith cochains of $K^{\cdot}$. For any A-module M one has the natural *base change maps*, $U^i(M): H^i(K^{\cdot}) \otimes_A M \to H^i(K^{\cdot} \otimes_A M)$ between the cohomologies of $K^{\cdot}$ and the complex $K^{\cdot} \otimes_A M$.

LEMMA A.3: *We have the following*:

(i) *If $b^{i+1}: B^{i+1} \to K^{i+1}$ is universally injective over A, then $U^i(M)$ is an isomorphism for all A-modules M.*

(ii) *If $U^i(\kappa)$ is surjective, then there exists an $f \in S$ such that b_f^{i+1} is universally injective.*

PROOF: Assume that b^{i+1} is universally injective, and let M be an A-module. Then $B^{i+1}(K)$ is flat over A by Cor. A.2, so from the exact sequence $0 \to Z^i \to K^i \to B^{i+1} \to 0$ we get a commutative diagram

$$\begin{array}{ccccccccc} 0 \to & Z^i \otimes_A M & \to & K^i \otimes_A M & \to & B^{i+1} \otimes_A M & \to 0 \\ & & & & \searrow^{d^i \otimes 1_M} & \downarrow{\scriptstyle b^{i+1} \otimes 1_M} & \\ & & & & & K^{i+1} \otimes_A M & \end{array}$$

with an exact row. It follows that the canonical map $Z^i \otimes_A M \to Z^i(K^{\cdot} \otimes_A M)$ is bijective. We have another commutative diagram with exact rows,

$$\text{(A.4)} \qquad \begin{array}{ccccccc} K^{i-1} \otimes M & \to & Z^i(K^{\cdot}) \otimes M & \to & H^i(K^{\cdot}) \otimes M & \to & 0 \\ \| & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle U^i(M)} & & \\ K^{i-1} \otimes M & \to & Z^i(K^{\cdot} \otimes M) & \to & H^i(K^{\cdot} \otimes M) & \to & 0. \end{array}$$

We just showed that φ is bijective, so by diagram chasing $U^i(M)$ is bijective.

Suppose now that $U^i(\kappa)$ is surjective. Consider the diagram (A.4) with $M = \kappa$. Here $U^i(\kappa)$ is surjective, thus φ is surjective by diagram chasing. In the commutative diagram below we have exact rows and φ is surjective,

$$\begin{array}{ccccccccc} & & Z^i(K^\cdot)\otimes\kappa & \to & K^i\otimes\kappa & \to & B^{i+1}(K^\cdot)\otimes\kappa & \to & 0 \\ & & \downarrow\varphi & & \| & & \downarrow\psi & & \\ 0 & \to & Z^i(K^\cdot\otimes\kappa) & \to & K^i\otimes\kappa & \to & B^{i+1}(K^\cdot\otimes\kappa) & \to & 0 \end{array}$$

Diagram chasing shows that ψ is injective; thus $b^{i+1}\otimes 1_\kappa$ is injective since it is equal to ψ followed by the inclusion map $B^{i+1}(K^\cdot\otimes\kappa)\subset K^{i+1}\otimes\kappa$. We now apply Prop. A.1 to obtain the assertion (ii). ■

THEOREM A.5: *(Property of exchange.) Same notation as above. Assume that $U^i(\kappa)$ is surjective. Then we have*

(i) *There exists an $f\in S$ such that $U^i(M)_f$ is an isomorphism for all A-modules M.*

(ii) *$U^{i-1}(\kappa)$ is surjective if and only if $H^i(K^\cdot)_p$ is flat over A.*

PROOF: Assertion (i) is an application of Lemma A.3. To prove (ii) note that the element $f\in S$ is chosen so that b_f^{i+1} is universally injective; hence $B^{i+1}(K^\cdot)_f$ is flat over A. Consider the following exact sequence:

$$0\to Z^i(K^\cdot)\xrightarrow{z} K^i\to B^{i+1}(K^\cdot)\to 0.$$

The flatness of $B^{i+1}(K^\cdot)_f$ implies that z_f is universally injective and that $Z^i(K^\cdot)_f$ is flat over A. Consider the following commutative diagram:

$$\begin{array}{ccccccccc} & & & K^i & & & & & \\ & & {}^{b^i}\nearrow & & \nwarrow^{z} & & & & \\ 0 & \to & B^i(K^\cdot) & \xrightarrow{a} & Z^i(K^\cdot) & \to & H^i(K^\cdot) & \to & 0. \end{array}$$

We set $f_p = f\otimes 1_{A_p}$ for any map f in the following. As z_p is universally injective, we see that b_p^i is universally injective if and only if a_p is universally injective, i.e., if and only if $H^i(K^\cdot)_p$ is flat over A by Cor. A.2. Assertion (ii) is now a consequence of Prop. A.1. ■

COROLLARY A.6: *Same notation as above. One has*

(i) *If $U^{i-1}(\kappa)$ and $U^i(\kappa)$ are surjective then there exists an $f\in S$ such that $H^i(K^\cdot)_f$ is A-flat and $U^i(M)_f$ is an isomorphism for all A-modules M.*

(ii) *If* $K^j = 0$ *for* $j < 0$ *and* $U^0(\kappa)$ *is surjective, then there exists an* $f \in S$ *such that* $H^0(K^{\cdot})_f$ *is A-flat and* $U^0(M)_f$ *is an isomorphism for all A-modules M.*

(iii) *Assume that* $U^i(M)$ *is an isomorphism for all A-modules M. Then* $H^i(K^{\cdot})$ *is A-flat if and only if* $U^{i-1}(M)$ *is an isomorphism for all A-modules M.*

(iv) *Assume that* $H^i(K^{\cdot} \otimes_A \kappa) = 0$. *Then there exists an* $f \in S$ *such that* $H^i(K^{\cdot} \otimes_A M)_f = 0$ *and* $U^{i-1}(M)_f$ *is an isomorphism for all A-modules M.*

(v) *If* $K^j = 0$ *for* $j < 0$ *and* $H^1(K^{\cdot} \otimes_A \kappa) = 0$, *then there exists an* $f \in S$ *such that* $H^0(K^{\cdot})_f$ *is A-flat and* $U^0(M)_f$ *is an isomorphism for all A-modules M.*

(vi) *Assume that* $H^i(K^{\cdot} \otimes_A \kappa) = 0$ *for* $i \gg 0$. *If* $H^i(K^{\cdot}) = 0$ *for* $i \geqq i_0$ *then* $H^i(K^{\cdot} \otimes_A \kappa) = 0$ *for* $i \geqq i_0$.

PROOF: Assertions (i)–(iii) follow immediately from the property of exchange.

(iv): $H^i(K^{\cdot} \otimes \kappa) = 0$ implies that $U^i(\kappa)$ is surjective. By Thm. A.5(ii) there exists an $g \in S$ such that $U^i(M)_g$ is an isomorphism for all A-modules M. In particular, $H^i(K^{\cdot}) \otimes \kappa = H^i(K^{\cdot} \otimes \kappa) = 0$. Hence $H^i(K^{\cdot})_p = 0$ by Nakayama's lemma (applied to the finite B_p-module $H^i(K^{\cdot})_p$). Consequently there exists an $h \in S$ such that $H^i(K^{\cdot})_h = 0$. Put $f = g \cdot h$ and apply Thm. A.5(ii) with this f.

(v): This follows now easily, and (vi) is proved by descending induction. ■

REMARK A.7: One usually expresses the property that $U^i(M)$ is an isomorphism for all A-modules M by saying that (*the formation of*) $H^i(K^{\cdot})$ *commutes with base change on A.*

Note also that $U^i(\kappa)$ is surjective if and only if $U^i(\kappa')$ is surjective, where $\kappa \subset \kappa'$ is an arbitrary field extension, since $\kappa \subset \kappa'$ is faithfully flat.

The statements above have obvious generalizations to the framework of locally noetherian schemes. One application is the derivation of the well-known facts about the higher direct images for a proper morphism (see [6, III] and [9, II.5]). Another is the analogues of these facts for relative local and global Ext's. We shall content ourselves with listing the properties that have found explicit use in the present paper.

PROPOSITION A.8: *Let* $p : X \to T$ *be a morphism of finite type between locally noetherian schemes and let* $K^{\cdot}$ *denote a complex of coherent* $\mathcal{O}_X$*-Modules that are flat over T. Then one has*

(i) *If the base change map $U^i(t)$ is surjective for all geometric points t of T then $H^i(K^{\cdot})$ commutes with base change on T.*

(ii) *If $H^i(K^{\cdot})$ commutes with base change on T, then $H^{i-1}(K^{\cdot})$ commutes with base change if and only if $H^i(K^{\cdot})$ is flat over T.*

REMARK A.9; Some further work shows that if the morphism p in Prop. A.8 is *projective* and if we just assume surjectivity of $U^i(t)$ for a *single* point t, then the conclusions of the proposition hold with T replaced by a suitable open neighborhood of t.

Prop. A.8 also follows from [2]; however, this further result does not, nor do Thm. A.5 (i) and Cor. A.6 (i), (ii), (iv), (v).

REFERENCES

[1] E. ARTIN: *Algebraic numbers and Algebraic functions*, Nelson, London, 1968.

[2] A. OGUS and G. BERGMAN: Nakayama's Lemma for half-exact Functors. *Proc. Amer. Math. Soc. 31* (1972) 67–74.

[3] P. DELIGNE and D. MUMFORD: The irreducibility of the space of curves of given genus. *Publ. Math. de l'I.H.E.S. 36* (1969) 75–110.

[4] P.M. EAKIN, JR: The Converse to a Well Known Theorem on Noetherian Rings. *Math. Ann. 177* (1968) 278–282.

[5] A. GROTHENDIECK: Séminaire de Géométrie Algébrique 1960–61. *I.H.E.S., Lecture Notes in Math. 224.* Springer-Verlag, 1971.

[6] A. GROTHENDIECK and J. DIEDONNE: Eléments de Géometrie Algébrique, Chap. I–IV, *Publ. Math. del'I.H.E.S. 4, 8, . . ., 32.*

[7] K. LØNSTED: The hyperelliptic locus with special reference to characteristic two. *Math. Ann. 222* (1976) 55–61.

[8] D. MUMFORD: *Lectures on curves on an algebraic surface.* Annals of Math. Studies 59, Princeton University Press, 1966.

[9] D. MUMFORD: *Abelian Varieties*, Oxford University Press, 1970.

[10] M. NAGATA: A type of subrings of a noetherian ring. *J. Math. Kyoto 8* (1968) 465–467.

[11] F. OORT: Hyperelliptic curves over number fields, in *Classification of algebraic varieties and compact complex manifolds. Lecture Notes in Math. 412.* Springer-Verlag, 1974.

[12] J.-P. SERRE: *Corps locaux.* Hermann, Paris, 1960.

[13] J.-P. SERRE: *Groupes algébriques et corps de classes.* Hermann, Paris 1959.

(Oblatum 22–IV–1977 & 5–IX–1978)

Knud Lønsted
Matematisk Institut
Universitetsparken 5
2100 Copenhagen, Denmark.

Steven L. Kleiman
Dept. of Mathematics, 2-278
M.I.T.
Cambridge, Mass. 02139
USA