M. CANTOR

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SOME PROBLEMS OF GLOBAL ANALYSIS ON ASYMPTOTICALLY SIMPLE MANIFOLDS

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Abstract

This paper establishes the setting for applying the techniques of global analysis to problems defined on the Riemannian manifold $(\mathbb{R}^n, g)$ where g is asymptotically Euclidean. It is shown that the necessary decomposition theorems for vector and tensor fields hold in certain weighted Sobolev spaces. Also the manifold and group structure of spaces of diffeomorphisms of $\mathbb{R}^n$ asymptotic to the identity at infinity are presented. These results are applied to establish unique solutions to the Euler equations for perfect fluid flow on $(\mathbb{R}^n, g)$. Also the manifold structure of the spaces of asymptotically Euclidean Riemannian metrics and conformal structures are studied.

0. Introduction

The techniques of global analysis have been successful in treating many classical problems of continuum mechanics, relativity, and geometry over compact manifolds (see for example, Marsden [13] and Fischer–Marsden [9]). However, several problems arise when one attempts to extend these techniques to the non-compact case. For example, it is well known that the manifold structure on spaces of C^r or Sobolev maps from a compact manifold to itself does not depend on any Riemannian structure on M (see Palais [17] or Penot [18]). This is useful when studying the space of all metrics on M. It is clear that the manifold structure of the Sobolev maps on a non-compact manifold does to some extent depend on the choice of metric on the manifold. For example, a continuous vector field on a compact manifold M will be square integrable with respect to any metric on

M. This property does not hold in the non-compact case; a vector field may be square integrable with respect to one metric and not square integrable with respect to another.

On the other hand, if two metrics on the same non-compact manifold are sufficiently close (say equal off a compact set) then they should generate the same differentiable structure for spaces of maps. This paper studies a special case of this situation where the base manifold is $\mathbb{R}^n$ and the metric g is sufficiently close to the standard Euclidean metric. The details of the construction of the manifold structures and of the group properties of various sets of diffeomorphisms of $(\mathbb{R}^n, g)$ are discussed in Section 2.

A second and perhaps more serious difficulty in applying the techniques of global analysis to non-compact domains is the lack of regularity of elliptic operators. Many of the problems entail showing some subset of the function space being considered (say the volume preserving diffeomorphisms as a subset of the diffeomorphisms) forms a smooth submanifold. This is done by showing the tangent space at each point of the subset splits appropriately. This in turn usually depends on the validity of the Fredholm alternative for elliptic operators (see Marsden [15] for a general discussion of this phenomenon). The Fredholm alternative in general fails for elliptic operators on Sobolev spaces over non-compact domains. However, this problem may be avoided by noting for certain pairs of weighted Sobolev spaces the elliptic operators act as isomorphisms. The asymptotic properties of these spaces correspond to standard physical assumptions on behavior at infinity for potential fields. For non-flat metrics on $\mathbb{R}^n$ these operators have non-constant coefficients when expressed in standard coordinates. Section 1 is devoted to showing that the sort of operators under study have the claimed isomorphism property. The results of this section depend in part on the work of Nirenberg and Walker [16] and the theorem of this section is related to their results.

Section 4 discusses the existence of weak Riemannian structures on the spaces discussed in Section 2, subject to the constraints imposed by requiring that the theorems of Section 1 are applicable. Sections 3, 5 and 6 contain applications of the above results. Section 3 discusses perfect fluid flow on $(\mathbb{R}^n, g)$. The techniques of this section are well known and so the proofs are only sketched.

Section 5 extends the slice theorem of Ebin [7] for the action of the diffeomorphism group on the space of asymptotically flat metrics on $\mathbb{R}^n$. Included in this section (Lemma 5.5.3) is a proof of the canonical decomposition of symmetric 2-tensors of Berger and Ebin [1] for tensor fields in the appropriate function space. This result was obtained independently by J. Marsden [6] who used similar techniques. The proof of part (1) of Theorem 5.5 was provided by the referee.

Section 6 contains the equivalent theorem for the space of conformally equivalent metrics. This theorem depends implicitly on the York Decomposition of symmetric 2-tensors (see York [19]). The compact case has recently been studied by Fischer and Marsden [10]. Their construction differs slightly from the one used in this paper. Rather than studying the action of the diffeomorphism group on the space of conformal structures, they consider the larger group of "conformeomorphisms" acting on the space of metrics.

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Throughout the paper the standard multi-index notion is used, that is

$$\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n, \quad |\alpha| = \sum \alpha_i \quad \text{and} \quad D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x^{\alpha_1}, \ldots, \partial x^{\alpha_n}}.$$

1. Elliptic operators with non-constant coefficients

Throughout this section we consider an operator $A = \Sigma_{|\alpha| \leq m} a_\alpha(x) D^\alpha$ where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index A will act on functions from $\mathbb{R}^n$ to $\mathbb{R}^k$ and A is assumed to be elliptic in the sense $\det|\Sigma_{|\alpha|=m} a_\alpha(x)\xi^\alpha| \neq 0$ for all $x \in \mathbb{R}^n$ and all nonzero ξ in $\mathbb{R}^n$.

DEFINITION 1.1: *Let* $|\ |_p$ *be the norm on* $L^p(\mathbb{R}^n, \mathbb{R}^k)$ *and* $\sigma(x) = (1 + |x|^2)^{1/2}$. *Then for* $1 \leq p \leq \infty$, $\delta \in \mathbb{R}$, *and* $s \in \mathbb{N}$, *define*

$$|f|_{p,s,\delta} = \sum_{|\alpha| \leq s} |(D^\alpha f)\sigma^{|\alpha|+\delta}|_p.$$

DEFINITION 1.2: *The completion of* $C_0^\infty(\mathbb{R}^n, \mathbb{R}^k)$ *with respect to* $|\ |_{p,s,\delta}$ *is denoted by* $\mathcal{M}^p_{s,\delta}(\mathbb{R}^n, \mathbb{R}^k)$.

Usually, when there is little chance of confusion, $\mathcal{M}^p_{s,\delta}(\mathbb{R}^n, \mathbb{R}^k)$ will be denoted by $\mathcal{M}^p_{s,\delta}$.

These spaces were introduced by the author [4], [5]. The use of the

$|\ |_{p,s,\delta}$ norms was suggested by a paper of Nirenberg and Walker [16]. Their paper contains results which are related to those of this section.

We shall prove the following theorems.

THEOREM 1.3: *Let $n > m$ and $A_\infty = \Sigma_{|\alpha|=m} \bar{a}_\alpha D^\alpha$ be an elliptic homogeneous operator with constant coefficients on $\mathbb{R}^n$. Let $A(x) = \Sigma_{|\alpha|\le m} a_\alpha(x) D^\alpha$ be an elliptic operator such that for $s \ge m$, $a_\alpha \in C^{s-m}(\mathbb{R}^n, \mathbb{R}^k)$ and for each multi-index γ with $0 \le |\gamma| \le s - m$*

$$\lim_{|x|\to\infty} \sup |(D^\gamma a_\alpha(x)) \sigma_{(x)}^{m-|\alpha|+|\gamma|}| < \epsilon \qquad \text{for } |\alpha| < m$$

$$\lim_{|x|\to\infty} \sup |D^\gamma (a_\alpha(x) - \bar{a}_\alpha) \sigma^{|\gamma|}(x)| < \epsilon \quad \text{for } |\alpha| = m.$$

Then if $p > n/(n-m)$, $0 \le \delta < -m + n(p-1)/p$ and ϵ is sufficiently small, A maps $M^p_{s,\delta}$ into $M^p_{s-m,\delta+m}$ continuously with closed range and finite dimensional kernel.

Furthermore suppose one of the following conditions hold

(i) $\sup |D^\gamma a_\alpha(x) \cdot \sigma_{(x)}^{m-|\alpha|+|\gamma|}| < \epsilon \qquad |\alpha| < m$

$\sup |(D^\gamma (a_\alpha(x) - \bar{a}_\alpha)) \sigma^{|\gamma|}(x)| < \epsilon \ \ |\alpha| = m$

for ϵ sufficiently small.

(ii) *There is a continuous curve c from $[0,1]$ into the space of bounded operators between $\mathcal{M}^p_{s,\delta}$ and $\mathcal{M}^p_{s-m,\epsilon+m}$ such that $c(0) = A_\infty$, $c(1) = A$ and for each $t \in [0,1]$, $c(t)$ is an injection, and satisfies the hypotheses of the theorem.*

Then A is an isomorphism.

By assuming more differentiability of the functions in the domain space of A, one can slightly relax the smoothness condition of the coefficients of A. This form of the isomorphism theorem will be quite useful in Sections 5 and 6.

THEOREM 1.4: *Let n, m, p, δ and $A_\infty = \Sigma_{|\alpha|\le m} \bar{a}_\alpha D^\alpha$ be as in Theorem* 1.3. *Let $s > n/p + m$ and $A(x) = \Sigma_{|\alpha|\le m} a_\alpha(x) \mathscr{D}^\alpha$ be an elliptic operator such that*

$$a_\alpha \in \mathcal{M}^p_{s-m,m-|\alpha|}, \qquad |\alpha| < m,$$

$$(a_\alpha - \bar{a}_\alpha) \in \mathcal{M}^p_{s-m,0}, \quad |\alpha| = m.$$

Then A maps $\mathcal{M}^p_{s,\delta}$ continuously into $\mathcal{M}^p_{s-m,\delta+m}$ with closed range and finite dimensional kernel.

Furthermore, suppose one of the following conditions hold:

(i) $\sum_{|\alpha|<m} |a_\alpha|_{p,s-m+|\alpha|,m-|\alpha|} + \sum_{|\alpha|=m} |a_\alpha - \bar{a}_\alpha|_{p,s-m,0} < \epsilon$
for ϵ *sufficiently small.*

(ii) *Condition* (*ii*) *of Theorem* 1.3 *holds.*

Then A *is an isomorphism of* $\mathcal{M}^p_{s,\delta}$ *and* $\mathcal{M}^p_{s-m,\delta+m}$.

COROLLARY 1.5: *If* $m = 2$, $k = 1$, A *satisfies the hypotheses of Theorem* 1.3 *or Theorem* 1.4 *and* $a_0(x) < \epsilon$ *for some* $\epsilon > 0$ (*sufficiently small*) *then* A *is an isomorphism.*

PROOF: This follows from conditions (i) and (ii) and the maximum principle.

Before proving the theorems we shall remark on a few points.

REMARK 1: For $m = 2$ and $n = 3$, one would expect from classical treatment of Poisson's equation that the domain spaces, $\mathcal{M}^p_{s,\delta}$, should include functions f such that $f \sim |x|^{-1}$, $Df \sim |x|^{-2}$, etc. It is easily seen the limits on p and δ are such that exactly these functions are included.

REMARK 2: Since for $\delta \geq 0$ the $|\,|_{p,s,\delta}$ norms are stronger than the regular Sobolev norms, the embedding theorems apply. Hence one can use Theorem 1.3 and 1.4 to establish the existence of $C^{k+\alpha}$ bounded solutions. For example, we have the following corollary:

COROLLARY 1.6: *Under the hypotheses of Theorem* 1.3 *with* $s > (n/p) + k$ *and* $m = 2$, *if* $g \in \mathcal{M}^p_{s-2,\delta+2}$ *we may find a solution to* $Au = g$ *with* $u \in C^k$.

In fact with sufficiently smooth and bounded coefficients of A *and* $g \in C^\infty_0$, *we may solve* $Au = g$ *with* $f \in C^\infty$.

We shall prove the theorem using a series of lemmas. We first consider A_∞.

LEMMA 1: *Let* A_∞, m, p, δ, *and* s *be as in Theorem* 1.3. *The* A_∞ *maps* $\mathcal{M}^p_{s,\delta}$ *onto* $\mathcal{M}^p_{s-m,\delta+m}$ *isomorphically.*

PROOF: The proof of this for the case $m = 2$ and A_∞ is the Laplacian is found in [5]. The more general case is proven identically. Surjectivity follows by noting A_∞ has a fundamental solution asymptotic to $|x|^{m-n}$ (see [11]). Injectivity follows from Theorem 2.1 of [16].

Continuity of A_∞ is also immediate and $(A_\infty)^{-1}$ is continuous because of the open mapping theorem for Banach spaces.

This next lemma is essentially due to Schauder (see [2], p. 238 for example).

LEMMA 2: *Suppose A_0 and A_1 are continuous linear maps between Banach spaces B and B'. Suppose further that*

(1) *A_0 is an isomorphism,*

(2) *There is a continuous curve c from $[0, 1]$ into the space of bounded operators from B to B' such that $c(0) = A_0$, $c(1) = A_1$, and for each $t \in [0, 1]$, $c(t)$ is an injection with closed range. Then A_1 is an isomorphism.*

PROOF: Let $S = \{t \in [0, 1] \mid c(t) \text{ is an isomorphism}\}$. We know $0 \in S$ and since the space of isomorphisms is open in the space of bounded operators, we know S is open. Using connectivity we need only show S is closed. Let t_i be a sequence in S with $t_i \to t_0$. We must show $t_0 \in S$.

First note since $c(t)$ is an injection with closed range, it follows from continuity of the inverse of $c(t)$ restricted to the range that for each $t \in [0, 1]$ there is a C_t such that for all $u \in B$

$$\|u\| \leq C_t \|c(t)u\|.$$

We wish to show there is a $C \in \mathbb{R}$ such that for all i sufficiently large and $u \in B$

$$\|u\| \leq C\|c(t_i)Au\| \tag{1}$$

We have

$$\begin{aligned}\|u\| &\leq C_{t_0}\|c(t_0)u\| \\ &\leq C_{t_0}(\|c(t_i)u\| + \|C(t_0)u - C(t_i)u\|) \\ &\leq C_{t_0}\|C(t_i)u\| + \||C(t_0) - C(t_i)\||\, \|u\|\end{aligned}$$

where $\||\ \||$ is the operator norm. Picking i sufficiently large so that $\||C(t_0) - C(t_i)\|| < 1$ we are done.

Now let $f \in B'$ and u_i a sequence in B so that $c(t_i)(u_i) = f$. For i and j sufficiently large we have

$$\|u_i - u_j\| \leq C_{t_0}\|c(t_0)(u_i - u_j)\|$$

$$\leq C_{t_0}\|c(t_i)u_i + (c(t_0) - c(t_i))u_i - c(t_j)u_j - (c(t_0) - c(t_j))u_j\|$$
$$\leq C_{t_0}\|c(t_0) - c(t_i)u_i - (c(t_j) - c(t_0))u_j\|$$
$$\leq C_{t_0}\||c(t_0) - c(t_i)\|| \, \|u_i\| + \||c(t_j) - c(t_0)\|| \, \|u_j\|.$$

But using (1) the $\{\|u_i\|\}$ is uniformly bounded by $c\|f\|$ and thus $\{u_i\}$ is a Cauchy sequence which converges to u_0. Finally, it is straightforward to check that $c(t_0)(u_0) = f$. Q.E.D.

PROOF OF THE THEOREM 1.3: Let $\mathscr{L}$ be the space of bounded linear transformations from $M^p_{s,\delta}$ to $\mathscr{M}^p_{s-m,\delta+m}$ with the uniform operator (norm) topology. For $E \in \mathscr{L}$, we denote

$$\|E\| = \sup_{|f|=1} |E(f)|_{p,s-m,\delta+m}$$

We shall show there is a $k > 0$ such that if $E(x) = \Sigma_{|\alpha|\leq m}\, e_\alpha(x)D^\alpha$ and for each γ $\sup|(D^\gamma e_\alpha(x)) \cdot \sigma_{(x)}^{m-|\alpha|+|\gamma|}| < M$ for $|\gamma| \leq s - m$ then

$$\|E\| < kM. \tag{2}$$

In particular, if $f \in \mathscr{M}^p_{s,\delta}$ we have

$$|Ef|_{p,s-m,\delta+m} \leq \sum_{|\beta|\leq s-m} |D^\beta(E(f)) \cdot \sigma^{\delta+m+|\beta|}|_p$$
$$\leq \sum_{|\beta|\leq s-m} \left| \sum_{|\alpha|=m} D^\beta(e_\alpha \cdot D^\alpha f)(x) \cdot \sigma^{\delta+m+|\beta|} \right|_p$$

Applying Leibnitz' rule and rearranging the summation yields

$$|Ef| \leq \sum_{|\beta|\leq s-m} \left| \sum_{|r|=0}^{m+|\beta|} \sum_{|\alpha|=\max(0,\, r-|\beta|)}^{\min(r,m)} \left(\binom{\beta}{r-\alpha} D^{\beta+\alpha-r} e_\alpha \cdot D^r f \right) \sigma^{\delta+m+|\beta|} \right|_p$$
$$\leq k_1 \sum_{|\beta|\leq s-m} \left| \sum_{|r|} \left(\left(\sum_{|\alpha|} D^{\beta+\alpha-r} e_\alpha \right) \sigma^{m+|\beta|-|r|} \right) D^r f \sigma^{\delta+|r|} \right|_p$$

Setting $\gamma = \beta + \alpha - r$, $|\gamma| \leq s - m$, the inequality follows if $\sup_{\mathbf{R}^n} |D^\gamma e_\alpha \sigma^{m-|\alpha|+|\gamma|}| < M$, which was assumed.

Now, it is well known that the set of isomorphisms is open in $\mathscr{L}$. Thus if condition (i) is satisfied for ϵ sufficiently small then setting $E = A - A_\infty$ we see from (2) the norm of E may be made small enough to guarantee A is an isomorphism.

The fact that A has finite dimensional kernel follows immediately from Theorem 4.1 of [14].

To show that A has closed range note that since the kernel of A is finite dimensional we may write $\mathcal{M}^p_{s,\delta} = \ker(A) \oplus W$ with W closed. Also $A(\mathcal{M}^p_{s,\delta}) = A(W)$ and for each f in the range of A there is a unique $w \in W$ with $A(w) = f$.

We shall now show there is a constant $C > 0$ such that for any $u \in W$

$$|u|_{p,s,\delta} \leq C|Au|_{p,s-m,\delta+m}. \tag{3}$$

If there were no such C then there is a sequence $\{u_i\} \subset W$ such that $|u_i|_{p,s,\delta} = 1$ and $|Au_i|_{p,s-m,\delta+m} \to 0$.

For each $R \geq 1$ let $\phi_R : \mathbb{R}^n \to \mathbb{R}$ satisfy

(1) $\phi_R(x) = 1$ if $|x| \leq R$

(2) $\phi_R(x) = 0$ if $|x| \geq 2R$

(3) $|D^\alpha \phi_R(x)| \leq 1$ for all α.

We write $u_i = \phi_R u_i + (1 - \phi_R)u_i$ and note the sequence $\{u_i\}$ is Cauchy if the sequences $\{\phi_R u_i\}$ and $\{(1 - \phi_R)u_i\}$ are Cauchy for some R. Let $B_R = \{x \in \mathbb{R}^n : |x| \leq R\}$.

For any R, the sequence $\{\phi_R u_i\}$ is uniformly bounded in $W^{p,s}(B_{2R})$ and so using the Rellich compactness theorem it has a subsequence (taken to be all of $\{\phi_R u_i\}$) converging to an element $u_R \in L^p(B_{2R})$. Since A is uniformly elliptic on B_{2R} we have the usual elliptic estimate and so there is a C_R such that

$$|\phi_R u_i - \phi_R u_j|_{W^{p,s}} \leq C_R(|A(\phi_R u_i) - A(\phi_R u_j)|_{W^{p,s-m}} + |\phi_R u_i - \phi_R u_j|_{L^p}).$$

But since $\{A(\phi_R u_i)\}$ is Cauchy in $W^{p,s-m}$ and $\{\phi_R u_i\}$ is Cauchy in L^p, it follows $\{\phi_R u_i\}$ is Cauchy in $W^{p,s}(B_{2R})$ and it follows that $\{\phi_R u_i\}$ is Cauchy in $\mathcal{M}^p_{s,\delta}$ for each R.

It follows from Lemma 1 that there is a $C_1 > 0$ such that for $v \in \mathcal{M}^p_{s,\delta}$

$$|v|_{p,s,\delta} \leq C_1|A_\infty v|_{p,s-m,\delta+m}.$$

Thus for each $R \geq 1$

$$\begin{aligned}|(1-\phi_R)u|_{p,s,\delta} \leq C_1|A_\infty((1-\phi_R)u)|_{p,s-m,\delta+m} &\leq C_1(|A((1-\phi_R)u)|_{p,s-m,\delta+m} \\ + |(A_\infty - A)((1-\phi_R)u)|_{p,s-m,\delta+m}) &\leq C_1(|A(1-\phi_R)_u)|_{p,s-m,\delta+m} \\ + \|A_\infty - A\|_{|x|>R}|u|_{p,s-m,\delta+m})&\end{aligned}$$

where $\|A_\infty - A\|_{|x|\geq R}$ is the norm of the difference of A_∞ and A when restricted to functions with support on $\{x \in \mathbb{R}^n : |x| \geq R\}$. It follows from the hypotheses of the theorem and inequality (2) that

$$\lim_{R\to\infty} \|A_\infty - A\|_{|x|\geq R} \leq k\epsilon.$$

Thus for $\epsilon < (C_1 k)^{-1}$ and R sufficiently large

$$|(1-\phi_R)u|_{p,s,\delta} \leq C_2 |A(1-\phi_R)u|_{p,s-m,\delta+m} \leq C_2(|(1-\phi_R)Au|_{p,s-m,\delta+m} + |A((1-\phi_R)u)-(1-\phi_R)Au|_{p,s-m,\delta+m}).$$

Note the sequence

$$\{A((1-\phi_R)u_i)-(1-\phi_R)Au_i\}_{i\in\mathbb{N}}$$

consists of functions with support in B_{2R} and is bounded in $W^{p,s-m+1}(B_{2R})$ (the highest terms cancel). Thus by passing to a subsequence we may assume it is Cauchy in $M^p_{s-m,\delta+m}$. It is clear $(1-\phi_R)Au_i$ is Cauchy and so from the above inequality we have $(1-\phi_R)u_i$ is Cauchy. Let $\lim_{i\to\infty} u_i = u_0 \in W$. Then $A(u_0)=0$ and $|u_0|_{p,s,\delta}=1$. But since $W \cap \ker(A) = \{0\}$ we have a contradiction and the inequality is established.

Now let $f_i \in A(\mathcal{M}^p_{s,\delta})$ and $f_i \to f_0$. Let $w_i \in W$ such that $A(w_i)=f_i$. Using (3) we see $w_i \to w_0 \in W$ and $A(w_0)=f_0$. Thus the range of A is closed.

Finally sufficiency of condition (ii) for A to be an isomorphism follows immediately from Lemmas 1 and 2. Q.E.D.

Before proving Theorem 1.4 we shall need two more lemmas:

LEMMA 3: *Let $p>1$, $s>n/p$, $0\leq \ell \leq s$, and $\delta_1, \delta_2 \geq 0$ and "$\cdot$" denote any bilinear map $R^m \times R^q \to R^r$. Then pointwise application of "$\cdot$" induces a continuous map*

$$\mathcal{M}^p_{s,\delta_1}(\mathbb{R}^n,\mathbb{R}^m) \oplus \mathcal{M}^p_{\ell,\delta_2}(\mathbb{R}^n,\mathbb{R}^q) \to \mathcal{M}^p_{\ell,\delta_1+\delta_2}(\mathbb{R}^n,\mathbb{R}^r).$$

PROOF: It suffices to show that for $|B|\leq \ell$ that $(f,g)\to \mathcal{D}^\beta(f\cdot g)$ is continuous from $\mathcal{M}^p_{s,\delta_1} \oplus \mathcal{M}^p_{\ell,\delta_2}$ to $\mathcal{M}^p_{0,\delta_1+\delta_2+|B|}$. Using Liebnitz' rule we have

$$|D^\beta(f\cdot g)\sigma^{\delta_1+\delta_2+|\beta|}|_p \leq C_1 \sum_{\gamma\leq\beta} |D^\gamma(f)\cdot D^{\gamma-\beta}(g)\sigma^{\delta_1+\delta_2+|\beta|}|_p$$

$$\leq C_2 \sum_{\gamma\leq\beta} |(D^\alpha(f)\sigma^{\delta_1+|\gamma|})(D^{\beta-\gamma}(g)\sigma^{\delta_2+|\beta-\gamma|}|_p.$$

It is well known that if $k+1>n/p$ then pointwise multiplication is continuous from $W^{p,k}\oplus W^{p,1}\to W^{P}$ (see Palais [15], Theorem 9.4). By assumption $D^{\gamma}f\in \mathcal{M}^{p}_{s-|\gamma|,\delta_1+|\gamma|}$. It follows easily that $(\sigma^{\delta_2+|\beta-\gamma|}D^{\beta-\gamma}g\in W^{p,1-|\beta-\gamma|}$. Also $(s-|\gamma|+1-|\beta-\gamma|)=s+(1-|\beta'|)\geq s>n/p$. Thus there is a constant C_2 such that

$$|D^{\beta}(f\cdot g)^{\delta_1+\delta_2+|\beta|}|_p\leq C_2\sum_{\gamma\leq\beta}|D^{\gamma}f\cdot\sigma^{\delta_1+|\gamma|}|_p|D^{\beta-\gamma}(g)\sigma^{\delta_2+|\beta-\gamma|}|_p$$
$$\leq C|f|_{p,s,\delta_1}|g|_{p,\ell,\delta_2}.\quad \text{Q.E.D.}$$

LEMMA 4: *Let $p>1$, $s>n/p+m$, $\delta\geq 0$ and $E(x)=\sum_{|\alpha|\leq m}e_\alpha(x)\mathcal{D}^{\alpha}$ be a differential operator with each $e_\alpha\in\mathcal{M}^{p}_{s-m,m-|\alpha|}$. Then there is a $C>0$ such that for each $f\in\mathcal{M}^{p}_{s,\delta}$*

$$|Ef|_{p,s-m,\delta+m}\leq C\Big(\sum_{|\alpha|\leq m}|e_\alpha|_{p,s-m,m-|\alpha|}\Big)|f|_{p,s,\delta}.$$

PROOF: For $f\in M^{p}_{s,\delta}$, $D^{\alpha}f\in M^{p}_{s-|\alpha|,\delta+|\alpha|}$. Also $e_\alpha\in M^{p}_{s-m,m-|\alpha|}$ and it follows from Lemma 3 that

$$|e_\alpha D^{\alpha}f|_{p,s-m,\delta+m}\leq C_\alpha|e_\alpha|_{p,s-m,m-|\alpha|}|D^{\alpha}f|_{p,s-|\alpha|,\delta+|\alpha|}$$
$$\leq C_\alpha|e_\alpha|_{p,s-m,m-|\alpha|}|f|_{p,s,\delta}.$$

Therefore

$$|Ef|_{p,s-m,\delta+m}\leq\sum_{|\alpha|\leq m}|e_\alpha D^{\alpha}f|_{p,s-m,\delta+m}$$
$$\leq\Big(\sum_{|\alpha|\leq m}C_\alpha|e_\alpha|_{p,s-m,m-|\alpha|}\Big)|f|_{p,s,\delta}$$
$$\leq\Big(\sum_{|\alpha|\leq m}|e_\alpha|_{p,s-m,m-|\alpha|}\Big)|f|_{p,s,\delta}.\quad\text{Q.E.D.}$$

PROOF OF THEOREM 1.4: It follows from Lemma 4 that A maps $M^{p}_{s,\delta}$ into $M^{p}_{s-m,\delta+m}$. Also from Lemma 4 if $E=A-A_\infty$ and $\|E\|_{\{|x|>R\}}$ is the operator norm restricted to functions with support on $\{x\colon x>R\}$. We see $\lim_{R\to\infty}\|E\|_{\{|x|>R\}}=0$. Thus the techniques of Theorem 1.3 apply. (In fact since $s-m>n/p$ the e_α are all continuous and we can apply Theorem 4.1 of [16] to see A has closed range). Q.E.D.

2. Groups of diffeomorphisms over asymptotically simple manifolds

DEFINITION 2.1: *We define an asymptotically simple manifold to order m to be a Riemannian manifold $M=(\mathbb{R}^n,g)$ where g is complete* *and such if e is the Euclidean metric* $(\mathsf{E}^n = (\mathbb{R}^n, e))$ then

$$\lim_{|x|\to\infty} \sup|(D^\alpha(g-e)(x))\sigma^\alpha(x)| = 0$$

for $|\alpha| \leq m$.

When considering $\mathcal{M}^p_{s,\delta}$ topologies for spaces of maps on M, we shall assume $m \geq s+2$. This will guarantee sufficient boundedness of the Christoffel symbols associated with the s^{th} covariant derivative associated with g.

Throughout this section we retain the notation of section 1, *i.e.* $|\ |_{p,s,\delta}$ will denote the $\mathcal{M}^p_{s,\delta}$ norm with respect to the Euclidean metric e.

Definition 2.2: *For* $V_x \in T_xM$ *set* $\|V\|^2_x = g_x(V_X, V_x)$. *Similarly set the norm of* $W_x \in T^r_qM$ generated by g to be $\|W_x\|$. dV *is volume form generated by* g. *Also for* $v \in C^\infty_0(T^r_qM)$ define for $p \geq 1$, $s \in N$, *and* $\delta \in \mathbb{R}$,

$$\|V\|_{p,s} = \sum_{|\alpha|\leq s} \left(\int \|D^\alpha V\|^p dV\right)^{1/p},$$

$$\|V\|_{p,s,\delta} = \sum_{|\alpha|\leq s} \left(\int \|(D^\alpha V)\sigma^{|\alpha|+\delta}\|^p dV\right)^{1/p}.$$

(*Note that* $\|\ \|_{p,s}$ *does not equal* $\|\ \|_{p,s,0}$.)

$\|V\|_{C^k} = \sum_{|\alpha|\leq k} \sup\|^\alpha V\|$ and set $\mathcal{M}^p_{s,\delta}(T^r_qM)$ (resp. $C^k(T^r_qM)$, $W^{p,s}(T^r_qM)$) to be the completion of $C^\infty_0(T^r_sM)$ with respect to $\|\ \|_{p,s,\delta}$ (resp. $\|\ \|_{C^k}$, $\|\ \|_{p,s}$).

If $f: M \to M$, recall that a section of a vector bundle $\pi: E \to M$ over f is a map $S: M \to E$ such that $\pi \circ S = f$. For such sections of tensor bundles over f we may define the $W^{p,s}$, C^r and $\mathcal{M}^p_{s,\delta}$ norms in the natural way. For example if $X: M \to T^r_qM$ covers f, *then*

$$\|X\|_{p,s,\delta} = \sum_{|\alpha|\leq s} \left(\int \|((f^*\nabla)^\alpha_x(p)\sigma^{|\alpha|+\delta}(p)\|_{f(p)} dV\right.$$

where $(f^*\nabla)$ is the induced connection (*i.e.* $f^*\nabla$ satisfies $(f^*\nabla_V)(W \circ f) = \nabla_{f^*(v)}W$ for $v \in T_pM$ and W a tensor field on M).

Remark: We might replace $\sigma(x)$ with $(1 + d(x, 0)^2)^{1/2}$ where d is the distance function generated by g. However, because of assumptions on g, norms using either weight function are equivalent.

In fact we have the following proposition.

PROPOSITION 2.3: *When M is asymptotically simple to order $m \geq \max\{k, s+2\}$ the $W^{p,s}$, C^k and $\mathcal{M}^p_{s,\delta}$ norms generated by g and those generated by the Euclidean metric e are equivalent.*

PROPOSITION 2.4: *If $s > (n/p) + k$ and $\delta \geq 0$, then $\mathcal{M}^p_{s,\delta}(T^r_q m) \subset C^k(T^r_q M)$, and the inclusion is continuous.*

PROOF: This follows immediately from the standard Sobolev inequality on $\mathbb{R}^n$ and Proposition 2.3.

Proposition 2.3 permits us to treat e as a 'background' metric for M. This is a familiar practice in Relativity.

DEFINITION 2.5: *For p, s, and δ as in Definition 2.2 and f a C^∞ map from M to M (not necessarily bounded) set*

$$\mathcal{M}^p_{s,\delta}(f) = \{h : M \to M : |f - h|_{p,s,\delta} < \infty\}.$$

In the following section we shall need to treat $\mathcal{M}^p_{s,\delta}(f)$ as a manifold of maps whose structure is compatible with the metric g.

DEFINITION 2.6: *For f as in Definition 2.5 we set for $h \in M^p_{s,\delta}(f)$,*

$$\bar{T}_h\mathcal{M}^p_{s,\delta}(f) = \Big\{V : M \to TM,\ \pi \circ V = h \text{ and } \sum_{|\alpha| \leq s} \Big(\int \|({}^\alpha V(x))\sigma^{\delta+|\alpha|}(x)\|^p_{f(x)} dV(x)\Big)^{1/p} < \infty\Big\}.$$

We shall show that $\mathcal{M}^p_{s,\delta}(f)$ has a manifold structure so that for $h \in \mathcal{M}^p_{s,\delta}(f)$ the tangent space $T_h\mathcal{M}^p_{s,\delta}(f)$ may be identified with $\bar{T}_h\mathcal{M}^p_{s,\delta}(f)$.

LEMMA 2.7: *Let* exp *be the time-one geodesic flow map of g. Then*

(1) *$f(x, V) = \exp_x(V) - x$ is a C^∞ bounded map from $\mathbb{R}^n \times \mathbb{R}^n$.*

(2) *The injectivity radius of g (*$\sup\{r : \exp_x$ *is injective on the ball* $B_x(0, r)$ *in* $T_xM\}$*) is strictly positive.*

PROOF: These results follow from standard comparison results noting that the geodesic spray associated with g is C^∞ bounded and falls to zero at ∞.

PROPOSITION 2.8: *Suppose $Df \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$. Then for each $h \in \mathcal{M}^p_{s,\delta}(f)$ there is a neighborhood U_h of 0 in $\bar{T}_h\mathcal{M}^p_{s,\delta}(f)$ such that $\overline{\exp}_h: U_h \to \mathcal{M}^p_s(f)$, $\overline{\exp}_h(V_h)(x) = \exp_{h(x)} V_h(x)$ is a C^∞ embedding U_h onto a neighborhood of h in $\mathcal{M}^p_{s,\delta}(f)$. Furthermore, if $h_1, h_2 \in \mathcal{M}^p_{s,\delta}(f)$ and then $\overline{\exp}_{h_2}^{-1} \circ \exp_{h_1}: U_{h_1} \to \bar{T}_{h_2}\mathcal{M}^p_{s,\delta}(f)$ is a diffeomorphism of Banach spaces onto its image.*

PROOF: Without loss of generality we use the Euclidean based norms. From Lemma 2.7 and the Sobolev embedding theorem we find U_h to be a neighborhood of vector fields covering h where C^0 length is less than the injectivity radius of g. The fact that $\overline{\exp}_h$ maps smoothly into $\mathcal{M}^p_{s,\delta}(f)$ follows the standard argument that composition on the left with a C^∞ bounded map is smooth, (for example, see Palais [15], Theorem 9.10). Finally, the smoothness of $\overline{\exp}_h^{-1}$ follows the inverse function theorem and the fact $|D\overline{\exp}_h|$ is bounded away from 0 on U_h. Q.E.D.

It follows that $\mathcal{M}^p_{s,\delta}(f)$ is a Banach manifold modelled on the $\mathcal{M}^p_{s,\delta}$ vector fields, also $\bar{T}_h\mathcal{M}^p_{s,\delta}(f) = T_h\mathcal{M}^p_{s,\delta}(f)$.

DEFINITION 2.9: *Let $p \geq 1$, $s \in N$, $\delta \in \mathbb{R}$, and I the identity map on $\mathbb{R}^n$. Then $\mathcal{D}^p_{s,\delta} = \{f \in \mathcal{M}^p_{s,\delta}(I) : f^{-1}$ exists on $f^{-1} \in \mathcal{M}^p_{s,\delta}(I)\}$.*

Using Proposition 2.9 we may endow $\mathcal{M}^p_{s,\delta}(I)$ with a Banach manifold structure. We shall assume that $\mathcal{M}^p_{s,\delta}(I)$ has this structure throughout the remainder of the paper.

THEOREM 2.10: *Let $p > 1$, $s > (n/p) + 1$, and $\delta \geq 0$. Then*
(1) *$\mathcal{D}^p_{s,\delta}$ is an open submanifold of $M^p_{s,\delta}(I)$.*
(2) *$\mathcal{D}^p_{s,\delta}$ is a topological group with respect to composition. Also right composition is smooth.*

PROOF: Both results are immediate from Proposition 2.3 and the analogous properties for maps on flat space. Q.E.D.

The next proposition will be used in the following sections:

PROPOSITION 2.11: *For $p > 1$, $s > (n/p) + 1$, and $\delta \geq 0$, $\eta \in \mathcal{D}^p_{s,\delta}$ and $\gamma \in \mathcal{M}^p_{s,\delta}(I)$. the map $W_\gamma \mapsto W_\gamma \circ \eta$ is a C^∞ map from $T_\gamma\mathcal{M}^p_{s,\delta}(I)$ into $T_{\gamma\circ\eta}\mathcal{M}^p_{s,\delta}(I)$.*

Note the requirement that η be a diffeomorphism seems to be essential here; even in the compact case the composition of $W^{p,k}$ maps need not be $W^{p,k}$ (see Ebin [7]).

3. Decomposition of vector fields and perfect fluid flow

Throughout this section we take $M = (\mathbb{R}^n, g)$ to be asymptotically simple to order m with $n \geq 3$. Also, we assume m to be greater than $s+2$ when discussing any $\mathcal{M}^p_{s,\delta}$ space of maps on M. T^*M is the cotangent space of M.

THEOREM 3.1: *If $p > n/(n-2)$, $0 \leq \alpha < -2 + n(1-1/p)$, and $s \geq 1$ then*

$$\mathcal{M}^p_{s,\alpha+1}(T^*M) = d(\mathcal{M}^p_{s+1,\alpha}(M, \mathbb{R})) \oplus \mathcal{I}^p_{s,\alpha+1}$$

where $\mathcal{I}^p_{s,\alpha+1} = \{v \in \mathcal{M}^p_{s,\alpha+1}(T^*M) \colon \delta v = 0\}$.

PROOF: We state a simple lemma which will also be useful in sections 5 and 6.

LEMMA 3.1.1: *Let E, F, G be Banach spaces and $f : E \to F$, $j : F \to G$ bounded linear maps. Then if $j \circ f$ is an isomorphism, we have $F = f(E) \oplus \ker(j)$.*

We apply this lemma to the case where $E = \mathcal{M}^p_{s+1,\alpha}(M, \mathbb{R})$, $F = \mathcal{M}^p_{s,\alpha+1}(T^*M)$ and $G = \mathcal{M}^p_{s-1,\alpha+2}(M, \mathbb{R})$ with $f = d$ and $j = \delta$. δd is the Laplace–Beltrami operator on real valued functions on M,

$$\delta d(f) = \frac{1}{\sqrt{g}} \sum_{i,k} \frac{\partial}{\partial x_i} \left(\sqrt{g} g^{ik} \frac{\partial f}{\partial x^k} \right)$$

where $g = \det(g_{ij})$ and g^{ij} is the (i, j) entry in the inverse matrix of g. Because of the asymptotic assumptions on (g_{ij}) it is easily seen that δd satisfies the hypothesis of Theorem 1.3. Furthermore, setting A_0 to $\Sigma^n_{i=1} \partial^2/\partial x_i^2$ we find for $A_t = A_0 + t(\delta d - A_0)$ that each A_t is an elliptic operator with no lowest order term and thus satisfies the maximal principle (see Bers, John, Schechter [2] p. 150f). In particular each A_t has no non-trivial solution falling to zero at infinity and so it is an injection on $\mathcal{M}^p_{s+1,\alpha}$. Thus condition (ii) holds and δd is an isomorphism. Q.E.D.

DEFINITION 3.2: *Let $\mu = \sqrt{g}\, dx^1 \ldots dx^n$ be the volume form on M. Then $\mathcal{F}^p_{s,\delta} = \{f \in \mathcal{D}^p_{s,\delta} \colon f^*(\mu) = \mu\}$.*

THEOREM 3.3: *For* $p > n/(n-2)$, $s > (n/p) + 1$ *and* $1 \leq \alpha < -1 + n(1 - 1/p)$, $\mathscr{F}^p_{s,\alpha}$ *is a smooth submanifold of* $\mathscr{D}^p_{s,\alpha}$ *and if* $\eta \in \mathscr{F}^p_{s,\delta}$, $W \in T_\eta \mathscr{F}^p_{s,\delta}$ *iff* $W \in T_\eta \mathscr{M}^p_{s,\delta}(I)$ *and* $\operatorname{div}(W \circ \eta^{-1}) = 0$.

This theorem is an immediate consequence of Theorem 3.1 and the fact the inverse image of a point under a submersion with splitting kernel is a submanifold. See Cantor [4] or Ebin–Marsden [8] where Theorem 3.3 is shown for flat and compact M respectively.

Recall that the equations for the velocity field of a perfect fluid (*i.e.* inviscid, incompressible, and homogeneous) are given by

$$(E) \qquad \begin{cases} \dfrac{\partial U_t}{\partial t} + \nabla_{U_t} U_t = -\operatorname{grad} p_t + f_t \\ \operatorname{div} U_t = 0 \\ U_0 \textit{ is given} \end{cases}$$

where p_t is the pressure and f_t is the external force.

For $f_t = 0$, it is shown by Ebin–Marsden [8], that these equations may be transformed to a spray on the appropriate space of volume preserving diffeomorphisms. Thus the existence and uniqueness of solutions to (E) follows from showing the spray is smooth. Since the transformation (given below) is only continuous (Prop. 2.10), one only gets continuous dependence on initial data. Using these techniques we may conclude.

THEOREM 3.4: *Let* $n > 2$, $p > n/(n-2)$, $s > n/p + 1$ *and* $1 \leq \alpha < -1 + n(1 - 1/p)$. *Then for any divergence free* $U_0 \in \mathscr{M}^p_{s,\alpha}(TM)$ *there exists a unique short-time solution of E,* $U_t \in \mathscr{M}^p_{s,\alpha}(TM)$ *starting at* U_0. *Furthermore the solution depends continuously on* U_0.

PROOF: (Outline)
Define $P: T\mathscr{D}^p_{s,\alpha}|_{\mathscr{F}^p_{s,\alpha}} \to T\mathscr{F}^p_{s,\alpha}$ by

$$P(V_\eta) = V_\eta - (\operatorname{grad} \Delta^{-1} \operatorname{div}(V_\eta \circ \eta^{-1})) \circ \eta$$

for $\eta \in \mathscr{F}^p_{s,\delta}$. The grad, Δ, and div operators are those induced on vector fields by the metric g, and are the duals of d and δ acting on one forms.

LEMMA 3.4.1: *P is a smooth bundle map.*

This is proved using Theorem 3.3 exactly as in Cantor [4] or Ebin–Marsden [8].

LEMMA 3.4.2: *If we define the spray* S: $T\mathcal{F}^p_{s,\alpha} \to T^2\mathcal{F}^p_{s,\alpha}$ *by* $S(V) = TP(Z \circ v)$, *where* Z *is the geodesic spray on* M, *then if* V_{Z_t} *is an integral curve of* S, $V_{Z_t} \circ Z_t^{-1}$ *is a solution to* E.

For a proof see Ebin–Marsden [8]. Q.E.D.

For physical problems over unbounded regions one would expect external forces which are gradients of potentials, (*e.g.* gravity). Assuming these are smooth, we need only be concerned that the force f has the appropriate asymptotic behavior.

DEFINITION 3.5: *For* $p \geq 1$, *and* $\delta \in \mathbb{R}$ *set* $\mathcal{M}^p_{\infty,\delta} = \bigcap_{s>n/p} \mathcal{M}^p_{s,\delta}$.

PROPOSITION 3.6: *If* $p > 1$ *and* $\delta \geq 0$, *then for any tensor bundle* V *over* M, $C_0^\infty(V) \subset \mathcal{M}^p_{\infty,\delta} \subset \dot{C}^\infty(V)$ *where* $C_0^\infty(V)$ *is the set of* C^∞ *tensors with compact support.*

PROOF: The first inclusion is obvious and the second inclusion is immediate from Proposition 2.4. Q.E.D.

PROPOSITION 3.7: *If* $p > 1$, $\delta \geq 0$ *and* $s > (n/p) + 1$, *then if* $f \in \mathcal{M}^p_{\infty,\delta}(TM)$ *the map* $\omega_f : \eta \to f \circ \eta$ *is a* C^∞ *vector field on* $\mathcal{D}^p_{s,\delta}$.

PROOF: Note $\pi \circ (f \circ \eta) = (\pi \circ f) \circ \eta = \eta$ and so $f \circ \eta \in T_\eta \mathcal{D}^p_{s,\delta}$. Smoothness follows from Proposition 2.1.1. Q.E.D.

By converting (E) to Lagrangian coordinates ($\ell_t = U_t \circ \eta_t$ with η_t the flow of U_t) we get:

$$(L) \qquad \begin{cases} \dfrac{d}{dt}\eta_t = \ell_t \\[2ex] \dfrac{d}{dt}\ell_t = -\operatorname{grad} p_t \circ \eta_t + f \circ \eta_t \\[2ex] \operatorname{div}(\ell_t \circ \eta_t^{-1}) = 0 \end{cases}$$

It follows that the spray associated with the force is the vertical lift to $T^2\mathcal{F}^p_{s,\delta}$ of the map $F : T\mathcal{F}^p_{s,\delta} \to T\mathcal{F}^p_{s,\delta}$, where $F(V_\eta) = V_\eta + f \circ \eta$. (See Ebin–Marsden [8]) This is clearly smooth and thus we have the following theorem.

THEOREM 3.8: *Let p, s, and δ be as in Theorem* 3.4 *and $f \in \mathcal{M}^p_{\infty,\delta}$. Then the spray given by $S(V) = TP(Z \circ V) + F^\ell(V)$ has unique short-time solutions (η_t, ℓ_t) to (L) and $U_t = \ell_t \circ \eta_t^{-1}$ solves E with force f (F^ℓ is the lift of F).*

REMARK: It is straightforward to extend Theorem 3.8 to consider time-dependent forces by requiring $t \to f_t$ be smooth in $\mathcal{M}^p_{\infty,\delta}$.

4. Riemannian structures on $\mathcal{D}^p_{s,\delta}$

In order to apply Theorem 1.3 to second order operators when $n \le 4$, we must assume $p > 2$. Then the natural class of spaces when using these methods to study physical problems are not Hilbert spaces. However, in many cases one may establish the existence of an appropriate weak inner product structure.

PROPOSITION 4.1:

(1) *If $n > 2$, $n/(n-2) < p \le 2$, $s > n/p + 1$ and $\delta \ge 0$, then $\mathcal{M}^p_{s,\delta}(\mathbb{R}^n, \mathbb{R}^m) \subset L^2(\mathbb{R}^n, \mathbb{R}^m)$ continuously*

(2) *If $n > 2$, $p > \max(2, n/(n-2))$, $s \ge 0$, and $n(p-1)/(2p) \le \delta$, then $\mathcal{M}^p_{s,\delta} \subset W^{2,s}$ continuously*

(3) *For $\delta \ge 0$ if $p < 2n/(n-2\delta)$ then*

$$\mathcal{M}^p_{s,\delta} \subset W^{2,s}.$$

(1) and (2) are found in Cantor [4], and (3) is found in Choquet-Bruhat and Marsden [6].

DEFINITION 4.2: *Let $M = (\mathbb{R}^n, g)$ be an asymptotically simple manifold, then if V and W are two tensor fields covering the map η, set*

$$(V, W)_\eta = \int_{\mathbb{R}^n} \bar{g}(\eta(x))(V(x), W(x))d\mu(\eta(x))$$

where $\bar{g}$ is the inner product induced on the tensor bundle by g and $d\mu$ is the volume form induced by g.

For $s \in N$, set

$$(V, W)_{\eta,s} = \sum_{|\alpha| \le s} (\nabla^\alpha V, \nabla^\alpha W)_\eta$$

Also set

$$N^s(\eta)(V) = (V, V)^{1/2}_{\eta,s}.$$

DEFINITION 4.3: *$T_\eta L^2_s$ is the completion of the C^∞_0 vector fields covering η with respect to $N^s(\eta)$.*

PROPOSITION 4.4: *Let M be an asymptotically simple manifold.*

(1) *If n, p, s, δ are as in part* (1) *or* (3) *of* 4.1 *then if $\eta \in \mathcal{M}^p_{s,\delta}(I)$, $T_\eta \mathcal{M}^p_{s,\delta}(I) \subset T_\eta L^2$ continuously*

(2) *If n, p, s, δ are as in part* (2) *of* 4.1 *then if $\eta \in \mathcal{M}^p_{s,\delta}(I)$, $T_\eta \mathcal{M}^p_{s,\delta}(I) \subset T_\eta W^{2,s}(I)$ continuously.*

PROOF: This is an immediate consequence of Propositions 4.1, 2.4.

Recall that a *weak Riemannian structure G* for an infinite dimensional Banach manifold W is a map assigning to each point $x \in W$ a bounded positive definite symmetric bilinear map on $T_x W$ such that the metric topology generated by G is weaker than the given topology on W.

THEOREM 4.5: *Let n, p, s, and δ be as in either part of Proposition* 4.4 *then the map $\eta \to (,)_\eta$ is a weak Riemannian structure on $\mathcal{D}^p_{s,\delta}$.*

PROOF: It is immediate from Proposition 4.4 that $(,)_\eta$ is bounded on $T_\eta \mathcal{D}^p_{s,\delta} \times T_\eta \mathcal{D}^p_{s,\delta}$ and that the L^2 topology is weaker than the $\mathcal{M}^p_{s,\delta}$ topology. Q.E.D.

REMARK: Since this structure involves composition, it is not in general smooth but only continuous. However, it is smooth when restricted to $\mathcal{F}^p_{s,\delta}$. This follows from the change of variables formula (see Ebin–Marsden [8]).

REMARK: It is possible to choose p, s, and δ so that the solutions to (E) established in Theorem 3.4 are globally square integrable. Thus these solutions are physical in the sense they have bounded energy.

The space of asymptotically flat metrics on $\mathbb{R}^n$

In this section we shall study the local structure of the orbit space of the action of the group $\mathcal{D}^p_{s,\delta}$ on the space of asymptotically flat metrics on $\mathbb{R}^n$. Included in this section is a proof of the canonical decomposition of Berger–Ebin [1] for symmetric 2-tensors (Lemma

5.3.3). Also it is shown there are no non-trivial Killing fields asymptotic to zero on an asymptotically simple manifold.

DEFINITION 5.1: *Let* $e = \delta_{ij}$ *be the Euclidean metric on* $\mathbb{R}^n$. *Let* $\mathcal{S}^p_{s,\delta}$ *be the set of symmetric 2-tensors in* $\mathcal{M}^p_{s,\delta}$ *and* $\mathcal{R}^p_{s,\delta} = \{$*symmetric 2-tensors* g *on* $\mathbb{R}^n : g - e \in \mathcal{S}^p_{s,\delta}$ *and* g *is positive definite*$\}$.

$\mathcal{R}^p_{s,\delta}$ is the set of Riemannian metrics which are asymptotic to the Euclidean metric, $\mathcal{R}^p_{s,\delta}$ has a natural Banach space structure for $s > n/p$:

PROPOSITION 5.2: *For* $p > 1$, $s > n/p$ *and* $\delta \in \mathbb{R}$, $\mathcal{R}^p_{s,\delta}$ *is an open cone in the affine Banach space* $\mathcal{S}^p_{s,\delta} + \{e\}$.

REMARK: Note that the sets of metrics g in $\mathcal{R}^p_{s,\delta}$ making $(\mathbb{R}^n, g)$ asymptotically simple is dense. It follows that on this dense subset at least that the set of diffeomorphisms asymptotic to the identity on $(\mathbb{R}^n, g)$ for $g \in \mathcal{R}^p_{s,\delta}$ does not depend on g (see Proposition 2.4). Thus throughout this section we shall denote by $\mathcal{D}^p_{s,\delta}$ the set of diffeomorphisms $\mathcal{M}^p_{s,\delta}$ near the identity with respect to e.

PROPOSITION 5.3: *Let* $p > 1$, $s > n/p + 1$, *and* $\delta \geq 0$. *Then* $\mathcal{D}^p_{s,\delta}$ *has a continuous action on* $\mathcal{R}^p_{s-1,\delta+1}$ *given by* $A : (\eta, g) \to \eta^*(g)$. *Also* A *is a* C^∞ *function of* g *and if* $g \in \mathcal{R}^p_{s-1+k,\delta}$ *then* $\eta \mapsto \eta^*(g)$ *is a* C^k *function on* $\mathcal{D}^p_{s,\delta}$. *(*$\eta^*(g)(p)$ *is an inner product on* $T_p\mathbb{R}^n$ *given by* $\eta^*(g)(p)(v, w) = g(\eta(p))(T\eta(p)v, T\eta(p)w)$.

PROOF: Setting $g = e + h$ with $h \in \mathcal{M}^p_{s-1,\delta+1}$ we have $\eta^*(g) = \eta^*(e) + \eta^*(h)$. In standard coordinates $\eta^*(e)_{ij} = (\partial \eta^k / \partial x_i)(\partial \eta^\ell / \partial x_j)\delta_{k\ell}$. Also $(\partial \eta^r / \partial x_s) = \delta^r_s + B^r_s$ where $B^r_s \in \mathcal{M}^p_{s-1,\delta+1}$. Thus $\eta^*(e)_{ij} - \delta_{ij} = 2B_{ij} + B^m_i B_{jm}$. Using the ring property for $\mathcal{M}^p_{s-1,\delta+1}$ functions (see Cantor [5]) and the fact $\eta \mapsto D\eta$ is smooth from $\mathcal{M}^p_{s,\delta}$ into $\mathcal{M}^p_{s-1,\delta+1}$ it follows that $\eta \mapsto \eta^*(e)$ is smooth.

Now $\eta^*(h)_{ij} = (\partial \eta^k / \partial x_i)(\partial \eta^\ell / \partial x_j)(h \circ \eta)_{k\ell}$. If $h \in M^p_{s-1+k,\delta+1}$ then the map $(h, \eta) \mapsto h \circ g$ as a function from $\mathcal{M}^p_{s-1,\delta+1} \times \mathcal{D}^p_{s,\delta}$ into $\mathcal{M}^p_{s-1,\delta+1}$ is C^∞ in h and C^k in η (see Cantor [5]). Q.E.D.

We now fix $p > n/(n-2)$, $s > n/p + 2$ and $0 \leq \delta < -2 + n(p-1)/p$. We consider the isotropy groups, orbits, and quotient spaces determined by the action.

DEFINITION 5.4: $\mathcal{R}^p_{\infty,\delta} = \bigcap_{s > n/p} \mathcal{R}^p_{s,\delta}$.

THEOREM 5.5: *Letting $\mathscr{D}^p_{s,\delta}$ act on $\mathscr{R}^p_{s-1,\delta+1}$, we have*

(0) *If $g \in \mathscr{R}^p_{\infty,\delta+1}$ the action of $\mathscr{D}^p_{s,\delta}$ on $\mathscr{R}^p_{s-1,\delta+1}$ is jointly C^∞ at (η, g) for any $\eta \in \mathscr{D}^p_{s,\delta}$.*

(1) *For any $g \in \mathscr{R}^p_{s-1,\delta+1}$ the only isometry of g in $\mathscr{D}^p_{s,\delta}$ is I.*

(2) *If $g \in \mathscr{R}^p_{s-1,\delta+1}$, $O_g = \{\eta^*(g) \colon \eta \in \mathscr{D}^p_{s,\delta}\}$ is a submanifold.*

(3) *If $g \in \mathscr{R}^p_{\infty,\delta+1}$, there is a submanifold W of $\mathscr{R}^p_{s-1,\delta+1}$ containing g and a neighborhood U of I in $\mathscr{D}^p_{s,\delta}$ such that the action is a homeomorphism of $U \times W$ onto a neighborhood of g in $\mathscr{R}^p_{s-1,\delta+1}$ and $O_g \cap W = \{g\}$.*

PROOF: We shall first prove a series of lemmas. Denote by B_R the set $\{x \in \mathbb{R}^n : |x| \leq R\}$. Recall s, p, and δ are as specified above definition 5.4.

LEMMA 5.5.1: *Let A be a uniformly elliptic operator with coefficients satisfying the conditions of theorem* 1.3 *or* 1.4.

Assume A has the unique continuation property. Then there is a constant C depending only the modulus of ellipticity of A, p, s, and the size of the coefficients of $A - A_\infty$ such that for all $R \geq 1$ and $u \in W^{p,s}_0(B_R)$ (= completion of $C^\infty_0(B_R)$ with respect to $|\ |_{p,s}$)

$$|u|_{p,s,\delta} \leq C|Au|_{p,s-m,\delta+m}$$

PROOF. We consider the case $s = m$. The general case follows easily. Let $R \geq 1$ and $u \in C^\infty_0(B_R)$. Now,

$$\text{(1)} \qquad \begin{aligned} |u|_{p,m,\delta} &= \sum_{|\alpha|\leq m} |D^\alpha u \sigma^{\delta+|\alpha|}|_{L^p(B_R)} \\ &= \sum_{|\alpha|\leq m} \left(\int_{B_R} |D^\alpha u(y) \cdot \sigma^{\delta+|\alpha|}(y)|^p dv\right)^{1/p} \\ &= \sum_{|\alpha|\leq m} R^{n/p}\left(\int_{B_1} |D^\alpha u(Rx) \cdot \sigma^{\delta+|\alpha|}(Rx)|^p dx\right)^{1/p} \\ &\leq R^{n/p} \sum_{|\alpha|\leq m} \sigma^{\delta+|\alpha|}(R)\left(\int_{B_1} |D^\alpha(u)(Rx)|^p dv\right)^{1/p} \end{aligned}$$

Let $g_R \colon B_1 \to B_R$ be given by $g_R(x) = Rx$. Then

$$\text{(2)} \qquad \begin{aligned} |u \circ g_R|_{W^{p,m}(B_1)} &= \sum_{|\alpha|\leq m} |D^\alpha(u \circ g_R)|_{L^p(B_1)} \\ &= \sum_{|\alpha|\leq m} R^{|\alpha|}|D^\alpha u(Rx)|_{L^p(B_1)} \end{aligned}$$

It follows from the assumptions and standard elliptic theory there is a constant C_1 such that for each R and $u \in C_0^\infty(B_1)$

$$|u|_{W^{p,m}(B_1)} \le C_1 |A \circ g_R(u)|_{L^p(B_1)}.$$

In particular

$$\begin{aligned}|u \circ g_R|_{W^{p,m}} &\le C_1 |A \circ g_R(u \circ g_{\bar{R}})|_{L^p} \\ &\le C_1 \Big(\int_{B_1} \Big| \sum_{|\beta| \le m} a_\beta(Rx) D^\beta(u \circ g_R) \Big|^p dv \Big)^{1/p} \\ &\le C_1 \Big(\int_{B_1} \Big| \sum_{|\beta| \le m} R^{|\beta|} a_\beta(Rx) D^\beta(u(Rx)) \Big|^p dv \Big)^{1/p}\end{aligned}$$

Now for all x and $R \ge 1$, $R^{|\beta|}/\sigma(\bar{R}x)^{|\beta|} \le C_2$ and so we get

$$(3) \qquad |u \circ g_R|_{W^{p,m}} \le C_1 C_2 \Big(\int_{B_1} \Big| \sum_{|\beta| \le m} a_\beta(Rx) D^\beta u(Rx) \sigma^{|\beta|}(Rx) \Big|^p \Big)^{1/p}$$

Note for $R > 1$, $\sigma^{|\alpha|}(R)/R^{|\alpha|} \le C_3$ for $|\alpha| \le m$. Thus using (1) and (2)

$$\begin{aligned}|u|_{p,m,\delta} &\le R^{n/p} \sum_{|\alpha| \le m} \frac{\sigma^{\delta + |\alpha|}(R)}{R^{|\alpha|}} R^{|\alpha|} |D^\alpha u(Rx)|_{L^p(B_1)} \\ &\le R^{n/p} \sigma^\delta(R) C_3 \sum_{|\alpha| \le m} R^{|\alpha|} |D^\alpha u(Rx)|_{L^p(B_1)} \\ &\le R^{n/p} \sigma^\delta(R) C^3 |u \circ g_R|_{W^{p,m}(B_1)}.\end{aligned}$$

Applying inequality (3) we get

$$\begin{aligned}|u|_{p,m,\delta} &\le R^{n/p} \sigma^\delta(R) C_1 C_2 C_3 \Big(\int_{B_1} \Big| \sum_{|\beta| \le m} a_\beta(Rx) D^\beta u(Rx) \sigma^{|\beta|}(Rx) \Big|^p dv \Big)^{1/p} \\ &\le R^{n/p} \sigma^\delta(R) C_1 C_2 C_3 \Big(\int_{B_1} \Big| \sum_{|\beta| \le m} a_\beta(Rx) D^\beta u(Rx) \frac{\sigma^{|\beta| + \delta}(Rx)}{\sigma^\delta(Rx)} \Big|^p dv \Big)^{1/p}\end{aligned}$$

Now

$$\frac{1}{\sigma^\delta(Rx)} = \frac{1}{(1 + |Rx|^2)^{\delta/2}} \le \frac{1}{\sigma^\delta(R)}$$

and so

$$|u|_{p,m,\delta} \le R^{n/p} C_1 C_2 C_3 \Big(\int_{B_1} \Big| \sum_{|\beta| \le m} a_\beta(Rx) D^\beta u(\overline{Rx}) \sigma^{|\beta| + \delta}(Rx) \Big|^p dv \Big)^{1/p}$$

$$\leq R^{n/p} C_1 C_2 C_3 \Big(\int_{B_1} \Big| \sum_{|\beta| \leq m} a_\beta(Rx) D^\beta u(\overline{Rx}) \cdot \sigma^{m+\delta}(Rx) \Big|^p dv \Big)^{1/p}$$

$$\leq C_1 C_2 C_3 \Big(\int_{B_{\bar{R}}} \Big| \sum_{|\beta| \leq m} a_\beta(y) D^\beta u(y) \cdot \sigma^{m+\delta}(y) \Big|^p dv \Big)^{1/p}$$

$$\leq C_1 C_2 C_3 |Au|_{p,0,\delta}$$

Now if $u \in W_0^{p,s}$ there is a sequence $u_i \in C_0^\infty(B_R)$ such that $|u_i - u|_{p,s,\delta} \to 0$ and $|Au - A(u_i)|_{p,s-m,\delta+m} \to 0$. Then for any $\epsilon > 0$ if i is sufficiently large we have

$$\begin{aligned} |u|_{p,s,\delta} &\leq |u_i|_{p,s,\delta} + |u - u_i|_{p,s,\delta} \\ &\leq C|Au_i|_{p,s-m,\delta+m} + |u - u_i|_{p,s,\delta} \\ &\leq C|Au|_{p,s-m,\delta+m} + |Au - Au_i|_{p,s-m,\delta+m} + |u - u_i|_{p,s,\delta} \\ &\leq C|Au|_{p,s-m,\delta+m} + 2\epsilon. \end{aligned}$$

Thus $|u|_{p,s,\delta} \leq C|Au|_{p,s-m,\delta+m}$. Q.E.D.

LEMMA 5.5.2: *Let A_∞ be as in Theorem 1.3. For $R > 1$, let $A_R(x) = A_\infty + \sum_{|\alpha| \leq m} b_{R,\alpha}(x) \mathscr{D}^\alpha$ where $b_{R,\alpha} \in W_0^{p,s-m+|\alpha|}(B_R)$. Also assume each A_R satisfies the hypotheses of Lemma 5.5.1 and that the moduli of infinity of the A_R are uniformly bounded away from zero. Then there is a C independent of R such that for all $u \in \mathscr{M}_{s,\delta}^p$.*

$$|u|_{p,s,\delta} \leq C|A_R u|_{p,s-m,\delta+m}.$$

PROOF: Let $\{\phi_R : R \geq 1\}$ be a family of functions $\phi_R : \mathbb{R}^n \to \mathbb{R}$ satisfying:

(1) $\phi_R \in C_0^\infty(B_{2R})$

(2) There is a constant D such that $|D^\alpha \phi_R(x)| \leq D$ for all $x \in \mathbb{R}^n$, $R > 1$, $|\alpha| \leq m$.

(3) $\phi_R(x) = 1$, $|x| \leq R$.

For $u \in \mathscr{M}_{s,\delta}^p$, we have $u = \phi_R u + (1 - \phi_R)u$. By Lemma 5.5.1 there is a constant C such that for every $R \geq 1$,

$$|\phi_R u|_{p,s,\delta} \leq C|A_R \phi_R u|_{p,s-m,\delta+m} \tag{1}$$

since $\phi_R u \in W_0^{p,s}(B_{2R})$.

Also $A_R((1 - \phi_R)u) = A_\infty(1 - \phi_R)u)$ and it follows from Theorem 1.4 that

$$\begin{aligned} |(1 - \phi_R)u|_{p,s,\delta} &\leq C_1|A_\infty(1 - \phi_R)u|_{p,s-m,\delta+m} \\ &\leq C_2|A_R(1 - \phi_R)u|_{p,s-m,\delta+m} \end{aligned}$$

Thus using the uniform bounds on the derivatives of ϕ_R.

$$\begin{aligned}|u|_{p,s,\delta} &\le |\phi_R u|_{p,s,\delta} + |(1-\phi_R)u|_{p,s,\delta}\\ &\le C_3(|A_R\phi_R u|_{p,s-m,\delta+m} + |A_R(1-\phi_R)u|_{p,s-m,\delta+m}\\ &\le C_4|A_R u|_{p,s-m,\delta+m}\end{aligned}$$

Q.E.D.

We now prove the canonical decomposition for symmetric 2-tensors over R^n. Recall that the Killing form K_g for a metric g carries vector fields into symmetric 2-tensors by the formula $k_g(X) = L_X g$ where L_X denotes Lie differentiation with respect to X.

LEMMA 5.5.3: *Let p, s, and δ be as stated above. Then if k_g and* div_g *are the Killing and divergence operators associated to $g \in \mathcal{R}^p_{s-1,\delta+1}$,*

$$\mathcal{S}^p_{s-1,\delta+1} = k_g(\mathcal{M}^p_{s,\delta}(\mathbb{R}^n, \mathbb{R}^n)) \oplus J_g$$

where $J_g = \{X \in \mathcal{S}^p_{s-1,\delta+1}\colon \operatorname{div}_g X = 0\}$.

PROOF: We apply Lemma 3.3.1 using the operators k_g and div_g. We need only that $\operatorname{div}_g\circ k_g.\ \mathcal{M}^p_{s,\delta}\to\mathcal{M}^p_{s-2,\delta+2}$ is an isomorphism. Following Berger–Ebin [1] we note $(1/2)\operatorname{div}_g$ is the formal adjoint of k_g and k_g has injective symbol. Thus $\operatorname{div}_g\circ k_g$ is elliptic. Since $s > n/p + 2$ it is not hard to show the coordinate expression for $\operatorname{div}_g\circ k_g$ satisfies the hypotheses of Theorem 1.4 where $A_\infty = \operatorname{div}_e\circ k_e$.

For $R \ge 1$ let $\{\phi_R\}$ be the functions described in the proof of lemma 5.5.2. Let $g_R = e + \phi_R(g-e)$ and $A_R = \operatorname{div}_{g_R}\circ k_{g_R}$. It is not hard to check that Lemma 5.5.2 applies to operators A_R and that $A_R\to\operatorname{div}_g\circ k_g$ in the operator norm as $R\to\infty$. Thus we find for each $X \in \mathcal{M}^p_{s,\delta}$

$$\begin{aligned}|X|_{p,s,\delta} &\le C_1|A_R X|_{p,s-2,\delta+2}\\ &\le C_1(|\operatorname{div}_g\circ k_g)X|_{p,s-2,\delta+2}\\ &\quad + \|A_R - \operatorname{div}_g\circ k_g\|\,|X|_{p,s,\delta}).\end{aligned}$$

Picking R sufficiently large so that $C_1\|A_R - \operatorname{div}_g\circ k_g\| < 1$ we find there is a $C > 0$ such that for all $X \in \mathcal{M}^p_{s,\delta}$

$$|X|_{p,s,\delta} \le C|(\operatorname{div}_g\circ k_g)X|_{p,s-2,\delta+2}.$$

It follows that $\operatorname{div}_g\circ k_g$ is an injection. In fact, if $g_t = e + t(g-e)$ we find that for each t, $\operatorname{div}_{g_t}\circ k_{g_t}$ is an injection, and that $t\to\operatorname{div}_{g_t}\circ k_{g_t}$ is a smooth map into the operators between $\mathcal{M}^p_{s,\delta}$ and $\mathcal{M}^p_{s-2,\delta+2}$. Since $\mathrm{div}_e \circ k_e$ is an isomorphism, Theorem 1.4 implies that $\mathrm{div}_g \circ k_g$ is an isomorphism. Q.E.D.

PROOF OF THE THEOREM.

PROOF OF (0): This follows immediately from Proposition 5.3.

PROOF OF (1): First note the set of infinitesimal isometries (the kernel of k_g) is trivial. This result is seen by noting $\ker(k_g) \subset \ker(\mathrm{div}_g \circ k_g)$ and that $\ker(\mathrm{div}_g \circ k_g) = \{0\}$ (see the proof of Lemma 5.5.3). Thus, the isometries of g are isolated.

We now claim that the isometry group of g has no nontrivial compact subgroup. If there were such a subgroup it follows from above it would have finite order and thus there would be an isometry $\psi \in \mathcal{D}^p_{s,\delta}(\psi \neq I)$ such that $\psi^k = I$ for some k. Also the convexity radius of $g(x)$ becomes infinite as $x \in \mathbb{R}^n$ goes to infinity, and the displacement function of ψ goes to zero. Thus for $|m|$ sufficiently large, the convexity radius at m is at least $4k$ times the displacement of ψ at m. Thus $m, \psi(m), \ldots, \psi^{k-1}(m)$ lie in some convex ball and therefore their center of mass (see [12] for definition) is well defined and unique, and hence a fixed point of ψ. It follows there is an entire neighborhood of fixed points, and any C^1 isometry which fixes a neighborhood is the identity. Thus the subgroup is trivial.

It follows that all of the orbits of any isometry $\psi \in \mathcal{D}^p_{s,\delta}$ must be unbounded. Otherwise ψ would generate a compact subgroup (see Lemmas 3 and 5 pp. 47–48 of [13]). However along an orbit the displacement function of an isometry remains constant and thus for ψ the displacement must vanish and $\psi = I$.

PROOF OF (2): We show the map $A_g : \mathcal{D}^p_{s,\delta} \to \mathcal{R}^p_{s-1,\delta+1}$ given by $A_g(\eta) = \eta^*(g)$ is an immersion and if the isometry group of g is trivial A_g is an embedding. If $\eta_1^*(g) = \eta_2^*(g)$ then $\eta_1 \circ \eta_2^{-1}$ is an isometry of g. Thus from part (1) A_g is injective. Also from Proposition 5.3 we have A_g is a C^∞ map. For $\eta_0 \in \mathcal{D}^p_{s,\delta}$ and η near η_0, we have $\eta \circ \eta_0^{-1}$ is near I and $\eta^*(g) = (\eta \circ \eta_0^{-1})^* \circ \eta_0^*(g)$. Thus it suffices to assume $\eta_0 = I$ and to show the image of $T_I A_g$ splits (see Lang [14]). Let $X \in T_I \mathcal{D}^p_{s,\delta}$ and η_t the flow of X. In the appendix of this paper it is shown $\eta_t \in \mathcal{D}^p_{s,\delta}$ and $T_I A_g(X) = (d/dt)\eta_t^*(g)|_{t=0} = k_g(X)$.

Thus $im(T_I A_g) = k_g(\mathcal{M}^p_{s,\delta}(TR^n))$ which by Lemma 5.5.3 is a closed direct summand of $\mathcal{S}^p_{s-1,\delta+1} = T_g \mathcal{R}^p_{s-1,\delta+1}$.

Proof of (3): It is clear that there is an open neighborhood B of 0 in J_g such that if $j \in B$ then $g + j \in \mathscr{R}^p_{s-1,\delta+1}$. Set $W = g + B$. W is an embedded submanifold of $\mathscr{R}^p_{s-1,\delta+1}$. Now setting $F: \mathscr{D}^p_{s,\delta} \times B \to \mathscr{R}^p_{s-1,\delta+1}$ to be $F(\eta, j) = \eta^*(j)$ we see F is differentiable at (I, g) and

$$T_{(I,g)}F: \mathscr{M}^p_{s,\delta} \times J_g \to \mathscr{S}^p_{s-1,\delta+1}$$

has the form

$$T_{(I,g)} = \begin{pmatrix} k_g & 0 \\ 0 & I \end{pmatrix}.$$

This is clearly an isomorphism by Lemma 5.5.3. Thus we may apply the inverse function theorem to obtain the result. Q.E.D.

Remark 1: One may construct slices for the action of $\mathscr{D}^p_{\infty,\delta}$ on $\mathscr{R}^p_{\infty,\delta+1}$ by taking intersections over s on $\mathscr{D}^p_{s,\delta}$ and $\mathscr{R}^p_{s-1,\delta+1}$ and taking the inverse limit topology. In this way we obtain Frechet manifolds. Then using the above theorem we obtain a slice W^s for each $s > n/p + 2$. If $W = \cap W^s$ it is immediate that W satisfies the statement of part (3) of the theorem. Parts (1) and (2) also hold.

Remark 2: By choosing $-1 + n(p-1)/p \leq \delta < -2 + n(p-1)/p$ we have that Theorem 5.5 holds for $\mathscr{R}^p_{s-1,\delta+1}$ and $\mathscr{M}^p_{s-1,\delta+1} \subset W^{2,s-1}$ (Proposition 4.1). Thus we may in this case use the construction found in Ebin [7], establish a weak Riemannian structure on $\mathscr{R}^p_{s-1,\delta+1}$ and define W using the geodesic spray associated to this structure. Also the summands in Lemma 5.5.3 are L^2 orthogonal.

6. The space of conformally equivalent metrics on $\mathbb{R}^n$

Throughout this section we shall adopt the definitions of Section 5.

Definition 6.1: *If g is a Riemannian metric on $\mathbb{R}^n$, denote the canonical volume form induced by g as $\mu_g = |g|^{1/2} dx^1 \wedge \cdots \wedge dx^n$ where $|g| = \det(g_{ij})$. The term $|g|^{1/2}$ is the volume density of g.*

Definition 6.2: *Set $\mathscr{V}^p_{s,\delta} = \{|g|^{1/2}: g \in \mathscr{R}^p_{s,\delta}\}$. We call $\mathscr{V}^p_{s,\delta}$ the set of volume densities of $\mathscr{R}^p_{s,\delta}$. (see Definition 5.1).*

Note that $\mathscr{V}^p_{s,\delta} \subset \{f: \mathbb{R}^n \to \mathbb{R}: f - 1 \in \mathscr{M}^p_{s,\delta}\}$, *i.e.* members of $\mathscr{V}^p_{s,\delta}$ are asymptotic to the constant 1 function.

LEMMA 6.3: *If $h \in \mathcal{R}^p_{s,\delta}$ for $s > n/p$, $\delta \geq 0$ then $|h|^{-1/n} \in \mathcal{V}^p_{s,\delta}$.*

PROOF: We need to construct a member of $\mathcal{R}^p_{s,\delta}$ whose density is $|h|^{-1/n}$. Let $\alpha_{ij} = |h|^{-2/n^2}\delta_{ij}$. Then $|\alpha|^{1/2} = |h|^{-1/n}$ and we need show that $\alpha_{ij} - \delta_{ij} = (|h|^{-2/n^2} - 1)\delta_{ij}$ belongs to $\mathcal{M}^p_{s,\delta}$. We may write $|h| = 1 + j$ with $j \in M^p_{s,\delta}$. Since h is continuous by Proposition 2.4, we need only check integrability of $(|h|^{-2/n^2} - 1)$ on the complement of a compact set. (It is clear that $|h|$ is bounded away from zero on any compact set). We pick $K \subset \mathbb{R}^n$ to be a compact set such that if $x \notin K$ then $|j(x)| < 1/2$. Then setting $\beta = 2/n^2 < 1$ we have for $x \notin K$,

$$|h(x)|^{-\beta} = (1 + j(x))^{-\beta} = 1 - \beta j(x) + \frac{\beta(\beta+1)}{2} j^2(x) \ldots$$

and

$$|h(x)|^{-\beta} - 1 = -\beta j(x) + \frac{\beta(\beta+1)}{2} j^2(x) + \cdots.$$

The above power series has radius of convergence equal to one and so the convergence is C^∞ uniform on $\mathbb{R}^n - K$ and thus by the triangle inequality it suffices to note each term belongs to $\mathcal{M}^p_{s,\delta}$. This follows from the ring property for $\mathcal{M}^p_{s,\delta}$ functions (see Cantor [5]). Q.E.D.

DEFINITION 6.4: *Define for $s > n/p$, $\delta \geq 0$, $\Phi : \mathcal{R}^p_{s,\delta} \to \mathcal{V}^p_{s,\delta} \otimes \mathcal{R}^p_{s,\delta}$ by $\Phi(h) = |h|^{-1/n} \otimes h$. Recall two metrics g and g' are said to be conformally related if $g' = fg$ where f is a real valued function.*

PROPOSITION 6.5: (1) *h and $\bar{h}$ in $\mathcal{R}^p_{s,\delta}$ are conformally related iff $\Phi(h) = \Phi(\bar{h})$.*

(2) *Φ maps into $\mathcal{V}^p_{s,\delta} \otimes \mathcal{R}^p_{s,\delta}$.*

PROOF: (1) Suppose $\bar{h} = fh$. Then $|\bar{h}| = f^n|h|$ and $|\bar{h}|^{-1/n} = f^{-1}|h|^{-1/n}$, and

$$\Phi(\bar{h}) = |\bar{h}|^{-1/n} \otimes \bar{h} = f^{-1}|h|^{-1/n} \otimes fh = \Phi(h).$$

If $\Phi(\bar{h}) = \Phi(h)$ then we have

$$1 \otimes |\bar{h}|^{-1/n}\bar{h} = 1 \otimes |h|^{-1/n}h \text{ and so } \bar{h} = \left(\frac{|h|}{|\bar{h}|}\right)^{-1/n} h.$$

(2) This follows from Lemma 6.3. Q.E.D.

Thus Φ may be thought of as mapping h to the conformal equivalent class of h. We have the following definition:

DEFINITION 6.6: *We call the image $\Phi(\mathscr{R}^p_{s,\delta})$ the set of asymptotically flat conformal structures on $\mathbb{R}^n$. Denote* $\mathscr{C}^p_{s,\delta} = \Phi(\mathscr{R}^p_{s,\delta})$ and $\mathscr{C}^p_{\infty,\delta} = \bigcap_{s>n/p} \mathscr{C}^p_{s,\delta}$.

DEFINITION 6.7: *Define the action of $\mathscr{D}^p_{s+1,\delta-1}$ on $\mathscr{C}^p_{s,\delta}$ by $A(\eta, \gamma) = \Phi(\eta^*(h))$ where $\Phi(h) = \gamma$.*

It is easy to see the action is well defined for if h and $\bar{h}$ are conformally equivalent, then also $\eta^*(h)$ and $\eta^*(\bar{h})$ are conformally related. Also by Proposition 5.2, it is clear that A maps into $\mathscr{C}^p_{s,\delta}$.

As in Section 5, we are interested in studying the orbits and transverse slices of the action given in Definition 5.6. If η_t is a smooth one-parameter family of diffeomorphisms in $\mathscr{D}^p_{s+1,\delta-1}$ with $\eta_0 = I$ and $(d/dt)\eta_t|_{t=0} = X \in M^p_{s+1,\delta-1}(\mathbb{R}^n, \mathbb{R}^n)$ we have $(d/dt)A(\eta_t, \gamma)|_{t=0} = (d/d_t)\Phi(\eta_t^*(h))|_{t=0} = L_x\Phi(h)$.

PROPOSITION 6.8: *For $h \in \mathscr{R}^p_{s,\delta}$ and $X \in T_I\mathscr{D}^p_{s,\delta}$*

$$L_X\Phi(h) = |h|^{-1/n}\left(\frac{-2}{n}\operatorname{div}_h X \otimes h + 1 \otimes k_h(x)\right)$$

where k_h is the Killing form generated by h ($k_h(X) = L_X h$) and div_h *is the divergence generated by h.*

PROOF:

$$\begin{aligned} L_X\Phi(h) &= L_X(|h|^{-1/n} \otimes h) \\ &= L_X(|h|^{-1/n}) \otimes h + |h|^{-1/n} \otimes L_X h. \end{aligned}$$

It is a standard fact of Riemmanian geometry that $L_X(|h|^{1/2}) = \operatorname{div}_h X|h|^{1/2}$ and thus

$$\begin{aligned} L_X\Phi(h) &= L_X(|h|^{1/2})^{-2/n} \otimes h + |h|^{-1/n} \otimes k_n(X) \\ &= -\frac{2}{n}(|h|^{1/2})^{-2/n-1}(\operatorname{div}_h X)|h|^{1/2} \otimes h + |h|^{-1/n} \otimes k_h(X) \\ &= -\frac{2}{n}|h|^{-1/n} \operatorname{div}_h X \otimes h + |h|^{-1/n} \otimes k_h(X). \end{aligned}$$

The result is immediate. Q.E.D.

COROLLARY 6.8.1:

$$L_X\Phi(h) = 0 \quad \textit{iff} \quad L_h(X) = |h|^{-1/n}\Big(k_h(X) - \frac{2}{n}\operatorname{div}_h X \cdot h\Big) = 0.$$

PROOF: We may write

$$L_X\Phi(h) = |h|^{-1/n}\Big(1 \otimes -\frac{2}{n}\operatorname{div}_h X \cdot h + 1 \otimes k_h(X)\Big) = 1 \otimes k_h(X).$$

Q.E.D.

PROPOSITION 6.9: $\operatorname{tr}_h L_h(X) = 0$ *where* k_h *is as in* 6.8.1 *and* $\operatorname{tr}_h(k) = \Sigma_{ab}\, h^{ab}k_{ab}$.

PROOF: $\operatorname{tr}_h L_h(X) = |h|^{-1/n}(\operatorname{tr}_h k_h(X) - (2/n)\operatorname{div} X \operatorname{tr}_h h)$. But $\operatorname{tr}_h(h) = n$ and $\operatorname{tr} k_h(X) = h^{ij}(X_{i|j} + X_{j|i}) = 2\operatorname{div}_h X$. Thus

$$\operatorname{tr}_h L_h(X) = |h|^{-1/n}\Big(2\operatorname{div}_h X - \frac{2}{n}\operatorname{div}_h X \cdot n\Big) = 0. \qquad \text{(Q.E.D}$$

We may identify an element of $\mathscr{V}^p_{s,\delta} \otimes M^p_{s,\delta}(T^2_0\mathbb{R}^n)$ with $M^p_{s,\delta}(T^2_0\mathbb{R}^n)$ via the bilinear map:

$$\psi(v \otimes h) = v \cdot h.$$

The fundamental properties of ψ and ϕ are summarized in the following proposition.

PROPOSITION 6.10:
(1) $(\psi \circ \Phi)^2(h) = (\psi \circ \Phi)(h)$.
(2) $L_h X = \psi(L_X\Phi(h))$.
(3) *Let* $s > n/p$ *and* $\delta \geq 0$ *then set* $\mathscr{T}^p_{s,\delta}(h) = \{g \in \mathscr{S}^p_{s,\delta} : \operatorname{tr}_h(g) = 0\}$. *We have* $T^p_{s,\delta}(h)$ *is a closed subspace and* $L_h : \mathscr{S}^p_{s,\delta} \to \mathscr{T}^p_{s,\delta}(h)$ *is a bounded linear map* (*see Definition* 5.1).

PROOF: (1) and (2) are clear. The fact the image of L_h lies in the trace-free tensors is Proposition 6.9. Thus we only need to show continuity. This follows from the formula for L_h given in Proposition 6.8 and this simple lemma:

LEMMA: *If* $\mathscr{M}^p_{s,\delta}(c) = \{f : \mathbb{R}^n \to R : f - c \in \mathscr{M}^p_{s,\delta}$ *for* $c \in \mathbb{R}\}$ *then if* $s > n/p$ *and* $\delta \geq 0$ *pointwise multiplication induces a jointly continuous map* $\mathscr{M}^p_{s,\delta}(c) \times \mathscr{M}^p_{s,\delta} \to \mathscr{M}^p_{s,\delta}$. Q.E.D.

Note that Φ does not map onto $\mathcal{M}^p_{s,\delta} \otimes \mathcal{R}^p_{s,\delta}$. In fact, if $v \otimes h \in \operatorname{Im} \Phi$ then $|h|^{-1/n} = v$. However, we shall show the image of Φ is a submanifold.

PROPOSITION 6.11: *$\mathscr{C}^p_{s,\delta}$ is a smooth submanifold of $\mathscr{V}^p_{s,\delta} \otimes \mathcal{R}^p_{s,\delta}$, for $s > n/p$, $\delta \geq 0$. Also for $1 \otimes h \in \mathscr{C}^p_{s,\delta}$, $T_{1\otimes h}\mathscr{C}^p_{s,\delta} \simeq \mathscr{T}^p_{s,\delta}(h)$.*

PROOF: $\mathscr{V}^p_{s,\delta} \otimes \mathcal{R}^p_{s,\delta}$ as a cone in a vector space is isomorphic to $\mathcal{R}^p_{s,\delta}$ and thus its tangent space at any point may be identified with $\mathscr{S}^p_{s,\delta}$ by mapping $1 \otimes \gamma \to \gamma$.

LEMMA 6.12: *For $g \in \mathcal{R}^p_{s,\delta}$, $S^p_{s,\delta} = (1/n)\operatorname{tr}(\mathscr{S}^p_{s,\delta}) \otimes g \oplus \mathscr{T}^p_{s,\delta}(g)$.*

PROOF: For $h \in \mathcal{R}^p_{s,\delta}$ then $h = (1/n)\operatorname{tr}(h)g + (h - 1/n)\operatorname{tr}(h)g)$ and also if $h = (1/n)(\operatorname{tr} \bar{h})g$ and $\operatorname{tr}(h) = 0$, then $\operatorname{tr}(h) = \operatorname{tr}(\bar{h}) = 0$ and $h = 0$; thus the subspaces have trivial intersection. The fact the two subspaces are closed follows immediately from Proposition 6.10. Q.E.D.

We define a map $j: \mathscr{V}^p_{s,\delta} \otimes \mathcal{R}^p_{s,\delta} \to \{f : \mathbb{R}^n \to \mathbb{R} \mid f - 1 \in \mathcal{M}^p_{s,\delta}\}$ by $j(v \otimes h) = |vh|$ where v is treated as a scalar. Then it is clear that $\mathscr{C}^p_{s,\delta} = j^{-1}(1)$, j is smooth and the tangent map at $1 \otimes h \in \mathscr{C}^p_{s,\delta}$ considered as a map on $\mathscr{S}^p_{s,\delta}$ is

$$TJ_{1\otimes h}(b) = \operatorname{trace}(h^{-1}b) = \operatorname{tr}_h b.$$

Thus $TJ_{1\otimes h}$ is a surjection with splitting kernel. Thus using standard methods (see Theorem 3.3) the proposition follows. Q.E.D.

REMARK: Note that the image of L_h which is the tangent space to the orbit of the action at $1 \otimes h$ is contained in the tangent space of $\mathscr{C}^p_{s,\delta}$ at $1 \otimes h$. Thus we may prove a slice theorem in $\mathscr{C}^p_{s,\delta}$.

We now state a slice theorem for the space of conformal structures on $\mathbb{R}^n$.

THEOREM 6.13: *Let p, s, and δ be as in Theorem 5.4. For $\gamma \in \mathscr{C}^p_{\infty,\delta}$ we have:*

(0) *The action of $\mathscr{D}^p_{s+1,\delta-1}$ on $\mathscr{C}^p_{s,\delta}$ is jointly C^∞ at γ.*
(1) *The isotropy group $I_\gamma = \{\eta : A(\eta, \gamma) = \gamma\}$ is trivial.*
(2) *The orbit $0_\gamma = \{A(\gamma, \eta) : \eta \in \mathscr{D}^p_{s+1,\delta-1}\}$ is a smooth submanifold of $\mathscr{C}^p_{s,\delta}$.*
(3) *There is a submanifold W of $\mathscr{C}^p_{s,\delta}$ and a neighborhood U of*

$I \in \mathscr{D}^p_{s+1,\delta-1}$ such that the action $A|_{U\times W}$ restricted to $U \times W$ is a homeomorphism onto a neighborhood of γ, with $0_\gamma \cap W = \{\gamma\}$.

PROOF: (0) This is clear. See Theorem 5.5.

For (1) and (2) we shall prove a lemma:

LEMMA 6.13.1: *For p, s, and δ as in Theorem 5.5, $h \in \mathscr{R}^p_{\infty,\delta}$*

$$\mathscr{T}^p_{s,\delta}(h) = L_h(\mathscr{M}^p_{s+1,\delta-1}(TM)) \oplus (\operatorname{Ker}(\operatorname{div}_h) \cap \mathscr{T}^p_{s,\delta}(h))$$

PROOF: This is proven exactly as Lemma 5.5.1 using the observation of J. York [17] that $\operatorname{div}_h \circ L_h$ is elliptic. Q.E.D.

Since the tangent to the orbits is the image of L_h the proof proceeds as part (1) and (2) of Theorem 5.5.

(3) For $h \in \mathscr{R}^p_{s,\delta}$, $J \in \mathscr{S}^p_{s,\delta}$ there is an $\epsilon > 0$ such that if $|t| < \epsilon$ then $h + tJ \in \mathscr{R}^p_{s,\delta}$. Let $\gamma = 1 \otimes g$ with $g \in \mathscr{R}^p_{s,\delta}$.

$$W = \Phi(\{g + tJ : |t| < \epsilon J \in \mathscr{S}^p_{s,\delta}, \operatorname{div}_g J = \operatorname{tr}_g J = 0\}).$$

Note $\Phi(g) = \gamma$ and $|g| = 1$, therefore, the tangent space of W may be determined by noticing members are of the form

$$\begin{aligned} \frac{d}{dt}(|g + tJ|^{-1/n} \otimes g + tJ) &= -\frac{1}{n}(\operatorname{tr}_g J) \otimes g + 1 \otimes J \\ &= 1 \otimes J \simeq J. \end{aligned}$$

Therefore, the tangent space of W is the complementary direct summand of the tangent space of the orbit as seen in Lemma 6.13.1. The proof now proceeds exactly as in Theorem 5.5. Q.E.D.

REMARK 1: It follows from the above theorem that in a neighborhood of γ distinct points of W lie on distinct orbits.

REMARK 2: Combining Lemmas 6.12.1 and 6.13.1 yields a decomposition of the symmetric 2-tensors into a pure trace part, a 'longitudinal' (or image of k) part, and a transverse-traceless part. This decomposition was first discovered by J. York [19].

Appendix

Flows of $\mathcal{M}^p_{s,\delta}$ vector fields

In this section we show for $p>1$, $s>n/p+1$, and $\delta\geq 0$ that the flow of an $\mathcal{M}^p_{s,\delta}$ vector field is a one parameter subgroup of $\mathcal{D}^p_{s,\delta}$ (the maps on $\mathbb{R}^n$ $\mathcal{M}^p_{s,\delta}$ close to the identity). Furthermore, it is shown the flow depends continuously on the vector field. This proof is based on the proof of a similar theorem found in Ebin–Marsden [8]. However, it is shown here how the required degree of differentiability may be lowered from $s>n/p+2$ to $s>n/p+1$ while retaining the structure of the proof.

Also, the following proof, suitably modified, may be used to show the smoothness of $W^{p,s}$ flows over $\mathbb{R}^n$. In this case, the flow is a subgroup of the diffeomorphisms of $\mathbb{R}^n$ which $W^{p,s}$ close to the identity. This result, using different methods was obtained by Bourguignon and Brezis for bounded regions (see the appendix of [3]).

LEMMA: *Let* $p>1$, $s>n/p$, $\delta\geq 0$ *and* $k\geq 0$. *Then if* $\eta\in\mathcal{D}^p_{s,\delta}\cap C^0(I)$ *and* $W\in\mathcal{M}^p_{s,\delta+k}$, *we have* $W\circ\eta\in\mathcal{M}^p_{s,\delta+k}$.

PROOF: We shall show by induction on $|B|$ that composition is continuous from $\mathcal{M}^p_{|B|,\delta+k+s-|B|}\times(\mathcal{D}^p_{s,\delta}\cap C^0(I)$ into $\mathcal{M}^p_{|B|,\delta+k+s-|B|}$. Set $\ell=s-|B|$.

For $|B|=0$, note the Jacobian satisfies $|J\eta(x)|\geq\epsilon>0$ for all $x\in\mathbb{R}^n$. Thus we have

$$\left(\int\left| W\circ\eta\cdot\sigma^{\delta+k+|\ell|}\right)^{1/p}\leq\sup_x\frac{C}{|J\eta(x)|}\left(\int|W\cdot\sigma^{\delta+k+|\ell|}|dv\right)^{1/p}<\infty$$

Assuming the theorem holds for $|B|=n\leq s$, it is sufficient to show $D(W\circ\eta)\in\mathcal{M}^p_{n,\delta+k+s-n}$ where $W\in\mathcal{M}^p_{n+1,\delta+k+s-(n+1)}$. But $D(W\circ\eta)=(Dw\circ\eta)\cdot D\eta$ and $DW\in\mathcal{M}^p_{n,\delta+k+s-n}$ so $DW\circ\eta\in\mathcal{M}^p_{n,\delta+k+s-n}$ by induction and $D\eta\in\mathcal{M}^p_{s-1,\delta+1}$. Thus it easily follows from the usual product theorems (see Cantor [5]) that $(DW\circ\eta)\cdot D\eta\in\mathcal{M}^p_{s-1,\delta+k+s-n}$. ■

THEOREM: *Let* $s>n/p+1$, $p>1$, $\delta\geq 0$

(i) *If V is an $\mathcal{M}^p_{s,\delta}$ vector field on $\mathbb{R}^n$, its flow η_t is a (local one parameter subgroup of $\mathcal{D}^p_{s,\delta}$.*

(ii) *The map* exp: $T_I\mathcal{D}^p_{s,\delta}\to\mathcal{D}^p_{s,\delta}$ *is continuous (but not C^1).*

PROOF: Since $V \in \mathcal{M}^p_{s,\delta} \subset C^1$, we know V has a unique flow $\eta_t \in C^1(I)$. We first show $\eta_t \in \mathcal{D}^p_{s-1,\delta}$.

Let $\omega_V(f) = V \circ f$. ω_V is a C^1 vector field on $\mathcal{D}^p_{s-1,\delta}$ and so there is a unique integral curve $\bar{\eta}(t)$ of w_V in $\mathcal{D}^p_{s-1,\delta}$ with $\bar{\eta}(0) = id$. But

$$\frac{d\bar{\eta}(t)}{dt} = \omega_V(\bar{\eta}(t)) = V \circ \bar{\eta}(t)$$

Thus, by uniqueness of the flow $\bar{\eta}(t) = \eta_t$. To finish the proof of (i) we show $D\eta_t \in \mathcal{M}^p_{s-1,\delta+1}$. Now $DV \in \mathcal{M}^p_{s-1,\delta+1}$ and $\eta_t \in \mathcal{D}^p_{s-1,\delta} \cap C^1(I)$ and it follows that $DV \circ \eta_t \in \mathcal{M}^p_{s-1,\delta+1}$, and $t \to DV \circ \eta_t$ is continuous into $\mathcal{M}^p_{s-1,\delta+1}$. We also have

$$\frac{d}{dt} D\eta_t = D\left(\frac{d}{dt}\eta_t\right) = D(V \circ \eta_t) = (DV \circ \eta_t) \cdot D\eta_t$$

Letting $B_t = DV \circ \eta_t$ we find that $D\eta_t$ satisfies a linear differential equation with *continuous* time dependence. Thus there is a unique integral curve of this equation in $\mathcal{M}^p_{s-1,\delta+1}$ with $D\eta_0 = e$ ($DI = e$). Thus as before the flow is in $\mathcal{M}^p_{s,\delta}$.

(ii) This follows immediately from the fact that the solution to a differential equation which varies continuously with respect to a parameter depends continuously on the parameter.

Finally, the fact that η_t remains in $\mathcal{D}^p_{s,\delta}$ for as long as the solution exists follows from the formula

$$\eta_t = \overbrace{(\eta_{t/r} \circ \eta_{t/r} \cdots \circ \eta_{t/r})}^{r}$$

and the fact that $\mathcal{D}^p_{s,\delta}$ is a topological group.

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Department of Mathematics
Duke University
Durham, North Carolina 27706