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A PROOF OF NOETHER'S FORMULA FOR THE ARITHMETIC GENUS OF AN ALGEBRAIC SURFACE

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1. The proof

Let X be a smooth, proper surface defined over an algebraically closed field k. Denote by $\chi(\mathcal{O}_X) = \Sigma_{i=0}^{2} (-1)^i \dim_k H^i(X, \mathcal{O}_X)$ its Euler-Poincaré characteristic, by $c_i = c_i(\Omega_X)$ the ith Chern class of its cotangent bundle, and by $\int$ the degree of a zero-dimensional cycle in the Chow ring A.X. The above invariants of X are related by the formula

$$12\chi(\mathcal{O}_X) = \int (c_1^2 + c_2), \tag{1}$$

due to Max Noether [9]. The formula is a special case of Hirzebruch's Riemann–Roch theorem (however, it is not a special case of the original Riemann–Roch theorem for a surface (see [13]), which states that a certain inequality holds).

Here we give a proof of (1) more in the spirit of Noether's original (see §2). First we realize X as the normalization of a surface X_0 in $\mathbf{P}^3$, with ordinary singularities. Then we obtain expressions for $\int c_1^2$, $\int c_2$, and $\chi(\mathcal{O}_X)$ in terms of numerical characters of X_0 and we verify that these expressions satisfy the relation (1).

By realizing X as the normalization of a surface X_0 with ordinary singularities in $\mathbb{P}^3$ we mean the following. Let $X \hookrightarrow \mathbb{P}^N$ be any embedding of X. Replacing it by the embedding determined by hypersurface sections of degree ≥ 2, we may assume that the projection $f: X \to \mathbf{P}^3$ of X from any generically situated linear space of codimension 4 has the following properties [10, p. 206, theorem 3]:

(A) Put $X_0 = f(X)$. The map $f: X \to X_0$ is finite and birational (hence it is equal to the normalization map).

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(B) X_0 has only ordinary singularities: a double curve Γ_0, which has t triple points (these being also triple for the surface) and no other singularities; a finite number of pinch points, these being the images of the points of ramification of f. The completion of the local ring of X_0 at a point y of Γ_0 looks like

(a) $k[\![t_1, t_2, t_3]\!]/(t_1t_2)$ for most points y of Γ_0 and at such points $\#f^{-1}(y) = 2$.

(b) $k[\![t_1, t_2, t_3$ $/(t_1t_2t_3)$ if y is triple, and then $\#f^{-1}(y) = 3$.

(c) $k[\![t_1, t_2, t_3]\!]/(t_2^2 - t_1^2t_3)$ if y is a pinch point and $\operatorname{char} k \neq 2$ (otherwise the ring is $k[\![t_1, t_1t_2, t_2^2 + t_2^3]\!]$) and $\#f^{-1}(t) = 1$.

In order to compute the invariants of X in terms of the numerical characters of X_0, we shall first make some observations concerning the scheme structure of the double curve Γ_0.

We let $\mathscr{C}_0 = \operatorname{Hom}_{\mathcal{O}_{X_0}}(f_*\mathcal{O}_X, \mathcal{O}_{X_0})$ denote the conductor of X in X_0 and put $\mathscr{C} = \mathscr{C}_0\mathcal{O}_X$. It follows that $f_*\mathscr{C} = \mathscr{C}_0$ holds. Moreover, using duality for the finite morphism f [see 13, III, appendix by D. Mumford, p. 71; also 7, V. 7], we obtain a canonical isomorphism

$$\mathscr{C} \cong \Lambda^2\Omega_X \otimes \mathscr{L}^{-n+4},$$

where $\mathscr{L} = f^*\mathcal{O}_{\mathbf{P}^3}(1)$ is the pullback of the tautological line bundle on $\mathbf{P}^3$ and n is the degree of X_0 in $\mathbf{P}^3$. In particular this shows that $\mathscr{C}$ is invertible.

Using (B) we see that the ideal $\mathscr{C}_0$ defines the reduced scheme structure on the double curve, call this scheme Γ_0 also. Now put $\Gamma = f^{-1}(\Gamma_0)$; thus Γ is defined on X by the ideal $\mathscr{C}$. This gives an equality in the Chow ring:

$$c_1 = c_1(\Omega_X) = (n-4)c_1(\mathscr{L}) - [\Gamma]. \tag{2}$$

The equality (2) allows us to compute $\int c_1^2$. First, let us introduce the following numerical characters of X_0, in addition to its degree n,

degree of $\Gamma_0 = m$,
$\#$ triple points of Γ_0 (or of X_0) $= t$,
grade (self-intersection) of Γ on $X = \lambda$,
$\#$ (weighted) pinch points $= \nu_2$.

By definition ν_2 is the degree of the ramification cycle of f on X; this cycle is defined by the 0th Fitting ideal $F^0(\Omega_{X/\mathbf{P}^3})$ of the relative differentials of f. (If $\operatorname{char} k \neq 2$, ν_2 is equal to the actual number of pinch points of X_0; if $\operatorname{char} k = 2$, ν_2 is twice the number of actual pinch points [11, p. 163, prop. 6].) From (2) then we get the expression

$$\int c_1^2 = (n-4)^2 n - 4(n-4)m + \lambda.$$

Here we used $\int c_1(\Gamma)^2 = n$ and $\int c_1(\mathscr{L})[\Gamma] = 2m$, which holds because the map $f|_\Gamma : \Gamma \to \Gamma_0$ has degree 2.

For a surface with ordinary singularities in $\mathbf{P}^3$ there is the *triple point formula*:

$$3t = \lambda - mn + \nu_2,$$

due to Kleiman [7, I, 39]. Substituting the resulting value of λ in the above formula for $\int c_1^2$, we find

$$\int c_1^2 = n(n-4)^2 - (3n-16)m + 3t - \nu_2. \tag{3}$$

Next we want to obtain an expression for $\int c_2$. Since there is an exact sequence

$$f^* \Omega_{\mathbf{P}^3} \to \Omega_X \to \Omega_{X/\mathbf{P}^3} \to 0,$$

Porteous' formula [6, p. 162, corollary 11] gives

$$\nu_2 = \int c_1^2 - c_2 + 4c_1 \cdot c_1(\mathscr{L}) + 6c_1(\mathscr{L})^2.$$

Using (2) and (3) we obtain

$$\int c_2 = n(n^2 - 4n + 6) - (3n-8)m + 3t - 2\nu_2. \tag{4}$$

The last invariant to be considered is $\chi(\mathcal{O}_X)$. We claim that the arithmetic genus $\chi(\mathcal{O}_X) - 1$ satisfies the postulation formula (see §2),

$$\chi(\mathcal{O}_X) - 1 = \binom{n-1}{3} - (n-4)m + 2t + g - 1, \tag{5}$$

where g denotes the (geometric) genus of Γ_0.

To prove (5) we consider the exact sequences $0 \to \mathscr{C} \to \mathcal{O}_X \to \mathcal{O}_\Gamma \to 0$ and $0 \to \mathscr{C}_0 \to \mathcal{O}_{X_0} \to \mathcal{O}_{\Gamma_0} \to 0$. Since f is finite, f_* is exact, and we have seen that $f_*\mathscr{C} = \mathscr{C}_0$ holds. Therefore, by additivity of χ, we obtain

$$\chi(\mathscr{C}_0) = \chi(f_*\mathcal{O}_X) - \chi(f_*\mathcal{O}_\Gamma) = \chi(\mathcal{O}_{X_0}) - \chi(\mathcal{O}_{\Gamma_0}),$$

hence

$$\chi(\mathcal{O}_X) = \chi(\mathcal{O}_{X_0}) + \chi(\mathcal{O}_\Gamma) - \chi(\mathcal{O}_{\Gamma_0}).$$

Moreover, since X_0 is a hypersurface of degree n in $\mathbf{P}^3$, $\chi(\mathcal{O}_{X_0}) = \binom{n-1}{3} + 1$ holds. Since Γ is a curve on a smooth surface, its arithmetic genus is given by the adjunction formula

$$-\chi(\mathcal{O}_\Gamma) = \tfrac{1}{2}\int ([\Gamma] + c_1)\cdot[\Gamma],$$

hence, using (2), we get

$$\chi(\mathcal{O}_\Gamma) = -(n-4)m.$$

Finally, the equality

$$\chi(\mathcal{O}_{\Gamma_0}) = 1 - g - 2t$$

holds because the difference in arithmetic and geometric genus due to a triple point with linearly independent tangents is equal to 2. This is seen as follows. Consider the local ring R of Γ_0 at a triple point, and let $R \to R'$ denote its normalization. By (B) the map on the completions looks like

$$\hat{R} = k[\![t_1, t_2, t_3]\!]/(t_1t_2, t_1t_3, t_2t_3) \to \hat{R}' = k[\![t]\!]^3.$$

The image of $\hat{R}$ in $\hat{R}'$ consists of triples (ψ_1, ψ_2, ψ_3) such that $\psi_i(0) = \psi_j(0)$, the cokernel of $\hat{R} \to \hat{R}'$ is isomorphic to k^2, and the map $\hat{R}' \to k^2$ is given by

$$(\psi_1, \psi_2, \psi_3) \mapsto (\psi_1(0) - \psi_2(0),\ \psi_1(0) - \psi_3(0))$$

(Similar computations show that a triple point with coplanar tangents would diminish the genus by 3.) Thus we have proved (5).

Consider the curve Γ; above each triple point of Γ_0 it has 3 ordinary double points. Hence the difference between its arithmetic and geometric genus is $3t$ (since Γ has no other singularities). We have observed that the map $f|_\Gamma : \Gamma \to \Gamma_0$ has degree 2; since its ramification locus is equal to that of f, the Riemann-Hurwitz formula now gives a formula

$$2m(n-4)-6t = 2(2g-2)+\nu_2.$$

Hence we can substitute for g in (5) and multiply by 12 to get

$$(6) \qquad 12\chi(\mathcal{O}_X) = 2n(n^2-6n+11)-6(n-4)m+6t-3\nu_2.$$

This equality, together with (3) and (4), now yields (1).

2. Historical note

Formula (1) was stated by Noether [9] as

$$(1') \qquad \pi^{(1)} = 12(p+1)-(p^{(1)}-1).$$

He established it by considering a model of the surface in $\mathbf{P}^3$. Previously [8] he had found formulae for the arithmetic genus p and the genus $p^{(1)}$ of a canonical curve in terms of the numerical characters of the model in $\mathbf{P}^3$. Now he showed that the expression he got for the difference $12(p+1)-(p^{(1)}-1)$ was equal to the expression for the invariant $\pi^{(1)}$ given by Zeuthen [14].

Clebsch [5] was the first to look for a class number of the birational class to which a surface belongs. He defined the *genus* of a surface as the number p_g of independent everywhere finite double integrals. He showed that for a model $f(x, y, z) = 0$ of the surface in $\mathbf{P}^3$, of degree n, with only double and cuspidal curves, these integrals are of the form $\int\int \phi/f'_z \, dx \, dy$, where ϕ is a polynomial of degree $n-4$ which vanishes on the singular curves of $f = 0$ (this result is attributed to Clebsch in [13, p. 157] but no reference is given). Noether [9] called the surfaces $\phi = 0$ *adjoints* to $f = 0$. He allowed more general singularities on $f = 0$. He proved that the number p_g of independent adjoints is a birational invariant of the surface (this result was announced by Clebsch in [5]). In [9] Noether developed the theory of adjoints for higher dimensional varieties as well.

Let S be a set of curves and points (with assigned multiplicities) in $\mathbf{P}^3$. Denote by $P(m, S)$ the number of conditions imposed on a surface of degree m by requiring it to pass through S. The number $P(m, S)$ is called the *postulation* of S with respect to surfaces of degree m. Cayley [3] was the first to consider $P(m, S)$ and give a formula for it, under certain restrictions on the set S. The restrictions were relaxed by Noether [8].

The work of Clebsch [5] led Cayley [2] to derive a *postulation formula* for the genus (and again this was generalized by Noether [8]). According to this formula the genus is the postulated number p_a of adjoints to a given model $f = 0$, hence equal to the number $\binom{n-1}{3}$ of all surfaces of degree $n-4$ minus the postulation $P(n-4, S)$, where S denotes the set of singular curves and points of $f = 0$. Zeuthen [14] uses Cayley's formula to show that p_a is a birational invariant.

Both Cayley [4] and Noether [9] found that p_a could be strictly less than the actual number, p_g, of adjoints. The breakthrough in understanding the difference $p_g - p_a$ was made by Enriques in 1896 [see 13, IV].

The next invariant $p^{(1)}$ that occurs in (1′) is what Noether called the *curve genus* of the surface. He defined it, via a model $f = 0$, as the genus of the variable intersection curve of the surface $f = 0$ with a general adjoint $\phi = 0$, *i.e.* of a canonical curve. He showed, by what amounts to applying the adjunction formula, that $p^{(1)} - 1$ is equal to the self-intersection $\int c_1^2$ of a canonical curve.

Zeuthen [14] studied the behaviour of a surface under birational transformation by methods similar to those he had applied to curves. He considered enveloping cones of a model of the surface in $\mathbf{P}^3$ and looked for numbers of such a cone that were independent of the particular vertex and of the particular model. He discovered the invariant $\pi^{(1)}$ (equal to $\int c_2$), and found a formula for it in terms of characters of the model, including the class n' (the class is the number of tangent planes that pass through a given point). Later Segre [12] studied pencils on a surface and found a formula for $\pi^{(1)} - 4$ in terms of characters of the pencil. The invariant $I = \pi^{(1)} - 4$ became known as the Zeuthen–Segre invariant of the surface, see also [1].

To deduce (1′) Noether used his earlier formula [8] for the class n' to eliminate n' in Zeuthen's formula for $\pi^{(1)}$. He showed that the resulting expression for $\pi^{(1)}$ was equal to his expression for $12(p+1) - (p^{(1)} - 1)$.

Added in proof

A proof of Noether's formula similar to the above has been given independently by P. Griffiths and J. Harris in their book "Principles of algebraic geometry" (Wiley Interscience, 1978).

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