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FUNCTIONAL EQUATION SATISFIED BY CERTAIN L-FUNCTIONS

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Introduction

The purpose of this paper is to establish the existence of a functional equation for certain L-functions of degree 4 attached to cusp forms on PGL_2 over a number field.

More precisely, let $°\pi$ be an irreducible (admissible) constituent of the space of cusp forms on PGL_2 over a number field (same definition as for GL_2, cf. [8]). We can write $°\pi$ as a tensor product of the local representations $°\pi_v$ which are class-one for almost all v.

Suppose $°\pi_v$ is a class-one representation. Let ρ be a four dimensional irreducible representation of $SL_2(\mathbf{C})$, the corresponding associated group, and let $\alpha_v^\wedge$ be the semi-simple conjugacy class in $SL_2(\mathbf{C})$ determined by $°\pi_v$ (cf. [10], also see §5). Then in [10], Langlands defines a local L-function attached to ρ, $°\pi_v$ and a complex number s as follows:

$$L(s, \rho, °\pi_v) = \det(I - \rho(\alpha_v^\wedge)q_v^{-s}).$$

Clearly $L(s, \rho, °\pi_v)$ depends only upon the classes of ρ and $°\pi_v$.

Now, let S be the finite set of places for which the corresponding representations are ramified. We put

$$L_S(s, \rho, °\pi) = \prod_{v \notin S} L(s, \rho, °\pi_v).$$

We also define a global coefficient $\gamma_S(s, \rho, °\pi)$ (see §5) which is closely related to the global root number $\epsilon(s, \rho, °\pi)$ introduced in [10]. The

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main result of this paper is the functional equation

$$L_S(1-s, \tilde{\rho}, {}^\circ\pi) = \gamma_S(s, \rho, {}^\circ\pi) L_S(s, \rho, {}^\circ\pi),$$

where $\tilde{\rho}$ denotes the contragradient representation of ρ (Theorem 5.9). This is in fact the functional equation for the L-function attached to ρ and ${}^\circ\pi$ which has been conjectured in the context of an arbitrary reductive group by Langlands [10]. This result together with a result of Langlands [11] concerning the meromorphicity of these L-functions, will prove the first conjecture in [10] for any cusp form on PGL_2, when ρ is as above. We extend this result to any cusp form on GL_2 for which the center acts according to an unramified quasi-character of the corresponding idele class group (Corollary 5.10).

We should mention that the same result will essentially follow from the works of P. Deligne, S. Gelbart, H. Jacquet and J.A. Shalika [3, 9, 16] (see §2.3 of Gelbart's paper 'Automorphic Forms and Artin's Conjecture' which is based on a talk given at the 1976 Bonn Conference on Modular Forms). However the approach is entirely different from the one explained in this paper.

Except for certain technical difficulties (particularly at infinite places), the same methods can be used to establish certain functional equations for the adjoint groups of type A, following the non-vanishing of some of their Fourier coefficients [15].

Interwoven with the proof of the functional equation is a description of the unique non-degenerate quotient of a certain induced representation (cf. [14]) of a simple algebraic group of type G_2 over a p-adic field (see Appendix).

In this paper, we have used the principle of applying the Eisenstein series to L-functions which is due to Langlands [11] (see also: R. Godement, Formes automorphes et produits eulérien; d'après R.P. Langlands; Séminaire Bourbaki, no. 349, 1968). For this reason we shall recall the important facts of the theory of Eisenstein series in §2.

In §3, we shall study certain properties of the Whittaker models for the algebraic groups of type G_2. Needless to say these results are true and can be proved by the same methods for an arbitrary split group over a p-adic field. This is based on certain unpublished results of W. Casselman and J.A. Shalika.

Most of the local computations are carried out in §4. In loose terms, we shall compute 'the local coefficients' which are basic to the definition of $\gamma_S(s, \rho, {}^\circ\pi)$, using a key lemma (lemma 4.4) and certain results of Jacquet [7]. Notice that these local coefficients differ from the Langlands' root numbers by a ratio of local L-functions. Again I should mention that these computations can be done for an arbitrary Chevalley group and it has been only for the sake of simplicity that we have limited ourselves to the case of G_2.

It is in §5 that we prove the functional equation (Theorem 5.9). We observe that the local coefficients are in fact defined by the local multiplicity one theorem of J.A. Shalika [15], together with a result of F. Rodier [14].

The problem of finding the poles of these L-functions still remains open.

I would like to express my gratitude to Professor Robert Langlands for his suggestion of the problem and for many helpful discussions during my stay at the Institute for Advanced Study, where the bulk of this paper was prepared.

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1. Notation and terminology

Let G be a simple algebraic group of type G_2 and let $\mathfrak{g}$ denote its Lie algebra. We assume that G splits over Q. We fix a Cartan subgroup T of G with Lie algebra $\mathfrak{h}$. We use B to denote a fixed Borel subgroup containing T. Put U for its unipotent radical.

Let Ψ denote the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. We use Δ, Ψ^+ and Ψ^- for simple, positive, and negative roots, respectively. Then Δ consists of two elements, α, the short root, and β, the long one. Other positive roots are $\alpha+\beta$, $2\alpha+\beta$, $3\alpha+\beta$, and $3\alpha+2\beta$. We have:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\gamma \in \Psi} \mathfrak{g}_\gamma$$

with root spaces $\mathfrak{g}_\gamma$, $\gamma \in \Psi$.

Let W be the Weyl group of G. W is generated by the reflections σ_γ, $\gamma \in \Phi$. We shall identify each σ_γ with an element of $N_G(T)$ through the isomorphism between W and $N_G(T)/C_G(T)$. We denote this element by w_γ.

Let F be a number field. For every place v of F, we shall write G_v for the group of F_v-rational points of G. We use G_F for the group of F-rational points.

For each v, we fix a maximal compact subgroup K_v of G_v relative to T_v, so that

$$G_v = K_v \cdot B_v.$$

To define K_v for the finite places, we fix a Chevalley lattice M for $\mathfrak{g}$ (cf. [1]). If $\mathbb{A}$ is the ring of adèles of F, we write $G_{\mathbb{A}}$ for the corresponding adèlized group. We use the same index for the corresponding subgroups.

By a character of a group we shall understand a homomorphism from the group into the complex numbers of absolute value one.

We call a character χ of $U_{\mathbb{A}}/U_F$ non-degenerate if its restriction to every non-trivial subgroup $U^w_{\mathbb{A}}$,

$$U^w = wUw^{-1} \cap U,$$

is nontrivial, $w \in W$. Clearly

$$\chi = \prod_{\gamma \in \Delta} \chi_\gamma,$$

where each χ_γ is a non-trivial character of $U^\gamma_{\mathbb{A}}/U^\gamma_F$. Here U^γ denotes the connected subgroup whose Lie algebra is $\mathfrak{g}_\gamma$, the root group for γ. Then χ_γ can be considered as a non-trivial character of $\mathbb{A}/F$ and

$$\chi_\gamma = \prod_v \chi_{\gamma,v}$$

with each $\chi_{\gamma,v}$ a non-trivial character of F_v. Furthermore for almost all v, the largest ideal for which $\chi_{\gamma,v}$ is trivial is the ring of integers O_v of F_v. Therefore we can write

$$\chi = \prod_v \chi_v$$

with

$$\chi_v = \prod_{\gamma \in \Delta} \chi_{\gamma,v}$$

a non-degenerate character of U_v.

Throughout this paper, we shall fix a non-degenerate character χ of U. Later in §5, we shall put certain conditions on χ.

We use $G^\wedge$ to denote the associated complex group for G which is defined in general by Langlands [10]. Then $G^\wedge$ is a complex group of type G_2.

Let $T^\wedge$ be a Cartan subgroup of $G^\wedge$ and let $L^\wedge$ be the root lattice of

$G^\wedge$ with respect to $T^\wedge$. We may identify $L^\wedge$ with the $\mathbb{Z}$-lattice generated by

$$\{H_\gamma \mid \gamma \in \Delta\}$$

in $\mathfrak{h}_{\mathbb{R}}$. Here H_γ is defined by

$$\gamma^\wedge(H) = \kappa(H, H_\gamma) \quad (\forall H \in \mathfrak{h}),$$

where κ denotes the Killing form on $\mathfrak{g}$ and

$$\gamma^\wedge = \frac{2\gamma}{(\gamma, \gamma)}.$$

In fact we identify the simple roots of $G^\wedge$ relative to $T^\wedge$ with H_γ, $\gamma \in \Delta$.

Let $\mathfrak{h}^\wedge$ be the Lie algebra of $T^\wedge$; then $\mathrm{Hom}_{\mathbb{Z}}(L^\wedge, \mathbb{Z})$ can be identified with a $\mathbb{Z}$-lattice in $\mathfrak{h}^\wedge$. Thus we may identify $\mathfrak{h}^\wedge$ with $\mathrm{Hom}_{\mathbb{Z}}(L^\wedge, \mathbb{C})$, which itself is isomorphic to $\mathrm{Hom}_{\mathbb{C}}(L^\wedge \otimes_{\mathbb{Z}} \mathbb{C}, \mathbb{C})$. Since $L^\wedge \otimes_{\mathbb{Z}} \mathbb{C}$ is equal to $\mathfrak{h}_{\mathbb{C}}$, the complexification of $h \otimes_{\mathbb{Q}} \mathbb{R}$, we conclude that $\mathfrak{h}_{\mathbb{C}}$ and $\mathfrak{h}^\wedge$ are dual to each other.

Finally, let G be a split group defined over a local field, and let π be an irreducible admissible representation of G (or corresponding algebra) on a complex vector space V. Fix a Borel subgroup B of G and denote its unipotent radical by U. Let χ be a character of U. By a Whittaker functional on V, we shall mean a continuous linear functional on V satisfying

$$\lambda(\pi(u)v) = \overline{\chi(u)}\lambda(v)$$

for all u in U and v in V. Then from [15] it follows that the space of such linear functionals is at most one-dimensional. If there is such a functional, we shall say that π is non-degenerate. Suppose π is non-degenerate. For each v in V, we define a complex function w_v on G by:

$$w_v(g) = \lambda(\pi(g^{-1})v).$$

The space of all such functions is called the Whittaker model of π. We denote this space by $\mathrm{W}(\pi)$. The elements of $W(\pi)$ will be called the Whittaker functions of π.

2. Eisenstein series and Fourier coefficients

Let P be a maximal parabolic subgroup of G containing B. We put M for a fixed Levi factor of P and we write:

$$P = MN,$$

where N is the unipotent radical of P with Lie algebra $\mathfrak{n}$. We shall identify M with the quotient P/N. We assume that $\mathfrak{n}$ is generated by the root spaces $\mathfrak{g}_{\alpha}$, $\mathfrak{g}_{\alpha+\beta}$, $\mathfrak{g}_{2\alpha+\beta}$, $\mathfrak{g}_{3\alpha+\beta}$, and $\mathfrak{g}_{3\alpha+2\beta}$. We also assume that M contains T. Let A be the center of M. As in [11] we put

$$^{\circ}G = M/A.$$

Then $^{\circ}G$ is the adjoint group of a split Lie algebra $^{\circ}\mathfrak{g}$ of type A_1. More precisely $^{\circ}G$ is isomorphic to PSL_2.

Let $^{\circ}T$ be the image of T in $^{\circ}G$. Put

$$^{\circ}U = U \cap M.$$

We may consider $^{\circ}U$ as unipotent radical of a Borel subgroup of $^{\circ}G$. Then the characters of $^{\circ}U$ are the restriction of those of U.

As usual we use the index v for the F_v-rational points of each of the groups mentioned here. We put

$$^{\circ}K_v = \overline{P_v \cup K_v},$$

where $\overline{P_v \cap K_v}$ denotes the image of $P_v \cap K_v$ under the natural projection,

$$^{\circ}K = \prod_v {^{\circ}K_V},$$

and

$$K = \prod_v K_v.$$

From now on, we shall identify $^{\circ}G$ with the group PSL_2.

Let φ^* be a cusp form on $GL_2(\mathbb{A})$ as defined in [8]. More precisely φ^* is a continuous function on $GL_2(\mathbb{A})/GL_2(F)$ which under the action of the global Hecke algebra generates an irreducible (admissible) constituent of $A_0(\omega)$, the subspace of the cusp forms for which the center of $GL_2(\mathbb{A})$ acts according to the quasi-character of ω of $\mathbb{A}^*/F^*$. We shall assume that ω is unramified.

φ^* is assumed to be slowly increasing at infinite places (cf. [8]).

Put

$$\varphi(g) = \varphi^*(g)\omega^{-1/2}(\det g) \quad g \in GL_2(\mathbb{A}).$$

Then φ is a cusp form on $PGL_2(\mathbb{A})$ and by restriction on $PSL_2(\mathbb{A})$. In our notation, φ is a function in $\mathbb{A}_0(\mathbb{1})$, where $\mathbb{1}$ denotes the trivial character of the group of the ideles of $\mathbb{A}$. In fact, every cusp form on $PGL_2(\mathbb{A})$ is a constituent of $A_0(1)$.

For every place v, we shall identify $^\circ K_v$ with the maximal compact subgroup of $PSL_2(F_v)$ induced from the standard maximal compact subgroup of $GL_2(F_v)$. We also identify $^\circ U$ with the unipotent radical of the standard parabolic subgroup of GL_2.

The restriction of the character χ of $U_{\mathbb{A}}/U_F$ to $^\circ U_{\mathbb{A}}/{}^\circ U_F$ is a non-degenerate character of $^\circ U_{\mathbb{A}}/{}^\circ U_{\mathrm{F}}$ which we still denote by χ. We consider

$$^\circ w(g) = \int_{^\circ U_{\mathbb{A}}/^\circ U_F} \varphi(gu)\overline{\chi(u)}du \tag{2.1}$$

with g in $^\circ G_{\mathbb{A}}$. In §5, we shall use $^\circ\chi$ to denote the restriction of χ to $^\circ U_{\mathbb{A}}/{}^\circ U_{\mathrm{F}}$.

We shall assume that

$$^\circ w = \bigotimes_v {}^\circ w_v,$$

where for every v, $^\circ w_v$ denotes a local Whittaker function which for almost all v is a class-one function, i.e.

$$^\circ w_v(k_v) = 1 \quad k_v \in {}^\circ K_v.$$

Then for $g = (g_v)$ in $^\circ G_{\mathbb{A}}$, we have

$$^\circ w(g) = \prod_v {}^\circ w_v(g_v),$$

where almost all the factors are equal to 1. The function $^\circ w$ is in fact defined for g in $\mathrm{GL}_2(\mathbb{A})$.

Let σ_v be the finite dimensional representation of $^\circ K_v$ on the span of $^\circ w_v$ by the elements of $^\circ K_v$.

The representation σ_v is in fact the restriction of the same representation when $^{\circ}w_v$ is considered as a function on $GL_2(F_v)$. σ_v can be extended to a representation of $P_v \cap K_v$ trivial on A_v and N_v.

For every place v, we shall fix a finite dimensional representation of K_v whose restriction to $P_v \cap K_v$ contains σ_v. When σ_v is the (one dimensional) trivial representation of $^{\circ}K_v$, we take this representation also to be trivial. We denote this representation by $\tilde{\sigma}_v$. We put

$$\sigma = \bigotimes_v \sigma_v$$

and

$$\tilde{\sigma} = \bigotimes_v \tilde{\sigma}_v$$

which are representations of $^{\circ}K$ and K, respectively.

Let P_σ denote the projection onto the space of σ. Clearly

$$P_\sigma = \bigotimes_v P_{\sigma_v}.$$

We define the following well-defined operator valued function (projection)

$$\tilde{\varphi}(m) = \int_{^{\circ}K} \varphi(k\bar{m})\tilde{\sigma}(k^{-1})dk \cdot p_\sigma \tag{2.2}$$

for m in $M_{\mathbb{A}}$. Here dk denotes the Haar measure on $^{\circ}K$ which is a product of local measures dk_v for which

$$\int_{^{\circ}K_v} dk_v = 1.$$

In the terminology of [5], this is a $\tilde{\sigma}$-function on $M_{\mathbb{A}}$. It can be extended to a $\tilde{\sigma}$-function on $G_{\mathbb{A}}$ as follows:

$$\tilde{\varphi}(g) = \tilde{\sigma}(k)\tilde{\varphi}(m),$$

where

$$g = kmn$$

with k in K, m in $M_{\mathbb{A}}$, and n in $N_{\mathbb{A}}$.

Let $\delta_{P,v}$ denote the modulus character of M with respect to $\mathfrak{n}$. More precisely, at each place v

$$\delta_{P,v}(mn) = |\det Ad_{\mathfrak{n}}(m)|_v.$$

We define the global δ_P by

$$\delta_P(m) = \prod_v \delta_{P,v}(m_v)$$

for $m = (m_v)$ in $M_{\mathbb{A}}$.

For a complex number s, we define

$$\Phi_s(g) = \delta_P^{s-1/2}(b)\tilde{\varphi}(g) \tag{2.3}$$

for

$$g = kb$$

with k in K and b in $B_{\mathbb{A}}$. Then Φ_s is a $\mathfrak{Z}$-finite function (cf. [5]), where $\mathfrak{Z}$ is the center of the universal enveloping algebra of

$$G_\infty = \prod_{v=\infty} G_v.$$

Now we define the *Eisenstein series* attached to φ as follows:

$$E(s;\tilde{\varphi};g;P) = \sum_{\gamma\in G_F/P_F} \Phi_s(g\gamma). \tag{2.4}$$

Here g is in $G_{\mathbb{A}}$.

The series converges absolutely for $\mathrm{Re}(s) < -\frac{1}{2}$ and defines a function which is holomorphic in s (whenever it converges absolutely). As a function on $G_{\mathbb{A}}$, it is a $\tilde{\sigma}$-function. Its restriction to G_∞ is smooth and $\mathfrak{Z}$-finite (cf. [5] and [12]). Clearly (3.4) is a right G_F-invariant function.

As a function of s, $E(s;\tilde{\varphi};g;P)$ can be continued to a meromorphic function on the whole complex plane.

We use W_M to denote $N_M(T)/C_M(T)$. Then W_M can be considered as a subgroup of W. It is equal to the Weyl group for ${}^\circ G$. We shall assume that each w_γ (see §1) has been chosen to lie in $G_Z \cap K_\infty$, where

$$K_\infty = \prod_{v=\infty} K_v.$$

We use w_0 to denote $w_{2\alpha+\beta}$. Modulo W_M, w_0 is the longest element in W.

For $\mathrm{Re}(s) < -\frac{1}{2}$, the integral

$$(2.5) \qquad \int_{N_{\mathbb{A}}} \Phi_s(gnw_0)dn$$

converges absolutely and can be continued to a meromorphic function (in s) on the whole complex plane (cf. [5]).

Now, we define the function $M(s)\tilde{\varphi}$ on $M_{\mathbb{A}}$ by

$$(2.6) \qquad \delta_P^{-s-1/2}(m)(M(s)\tilde{\varphi})(m) = \int_{N_{\mathbb{A}}} \Phi_s(mnw_0)dn$$

for m in $M_{\mathbb{A}}$. The function $M(s)\tilde{\varphi}$ is in fact a cuspidal function on ${}^\circ G_{\mathbb{A}}$. It is a $\tilde{\sigma}$-function and ${}^\circ\mathfrak{Z}$-finite (${}^\circ\mathfrak{Z}$ denotes the center of the universal enveloping algebra of ${}^\circ G_\infty$).

The functional equation for (2.4) can be written as follows (cf. [5] and [12]):

$$(2.7) \qquad E(-s; M(s)\tilde{\varphi}; g; P) = E(s; \tilde{\varphi}; g; P),$$

where g is in $G_{\mathbb{A}}$.

REMARK: If we denote the function defined by the integral in (2.5) by $\tilde{\Psi}_s$, then the left hand side of (2.7) is equal to

$$(2.8) \qquad \sum_{\gamma \in G_F/P_F} \tilde{\Psi}_s(g\gamma).$$

We need the following notations.

For g in $G_{\mathbb{A}}$ (resp. G_v), we write (Iwasawa decomposition)

$$g = k(g)b(g)$$

with $k(g)$ in K (resp. K_v) and $b(g)$ in $B_{\mathbb{A}}$ (resp. B_v). Also for b in $B_{\mathbb{A}}$ (resp. B_v), we write

$$b = t(b)u(b)$$

with $t(b)$ in $T_{\mathbb{A}}$ (resp. T_v) and $u(b)$ in $U_{\mathbb{A}}$ (resp. U_v). Hence we can write

$$g = k(g)t(g)u(g),$$

where

$$t(g) = t(b(g)),$$

and

$$u(g) = u(b(g)).$$

We also write

$$\begin{aligned} g &= k(g)p(g) \\ &= k(g)m(g)n(g) \end{aligned}$$

with $k(g)$ in K (resp. K_v), $p(g)$ in $P_{\mathbb{A}}$ (resp. P_v), $m(g)$ in $M_{\mathbb{A}}$ (resp. M_v), and $n(g)$ in $N_{\mathbb{A}}$ (resp. N_v).

The convergent integral

$$\int_{U_{\mathbb{A}}/U_F} E(s;\tilde{\varphi};gu;P)\overline{\chi(u)}\,du \tag{2.9}$$

is called a *Fourier coefficient* of $E(s;\tilde{\varphi};g;P)$. We denote (2.9) by $\underline{E}_\chi(s;\tilde{\varphi};g;P)$. Then

$$\underline{E}_\chi(s;\tilde{\varphi};gu;P) = \chi(u)\underline{E}_\chi(s;\tilde{\varphi};g;P)$$

for u in $U_{\mathbb{A}}$.

LEMMA 2.1: *For* $\mathrm{Re}(s) < -\frac{1}{2}$, $\underline{E}_\chi(s;\tilde{\varphi};g;P)$ *is equal to*

$$\int_{u\in {}^\circ U_{\mathbb{A}}/{}^\circ U_F}\int_{u'\in N_{\mathbb{A}}} \Phi_s(gu'w_0u)\overline{\chi(uu')}\,du'du. \tag{2.10}$$

PROOF: For $\mathrm{Re}(s) < -\frac{1}{2}$, we have

$$\int_{U_{\mathbb{A}}/U_F} E(s;\varphi;gu;P)\overline{\chi(u)}\,du = \int_{u\in U_{\mathbb{A}}/U_F}\sum_{\gamma\in G_F/P_F} \Phi_s(gu\gamma)\overline{\chi(u)}\,du. \tag{2.10.1}$$

Then (2.10.1) is equal to

$$\begin{aligned} &\int_{U_{\mathbb{A}}/U_F}\sum_{\gamma\in U_F\backslash G_F/P_F}\sum_{\delta\in U_F/\gamma P_F\gamma^{-1}\cap U_F} \Phi_s(gu\delta\gamma)\overline{\chi(u)}\,du \\ &= \sum_{\gamma\in U_F\backslash G_F/P_F}\int_{U_{\mathbb{A}}/\gamma P_F\gamma^{-1}\cap U_F} \Phi_s(gu\gamma)\overline{\chi(u)}\,du. \end{aligned} \tag{2.10.2}$$

Using the Bruhat decomposition in (2.10.2) we have

$$\sum_{w\in W/W_M} \int_{U_A/wP_Fw^{-1}\cap U_F} \Phi_s(guw)\overline{\chi(u)}\,du. \tag{2.10.3}$$

In fact, w has been chosen to lie in $G_Z \cap K_\infty$. We put

$$V = wPw^{-1} \cap U$$

and

$$V_w = V \cap wNw^{-1}.$$

Then

$$V_w = U \cap wNw^{-1}.$$

Also let N^V be the quotient of U by V.

A single term in (2.10.3) is equal to

$$\int_{u'\in N_A^V} \int_{u\in V_A/V_F} \Phi_s(gu'uw)\overline{\chi(u)\chi(u')}\,du\,du'$$

for some w. The integral over V_A/V_F factors through the integral

$$\int_{(V_w)_A/(V_w)_F} \overline{x(u)}\,du$$

which vanishes if V_w is non-trivial. Hence there is only one non-zero term which corresponds to the class of w_0 modulo W_M. This completes the proof of the lemma.

3. Whittaker models

The purpose of this section is to study the Whittaker models of certain classes of induced representations. As we shall see later, these are crucial in the definition of the corresponding local root numbers.

Let ${}^\circ\pi$ be the irreducible (admissible) representation of the global Hecke algebra of GL_2 on the space generated by φ. Then ${}^\circ\pi$ can be written as

$${}^\circ\pi = \bigotimes_v {}^\circ\pi_v,$$

where the tensor product is defined as in [8]. Let ${}^\circ V_v$ be the space of ${}^\circ\pi_v$. For almost all v, the representation ${}^\circ\pi_v$ is a class one representation. For every v, let $W({}^\circ\pi_v)$ be the Whittaker model of ${}^\circ\pi_v$.

The same is true for ${}^\circ\pi^*$, the representation generated by φ^*, in particular,

$$^\circ\pi^* = \bigotimes_v {}^\circ\pi_v^*.$$

In this section we shall assume that v is finite.

$^\circ\pi_v$ can be considered as a representation of P_v (trivial on A_v and N_v). We shall study the Whittaker model of the following induced representation

$$\Pi_v = \operatorname*{ind}_{P_v \uparrow G_v} {}^\circ\pi_v \otimes \delta^s_{P,v}.$$

The space V_v of this representation consists of all the left K_v-finite functions f_v from G_v into the space $W(^\circ\pi_v)$ which satisfy

$$f_v(gp) = {}^\circ\pi_v(p^{-1})\delta_{P,v}^{s-1/2}(p)f_v(g) \quad (p \in P_v).$$

The representation Π_v is given by left inverse translations.

We use $(f_v(g), m)$ to denote the value of the Whittaker function $f_v(g)$ at a point m in M_v.

We need the following lemma.

LEMMA 3.1: *Let N be a unipotent group over a non-archimedean field. Then there exists an increasing sequence $\{N_i\}$ of open compact subgroups of N which exhausts N.*

PROOF: N can be imbedded in the subgroup of $n \times n$ upper triangular matrices (some $n > 0$) for which the lemma holds.

Let $\{N_{v,i}\}$ be a filtration of N_v as in Lemma 3.1. The following proposition is essentially due to Casselman–Shalika [2], and I am indebted to H. Jacquet for mentioning it to me.

PROPOSITION 3.2: *Given f_v in V_v, there exists an integer $i(f_v)$ so that the integral*

$$\int_{N_{v,i}} (f_v(nw_0), e)\overline{\chi_v(n)}\,dn \tag{3.1}$$

has a value $\lambda(f_v)$ independent of i if $i \geq i(f_v)$.

PROOF: Let $V_\chi(U_v)$ denote the subspace generated by all the functions in V_v of the form

$$\Pi_v(u)f_v - \overline{\chi_v(u)}f_v$$

with u in U_v and f_v in V_v. Then every f_v in V_v can be written as

$$f_v = f_{1,v} + f_{2,v},$$

where $f_{1,v}$ is in $V_\chi(U_v)$ and $f_{2,v}$ has support in $N_v w_0 P_v$ (cf. [2], see also §2 in [9]). We may assume that

$$f_{1,v} = \Pi_v(u_0) f_{0,v} - \overline{\chi_v(u_0)} f_{0,v},$$

where u_0 is in U_v and $f_{0,v}$ is in V_v. Then for the large values of i, the integral vanishes for $f_{1,v}$ and converges for $f_{2,v}$ since $f_{2,v}$ has compact support modulo P. This completes the proof of the proposition.

COROLLARY 3.3: *The linear functional λ defined by*

$$\lambda(f_v) = \int_{N_v} (f_v(nw_0), e)\overline{\chi_v(n)}\, dn \tag{3.2}$$

(in the sense of Proposition 3.2) is a Whittaker functional for the space of Π_v.

COROLLARY 3.4: *As a function of s, the integral (3.2) is entire.*

COROLLARY 3.5: *There exists a function f_v in V_v for which $\lambda(f_v)$ is non-zero.*

Let $({}^\circ\tilde{\pi}_v, {}^\circ\tilde{V}_v)$ denote the contragradient representation of $({}^\circ\pi_v, {}^\circ V_v)$. We use

$$\langle\langle\ ,\ \rangle\rangle$$

to denote the pairing on ${}^\circ V_v \times {}^\circ\tilde{V}_v$. Let $\tilde{v}$ be an arbitrary vector in ${}^\circ\tilde{V}_v$. Then:

PROPOSITION 3.6: *The integral*

$$F_v(s) = \int_{N_v} \langle\langle f_v(gnw_0), \tilde{v}\rangle\rangle dn \quad (f_v \in V_v)$$

converges absolutely for $\mathrm{Re}(s)$ *small enough and it can be extended to a meromorphic function on the whole complex plane. Furthermore there exists a polynomial $P(q_v^s)$ (q_v denotes the number of the elements* *in the residual field) so that as a function of s*

$$P(q_v^s)\,F_v(s)$$

is entire.

PROOF: In the case in hand, this can be proved using the same technique as in [4]. In general this is a result of Harish–Chandra [6].

Now, for f_v in V_v we shall define

$$f'_v(g) = \int_{N_v} f_v(gnw_0)dn. \tag{3.3}$$

We put

$$°\pi'_v(m) = °\pi_v(w_0^{-1}mw_0);$$

then f'_v belongs to the space V'_v of the representation

$$\Pi'_v = \operatorname*{ind}_{P_v \uparrow G_v} °\pi'_v \otimes \delta_{P,v}^{-s} \quad (\forall' s \in \mathbb{C}).$$

Let λ' denote the primed analogue of the Whittaker functional defined by (3.2). More precisely,

$$\lambda'(f'_v) = \int_{N_v} (f'_v(nw_0), e)\overline{\chi_v(n)}\,dn \tag{3.4}$$

with f'_v in V'_v.

Now, let f'_v be a function defined by (3.3). Then (3.4) defines another Whittaker functional (which we still denote by λ') on V_v by

$$\lambda'(f_v) = \lambda'(f'_v) \tag{3.5}$$

Later in §5, we shall show that λ is in fact a non-zero Whittaker functional on Π_v (Theorem 5.5). Therefore from [14] and [15] it follows that λ and λ' are proportional and the computations of the next section will show that the coefficient of proportionality is directly related to the Langlands' root number attached to a four dimensional representation of SL_2.

More precisely, let ρ be a four dimensional representation of SL_2, the associated group for ${}^\circ G$. Assume at each place v the local L-function $L(s, \rho, {}^\circ\pi_v)$ is defined. As we shall see later these coefficients of proportionality will lead to certain local coefficients which we denote by $\gamma(s, \rho, {}^\circ\pi_v, {}^\circ\chi_v)$ (see §5). Then the Langlands' root numbers are defined by

$$\epsilon(s, \rho, {}^\circ\pi_v, {}^\circ\chi_v) = \gamma(s, \rho, {}^\circ\pi_v, {}^\circ\chi_v) \cdot \frac{L(1-s, \tilde{\rho}, {}^\circ\pi_v)}{L(s, \rho, {}^\circ\pi_v)},$$

where $\tilde{\rho}$ denotes the contragredient representation of ρ.

4. Local coefficients

In this section, we shall explicitly compute the coefficients of proportionality, which was mentioned at the end of §3, for certain classes of representations.

Let v and $\tilde{v}$ be two fixed vectors in the space of σ_v and that of the contragredient representation of $\tilde{\sigma}_v$, respectively. We use

$$\langle\ ,\ \rangle$$

for the natural pairing between $\tilde{\sigma}_v$ and its contragredient. We shall start with the following trivial lemma.

LEMMA 4.1: *The function*

$$f_{s,v}(g) = \delta_{P,v}^{s-1/2}(p_0) \int_{{}^\circ K_v} \langle \tilde{\sigma}_v(k_0)\sigma_v(k_v^{-1})v, \tilde{v}\rangle \cdot {}^\circ\pi_v(p_0^{-1}k_v^{-1}){}^\circ w_v dk_v \tag{4.1}$$

with

$$g = k_0 p_0$$

is a well defined function on G_v. Furthermore $f_{s,v}$ belongs to V_v and every function in V_v is a finite linear combination of the functions of this type (different σ_v, $\tilde{\sigma}_v$ and ${}^\circ w_v$).

For $f_{s,v}$ as above, we put

$$w_{s,v}(g) = \lambda(\Pi_v(g^{-1})f_{s,v}).$$

Then

$$(4.2)\qquad w_{s,v}(g)=\int_{N_v}\int_{{}^\circ K_v}\delta_{P,v}^{s-1/2}(b(gnw_0))\langle\tilde{\sigma}_v(k(gnw_0))\sigma_v(k_v^{-1})v,\tilde{v}\rangle \cdot {}^\circ w_v(k_v\cdot\overline{b(gnw_0)})\overline{\chi_v(n)}\,dk_v dn$$

with the same notation as in §2, i.e.

$$gnw_0=k(gnw_0)\cdot b(gnw_0)$$

with the obvious meanings for $k(gnw_0)$ and $b(gnw_0)$.

We shall use $\underline{w}_{s,v}$ to denote the corresponding operator valued function, i.e.

$$\underline{w}_{s,v}(g)=\int_{N_v}\int_{{}^\circ K_v}\delta_{P,v}^{s-1/2}(b(gnw_0))^\circ w_v(k_v\cdot\overline{b(gnw_0)})\overline{\chi_v(n)} \times\tilde{\sigma}_v(k(gnw_0))\sigma_v(k_v^{-1})p_{\sigma_v}\cdot dk_v dn.$$

For each $f_{s,v}$, the function $f'_{s,v}$ defined by (3.3) is equal to

$$(4.3)\qquad f'_{s,v}(g)=\int_{N_v}\int_{{}^\circ K_v}\delta_{P,v}^{s-1/2}(b(gnw_0))\langle\tilde{\sigma}_v(k(gnw_0))\sigma_v(k_v^{-1})v,\tilde{v}\rangle \cdot {}^\circ\pi_v(b(gnw_0)^{-1}\cdot k_v^{-1})^\circ w_v dk_v dn.$$

We still use $f'_{s,v}$ to denote the analytic continuation of (4.3) by means of Proposition 3.6. We put

$$w'_{s,v}(g)=\lambda'(\Pi'_v(g^{-1})f'_{s,v})$$

for $f'_{s,v}$ defined by (4.3).

Now we shall assume that ${}^\circ\pi_v$ is not supercuspidal (when v is a finite place). Then there is a quasi-character η_v of ${}^\circ T_v$ so that ${}^\circ\pi_v$ can be realized as a quotient of the space of the left ${}^\circ K_v$-finite functions ${}^\circ f_v$ on ${}^\circ G_v$ which satisfy

$$(4.4)\qquad {}^\circ f_v(gtu)=\eta_v(t)^\circ\delta_v^{-1/2}(t)^\circ f_v(g)$$

with t in ${}^\circ T_v$, u in ${}^\circ U_v$, and g in ${}^\circ G_v$. Here ${}^\circ\delta_v$ denotes the modulus character of ${}^\circ T_v$ and the corresponding Hecke algebra acts by convolutions (cf. [8]).

We use w_1 to denote w_β which we realize as the longest element in

W_M. Then, there is a function ${}^\circ f_v$ so that

$$(4.5) \qquad {}^\circ w_v(g) = \int_{{}^\circ U_v} {}^\circ f_v(guw_1)\overline{\chi_v(u)}\,du,$$

where g is in ${}^\circ G_v$.

We shall consider η_v as a character of T_v in the obvious manner (i.e. trivial on A_v).

The goal of this section is to compute explicitly $w'_{s,v}$ in terms of $w_{s,v}$.

Following Jacquet [7], we define the following operator valued function on G_v

$$(4.6) \qquad \underline{h}_{s,v}(g) = \eta_v \cdot {}^\circ\delta_v^{-1/2}\delta_{P,v}^{s-1/2}(t(g))\tilde{\sigma}_v(k(g))P(\tilde{\sigma}_v, \eta_v),$$

Where $P(\tilde{\sigma}_v, \eta_v)$ denotes the projection onto the subspace of the vectors v which satisfy

$$\tilde{\sigma}_v(tu)v = \eta_v(t)v$$

with t in $K_v \cap T_v$ and u in $K_v \cap U_v$. The corresponding Whittaker function is defined by

$$(4.7) \qquad \underline{w}_{s,v}(g) = \int_{U_v} \underline{h}_{s,v}(guw_2)\overline{\chi_v(u)}\,du$$

with

$$w_2 = w_1 w_0,$$

the longest element in W. As a function of s, $\underline{w}_{s,v}$ is entire (cf. [7]).

We shall consider the following operator valued function

$$(4.8) \qquad \underline{f}'_{s,v}(g) = \int_{N_v}\int_{{}^\circ K_v} \delta_{P,v}^{s-1/2}(b(gnw_0))^\circ w_v(k_v\overline{b(gnw_0)})\tilde{\sigma}_v(k(gnw_0)) \cdot \tilde{\sigma}_v(k_v^{-1}) \cdot p_{\sigma_v} dk_v dn,$$

where the integral is convergent for $\mathrm{Re}(s)$ sufficiently small. Then (4.3) implies that

$$(f'_{s,v}(g), e) = \langle \underline{f}'_{s,v}(g)v, \tilde{v}\rangle.$$

We need the following lemma.

LEMMA 4.2: *The non-zero operator*

$$\int_{{}^\circ K_v} {}^\circ f_v(k_v)\tilde{\sigma}_v(k_v^{-1})dk_v \cdot p_{\sigma_v}$$

sends the space of $\tilde{\sigma}_v$ *into the range of*

$$p(\tilde{\sigma}_v, \eta_v) \cdot p_{\sigma_v}.$$

We use $H({}^\circ f_v, \tilde{\sigma}_v)$ to denote the operator introduced in Lemma 4.2.

LEMMA 4.3: *For* $\mathrm{Re}(s)$ *sufficiently small, we have*

$$w'_{s,v}(g) = \underline{w}_{-s,v}(g) \int_{N_v} \underline{h}_{s,v}(nw_0)dn \cdot H({}^\circ f_v, \tilde{\sigma}_v). \tag{4.9}$$

PROOF: Substitution of (4.5) into (4.8), followed by a simple change of variables, shows that for $\mathrm{Re}(s)$ sufficiently small

$$\begin{aligned} \underline{f}'_{s,v}(g) = \int_{N_v}\int_{{}^\circ U_v} & \delta_{P,v}^{s-1/2}(b_0)\eta_v{}^\circ\delta_v^{-1/2}(t(b_0uw_1))\overline{\chi_v(u)} \\ & \cdot \tilde{\sigma}_v(k_0)\tilde{\sigma}_v(k(\bar{b}_0uw_1))dudn \cdot H({}^\circ f_v, \tilde{\sigma}_v) \end{aligned} \tag{4.9.1}$$

with

$$b_0 = b(gnw_0)$$

and

$$k_0 = k(gnw_0).$$

Using

$$k(gnw_0uw_1) = k_0 \cdot k(b_0uw_1),$$

$$\delta_{P,v}(uw_1) = 1 \quad (u \in {}^\circ U_v)$$

and Lemma 4.2, (4.9.1) reduces to

$$\int_{N_v}\int_{{}^\circ U_v} \underline{h}_{s,v}(gnw_0uw_1)\overline{\chi_v(u)}\,dudn \cdot H({}^\circ f_v, \tilde{\sigma}_v)$$

which is equal to

$$\int_{N_v}\int_{{}^\circ U_v} \underline{h}_{s,v}(guw_1nw_0)\overline{\chi_v(u)}\,dudn \cdot H({}^\circ f_v, \tilde{\sigma}_v). \tag{4.9.2}$$

Therefore for $\mathrm{Re}(s)$ sufficiently small (4.9.2) implies that

(4.9.3)

$$\underline{f}'_{s,v}(g)=\int_{^\circ U_v}\underline{h}_{-s,v}(guw_1)\overline{\chi_v(u)}\,du\cdot\int_{N_v}\underline{h}_{s,v}(nw_0)dn\cdot H(^\circ f_v,\tilde{\sigma}_v).$$

Then using Proposition 3.6 and (4.9.3), we can define $\underline{f}'_{s,v}$ for (almost) all s and the lemma follows immediately.

To simplify the right hand side of (4.9), we need the following results from the first part of [7].

For a local field K (archimedean or non-archimedean), let H denote the group $SL_2(K)$ and let M_H be the maximal compact subgroup of H as in [7]. We put A_H and N_H for the diagonal and the standard unipotent subgroups of H, respectively.

Let $\mathfrak{D}$ be a finite dimensional representation of M_H on the Hilbert space $H(\mathfrak{D})$. We use η to denote a character of K^*, the multiplicative subgroup of K. It can be considered as a character of A_H by means of

$$\eta\left(\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix}\right)=\eta(a).$$

We use $\mathfrak{S}(K^2)$ to denote the space of the Schwartz-Bruhat functions on $K\times K$ and $\mathrm{End}(H(\mathfrak{D}))$ for the algebra of endomorphisms of $H(\mathfrak{D})$. Let s be a complex number and let Ψ be a function in

$$\mathfrak{S}(K^2)\otimes\mathrm{End}(H(\mathfrak{D})).$$

We define (Proposition 1.7 of [7])

$$L_\Psi(g,\eta,s)=\int_{K^*}\Psi(te_1\cdot{}^tg)\overline{\eta(t)}|t|^s d^*t, \tag{4.10}$$

with g in H and

$$e_1=(1,0).$$

Then

$$L_\Psi(g,\eta,s)=L(g,\mathfrak{D},\eta,s)L_\Psi(e,\eta,s) \tag{4.11}$$

with

$$L(g,\mathfrak{D},\eta,s)=\mathfrak{D}(m)\eta(a)|a|^{-s}P(\mathfrak{D},\eta), \tag{4.12}$$

if

$$g = man$$

with m in M_H, a in A_H, and n in N_H. Here $P(\mathfrak{D}, \eta)$ denotes the projection into the subspace of the vectors v which satisfy

$$\mathfrak{D}(an)v = \eta(a)v$$

with a in $A_H \cap M_H$ and n in $N_H \cap M_H$. Let τ be the additive character of K fixed in Section 1 of [7]. We put

$$\hat{\Psi}(x, y) = \int_{K^2} \Psi(u, v)\tau(xv - yu)dudv$$

and

$$w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Finally the operator

$$\text{Ч}(\mathfrak{D}, \eta, s),$$

introduced in Corollary (1.10) of [7], satisfies the following relations

$$L_{\hat{\Psi}}(e, \bar{\eta}, -s + 1) = \text{Ч}(\mathfrak{D}, \eta, s)L_{\Psi}(e, \eta, s + 1), \tag{4.13}$$

$$\text{Ч}(\mathfrak{D}, \bar{\eta}, -s) \cdot \text{Ч}(\mathfrak{D}, \eta, s) = P(\mathfrak{D}, \eta) \tag{4.14}$$

and

$$P(\mathfrak{D}, \bar{\eta})\text{Ч}(\mathfrak{D}, \eta, s)P(\mathfrak{D}, \eta) = \text{Ч}(\mathfrak{D}, \eta, s). \tag{4.15}$$

Let $\gamma(\eta, s)$ be the coefficient (cf. [17]) defined by

$$\int_{K^*} \Psi(t)\overline{\eta(t)}|t|^{1-s}d^*t = \gamma(\eta, s) \int_{K^*} \hat{\Psi}(t)\eta(t)|t|^s d^*t, \tag{4.16}$$

where Ψ is a Schwartz function on K and

$$\hat{\Psi}(t) = \int_K \Psi(x)\overline{\tau(tx)}\, dx.$$

We shall prove the following important lemma.

LEMMA 4.4: *For* Re(s) *sufficiently small, one has*

$$\text{(4.17)}\quad \text{Ч}(\mathfrak{D}, \bar{\eta}, -s)\int_{N_H} L(nw, \mathfrak{D}, \eta, s+1)dn = \gamma^{-1}(\bar{\eta}, s)P(\mathfrak{D}, \eta).$$

PROOF: We have

$$\text{(4.17.1)}\qquad \int_{N_H} L_\Psi(nw, \eta, s+1)dn = \int_{N_H} L(nw, \mathfrak{D}, \eta, s+1)dn \cdot L_\Psi(e, \eta, s+1).$$

Then straight forward computations imply that the left hand side of (4.17.1) is equal to

$$\int_{K\times K^*} \Psi(x, t)\overline{\eta(t)}|t|^s d^*t dx.$$

Using (4.16), this is equal to

$$\text{(4.17.2)}\qquad \eta(-1)\gamma(\eta, 1-s)\int_{K^*} \hat{\Psi}(t, 0)\eta(t)\|t|^{1-s}d^*t = \gamma^{-1}(\bar{\eta}, s)L_\Psi(e, \bar{\eta}, -s+1).$$

The lemma follows if we apply (4.13) and (4.14) to (4.17.1) and the right hand side of (4.17.2).

REMARK: The lemma is still true if we assume that η is a quasi-character and replace $\bar{\eta}$ by η^{-1}.

For every root γ, there is an isomorphism $\varkappa_\gamma$ from SL_2 onto the group G^γ whose Lie algebra is generated by $H_\gamma \in \mathfrak{h}$, $X_\gamma \in \mathfrak{g}_\gamma$ and $X_{-\gamma} \in \mathfrak{g}_{-\gamma}$ (G is universal). It sends $\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}1 & 0\\ x & 1\end{smallmatrix}\right)$, and $\left(\begin{smallmatrix}t & 0\\ 0 & t^{-1}\end{smallmatrix}\right)$ to $\exp(xX_\gamma)$, $\exp(xX_{-\gamma})$, and $h_\gamma(t)$, respectively. There is also an isomorphism (conjugation by w) between G^γ and $G^{w(\gamma)}$ for every w in W.

For every place v, we have the Iwasawa decomposition

$$G_v^\gamma = K_v^\gamma \cdot T_v^\gamma \cdot U_v^\gamma$$

with

$$K_v^\gamma = G_v^\gamma \cap K_v$$

and

$$T_v^\gamma = \{h_\gamma(t); t \in F_v^*\}.$$

We put

$$\nu = \sum_{\gamma \in \Psi^+ - \{\beta\}} \gamma$$

and

$$\langle \gamma, \delta \rangle = 2 \frac{(\gamma, \delta)}{(\delta, \delta)}$$

for a pair of roots γ and δ.

As we mentioned before, there is no harm in assuming that η_v is trivial on the center and therefore η_v is a quasi-character of T_v. Notice that

$$\eta_v(h_\gamma(t)) = \eta_v^{\langle \beta, \gamma \rangle}(t) \quad (\forall \gamma \in \Psi),$$

by means of which we may consider each $\eta_v^{\langle \beta, \gamma \rangle}$ as a quasi-character of F_v^*.

For a pair of roots γ and δ, we define

$$\underline{h}_{s,v}^{\delta,\gamma}(g) = \tilde{\sigma}_v(k(g)) \cdot \eta_v^{\langle \beta, \gamma \rangle} \cdot \alpha_v^{s\langle \nu, \gamma \rangle - 1}(t) \cdot \underline{P}(\tilde{\sigma}_v, \eta_v) \tag{4.18}$$

for g in G_v^δ with

$$g = k(g) h_\delta(t) u(g).$$

Here α_v denotes the modulus character (absolute value) and $\underline{P}(\tilde{\sigma}_v, \eta_v)$ is the corresponding projection. We put

$$A_{s,v}^{\delta,\gamma} = \int_{U_v^\delta} \underline{h}_{s,v}^{\delta,\gamma}(u w_\delta) du. \tag{4.19}$$

We need the following lemma.

LEMMA 4.5: *For* Re(s) *sufficiently small, the following relation holds.*

$$\int_{N_v} \underline{h}_{s,v}(n w_0) dn = A_{s,v}^{\alpha,\alpha+\beta} \cdot A_{s,v}^{\beta,3\alpha+2\beta} \cdot A_{s,v}^{\alpha,2\alpha+\beta} \cdot A_{s,v}^{\beta,3\alpha+\beta} \cdot A_{s,v}^{\alpha,\alpha}. \tag{4.20}$$

PROOF: In fact the ordering introduced here is the one in [4] (see also [11]). Set

$$\bar{\theta}_0 = \Psi^- - \{-\beta\};$$

then (4.20) is equal to

$$\tilde{\sigma}_v(w_0)\int_{N_v(\bar{\theta}_0)} \underline{h}_{s,v}(\bar{n})d\bar{n},$$

where $N_v(\bar{\theta}_0)$ denotes the unipotent subgroup generated by $\bar{\theta}_0$. If we put

$$\bar{\theta}_1 = \Psi^- - \{-\beta, -(\alpha+\beta)\}$$

and denote the unipotent subgroup generated by $\bar{\theta}_1$ by $N_v(\bar{\theta}_1)$, we can write

$$\begin{aligned}(4.20.1)\qquad \int_{N_v} \underline{h}_{s,v}(nw_0)dn = \tilde{\sigma}_v(w_0 w_{\alpha+\beta}) \cdot \int_{U_v^{\alpha+\beta}} &\tilde{\sigma}_v(k(uw_{\alpha+\beta}))\\ &\cdot \eta_v \delta_{P,v}^{s}\, {}^\circ\delta_{\alpha+\beta,v}^{-1/2}(t(uw_{\alpha+\beta}))du\\ &\cdot \int_{N_v(\bar{\theta}_1)} \underline{h}_{s,v}(\bar{n})d\bar{n}.\end{aligned}$$

Here ${}^\circ\delta_{\alpha+\beta,v}$ is defined by

$${}^\circ\delta_{\alpha+\beta,v}(h_{\alpha+\beta}(t)) = |t|^2.$$

If we conjugate $k(uw_{\alpha+\beta})$ with $w_0 w_{\alpha+\beta}$ and use the isomorphism between $G_v^{\alpha+\beta}$ and G_v^{α} (through this conjugation), we shall see that (4.20.1) is equal to

$$A_{s,v}^{\alpha,\alpha+\beta} \cdot \sigma_v(w_0 w_{\alpha+\beta}) \cdot \int_{N_v(\bar{\theta}_1)} \underline{h}_{s,v}(\bar{n})d\bar{n}.$$

Then applying the same argument to

$$\tilde{\sigma}_v(w_0 w_{\alpha+\beta}) \cdot \int_{N_v(\bar{\theta}_1)} \underline{h}_{s,v}(\bar{n})d\bar{n},$$

by means of the ordering introduced in the statement of the lemma, completes the proof inductively.

At each finite place v, we shall fix a non-trivial character τ_v of F_v so that the largest ideal for which τ_v is trivial is the ring of integers O_v of F_v. Then

$$\chi_{\gamma,v}(\exp(xX_\gamma)) = \tau_v(\mu_{\gamma,v}x) \quad (\gamma \in \Delta)$$

with $\mu_{\gamma,v}$ in F_v^*. Notice that $\mu_{\gamma,v}$ is a unit for almost all v. When v is an infinite place, we shall fix τ_v as in [7].

For a simple root δ, we put

$$B_{s,v}^{\delta,\gamma} = \Upsilon_\delta(\tilde{\sigma}_v, \eta_v^{-\langle\beta,\gamma\rangle}, \langle\nu,\gamma\rangle \cdot s). \tag{4.21}$$

LEMMA 4.6: *With $B_{s,v}^{\delta,\gamma}$ as in* (4.21) *we have*

$$\begin{aligned}\underline{w}_{-s,v}(g) = |\mu_{\alpha,v}^2 \cdot \mu_{\beta,v}|^{10s} \underline{w}_{s,v}(g) \cdot B_{s,v}^{\alpha,\alpha} \cdot B_{s,v}^{\beta,3\alpha+\beta} \\ \cdot B_{s,v}^{\alpha,2\alpha+\beta} \cdot B_{s,v}^{\beta,3\alpha+\beta} \cdot B_{s,v}^{\alpha,\alpha+\beta}.\end{aligned} \tag{4.22}$$

PROOF: Put

$$\bar{U} = w_2 U w_2^{-1}.$$

We use the ordering introduced in Lemma 4.5 for the set

$$\Psi^+ - \{\beta\}.$$

Then if we put γ_1 for $\alpha+\beta$, γ_5 will be equal to α.

Now, we inductively define the following quasi-characters of T_v:

$$\eta_{i+1} = w_{\delta(i)}\eta_i \quad 1 \leq i \leq 5$$

and

$$\lambda_{i+1} = w_{\delta(i)}\eta_i \quad 1 \leq i \leq 5$$

with the obvious action of W on the character group. The index $\delta(i)$ is defined by

$$\delta(i) = \begin{cases} \alpha & i \text{ odd} \\ \beta & i \text{ even,} \end{cases}$$

and

$$\begin{aligned}\eta_1 &= w_0\eta_v \\ &= \eta_v,\end{aligned}$$

and finally

$$\begin{aligned}\lambda_1 &= w_0\delta_{P,v}^s \\ &= \delta_{P,v}^{-s}.\end{aligned}$$

Then from the functional equation of Jacquet ([7], Proposition 3.3), it follows that

$$E_{\bar{U}}(g, \tilde{\sigma}_v, \eta_i, \lambda_i, \chi_v) = \eta_v^{\langle\beta,\gamma_i\rangle}(\mu_{\delta(i),v})|\mu_{\delta(i),v}|^{\langle\nu,\gamma_i\rangle s} \cdot E_{\bar{U}}(g, \tilde{\sigma}_v, \eta_{i+1}, \lambda_{i+1}, \chi_v) \cdot \Psi_{\delta(i)}(\tilde{\sigma}_v, \eta_v^{-\langle\beta,\gamma_i\rangle}, \langle\nu, \gamma_i\rangle s) \tag{4.22.1}$$

for $i = 1, \ldots, 5$. But

$$E_{\bar{U}}(gw_2^{-1}, \tilde{\sigma}_v, \eta_1, \lambda_1, \chi_v) = \underline{w}_{-s,v}(g) \tag{4.22.2}$$

and

$$E_{\bar{U}}(gw_2^{-1}, \tilde{\sigma}_v, \eta_6, \lambda_6, \chi_v) = \underline{w}_{s,v}(g), \tag{4.22.3}$$

and the lemma follows if we consider (4.22.1) for $1 \leq i \leq 5$ and combine them together with (4.22.2) and (4.22.3).

If we apply Lemmas 4.4, 4.5, and 4.6 to Lemma 4.3, we conclude

THEOREM 4.7: *Let $H({}^\circ f_v, \tilde{\sigma}_v)$ be defined as in Lemma 4.2; then*

$$\int_{N_v} \underline{f}'_{s,v}(gnw_0)\overline{\chi_v(n)}\, dn = |\mu_{\alpha,v}^2 \cdot \mu_{\beta,v}|^{10s} \prod_{i=1}^{5} \gamma^{-1}(\eta_v^{-\langle\beta,\gamma_i\rangle}, -\langle\nu, \gamma_i\rangle s) \cdot \underline{w}_{s,v}(g) \cdot H({}^\circ f_v, \tilde{\sigma}_v). \tag{4.23}$$

COROLLARY 4.8: *With notation as before, we have*

$$\underline{w}'_{s,v}(g) = |\mu_{\alpha,v}^2 \cdot \mu_{\beta,v}|^{10s} \prod_{i=1}^{5} \gamma^{-1}(\eta_v^{-\langle\beta,\gamma_i\rangle}, -\langle\nu, \gamma_i\rangle s) \cdot \underline{w}_{s,v}(g). \tag{4.24}$$

We shall conclude this section by establishing a result similar to that of Corollary 3.5 for the archimedean places.

Suppose v is an archimedean place. As usual, let $C_c^\infty(G_v)$ denote the space of smooth functions on G_v with compact support. We assume $C_c^\infty(G_v)$ has the Schwartz topology.

For every function Ψ in $C_c^\infty(G_v)$, the integral

$$h_\Psi(g) = \int_{B_v} \delta_{P,v}^{-s+1/2}(b)\eta_v^{-1} \cdot {}^\circ\delta_v^{1/2}(t(b))\Psi(gb)d_\ell(b) \tag{4.25}$$

defines a function which satisfies

$$h_\Psi(gtu) = \eta_v \circ \delta_v^{-1/2} \delta_{P,v}^{s-1/2}(t) h_\Psi(g) \tag{4.26}$$

with t in T_v and u in U_v. Here $d_\ell(b)$ denotes the left invariant Haar measure on B_v.

The map sending Ψ to h_Ψ is a surjective homomorphism onto the space of smooth functions on G_v which satisfy (4.26)

Let h be such a function; then the integral

$$\int_{U_v} h(guw_2)\overline{\chi_v(u)}\, du$$

converges absolutely for $\mathrm{Re}(s)$ sufficiently small (cf. [7]), and defines a function $w(g)$ on G_v.

LEMMA 4.0: *There is a function Ψ in $C_c^\infty(G_v)$ for which the function*

$$w_\Psi(g) = \int_{U_v} h_\Psi(guw_2)\overline{\chi_v(u)}\, du$$

(*which is defined by this integral for* $\mathrm{Re}(s)$ *sufficiently small*) *does not vanish identically.*

PROOF: For $\mathrm{Re}(s)$ sufficiently small we can write

$$w_\Psi(g) = \int_{U_v}\int_{B_v} \delta_{P,v}^{-s+1/2}(b)\eta_v^{-1} \circ \delta_v^{1/2}(t(b)) \cdot \overline{\chi_v(u)}\Psi(guw_2b)d_\ell(b)du.$$

Then we may choose Ψ with support in $U_v w_2 B_v$ (which is open in G_v) so that $w_\Psi(e)$ is not zero. This completes the lemma.

LEMMA 4.10: *For* $\mathrm{Re}(s)$ *sufficiently small, the integral*

$$w_\Psi(e) = \int_{U_v}\int_{B_v} \delta_{P,v}^{-s+1/2}(b) \cdot \eta_v^{-1} \circ \delta_v^{1/2}(t(b))\overline{\chi_v(u)}\Psi(uw_2b)d_\ell(b)du$$

$$(\Psi \in C_c^\infty(G_v))$$

defines a distribution on G_v.

PROOF: We define the following function on G_v:

$$f(g) = \delta_{P,v}^{-s+1/2}(b)\eta_v^{-1} \circ \delta_v^{1/2}(t(b))\overline{\chi_v(u)}$$

if

$$g = uw_2b \in U_v w_2 B_v$$

and zero otherwise. Then $w_\Psi(e)$ can be written as

$$w_\Psi(e) = \int_{G_v} f(g)\Psi(g)dg \quad (\Psi \in C_c^\infty(G_v))$$

which is clearly a distribution since f is locally integrable for $\mathrm{Re}(s)$ sufficiently small. In fact, if ω is a compact subset of G_v, there exists a positive function h in $C_c^\infty(G_v)$ with

$$h(x) = 1 \quad (x \in \omega).$$

Then, it is clear that

$$\int_\omega |f(g)|dg \leq \int_{B_v}\int_{U_v} h(uw_2b)|\delta_{P,v}^{-s+1/2}(b)\cdot \eta_v^{-1}\circ\delta_v^{1/2}(t(b))|du d_\ell(b).$$

But for the small values of $\mathrm{Re}(s)$ the right hand side of this inequality is finite which implies the lemma.

The functions $h_{s,v}(g)$ (different $\tilde{\sigma}_v$, v and $\tilde{v}$) will generate the subspace of left K_v-finite functions on G_v which satisfy (4.26). Since the subspace of left K_v-finite functions with compact support on G_v is dense in $C_c^\infty(G_v)$, we conclude the following corollary.

COROLLARY 4.11: *Let v be an archimedean place; then for* $\mathrm{Re}(s)$ *sufficiently small, there is a function $\underline{h}_{s,v}$, defined by* (4.6), *for which $\underline{w}_{s,v}$ is not identically zero.*

5. Functional equation

From now on, we shall assume that the character χ has been chosen so that

$$\mu_{\alpha,v} = \mu_{\beta,v} \quad (\forall v).$$

The complex dual group for ${}^\circ G$ (or PGL_2), i.e. ${}^\circ G^\wedge$, is $SL_2(\mathbb{C})$.

For an unramified place (i.e. when ${}^\circ\pi_v$ is class-one and $\mu_{\alpha,v}$ is a unit in O_v), η_v is unramified and therefore we can write

$$\eta_v(h_\beta(t)) = |t|_v^{2s_1}$$

with a complex number s_1 (s_1 depends on v). Then the matrix

$$t_v^\wedge = \begin{pmatrix} q_v^{-s_1} & 0 \\ 0 & q_v^{s_1} \end{pmatrix}$$

determines a semi-simple conjugacy class $\alpha_v^\wedge$ in $SL_2(\mathbb{C})$ (see the introduction). q_v is the number of elements in the residue field.

Let ρ be a four dimensional irreducible representation of $SL_2(\mathbb{C})$ (e.g. restriction of the third symmetric power of the standard representation of $GL_2(\mathbb{C})$). In fact up to equivalence ρ is unique. The highest weight for ρ is equal to 3δ where δ is the fundamental weight for $SL_2(\mathbb{C})$. Then the local L-function $L(s, \rho, {}^\circ\pi_v)$ defined by Langlands in [10] is equal to

$$L(s, \rho, {}^\circ\pi_v) = \det(I - \rho(t_v^\wedge)q_v^{-s})^{-1}. \tag{5.1}$$

In fact

$$L(s, \rho, {}^\circ\pi_v) = \prod_{i=1,2,4,5} (I - q_v^{\langle\beta,\gamma_i\rangle s_1} \cdot q_v^{-s})^{-1} \tag{5.2}$$

with notation as in Lemma 4.6 (cf. [11]).

Let us assume that

$$\omega = \prod_v \omega_v$$

is an unramified quasi-character. This means that there exists a complex number s_2 such that

$$\omega\left(\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}\right) = |z|^{2s_2} \quad (z \in \mathbb{A}^*).$$

Then

$$L(s, \rho, {}^\circ\pi_v) = L(s + 3s_2, \rho^*, {}^\circ\pi_v^*), \tag{5.3}$$

where ρ^* is the third symmetric power representation of $GL_2(\mathbb{C})$ (the associated group for GL_2) and ${}^\circ\pi_v^*$ is defined to be the v-th component of ${}^\circ\pi^*$, the space generated by φ^* (see §2).

To proceed, we need the following lemma which is a simple consequence of Lemma 2.1.

LEMMA 5.1: *Let* $\underline{E}_\chi(s;\tilde{\varphi};g;P)$ *be the Fourier coefficient of* $E(s;\tilde{\varphi};g;P)$ *defined by* (2.9), *then for* $\mathrm{Re}(s)<-1/2$

$$\underline{E}_\chi(s;\tilde{\varphi};g;P)=\bigotimes_v w_{s,v}(g_v)$$

with $g=(g_v)$ *in* $G_{\mathbb{A}}$.

REMARK: For almost all v, g_v is in K_v and $\underline{w}_{s,v}$ is a class-one function; thus for such places, we have

$$\underline{w}_{s,v}(g_v)=\underline{w}_{s,v}(e)$$

and the tensor product is defined without ambiguity.

The space of $\tilde{\sigma}_v$ (resp. its contragredient) is generated by the vectors of the form $\otimes_v e_v$ (resp. $\otimes_v \tilde{e}_v$) with e_v (resp. $\tilde{e}_v$) in the space of $\tilde{\sigma}_v$ (resp. its contragredient) where for almost all v, $\langle e_v, \tilde{e}_v\rangle$ is equal to 1. We put

$$\left\langle \bigotimes_v e_v, \bigotimes_v \tilde{e}_v\right\rangle=\prod_v \langle e_v, \tilde{e}_v\rangle.$$

We shall fix two such vectors $\otimes_v e_v$ and $\otimes_v \tilde{e}_v$ for which

$$\langle e_v, \tilde{e}_v\rangle = 1$$

whenever ${}^\circ\pi_v$ is a class-one representation. We set

$$E_\chi(s;\tilde{\varphi};g;P)=\prod_v \langle \underline{w}_{s,v}(g_v)e_v, \tilde{e}_v\rangle$$

for $g=(g_v)$ in $G_{\mathbb{A}}$.

Now we shall explain the analytic continuation of $E_\chi(s;\tilde{\varphi};g;P)$ as a function of s. First we shall mention a result of W. Casselman and J.A. Shalika [2] which unfortunately has not been published yet. We use

$$L(1_v, s)=(1-q_v^{-s})^{-1}$$

to denote the local Hecke L-function attached to the trivial character 1_v of F_v^*. Let $\tilde{\rho}$ denote the contragredient of ρ. As before put

$$w_{s,v}(g)=\langle \underline{w}_{s,v}(g)e_v, \tilde{e}_v\rangle.$$

THEOREM 5.2: (W. Casselman–J.A. Shalika). *Assume v is unramified; then*

$$w_{s,v}(e) \cdot L(1_v, -10s+1) \cdot L(-5s+1, \tilde{\rho}, {}^\circ\pi_v) = {}^\circ w_v(e) \tag{5.4}$$

where $L(-5s+1, \tilde{\rho}, {}^\circ\pi_v)$ is defined by (5.1).

Now as in introduction, let S be the finite set of ramified places (including infinite places); then it follows from [10] that the product

$$L_S(s, \tilde{\rho}, {}^\circ\pi) = \prod_{v \notin S} L(s, \tilde{\rho}, {}^\circ\pi_v) \tag{5.5}$$

is convergent for Re(s) sufficiently large. We put

$$L_S(1, s) = \prod_{v \notin S} L(1_v, s) \tag{5.6}$$

which is again convergent for large values of Re(s). We have

COROLLARY 5.3: *As a function of s, the product (which converges for* Re(s) *sufficiently small)*

$$E_\chi(s; \tilde{\varphi}; g; P) \cdot L_S(1, -10s+1) \cdot L_S(-5s+1, \tilde{\rho}, {}^\circ\pi)$$

can be continued to an entire function on the whole complex plane.

PROOF: This is an easy consequence of Theorem 5.2, Corollary 3.4 and the relation

$$\prod_v {}^\circ w_v(e) = {}^\circ w(e).$$

REMARK: Corollary 5.3 may also be considered as a consequence of the last section in [7].

As a consequence of relation (2.7) and Corollary 5.3 we have:

THEOREM 5.4: *As a function of s, the Fourier coefficient $E_\chi(s; \tilde{\varphi}; g; P)$ can be continued to a meromorphic function on the whole complex plane. Furthermore*

$$E_\chi(-s; M(s)\tilde{\varphi}; g; P) = E_\chi(s; \tilde{\varphi}; g; P). \tag{5.7}$$

REMARK: For $\mathrm{Re}(s) > \frac{1}{2}$, we have

$$(5.8) \qquad E_\chi(-s; M(s)\tilde{\varphi}; g; P) = \prod_v \langle \underline{w}'_{s,v}(g_v)e_v, \tilde{e}_v\rangle$$

with $\underline{w}'_{s,v}$ defined as in §4.

THEOREM 5.5: *At each place v, the linear functional λ' defined by (3.4) is non-zero. Therefore there exists a complex function*

$$c(s, \rho, {}^\circ\pi_v, \chi_v),$$

meromorphic in s, such that

$$(5.9) \qquad w_{s,v}(g) = c(s, \rho, {}^\circ\pi_v, \chi_v)w'_{s,v}(g) \quad (g \in G_v).$$

Furthermore $c(s, \rho, {}^\circ\pi_v, \chi_v)$ depends only on the class of ${}^\circ\pi_v$.

PROOF: For $\mathrm{Re}(s) > \frac{1}{2}$, it follows from Corollary 3.5, Corollary 4.11, functional equation (5.7), and the relation (5.8) that λ' is non-zero. The existence of $c(s, \rho, {}^\circ\pi_v, \chi_v)$ follows from Theorem 2 of [14]. It clearly depends only upon the class of ${}^\circ\pi_v$. Since $w_{s,v}(g)$ and $w'_{s,v}(g)$ are at most meromorphic, it follows that $c(s, \rho, {}^\circ\pi_v, \chi_v)$ is a meromorphic function of s.

COROLLARY 5.6: *As a function of s, $c(s, \rho, {}^\circ\pi_v, \chi_v)$ is holomorphic whenever λ' is non-zero.*

PROOF: For a fixed s, there exists a function $w'_{s,v}$ for which $w'_{s,v}(g_0)$ is not zero for some g_0 in G_v. Then if $c(s, \rho, {}^\circ\pi_v, \chi_v)$ had a pole at s, it would appear as a pole for $w_{s,v}(g_0)$ which is a contradiction to Corollary 3.4.

REMARK: Since $w'_{s,v}$ has already some poles coming from the intertwining operators, it is not true that $c(s, \rho, {}^\circ\pi_v, \chi_v)$ has no zeros.

Let ${}^\circ\chi$ be the restriction of χ to ${}^\circ U_{\mathbb{A}}$, then by the assumption on χ, we may assume that $c(s, \rho, {}^\circ\pi_v, \chi_v)$ depends on ${}^\circ\chi_v$ and write $c(s, \rho, {}^\circ\pi_v, {}^\circ\chi_v)$ for $c(s, \rho, {}^\circ\pi_v, \chi_v)$.

PROPOSITION 5.7: *Assume $^\circ\pi_v$ can be realized as a quotient of the space of the left $^\circ K_v$-finite functions $^\circ f_v$ on $^\circ G_v$ which satisfy*

$$^\circ f_v(gtu) = \eta_v(t)^\circ\delta_v^{-1/2}(t) f_v^\circ(g) \quad (t \in {}^\circ T_v, u \in {}^\circ U_v, g \in {}^\circ G_v),$$

where η_v is a quasi-character of $^\circ T_v$ (this includes all the irreducible and admissible representations of the Hecke algebra of $^\circ G_v$ except when v is finite and the representation is supercuspidal). Then

$$c(s, \rho, {}^\circ\pi_v, {}^\circ\chi_v) = \prod_{i=1}^{5} |\mu_{\alpha,v}|^{-\langle\nu,\gamma_i\rangle s} \gamma(\eta_v^{-\langle\beta,\gamma_i\rangle}, -\langle\nu, \gamma_i\rangle s) \tag{5.10}$$

with $\mu_{\alpha,v}$, ν, γ and the ordering defined as before.

PROOF: This is a simple consequence of Corollary 4.8.

COROLLARY 5.8: *As a function of s, the product*

$$\prod_{i=1}^{5} \gamma(\eta_v^{-\langle\beta,\gamma_i\rangle}, -\langle\nu, \gamma_i\rangle s)$$

is holomorphic whenever λ' is non-zero.

Now, for $v \in S$, put

$$\gamma(s, \rho, {}^\circ\pi_v, {}^\circ\chi_v) = c\left(\frac{s-1}{5}, \rho, {}^\circ\pi_v, {}^\circ\chi_v\right) \cdot \gamma^{-1}(1_v, 2-2s), \tag{5.11}$$

where $\gamma(1_v, 2-2s)$ is defined by (4.16), and define

$$\gamma_S(s, \rho, {}^\circ\pi) = \prod_{v\in S} \gamma(s, \rho, {}^\circ\pi_v, {}^\circ\chi_v). \tag{5.12}$$

This product, as we shall see, is independent of $^\circ\chi$. The main result of this paper is the following theorem.

THEOREM 5.9 (*functional equation*). *Let $^\circ\pi$ be a cusp form on $PGL_2(\mathbb{A})$. Denote by S the corresponding set of ramified places. Define $\gamma_S(s, \rho, {}^\circ\pi)$ and $L_S(s, \rho, {}^\circ\pi)$ as before. Then*

$$L_S(1-s, \tilde{\rho}, {}^\circ\pi) = \gamma_S(s, \rho, {}^\circ\pi) L_S(s, \rho, {}^\circ\pi). \tag{5.13}$$

In particular

$$\prod_{v \in S} \gamma(s, \rho, {}^\circ\pi_v, {}^\circ\chi_v)$$

is independent of ${}^\circ\chi$.

PROOF: For $\mathrm{Re}(s)$ sufficiently small,

$$E_\chi(s; \tilde{\varphi}; g; P) L_S(1, -10s) \cdot L_S(-5s, \tilde{\rho}, {}^\circ\pi) \tag{5.13.1}$$

is equal to

$$\prod_{v \in S} w_{s,v}(g_v) \cdot \prod_{v \notin S} w_{s,v}(g_v) \cdot L(1_v, -10s) \cdot L(-5s, \tilde{\rho}, {}^\circ\pi_v). \tag{5.13.2}$$

For any v, infinite or finite in which case we assume that ${}^\circ\pi_v$ is not supercuspidal, Proposition 5.7 implies

$$\begin{aligned} w_{s,v}(g_v) \prod_{i=1}^{5} L(\eta_v^{-\langle\beta,\gamma_i\rangle s}, -\langle\nu, \gamma_i\rangle s) &= \prod_{i=1}^{5} |\mu_{\alpha,v}|^{-\langle\nu,\gamma_i\rangle s} \cdot \epsilon(\eta_v^{-\langle\beta,\gamma_i\rangle}, -\langle\nu, \gamma_i\rangle s) \\ &\quad \cdot \prod_{i=1}^{5} L(\eta_v^{\langle\beta,\gamma_i\rangle}, 1 + \langle\nu, \gamma_i\rangle s) \cdot w'_{s,v}(g_v). \end{aligned}$$

Here $\epsilon(\theta, s)$ (local root number), for a quasi-character θ of F_v^*, is defined by

$$\epsilon(\theta, s) = \gamma(\theta, s) \frac{L(\theta, s)}{L(\theta^{-1}, 1 - s)}.$$

Using (5.13.3) for unramified places, (5.13.2) can be written as follows:

$$\prod_{v \in S} w'_{s,v}(g_v) \cdot \prod_{v \in S} c(s, \rho, {}^\circ\pi_v, {}^\circ\chi_v) \cdot \prod_{v \notin S} w'_{s,v}(g_v) \cdot L_S(1, 1 + 10s) \cdot L_S(1 + 5s, \rho, {}^\circ\pi).$$

If we use analytic continuation of (5.13.1) to the large values of $\mathrm{Re}(s)$, we conclude that

(5.13.4)

$$\prod_v w'_{s,v}(g_v) \cdot \prod_{v \in S} c(s, \rho, {}^\circ\pi_v, {}^\circ\chi_v) \cdot L_S(1, 1+10s) \cdot L_S(1+5s, \rho, {}^\circ\pi)$$

$$= \prod_v w'_{s,v}(g_v) \cdot L_S(1, -10s) \cdot L_S(-5s, \tilde{\rho}, {}^\circ\pi).$$

It is well known that

$$L_S(1, -10s) = \prod_{v \in S} \gamma(1_v, -10s) \cdot L_S(1, 1+10s).$$

Now, if we cancel

$$\prod_v w'_{s,v}(g_v) \cdot L(1, -10s)$$

from both sides of (5.13.4), and change s to $s - 1/5$ we get (5.13) and therefore the theorem.

We put

(5.14) $$\gamma(s, \rho^*, {}^\circ\pi_v^*, {}^\circ\chi_v) = \gamma(s + 3s_2, \rho, {}^\circ\pi_v, {}^\circ\chi_v)$$

and

(5.15) $$\gamma_S(s, \rho^*, {}^\circ\pi^*) = \gamma_S(s + 3s_2, \rho, {}^\circ\pi).$$

Then we have:

COROLLARY 5.10: *Let ${}^\circ\pi^*$ be a cusp form on $GL_2(\mathbb{A})$. Assume its restriction to the center of $GL_2(\mathbb{A})$ is unramified. Denote by S the corresponding set of ramified places. Define $\gamma_S(s, \rho^*, {}^\circ\pi^*)$ as above and put*

$$L_S(s, \rho^*, {}^\circ\pi^*) = \prod_{v \notin S} L(s, \rho^*, {}^\circ\pi_v^*),$$

where the local factors are defined as before. Then

(5.16) $$L_S(1-s, \tilde{\rho}^*, {}^\circ\pi^*) = \gamma_S(s, \rho^*, {}^\circ\pi^*) L_S(s, \rho^*, {}^\circ\pi^*),$$

where $\tilde{\rho}^*$ denotes the contragredient representation of ρ^*.

PROPOSITION 5.11: *Let v be any place, infinite or finite in which case we assume that ${}^\circ\pi_v^*$ is not supercuspidal. Then*

$$\gamma(s, \rho^*, {}^\circ\pi_v^*, {}^\circ\chi_v) = \prod_{i=1,2,4,5} |\mu_{\alpha,v}|^{-\langle\nu,\gamma_i\rangle(s+3s_2)} \cdot \gamma(\eta_v^{-\langle\beta,\gamma_i\rangle}, -\langle\nu, \gamma_i\rangle(s+3s_2)).$$

Appendix

In [13], Langlands proved that, when v is an infinite place and $\mathrm{Re}(s)$ is sufficiently small, the image of

$$\Pi_v = \operatorname*{ind}_{P_v \uparrow G_v} {}^\circ\pi_v \otimes \delta_{P,v}^s$$

under the intertwining operator

$$\int_{N_v} f_v(gnw_0)dn \quad (f_v \in V_v, g \in G_v),$$

which is a subspace of

$$\Pi'_v = \operatorname*{ind}_{P_v \uparrow G_v} {}^\circ\pi'_v \otimes \delta_{P,v}^{-s},$$

is in fact irreducible.

Let us assume the following conjecture.

CONJECTURE: *Assume v is a finite place; then for* $\mathrm{Re}(s)$ *sufficiently large, the image of Π_v under the above intertwining operator is irreducible (by means of analytic continuation).*

COROLLARY TO CONJECTURE: *Assume ${}^\circ\pi_v$ is a component of a cusp form on $PGL_2(\mathbb{A})$ (v a finite place). Then for* $\mathrm{Re}(s)$ *sufficiently large, the image of the space of Π_v under the above intertwining operator, is equivalent to the unique non-degenerate quotient of Π_v.*

At any rate we should have:

PROPOSITION: *Assume* $^\circ\pi_v$ *is a component of a cusp form on* $PGL_2(\mathbb{A})$ *(*v *finite or infinite). Then for* $\mathrm{Re}(s)$ *sufficiently large, the image of the space of* Π_v *under the above intertwining operator is non-degenerate.*

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