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EISENSTEIN SERIES AND FIELDS OF DEFINITION

M. Karel*

Introduction

In this paper we prove two main results about Eisenstein series and then use these to obtain information about fields of definition for algebraic models of arithmetic quotients of bounded symmetric domains. Our results are valid for tube domains with zero-dimensional rational boundary component; however, for the sake of simplicity, in this introduction we consider only the case of a congruence subgroup Γ of $\mathrm{Sp}(2n, \boldsymbol{Z})$ acting on Siegel's half space $D = \mathrm{Sp}(2n, \boldsymbol{R})/U(n)$ of quadratic forms on $\boldsymbol{C}^n$ with positive definite imaginary part.

Firstly, we attach Eisenstein series to the various point-cusps (§2) and show that all their Fourier coefficients lie in a cyclotomic number field. For congruence subgroups of $\mathrm{SL}(2, \boldsymbol{Z})$, this generalizes results of H. Petersson [30]. Also, our results include some results of H. Klingen, [25], for congruence subgroups of the Hilbert-Blumenthal groups. Our method is reduction to the special case previously treated by Tsao, [36], and by Baily, [7]. The proof is completed in §5.

Secondly, we characterize those arithmetic subgroups Γ such that the field k_Γ of Γ-automorphic functions is generated by weighted homogeneous quotients of Eisenstein series. Namely, we show in §6 that the Eisenstein series generate k_Γ if and only if Γ is maximal among those arithmetic subgroups with a given set of cusps. We call such Γ *saturated.* Using this characterization, L.-C. Tsao has shown how to recover some results of *U.* Christian; see §6. Our proof is a recasting of an argument used by Siegel and, more recently, by Baily in [4].

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As a consequence of the above results, we obtain a cyclotomic field of definition k for the Satake compactification $\mathfrak{V}(\Gamma)$ of $\Gamma\backslash D$ whenever Γ is a saturated congruence subgroup. The field k is natural in the following sense: we may identify the homogeneous co-ordinate ring of $\mathfrak{V}(\Gamma)$ with a subalgebra of Γ-automorphic forms, and then the k-rational functions are precisely the automorphic forms with all their Fourier coefficients in k. Furthermore, certain morphisms are defined over $\boldsymbol{Q}_{ab}$ and we use this to show that the cusps of Γ and the Shimura variety $\Gamma\backslash D$ are also defined over $\boldsymbol{Q}_{ab}$.

In the case of Siegel's upper half-space, for example, Shimura has more precise results about fields of definition; see [32] and [33]. Furthermore, Shimura's methods apply to classical domains not treated in this paper. However, computations of Tsao show that our methods apply to the exceptional domain of complex dimension twenty-seven, which has not been treated by Shimura's methods; see §6. Hopefully, some of our techniques will be of use for a unified treatment of these problems on a larger class of domains.

The research leading to this paper was done largely while the author was visiting the Institute for Advanced Study in Princeton. It is a pleasure to acknowledge the hospitality of the Institute and to thank A. Borel, G. Prasad, M. Schlessinger and L.-C. Tsao for fruitful conversations and correspondence. Special thanks are due to W.L. Baily, Jr., who read a preliminary version of this paper and offered suggestions that led to several improvements.

§0. Notation and Conventions

0.1: As usual, $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$ and $\boldsymbol{C}$ denote, respectively, the ring of integers and the fields of rational, real and complex numbers. If $n \in \boldsymbol{Z}$ let $\boldsymbol{Z}(>n) = \{j \in \boldsymbol{Z} : j > n\}$. Let $\boldsymbol{Z}_p$ be the ring of p-adic integers and let $\hat{\boldsymbol{Z}} = \Pi_p \boldsymbol{Z}_p$ with p ranging over the rational primes. Let $\boldsymbol{Q}_{ab}$ be the compositum of all cyclotomic fields in $\boldsymbol{C}$. Fix a square root $(-1)^{1/2}$ of -1 in $\boldsymbol{C}$.

0.2: Suppose that F is a field. If F' is a finite algebraic extension of F, let $\mathcal{R}_{F'/F}$ be the ground field restriction functor of Weil [40; 1.3]. If E is any normal algebraic extension of F, let $\mathrm{Gal}(E/F)$ be the Galois group of the extension and endow it with the Krull topology.

Suppose that F is a number field and let $\boldsymbol{V}(F)$, $\boldsymbol{V}_\infty(F)$ and $\boldsymbol{V}_{na}(F)$ be, respectively, the sets of all praces of F, all archimedean places of F and all non-archimedean places of F. If $v \in \boldsymbol{V}(F)$, then F_v is one of the completions of F that constitute the equivalence class v. We let ${}_F\boldsymbol{A}$ be the ring of F-adèles and if x is an F-idèle we let $|x|_{\boldsymbol{A}}$ be the idèle modulus of x; cf. [41; Chap. IV].

0.3: Suppose that G is a group. If $g, x \in G$, let $\operatorname{Int}(g)x = g \cdot x \cdot g^{-1}$. If H is a subgroup of G, let $\mathscr{Z}_G(H)$ (*resp.* $\mathscr{N}_G(H)$) be the centralizer (*resp.* normalizer) of H in G. Thus, $\mathscr{C}_G = \mathscr{Z}_G(G)$ is the center of G. If M is a subset of G, we let ${}^xM = \operatorname{Int}(x)M$ and $M^x = x^{-1} \cdot M \cdot x$ for $x \in G$. Let $\mathscr{D}G = [G, G]$ be the commutator subgroup of G.

A *character* χ of G is a homomorphism from G to the multiplicative group $\boldsymbol{C}^\times$; χ is a *unitary character* if it takes its values in the unit circle $\{z \in \boldsymbol{C}: z\bar{z} = 1\}$. Let e be the unitary character $x \mapsto \exp(2\pi(-1)^{1/2} \cdot x)$ of $\boldsymbol{R}$.

In case G is a topological group, G^0 is the connected component of G containing the identity element 1_G. A character of a topological group is understood to be continuous.

0.4: All our algebraic groups are linear, and we follow the notation of Borel's book, *Linear Algebraic Groups*, [12]. In particular, if R is a subring of a field F and if G is an F-group, then $G(R)$ is the group of R-rational points of G. We let $G_v = G(F_v)$ if v is a place of F; $\operatorname{Lie}(G)$ is the Lie algebra of G. Each object defined for abstract groups or topological groups in 0.3 has an obvious analogue for algebraic groups.

0.5: The phrase 'for almost all' will mean 'for all but finitely many'.

§1. Preliminaries

1.1: The situation that we investigate generalizes the following familiar one. The symplectic group $G = \operatorname{Sp}(2n)$ has real points $G(\boldsymbol{R})$ acting by linear fractional transformations on the Siegel half-space D_n of n-rowed symmetric matrices with positive definite imaginary part; the subgroup of real affine transformations $x \mapsto A(x) + B$ is the group $P(\boldsymbol{R})$ of real points of a parabolic $\boldsymbol{Q}$-subgroup P of G; an element of P, expressed as a 2-by-2 tableau of four n-by-n blocks, has lower left corner zero.

With respect to the maximal $\boldsymbol{Q}$-split torus T of diagonal elements in G, the root system is of Cartan-Killing type C, and P is the standard parabolic subgroup associated to the set $\boldsymbol{\Theta}$ of short simple roots, as in [14; 4.2]. Namely, P is generated by the positive root groups and by the centralizer Z_Θ of T_Θ, the largest torus on which all roots in Θ are trivial. Elements of T_Θ, expressed as tableaux, have scalar matrices as upper left and lower right corners.

Let I_n be the n-rowed identity matrix, so $o_n = (-1)^{1/2} I_n$ lies in the half-space D_n. The isotropy group of o_n in $G(\boldsymbol{R})$ is a maximal compact subgroup isomorphic to the unitary group $U(n, \boldsymbol{R})$. Let $K_\infty = U(n)$. Then the canonical automorphy factor

$$\mathfrak{z} : G(\boldsymbol{R}) \times D_n \to K_\infty(\boldsymbol{C}) \cong GL(n, \boldsymbol{C})$$

maps (g, x) to $cx + d$ if $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in block form; see [11; p. 202]. In [34], Siegel considered the automorphy factors $j(g, x) = \det(\mathfrak{z}(g, x))^{-\ell}$ for even integers $\ell > n + 1$. He showed the Eisenstein series

$$E_\ell(x) = \sum_\gamma j(\gamma, x), \qquad (\gamma \in \Gamma \cap P \backslash \Gamma), \tag{1}$$

where $\Gamma = \mathrm{Sp}(2n, \boldsymbol{Z})$, converges normally to a Γ-automorphic form. It is easy to see that E_ℓ has a Fourier expansion

$$E_\ell(x) = \sum_\lambda a(\lambda) \cdot \lambda(x), \tag{2}$$

where λ runs through a certain lattice of $\boldsymbol{Q}$-rational characters on U, the unipotent radical of P. Siegel showed that each $a(\lambda)$ is a rational number.

Replacing the full Siegel modular group $\Gamma = \mathrm{Sp}(2n, \boldsymbol{Z})$ by the congruence subgroup $\Gamma(m)$ of $2n$-rowed symplectic matrices congruent to the identity matrix modulo m, we construct additional Eisenstein series and show that their Fourier coefficients $a(\lambda)$ are cyclotomic numbers. It is convenient to use an adèlic setting, as in [32], for example, so we actually work with a compact open subgroup K of the group $G(\boldsymbol{A}_{na})$ of finite adèles instead of working directly with $\Gamma(m)$.

For us, a subgroup Δ of $G(\boldsymbol{Q})$ is a *congruence subgroup* if

$$\Delta = (G(\boldsymbol{R})^o \cdot K) \cap G(\boldsymbol{Q}).$$

For example, we realize $\Gamma(m)$ in this way by choosing $K = K(m)$, the subgroup of $2n$-rowed symplectic matrices g with entries g_{ij} in $\hat{\boldsymbol{Z}}$ satisfying $g_{ij} - \delta_{ij} \in m\hat{\boldsymbol{Z}}$, where both m and the Kronecker symbol δ_{ij} are viewed as rational adèles. Similarly, Hecke's congruence subgroups $\Gamma_0(m) \subseteq SL(2, \boldsymbol{Z})$ arise by taking as K the subgroup

$$(3) \qquad L_0(m) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \hat{\boldsymbol{Z}}) : c \in m\hat{\boldsymbol{Z}}\}.$$

This discussion also extends to Hilbert-Siegel modular groups: one applies to Sp$(2n)$ Weil's ground field restriction functor $\mathscr{R}_{k/\boldsymbol{Q}}$, with k a totally real number field, to obtain G; then T and P are naturally derived for G from the corresponding subgroups in Sp$(2n)$. We identify $G(\boldsymbol{A})$ with Sp$(2n, {}_k\boldsymbol{A})$ as in [40; 1.33] and thus obtain a subgroup $K(m)$ for each integral adèle m of k. If $\mathfrak{m}$ is the integral *ideal* of k that corresponds to the *adèle* m, and if $n = 1$ so that we are in the Hilbert modular case, then $K \cdot G(\boldsymbol{R})^0 \cap G(\boldsymbol{Q})$ is the group $\Gamma(\mathfrak{m})$ of [25].

Later we shall refer to the choices of G, T, P and K just described. However, our methods work in a broader setting to be described in detail in §1.2. We start with a reductive algebraic group G defined over $\boldsymbol{Q}$and such that the connected Lie group $G(\boldsymbol{R})^0$ acts transitively on a bounded symmetric domain D. For simplicity we assume that G modulo its center is $\boldsymbol{Q}$-simple. Also, we must require, roughly, that D have product structure compatible with the complex structure of D and with the $\boldsymbol{Q}$-structure of G. More precisely, we require that D be a tube $\boldsymbol{R}^n \times (-1)^{1/2}\mathfrak{C} \subseteq \boldsymbol{C}^n$ over a positive open homogeneous cone $\mathfrak{C}$ in $\boldsymbol{R}^n$ and that the real translations of $D \subseteq \boldsymbol{C}^n$ generate an algebraic $\boldsymbol{Q}$-subgroup U of G. This requirement insures that each automorphic form on D has a Fourier expansion like (2). Baily has called D a *rational tube domain* and has shown that a bounded symmetric domain D under the action of $G(\boldsymbol{R})^0$ is a rational tube domain if and only if the relative $\boldsymbol{Q}$-root system of G is of Cartan-Killing type C; see [7], [28; 4.4, Remark, p. 294] and [9; 2.9].

Let P be the normalizer of U in G, so P is a parabolic $\boldsymbol{Q}$-subgroup with unipotent radical U. To define Eisenstein series we must assume that

$$P(\boldsymbol{Q}_p)G(\boldsymbol{Z}_p) = G(\boldsymbol{Q}_p)$$

for each finite rational prime p. This is actually not restrictive, but rather amounts to a convenient choice of co-ordinates; see 1.3.0. Generally, it is best to have all our assumptions stated in terms of G and its subgroups rather than in terms of the domain D. In particular, instead of fixing a holomorphic factor of automorphy on D analogous to $(\det \circ \mu)^{-\ell}$, we fix a rational character analogous to the character $\det^{-\ell}$ on $GL(n, \boldsymbol{C})$ in the example above.

1.2. We are now prepared to introduce the configuration to be studied. It is a quintuple $\mathfrak{A} = (\mathrm{G}, \mathrm{T}, \mathrm{P}, \rho, K)$, where G is a connected reductive matrix group defined over $\boldsymbol{Q}$, T is a maximal $\boldsymbol{Q}$-split torus, P is a maximal parabolic $\boldsymbol{Q}$-subgroup of G containing T, ρ is a rational character of P defined over $\boldsymbol{Q}$, and K is a compact open subgroup of $G(\hat{\boldsymbol{Z}}) = \Pi_p G(\boldsymbol{Z}_p)$, with p running through the finite rational primes.

We make the following assumptions:

(A0) *For each finite rational prime p,* $\mathscr{D}G(\boldsymbol{Z}_p)$ *is a good maximal compact subgroup of* $\mathscr{D}G(\boldsymbol{Q}_p)$ *in the sense of Bruhat-Tits* [16; 4.4.1]. In particular, for each parabolic $\boldsymbol{Q}$-subgroup P' of G there is an Iwasawa decomposition

$$G(\boldsymbol{Q}_p) = P'(\boldsymbol{Q}_p)G(\boldsymbol{Z}_p).$$

(A1) *The symmetric space D of maximal compact subgroups of* $G(\boldsymbol{R})$ *carries a* $G(\boldsymbol{R})^0$*-invariant complex structure that makes it an hermitian symmetric space of non-compact type.*

(A2) *The relative* $\boldsymbol{Q}$*-root system* $\Sigma(T, G)$ *of T in G,* [14; 5.1], *is connected and reduced (i.e., the double of a root is never a root), hence of type C, see* [9; 2.9].

(A3) *For some ordering of* $\Sigma(T, G)$, *P is the standard parabolic* $\boldsymbol{Q}$*-subgroup associated to the subset* Θ *of all short simple roots of* $\Sigma(T, G)$ *as in* [14; 4.2]. That is, the largest subtorus T_Θ of T on which all roots in Θ are trivial has centralizer Z_Θ, which is a Levi component of P.

(A4) $\rho(b) > 0$ *for each* $b \in P(\boldsymbol{R}) \cap G(\boldsymbol{R})^0$.

Let M be a maximal compact subgroup of $G(\boldsymbol{R})$. Then owing to (A0) there is a function $\psi = \psi_M$ defined on $G(\boldsymbol{A})$ by $\psi(g\omega) = |\rho(g)|_{\boldsymbol{A}}$ for $g \in P(\boldsymbol{A})$, $\omega \in MG(\hat{\boldsymbol{Z}})$.

(A5) *The series*

$$\sum_{\gamma} \psi(\gamma g), \qquad (\gamma \in P(\boldsymbol{Q}) \backslash G(\boldsymbol{Q}), \quad g \in G(\boldsymbol{A}))$$

converges normally on $G(\boldsymbol{A})$.

1.3: Now we discuss the various assumptions in greater detail.

1.3.0: We owe to A. Borel a proof of the existence of co-ordinates on G satisfying (A0). More precisely, one has the following.

Let F be a global field with ring of integers $\mathfrak{o}_F$. If v is a place of F, let F_v be the corresponding completion and $\mathfrak{o}_v$ the valuation ring in F_v. Now suppose that G is an *arbitrary semi-simple* affine F-group, and let T be a maximal F-split torus of G. Then G has an embedding, which is defined over F, into some $GL(n)$, and which satisfies for each finite place v of F:

(i) $G(\mathfrak{o}_v)$ is a special maximal compact subgroup of $G(F_v)$.

(ii) $T(\mathfrak{o}_v)$ is the greatest compact subgroups of $T(F_v)$.

This result extends to reductive groups by an easy argument on lattices over F, as in [41; V, §2]. We leave this to the reader. Note that (ii) is not used until §5.

1.3.1: Assumptions (A1) and (A2) are basic, (A1) concerning the geometry of the domain D and (A2) concerning the relation between the geometry of D and the $\boldsymbol{Q}$-structure of G. By contrast, (A3) simply describes P. Both (A4) and (A5) are needed when we define the Eisenstein series. Since we obtain a holomorphic factor of automorphy by composing the canonical automorphy factor (or, rather, a Cayley transform of it) with ρ, specifying ρ amounts to specifying the weight of the Eisenstein series. Thus, (A4) corresponds to Siegel's requirement that the weight be even and (A5) corresponds to the requirement that the weight be high enough to give convergence of Eisenstein series. By [21; Theorem 3], we obtain a suitable character ρ from any rational character ρ_0 defined over $\boldsymbol{Q}$ simply by choosing a suitable power of ρ_0.

Each of our assumptions is compatible with passage to the boundary. In particular, we have avoided restricting ourselves to powers of the functional determinant as automorphy factors because the automorphy factor induced on a boundary component by such a power need not be of the same type; see [9].

1.4. Roots, Jordan Algebras and Cartan Involutions: We now take a closer look at how the geometry of D relates to the $\boldsymbol{Q}$-structure of G. It is easy to check that the unipotent radical U of P is abelian, [14; 2.5], hence U carries naturally the structure of a vector space over $\boldsymbol{Q}$; see [14; 3.17]. We denote by $\mathrm{Int}_G(T_\Theta)$ the image of T_Θ, the greatest torus on which all short simple roots of $\Sigma(T, G)$ are trivial, in the group $\mathrm{Int}(G)$ of inner automorphisms of G. The proof of the following lemma was suggested by G. Prasad; see also [14; 3.17]. We use this lemma in §5.

1.4.1. Lemma: *If G satisfies assumptions* (A1) *and* (A2), *then there* *exists a one-parameter* $\boldsymbol{Q}$*-subgroup* $\mu : GL(1) \to \mathrm{Int}_G(T_\Theta)$ *such that for each* $s \in GL(1)$*, the action of* $\mu(s)$ *on* U *is just scalar multiplication by* s.

The action Int of T_Θ on U is given by the character χ corresponding to the long simple $\boldsymbol{Q}$-root. The action of T_Θ on the opposite horocycle U^- is given by χ^{-1}. Since $\mathscr{D}G$ is $\boldsymbol{Q}$-simple and since U and U^- generate a normal $\boldsymbol{Q}$-subgroup of $\mathscr{D}G$, it follows that U and U^- generate $\mathscr{D}G$. Therefore, Int is trivial on the kernel of χ, so χ induces an isomorphism from $\mathrm{Int}_G(T_\Theta)$ to $GL(1)$. Let $\mu : GL(1) \to \mathrm{Int}_G(T_\Theta)$ be the inverse morphism.

1.4.2: as indicated in 1.1, assumptions (A1) and (A2) imply that D can be realized as a tube $\boldsymbol{R}^m \times (-1)^{1/2}\mathfrak{C}$ over some homogeneous positive open cone $\mathfrak{C}$ in $\boldsymbol{R}^m$. Let L and B be the algebraic subgroups of G generated by the real linear maps preserving D and by the real translations, respectively. Then L is a Levi complement to V in the parabolic $\boldsymbol{R}$-subgroup $L \cdot V$. Since $\mathrm{Int}_G(L)$ is the algebraic subgroup of $GL(m)$ generated by elements of $GL(m, \boldsymbol{R})$ that preserve $\mathfrak{C}$, the relative $\boldsymbol{R}$-root system of L has Cartan-Killing type A; see [1; II, §3.8]. It follows that $L \cdot V$ is conjugate to P and that L is conjugate to $Z_\Theta = \mathscr{Z}(T_\Theta)$ by an element of $G(\boldsymbol{R})$. This enables us to embed D in $U(\boldsymbol{C})$ so that, writing the group law additively, each $u \in U(\boldsymbol{R})$ acts on $D \subset U(\boldsymbol{C})$ by taking $x \in D$ to $x + u$, and so that the action of $Z_\Theta(\boldsymbol{R})$ on D is $\boldsymbol{R}$-linear. In fact, the action of $z \in Z_\Theta(\boldsymbol{R})$ on D is conjugation:

$$x + z(u) = zuz^{-1}(x), \qquad (x \in D,\ u \in U(\boldsymbol{R}))$$

We remark that, presupposing (A2), then assumption (A1) is equivalent to requiring that $P(\boldsymbol{R})$ have an open orbit $\mathrm{Int}(P(\boldsymbol{R})) \cdot e$ under conjugation on $U(\boldsymbol{R})$ and that the stabilizer of e in $Z_\Theta(\boldsymbol{R})$ be maximal compact; see [1; II, §1.3, Prop. 6].

1.4.3: By the work of Vinberg [39] and Koecher, one knows that, corresponding to the cone $\mathfrak{C}$ and to any choice of base point $e \in \mathfrak{C}$, U carries a unique formally real Jordan algebra structure J defined over $\boldsymbol{Q}$, with identity element e, and characterized by

$$\mathfrak{C} = \exp_J(U(\boldsymbol{R})) = \{a^2 : a \text{ is } J\text{-invertible in } U(\boldsymbol{R})\},$$

where $\exp_J$ is the Jordan algebra exponential; see [1; II, §2]. The geodesic symmetry $\sigma = \sigma_e$ of the Riemannian symmetric space $\mathfrak{C}$, with respect to the base point e, is the birational map of U that sends each $x \in \mathfrak{C}$ to its J-inverse, which we denote simply $J(x)$. Moreover, σ induces an analytic involutive automorphism of Lie($Z_{\boldsymbol{\theta}}$) and hence of $Z_{\boldsymbol{\theta}}$, the Cartan involution $\sigma^{\#}$, and $\sigma^{\#}$ is $\boldsymbol{Q}$-rational, [1; II, §3]. We owe this remark to Baily.

Similarly, the geodesic symmetry ι of D with respect to the point $o = (-1)^{1/2}e$ induces a Cartan involution $\iota^{\#}$ of G. By [24; Prop. 3.5, p. 177], the two involutions, $\sigma^{\#}$ and $\iota^{\#}$ coincide on $Z_{\boldsymbol{\theta}} \cap \mathscr{D}G$. In particular, $\iota(Z) = -J(Z)$ if $Z \in D$.

Referring back to the example described in 1.1, $G = \mathrm{Sp}(2n)$, we see that via its decomposition into root spaces, U, the unipotent radical of P, is naturally identified with the vector space of n-rowed symmetric matrices. The cone $\mathfrak{C}$ in this case is just the set of positive definite quadratic forms, and the Jordan algebra corresponding to this cone with the identity matrix e of rank n as base point has product $x \circ y = \frac{1}{2}(xy + yx)$, where xy is the ordinary matrix product of x and y. The geodesic symmetry at e is realised by the element w of $G(\boldsymbol{Q})$ given by the tableau $\left(\begin{smallmatrix} 0 & e \\ -e & 0 \end{smallmatrix}\right)$. Moreover, conjugation by w takes each diagonal matrix $t \in T$ to its inverse. These facts generalize.

We claim that, for suitable choice of base point e, the geodesic symmetry is realized by an element w in $\mathscr{D}G(\boldsymbol{Q}) \cap G(\boldsymbol{R})^0$ and that $\mathrm{Int}(w)$ acts as inversion on $T \cap \mathscr{D}G$. Indeed, let $w = e \cdot \iota^{\#}(e) \cdot e$. For $x \in D$, then $w(x) = e - J(e - J(x + e))$. By Hua's identity, [35; 3.9 (16), p. 43], $w(x) = -J(x)$, as required. By [1, II, §3.7] we can find a maximal $\boldsymbol{Q}$-split torus S in $\mathscr{D}G \cap Z_{\boldsymbol{\theta}}$ that is stable under the Cartan involution $\iota^{\#}$ and such that $\iota^{\#}(t) = t^{-1}$ for each $t \in S$. Now S and $T \cap \mathscr{D}G$ are conjugate by some element $g \in Z_{\boldsymbol{\theta}}(\boldsymbol{Q}) \cap \mathscr{D}G$: $T \cap \mathscr{D}G = gSg^{-1}$. Since the centralizer of S meets all connected components of $H(\boldsymbol{R})$, where $H = Z_{\boldsymbol{\theta}} \cap \mathscr{D}G$, cf. [14; 14.4], we may assume that $g \in H(\boldsymbol{Q}) \cap H(\boldsymbol{R})^0$. Then the base point $g(e) = geg^{-1} \in U(\boldsymbol{Q}) \cap \mathfrak{C}$ corresponds to the Cartan involution $g\iota^{\#}g^{-1}$, which acts as inversion on $T \cap \mathscr{D}G$, as required.

1.5: For our purposes it is best to circumvent Harish-Chandra's realization of D and, instead, work directly with the embedding of D in $U(\boldsymbol{C})$ described above. We can do this by using an idea of Borel; see [35; 2.2.], also [4; 4.3] in a special case.

1.5.1: We can describe the action of $G(\boldsymbol{R})^0$ on D algebraically as follows. Let P^- be the parabolic subgroup opposite to P, so $P \cap P^- = Z_{\boldsymbol{\theta}}$. Then the multiplication map $U \times P^- \to G$ is injective and has Zariski-dense image UP^-. Therefore, if $g \in G$ then $g \cdot x \cdot P^- \cap U$ is non-empty for x in a Zariski-open subset of U and must then consist of a single point which we call $A_g(x)$. It is easy to see that $A_g(A_{g'}(x)) = A_{gg'}(x)$ in the obvious sense.

If $x \in U$, $u \in U$ and $z \in Z_\Theta$, then $A_u(x) = x + u$ while $A_z(x) = zxz^{-1}$. Thus, the birational map A_g is everywhere defined if $g \in P$. If, moreover, $g \in P \cap G(\boldsymbol{R})^o$, then A_g coincides with the usual action of g on D.

To show that A_g gives the usual action of g on D for each $g \in G(\boldsymbol{R})^0$, it suffices to check this for $g = w$ and to show that w and $P(\boldsymbol{R})^o$ generate $G(\boldsymbol{R})^o$. Since $P(\boldsymbol{R})^o$ is an open subgroup of $P(R)$ and the map $(x, y) \mapsto xwy$ of $P(\boldsymbol{R}) \times P(\boldsymbol{R})$ into $G(\boldsymbol{R})$ is submersive, the double coset $P(\boldsymbol{R})^o \cdot w \cdot P(\boldsymbol{R})^o$ is open in $G(\boldsymbol{R})^o$. Therefore, w and $P(\boldsymbol{R})^o$ generate an open, hence closed, subgroup of the connected Lie group $G(\boldsymbol{R})^o$, so they generate $G(\boldsymbol{R})^o$. Let $x \in D$. Then the Jordan algebra inverse $J(x)$ exists and we must show that A_w takes x to $J(x)^{-1}$, i.e., $wx \in J(x)^{-1}P^-$ or, equivalently, $wJ(x)wxw^{-1} \in P$. It suffices then to show that the element

$$Q(x) = xwJ(x)wxw^{-1}$$

lies in Z_Θ, i.e., that $Q(x)$ acts linearly on D. Write the group law on U additively. Then w takes $y \in D$ to $-J(y)$. Consequently, $Q(x)$ takes y to $x - J(J(x) - J(x - J(y)))$. By Hua's identity, [35; 3.9 (16), p. 43], $x - J(J(x) - J(x - J(y))) = xyx$, where the Jordan algebra product $xyx = (xy)x = x(yx)$. Clearly, $Q(x)$ is linear, as required.

1.5.2: We can now define a rational factor of automorphy

$$\mathfrak{z} : G \times U \to Z_\Theta$$

by $gx \in A_g(x) \cdot \mathfrak{z}(g, x) \cdot U^-$ whenever $A_g(x)$ is defined, $(g \in G, x \in U)$. The cocycle relation

$$\mathfrak{z}(gg', x) = \mathfrak{z}(g, A_{g'}(x))\mathfrak{z}(g', x)$$

follows immediately because

$$\begin{aligned} A_{gg'}(x)\mathfrak{z}(gg', x)U^- &= gg'xU^- \\ &= gA_{g'}(x)U^-\mathfrak{z}(g', x) \\ &= A_g(A_{g'}(x))\mathfrak{z}(g, A_{g'}(x))U^-\mathfrak{z}(g, x) \\ &= A_g(A_{g'}(x))\mathfrak{z}(g, A_{g'}(x))\mathfrak{z}(g, x)U^-. \end{aligned}$$

For each $g \in G(\boldsymbol{R})^o$ the map $x \mapsto \mathfrak{z}(g, x)$ from D to $Z_\Theta(\boldsymbol{C})$ is everywhere defined and rational, hence holomorphic.

As one can verify by using a suitable Cayley transform, [28; §6], our rational automorphy factor $\mathfrak{z}$ is the canonical automorphy factor described, for example, in [11; §4].

For a tableau $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{Sp}(2n)$, and $x \in D_n$, we have $\mathfrak{z}(g, x) = cx + d$, an element of $GL(n, \boldsymbol{C})$.

1.5.3: To obtain the usual $\boldsymbol{C}^\times$-valued holomorphic automorphy factors, such as powers of the functional determinant, we compose $\mathfrak{z}$ with any rational character of Z_Θ (or, equivalently, of P); see [7; p. 28]. Let $j = \rho \circ \mathfrak{z}$.

We obtain one such character of P as follows. Let $\det_J(x)$ denote the generic norm of an element $x \in U$ with respect to the Jordan algebra structure J. Define the character $\chi_J : P \to GL(1)$ by setting

$$\chi_J(g) = \det_J(A_g(e)), \quad (g \in P)$$

Then $\chi_J(g)\det_J(x) = \det_J(g(x))$ for each $g \in G(\boldsymbol{R})^o \cap Z_\Theta$ and each x in D; see [35; 1.5].

Define $\mathfrak{z}_J(g, x) = \chi_J \circ \mathfrak{z}(g, x)$ for $g \in G$, $x \in U$, provided $\mathfrak{z}(g, x)$ exists. If $x \in D$, then $\mathfrak{z}_J(w, x) = \det_J(x)^{-2}$ because $\mathfrak{z}(w, x) = w^{-1}Q(x)w$ maps $y \in U$ to $J(x)yJ(x)$, and $\chi_J(\mathfrak{z}(w, x)) = \det_J(J(x)^2) = \det_J(x)^{-2}$.

Since $\det_J$ is positive on the cone $\mathfrak{C}$, and since $Z_\Theta \cap G(\boldsymbol{R})^o$ preserves $\mathfrak{C}$, it follows that $\chi_J(b)$ is positive for each b in $P \cap G(\boldsymbol{R})^o$. That is, all the powers of χ_J satisfy assumption (A4).

This much is enough for our present needs, but it is also true, in case $\mathscr{D}G$ is simply connected, that χ_J is the square of a rational character $\rho_{1/2}$ defined over $\boldsymbol{Q}$. First note that the simple connectivity of $\mathscr{D}G$ implies that $\mathscr{D}G(\boldsymbol{R})$ is a connected Lie group; this is well known and follows easily from [24; VI, Th. 1.1(iii)] by applying [24; VIII, Th. 7.2] to complex conjugation on a maximal compact subgroup H of $G(\boldsymbol{C})$ such that $H \cap G(\boldsymbol{R})$ is a maximal compact subgroup of $G(\boldsymbol{R})$. We owe the above remark to Baily. Let S be a maximal torus of Z_Θ defined over $\boldsymbol{Q}$ and containing T. Note that $S \cap \mathscr{D}Z_\Theta$ is connected since it centralizes its connected component $(S \cap \mathscr{D}Z_\Theta)^o$, which is a maximal torus of $\mathscr{D}Z_\Theta$. Since $\mathscr{D}G(\boldsymbol{R})$ is connected, χ_J must be positive on $Z_\Theta(\boldsymbol{R})$ and its restriction to S must therefore be the square of a character ρ_0 defined over $\boldsymbol{Q}$. Since ρ_0 is invariant under conjugation by elements of Z_Θ, the argument on page 16-04 of [17] shows that ρ_0 is trivial on the torus $S \cap \mathscr{D}Z_\Theta$; therefore, ρ_0 extends to a character $\rho_{1/2}$, which is defined over $\boldsymbol{Q}$, on Z_Θ, as required.

1.5.4. Root spaces and the Peirce decomposition: Having established an algebraic action of G on U, we can see that the decomposition of U into root spaces for the maximal $\boldsymbol{Q}$-split torus T is also a Peirce decomposition for the Jordan algebra structure J on U. We may assume that G is a simply connected semi-simple group. In case $G = \mathrm{Sp}(2n)$ with T as in 1.1, the identification of Peirce and root-space decompositions is apparent. In general, we choose a maximal set of orthogonal idempotent elements $e_1, \ldots, e_n$ in $U(\boldsymbol{Q})$ corresponding to the maximal $\boldsymbol{Q}$-split torus T as in [1; II, §3.7]. For each e_i there is a one-parameter $\boldsymbol{Q}$-subgroup $a_i : GL(1) \to T$ such that $e_i = \frac{1}{2}a_i'(1)e_i$ where $a_i' = (\mathrm{d}/\mathrm{d}s)a_i(s)$; indeed, by [1; II, §3.2], there is such a map into $\mathrm{Int}_{Z_\Theta}(T)$, and we can lift it back to T. Furthermore, the a_i's are a basis of the vector space of one-parameter $\boldsymbol{Q}$-subgroups. Let $\alpha_1, \ldots, \alpha_n$ be the dual basis of the lattice of rational characters of T defined over $\boldsymbol{Q}$; this is legitimate because $\mathscr{D}Z_\Theta$ is simply connected. If $\tau \in T$, then $\tau = \prod_{i=1}^n a_i(s_i)$ with $s_i = \alpha_i(\tau)$.

Let $U = \bigoplus_{i \le j} U_{ij}$ be the Peirce decomposition with respect to the set of idempotents $\{e_1, \ldots, e_n\}$. That is, $u \in U_{ij}$ if $e_k \cdot u = \frac{1}{2}(\delta_{ik} + \delta_{jk})u$, (Kronecker δ). By definition, $a_k(s)$ acts on U as $\exp(2\log(s)e_k)$ so, if $u \in U_{ij}$ then $a_k(s)u = s^d \cdot u$ with $d = \frac{1}{2}(\delta_{ik} + \delta_{jk})$. In particular, $\tau(u) = \alpha_i(\tau)\alpha_j(\tau)u$ if $u \in U_{ij}$. Thus, the characters $\alpha_i + \alpha_j$ are roots of T in U. The identification of Peirce and root-space decompositions is clear.

The roots of T in Z are calculated in [1; II, §3.8]; they are the characters $\alpha_i - \alpha_j$ with $i \neq j$. Therefore, one basis of simple roots is $\{\alpha_1 - \alpha_2, \alpha_2 - \alpha_3, \ldots, \alpha_n - \alpha_{n-1}, 2\alpha_n\}$.

1.6. Exponential sums: Let F be a non-archimedean local field with valuation denoted ord, valuation ring $\mathfrak{o}$ and prime ideal $\mathfrak{p}$. Let π be a generator for $\mathfrak{p}$ and let χ be a non-trivial character of $F/\mathfrak{o}$. Suppose that the characteristic of F is zero. Let W be a finite-dimensional vector space over F, let $E = \mathrm{End}_F(W)$ be the ring of F-linear endomorphisms of W, and let H' be a compact open subgroup of the F-rational points $H(F)$ of some F-subgroup H of $SL(W)$. Suppose that $W(\mathfrak{o})$ is an F-lattice in W, cf. [41; Ch. III, §2, Definition 2], and that $W(\mathfrak{o})$ is H'-stable. Let τ be an F-linear functional on W and suppose that, on $W(\mathfrak{o})$, τ takes values in $\mathfrak{o}$. Then, for any $w \in W(\mathfrak{o})$, define $f_w : E \to F$ by setting $f_w(T) = \tau(T(w))$ for each $T \in E$.

Let $E(\mathfrak{o})$ be the stabilizer of $W(\mathfrak{o})$ in E. If $a \in H$, let ${}_aH$ be the linear subspace of E such that ${}_aH + a$ is the embedded tangent space of H at a. Let $\mathfrak{p}(a) = {}_aH(F)$ and, for $w \in W(\mathfrak{o})$, put

$$m_a(w) = \inf_{T\in E(\mathfrak{o})\cap\mathfrak{p}(a)}(\mathrm{ord}(f_w(T)), \tag{1}$$

$$m(w) = \sup_{a\in H'}(m_a(w)). \tag{2}$$

If $g, h \in E$, $\nu \in \boldsymbol{Z}$, write $h \equiv g (\mathrm{mod}\, p^\nu)$ whenever $h - g \in \pi^\nu E(\mathfrak{o})$. Suppose that for each $a \in H'$, $f_w \not\equiv 0$ on $\mathfrak{p}(a)$. From [3; Prop. 1] it follows that the map $a \mapsto m_a(w)$ is locally constant. Since H' is compact and open it is possible to find $k \in \boldsymbol{Z}$ and $a_1, \ldots, a_r \in H$ such that H' is covered by the pairwise disjoint family of open discs $a_i + \pi^k E(\mathfrak{o})$ and such that $a \mapsto m_a(w)$ is constant on each disc. Let $\kappa(w)$ be the smallest such k. If $f_w \equiv 0$ on some $\mathfrak{p}(a)$, then $m(w) = \infty$, and we set $\kappa(w) = 0$.

1.6.1. LEMMA: *There is a constant ν_0 independent of $w \in W(\mathfrak{o})$ and such that, for each integer $\nu > \nu_0 + 2m(w) + \kappa(w)$,*

$$\sum_g \chi(\pi^{-\nu} f_w(g)) = 0, \qquad (g \in H' \bmod \mathfrak{p}^\nu).$$

This is [3; Theorem 2] except that we have replaced $H \cap W(\mathfrak{o})$ by the compact open subset H'. The proof is the same, using a version of Hensel's Lemma to linearize the problem.

1.6.2. DEFINITION: A vector $v \in W(\mathfrak{o})$ is said to be *primitive with respect to $W(\mathfrak{o})$ if $v \notin \pi W(\mathfrak{o})$.*

1.7: Suppose that F is a number field, v is a place of F and $x \in F_v^\times$. Choose an additive Haar measure on F_v and let $|x|_v$ be the modulus of multiplication by x. If v is a non-archimedean place, then the ring of integers of F_v is

$$\mathfrak{o}_v = \{x \in F_v : |x|_v \leq 1\}.$$

We normalize additive Haar measure $d_v x$ on F so that the volume of $\mathfrak{o}_v$ is 1. If $F_v = \boldsymbol{R}$, we let $d_v x$ be ordinary euclidean measure; if $F_v = \boldsymbol{C}$, we let $d_v z = (-1)^{1/2}\, \mathrm{d}z \cdot \mathrm{d}\bar{z}$.

For any $x \in {}_F\boldsymbol{A}$ and any place v of F, let x_v be the projection of x on F_v. If T is a set of places of F, let

$$\boldsymbol{A}_T = \{x \in {}_F\boldsymbol{A} : x_v = 0 \quad \text{if } v \notin T\}.$$

Then $\boldsymbol{A}_T$ is the union of open sets carrying the convergent product measure $\Pi_{v\in T}\, d_v x_v$, so there is a unique additive Haar measure $d_T x$

induced on $\boldsymbol{A}_T$ by $\Pi_{v\in T}\, d_v x_v$; see [40; 2.1.2]. If T and T' are disjoint subsets of $V(F)$, then $\boldsymbol{A}_T \times \boldsymbol{A}'_T = \boldsymbol{A}_{T\cup T'}$ and $d_T x \cdot d_{T'} x = d_{T\cup T'} x$. In case the symbol u is generic for an element of the unipotent radical U of P, we let $d_T u$ be the measure induced on $U_T = U(\boldsymbol{A}_T)$ by the coordinates of G.

Suppose that F/k is a finite extension of fields. Then the usual trace and norm from F to k extend to maps $\mathrm{Tr}_{F/K} : {}_F\boldsymbol{A} \to {}_k\boldsymbol{A}$ and $N_{F/K} : {}_F\boldsymbol{A} \to {}_k\boldsymbol{A}$, respectively.

Each character ψ of the additive group ${}_k\boldsymbol{A}$ restricts to a character ψ_v of $k_v = \boldsymbol{A}_{\{v\}}$. Let $\boldsymbol{\epsilon}$ be the unitary character of $\boldsymbol{A}$ that is trivial on $\boldsymbol{Q}\hat{\boldsymbol{Z}}$ and restricts to $\boldsymbol{e}$ on $\boldsymbol{Q}_\infty$; see 0.3. Define a unitary character $\boldsymbol{\epsilon}_F$ of ${}_F\boldsymbol{A}$ by letting $\boldsymbol{\epsilon}_F = \boldsymbol{\epsilon} \circ \mathrm{Tr}_{F/\boldsymbol{Q}}$.

§2. Eisenstein Series

2.0: In this paragraph we define the point-cusps, attach an Eisenstein series of given weight to each of them, and show that the generic Fourier coefficients of these Eisenstein series are integrals over $U(\boldsymbol{A})$. The Fourier expansion is taken with respect to the cusp at infinity, i.e., it is taken along U.

2.1: We begin by recalling some standard material about boundary components. With $\mathfrak{A} = (G, T, P, \rho, K)$ as in 1.2, we put $G^\infty = G(\boldsymbol{R})^o$ considered as a subgroup of $G(\boldsymbol{A})$, $K^+ = G(\boldsymbol{R})^o \cdot K\mathscr{C}_G(\boldsymbol{A}) \subseteq G(\boldsymbol{A})$, and $\Gamma = G(\boldsymbol{Q}) \cap K^+$, so Γ is an arithmetic subgroup of $G(\boldsymbol{Q})$ and $G(\boldsymbol{R})^o$. We embed D in its compact dual; see [24; V, §2]. Then the natural boundary of D decomposes into analytic subsets called boundary components; see [9; 1.5]. These are permuted by the natural action of $G(\boldsymbol{R})^o$ on the boundary. The normalizer $N_{\mathscr{F}}$ of a boundary component $\mathscr{F}$ is a parabolic subgroup of G^∞. That is, $N_{\mathscr{F}} = G^\infty \cap P_{\mathscr{F}}$ for some parabolic $\boldsymbol{R}$-subgroup $P_{\mathscr{F}}$ of G; $P_{\mathscr{F}}$ is uniquely determined. If $P_{\mathscr{F}}$ is defined over $\boldsymbol{Q}$, then the boundary component $\mathscr{F}$ is called *rational.*

To each parabolic $\boldsymbol{R}$-subgroup Q is attached a unique boundary component $\mathscr{F} = D(Q)$ such that $Q = P_{\mathscr{F}}$. We let $G(\boldsymbol{R})^o$ act on the set of $\boldsymbol{R}$-subgroups of G by conjugation. The correspondence between parabolic $\boldsymbol{R}$-subgroups of G and boundary components is then equivariant with respect to the natural action of $G(\boldsymbol{R})^o$ on the boundary. By [10; §7] there are only finitely many orbits for the action of Γ on the set of rational boundary components. These orbits are the *cusps* of Γ, and to each cusp $\mathrm{Int}(\Gamma) \cdot \mathscr{F}$ is attached a space

$\Gamma \cap P_{\mathscr{F}} \backslash \mathscr{F}$. These spaces are assembled to form the boundary for Satake's compactification of $\Gamma \backslash D$; see [9; §4].

The *zero-dimensional cusps* or *point-cusps* correspond to Γ-conjugacy classes among the parabolic $\boldsymbol{Q}$-subgroups that are $G(\boldsymbol{Q})$-conjugate to P. Since P is its own normalizer, the *point-cusps* also correspond to double cosets $P(\boldsymbol{Q})x\Gamma$ with $x \in G(\boldsymbol{Q})$. We let

$$\Xi^o = P(\boldsymbol{Q}) \backslash G(\boldsymbol{Q}) / \Gamma,$$

and identify it with the set of point-cusps.

2.2: Translation of the above material to adèlic language is straightforward. Let K_∞ be the normalizer of a maximal compact subgroup of $G(\boldsymbol{R})$, so D is identified with $G(\boldsymbol{R})/K_\infty$. Then $X = G(\boldsymbol{Q}) \backslash G(\boldsymbol{A}) / (K_\infty \times K)$ has finitely many connected components, each of them identified with an arithmetic quotient $g\Gamma g^{-1} \backslash D$, where $g \in G(\boldsymbol{Q})$; see [10; §2]. The connected component ${}^oX = G(\boldsymbol{Q}) \backslash G(\boldsymbol{Q}) G(\boldsymbol{R}) K / K_\infty K$ is then identified with $\Gamma \backslash D$.

In case G', the derived subgroup of G, is simply connected, then the connected components of X correspond to those of $T'(\boldsymbol{A}) / T'(\boldsymbol{Q}) T'(\boldsymbol{R}) \nu(K)$, where $T' = G/G'$ and $\nu : G \to T'$ is the quotient map; see [20; §2]. For example, if G is obtained from $GL(2)$ by ground field restriction from a totally real number field k, then $T' = \mathscr{R}_{k/\boldsymbol{Q}}(GL(1))$. Hilbert's modular group $G(\boldsymbol{R})^o \cap G(\boldsymbol{Z})$ corresponds to $K = G(\hat{\boldsymbol{Z}})$; ν is the determinant and $\nu(K)$ is the set of integral idèles of k with components equal 1 at all infinite places; thus, the number of connected components of X is the ideal class number of k.

2.2.1: We let $\Xi = P(\boldsymbol{Q}) \backslash G(\boldsymbol{A}) / G(\boldsymbol{R}) K$; it represents all the point-cusps of X and is finite by a lemma of Godement, [10; 7.2]. The point-cusps of the connected component ${}^oX = \Gamma \backslash D$ of X then correspond to the double cosets $G(\boldsymbol{Q}) \cdot x \cdot K^+$ with $x \in G(\boldsymbol{Q}) K^+$, and we identify ${}^o\Xi$ with $G(\boldsymbol{Q}) \backslash G(\boldsymbol{Q}) K^+ / K^+$.

Suppose, again, that the derived group of G is simply connected. Then it is easy to check that the derived group of $Z_{\boldsymbol{\Theta}}$ is also simply connected; hence by strong approximation and the fact that $\mathscr{D} Z_{\boldsymbol{\Theta}}(\boldsymbol{R})$ is connected

$$Z_{\boldsymbol{\Theta}}(\boldsymbol{A}) = T_{\boldsymbol{\Theta}}(\boldsymbol{A}) Z_{\boldsymbol{\Theta}}(\boldsymbol{Q}) Z_{\boldsymbol{\Theta}}(\hat{\boldsymbol{Z}}) Z_{\boldsymbol{\Theta}}(\boldsymbol{R})^o. \tag{1}$$

Now $P(\boldsymbol{A}) G(\hat{\boldsymbol{Z}}) \supseteq G(\boldsymbol{A}_{na})$ by assumption, and strong approximation for U allows us to replace $Z_{\boldsymbol{\Theta}}$ by P everywhere in (1). Therefore, we can choose representatives for the double cosets $P(\boldsymbol{Q})\backslash G(\boldsymbol{Q})K^+/K^+$ from the set $T_{\boldsymbol{\Theta}}(\boldsymbol{A})G(\hat{\boldsymbol{Z}})$.

In particular, if $K = G(\hat{\boldsymbol{Z}})$, then we may choose representatives for $P(\boldsymbol{Q})\backslash G(\boldsymbol{Q})K^+/K^+$ from $T_{\boldsymbol{\Theta}}(\boldsymbol{A})$. Let $T'' = \mathrm{Int}_G(T_{\boldsymbol{\Theta}})$. Then the cusps correspond to cosets $tT''(\boldsymbol{Q})T''(\boldsymbol{R})T''(\hat{\boldsymbol{Z}})$ with $t \in T''(\boldsymbol{A})$. For the Hilbert modular group attached to a totally real field k, we have $T'' = \mathscr{R}_{k/\boldsymbol{Q}}(GL(1))$; it follows that ${}^oX$ has number of cusps equal to the class number of k.

As another example, take $G = SL(2)$ and $K = K_0(p)$ as in 1.1 (3). The domain D is just the upper half-plane, and the set of rational boundary components in $\boldsymbol{Q} \cup \{\infty\}$. There are two cusps; one is the orbit $\boldsymbol{Q} \cap \boldsymbol{Z}_p$ of $\Gamma_0(p)$ corresponding to the double coset $P(\boldsymbol{Q})K^+$ consisting of $g = (g_v)_v \in SL(2, \boldsymbol{A})$, $g_p = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)$ with $|c|_p < |d|_p$.

2.2.2: If R is a commutative ring, let $S(R)$ be the R-module of R-valued functions on Ξ and let ${}^oS(R)$ be the R-submodule of functions supported on ${}^o\Xi$. When we wish to emphasize dependence on the configuration $\mathfrak{A}$, we write $\Xi_{\mathfrak{A}}$, $S_{\mathfrak{A}}$, etc.

2.3. Adelic Eisenstein series: By assumption (A0) of 1.2, if $g \in G(\boldsymbol{A})$, then $g = b \cdot \omega \cdot g$ for some choice of $b \in P(\boldsymbol{A})$, $\omega \in G(\hat{\boldsymbol{Z}})$ and $g^\infty \in G^\infty$. Let $N(g) = N_{\mathfrak{A}}(g) = |\rho(b)|_{\boldsymbol{A}}$. This well-defined because, by assumption (A4) of 1.2, we have $\rho(b) > 0$ if $b \in P(\boldsymbol{R}) \cap G(\boldsymbol{R})^o$. Suppose $s \in S(\boldsymbol{C})$. We let $\phi(g) = j(g^\infty, o)N(g)$, $c = s\phi$ and

$$\Phi_s(g) = \sum_{\gamma} c(\gamma g), \quad (\gamma \in P(\boldsymbol{Q})\backslash G(\boldsymbol{Q})). \tag{1}$$

By assumption (A5) of 1.2, the series (1) converges normally. Clearly, Φ_s is invariant under $G(\boldsymbol{Q})$ on the left and under K on the right. Furthermore, if $g \in G(\boldsymbol{A})$ and $m \in K_\infty$, then

$$\Phi_s(gm) = j(m, o)\Phi_s(g). \tag{2}$$

Corresponding to the Eisenstein series Φ_s on $G(\boldsymbol{A})$ there is an automorphic form E_s on the domain D. If $z \in D$, choose $g \in G^\infty$ such that $z = g(o)$ and let

$$E_s(z) = j(g, o)^{-1}\Phi_s(g).$$

Using (2) one checks easily that E_s is well-defined. Also, E_s is holomorphic; see below. The set of E_s, with $s \in S(\boldsymbol{C})$, forms a subspace of the space of automorphic forms with respect to (Γ, j) i.e., $E_s(\gamma z) = j(\gamma, z)^{-1}E_s(z)$ for $\gamma \in \Gamma$, $z \in D$. We call this subspace $\mathfrak{E}_{\mathfrak{A}}$ *the space of holomorphic Eisenstein series on X attached to* $\mathfrak{A}$, and we let ${}^o\mathfrak{E}_{\mathfrak{A}}$ be the space of all E_s with $s \in {}^oS(C)$.

We now verify that when $G = SL(2)$ and Γ is a principal congruence subgroup of $SL(2, \boldsymbol{Z})$, then the space ${}^o\mathfrak{E}_{\mathfrak{A}}$ of holomorphic Eisenstein series on ${}^oX = \Gamma\backslash D$ is spanned by certain Eisenstein series defined by Hecke and by Kloosterman; see [26].

For any G and any $a \in G(\boldsymbol{Q})$, let $\Delta(a) = P(\boldsymbol{Q})aK^+$. If $z \in D$, we choose $g \in G^\infty$ with $z = g(o)$. Note that $\gamma g \in \Delta(a)$ if and only if $\gamma \in P(\boldsymbol{Q})a\Gamma = G(\boldsymbol{Q}) \cap \Delta(a)$. Therefore,

$$(4)\qquad E_s(z) = \sum_{\Delta(a)} \Big(\sum_{\gamma} s(\gamma)N(\gamma)j(\gamma, z)\Big), \quad (\gamma \in P(\boldsymbol{Q})\backslash P(\boldsymbol{Q})a\Gamma).$$

The series on the right of (4) converges normally, so E_s is holomorphic. Let $\Gamma_a = \Gamma \cap a^{-1}Pa$. The map taking $\Gamma_a \cdot \gamma \in \Gamma_a\backslash\Gamma$ to $P(\boldsymbol{Q})a\gamma \in P(\boldsymbol{Q})\backslash P(\boldsymbol{Q})a\Gamma$ is bijective. Both s and N are Γ-invariant on the right. Thus,

$$(5)\qquad E_s = \sum_{\Delta(a)} s(a)N(a)\Big(\sum_{\gamma} j(a\gamma, z)\Big), \quad (\gamma \in \Gamma_a\backslash\Gamma).$$

If $G = SL(2)$, then the inner sum is the classical Eisenstein series attached to the cusp $\mathrm{Int}(\Gamma) \cdot P^a$. Note that $N(a) \in \boldsymbol{Q}$ and $N(1_G) = 1$.

2.4. Fourier expansion on $U(\boldsymbol{R})$: Recall that $U(\boldsymbol{R})$ acts on $D \subset U(\boldsymbol{C})$ via real translations. Suppose that Γ is an arithmetic subgroup of G. Then $\Gamma \cap U(\boldsymbol{R})$ is a lattice in the vector space $U(\boldsymbol{R})$. Suppose that h is a $\Gamma \cap U(\boldsymbol{R})$-invariant holomorphic function on D. Let L be the lattice dual to $\Gamma \cap U(\boldsymbol{R})$, i.e., L is the lattice of all $\boldsymbol{Q}$-linear forms on $U(\boldsymbol{C})$ that take integer values on $\Gamma \cap U(\boldsymbol{R})$. Then

$$(1)\qquad h(z) = \sum_{\lambda \in L} a(\lambda)e(\lambda z), \quad (z \in D).$$

In particular, the function E_s has such a Fourier expansion and we let $a_s(\lambda)$ or $a(\lambda)$ denote its coefficients.

2.5. Fourier expansion on $U(\boldsymbol{A})$: Recall that ϵ is the unitary character on $\boldsymbol{A}$ that is trivial on $\boldsymbol{Q}\hat{\boldsymbol{Z}}$ and that satisfies $\epsilon_\infty = e$. For each $g \in G(\boldsymbol{A})$ and each $r \in U(\boldsymbol{Q})$, then $\Phi_s(rg) = \Phi_s(g)$, so the function

$u \mapsto \Phi_s(u)$ on $U(\mathbf{A})$ has Fourier expansion

(1) $$\Phi_s(u) = \sum_{\lambda} b(\lambda)\epsilon(\lambda u), \quad (u \in U(\mathbf{A})),$$

where $\lambda : u \mapsto \lambda u$ ranges over all the $\mathbf{Q}$-linear $\mathbf{A}$-valued forms on the free $\mathbf{A}$-module $U(\mathbf{A})$. Since Φ_s is invariant under right translation by elements of $K \cap U(\mathbf{A})$, we have $b(\lambda) = 0$ unless $\epsilon \circ \lambda \equiv 1$ on $K \cap U(\mathbf{A})$, i.e., unless λ takes $K \cap U(\mathbf{A})$ into $\hat{\mathbf{Z}}$. If $\Gamma = G(\mathbf{Q}) \cap K^+$, then under restriction to U, such λ map to forms in L.

Conversely, if we first fix a basis for the lattice $U(\mathbf{R}) \cap \Gamma$, then we can extend each $\lambda \in L$ uniquely to a $\mathbf{Q}$-linear map $\lambda' : U(\mathbf{A}) \to \mathbf{A}$ such that $\lambda'(K \cap U(\mathbf{A})) \subset \hat{\mathbf{Z}}$. We can therefore compare the two Fourier expansions, 2.4 (1) for E_s and 2.5 (1) for Φ_s. Suppose that λ is a $\mathbf{Q}$-linear form on $U(\mathbf{A})$. If $\lambda \in L$, i.e., $\epsilon \circ \lambda \equiv 1$ on $K \cap U(\mathbf{A})$, then

(2) $$b(\lambda) = \mathbf{e}(\lambda o)a(\lambda).$$

Otherwise, $b(\lambda) = 0$. Rather than study $a(\lambda)$, we study $b(\lambda)$.

2.6. Bruhat decomposition: The results of 1.5.4 lead to explicit representations for the double-coset decomposition $P(\mathbf{Q})\backslash G(\mathbf{Q})/P(\mathbf{Q})$. Recall that there is a maximal set $e_1, \ldots, e_n$ of mutually orthogonal $\mathbf{Q}$-idempotents of the Jordan algebra U. Let $r_i = e_i\iota^*(e_i)e_i$, where ι^* is the Cartan involution. Then $Ad(r_i)$ restricted to Lie(T) is reflection in the hyperplane that annihilates α_{n-i}, ($i = 1, \ldots, n$). We claim that r_i and r_j commute modulo the center of G, i.e., $r_ir_jr_i^{-1}r_j^{-1} \in \mathscr{C}_G$. For this it suffices to show that

(1) $$\iota^*(e_i)e_j(x) = e_j\iota^*(e_i)(x)$$

for each $x \in D$; but (1) follows from the Jordan algebra identity

(2) $$J(e_i + J(x + e_j)) = (x + e_j)J(e_ix + e),$$

which is easily established. Similarly, from (1) we have for any subset $I \subset \{1, 2, \ldots, n\}$

$$r_I = e_I\iota^*(e_I)e_I \cdot g$$

for some $g \in \mathscr{C}_G$, where $r_I = \Pi_{j \in I}\, r_j$ and $e_I = \Sigma_{j \in I}\, e_j$. Let $w_i = r_1r_2 \ldots r_i$. Then w_n coincides with w modulo $\mathscr{C}_G$, i.e., $\mathrm{Int}_G(w_n) = \mathrm{Int}_G(w)$.

The relative $\boldsymbol{Q}$-Weyl group $W(T, G) = \mathcal{N}_G(T)/\mathcal{Z}_G(T)$ of G is generated by the permutations and sign changes of the strongly orthogonal $\boldsymbol{Q}$-roots $2\alpha_1, \ldots, 2\alpha_n$; see 1.5.4. Let W_Θ be the subgroup of $W(T, G)$ consisting of the permutations of $2\alpha_1, \ldots, 2\alpha_n$, and let Z_Θ be the centralizer in G of T_Θ. Then the relative $\boldsymbol{Q}$-Weyl group of Z_Θ can be identified with W_Θ. Clearly,

$$W(T, G) = \coprod_{i=0}^{n} W_\Theta \cdot w_i \cdot W_\Theta \tag{3}$$

By [14; 5.15], it follows that

$$G(\boldsymbol{Q}) = \coprod_{i=0}^{n} P(\boldsymbol{Q}) \cdot w_i \cdot P(\boldsymbol{Q}). \tag{4}$$

Let U_i be the Jordan subalgebra generated by root spaces corresponding to the roots $\alpha_r + \alpha_s$ with both $r, s > n - i$. Let $\perp U_i$ be the vector subspace generated by the other root spaces that occur in U, i.e., corresponding to $\alpha_r + \alpha_s$ with either $r \leq n - i$ or $s \leq n - i$. For $i = 1, \ldots, n$ we have $w_i(\perp U_i)w_i^{-1} \subset P$, hence

$$P(\boldsymbol{Q}) \cdot w_i \cdot P(\boldsymbol{Q}) = P(\boldsymbol{Q}) \cdot w_i \cdot U_i(\boldsymbol{Q}) \cdot Z_\Theta(\boldsymbol{Q}). \tag{5}$$

2.7. DEFINITION: The *$\boldsymbol{Q}$-rank $rk_{\boldsymbol{Q}}(\lambda)$* of a non-zero $\boldsymbol{Q}$-linear form λ on U is the smallest positive integer m such that, for each $g \in P(\boldsymbol{Q})$, the restriction of $\lambda \circ \mathrm{Int}(g)$ to $\perp U_i$ is non-zero for $0 \leq i < m$. If $rk_{\boldsymbol{Q}}(\lambda) = n$, then we call both λ and $a(\lambda)$ *generic*. We put $rk_{\boldsymbol{Q}}(0) = 0$.

2.8. *If λ is a generic character on U, then*

$$b(\lambda) = \int_{U(\boldsymbol{Q})\backslash U(\boldsymbol{A})} \left[\sum_{\gamma \in U(\boldsymbol{Q})} c(w\gamma u)\right] \epsilon(-\lambda u)\, d\dot{u}, \tag{1}$$

where $d\dot{u}$ *is the Haar measure on* $U(\boldsymbol{Q})\backslash U(\boldsymbol{A})$ *with total volume* 1.

Let $g \in G(\boldsymbol{A})$. Let $C(i) = P(\boldsymbol{Q})\backslash P(\boldsymbol{Q})w_iP(\boldsymbol{Q})$ for $i = 1, \ldots, n$. By 2.6 (4), $P(\boldsymbol{Q})\backslash G(\boldsymbol{Q}) = \coprod_{i=1}^{n} C(i)$. Therefore, $\Phi = \Sigma_{i=1}^{n} \Phi_i$, where

$$\Phi_i(g) = \sum_{\gamma \in C(i)} s(\gamma g)\phi(\gamma g).$$

Since $\mathrm{Int}_G(w_n) = \mathrm{Int}_G(w)$ normalizes $Z_\Theta(\boldsymbol{Q})$ and since $\mathrm{Int}(w)U$ intersects P trivially, we may identify $C(n)$ with $w \cdot U(\boldsymbol{Q})$. Thus, it suffices to show that, unless $i = n$,

$$\int_{U(\boldsymbol{Q})\backslash U(\boldsymbol{A})} \Phi_i(u)\epsilon(-\lambda u)\, d\dot{u} = 0.$$

Fix $i < n$ and look at one term of $\Phi_i(g)$, say $c(w_i\gamma g) = s(w_i\gamma g)\phi(w_i\gamma g)$ with some $\gamma \in P(\boldsymbol{Q})$. It is enough to show that the integral

$$\mathcal{I} = \int_{U(\boldsymbol{Q})\backslash U(\boldsymbol{A})} c(w_i\gamma u)\epsilon(-\lambda u)\, d\dot{u}$$

vanishes. Choose $x \in \perp U_i$ such that $\epsilon(-\lambda \circ \mathrm{Int}(\gamma^{-1})(x)) \neq 1$, and let $v = \gamma^{-1}x\gamma$. Then $c(w_i\gamma uv) = c(w_i\gamma u)$ for all $u \in U$, so $\mathcal{I} = \epsilon(-\lambda v)\mathcal{I}$. It follows that $\mathcal{I} = 0$, as required.

2.8.1. COROLLARY: *If λ is a generic character of U, then*

$$a_s(\lambda) = e(-\lambda o)\int_{U(\boldsymbol{A})} c(wu)\epsilon(-\lambda u)\, du,$$

where the Haar measure du *is normalized so that the volume of a fundamental domain for $U(\boldsymbol{Q})$ in $U(\boldsymbol{A})$ is* 1.

This follows at one because the integral

$$b(\lambda) = \int_{U(\boldsymbol{Q})\backslash U(\boldsymbol{A})} \left[\sum_{\gamma\in U(\boldsymbol{Q})} c(w\gamma u)\right]\epsilon(-\lambda u)\, d\dot{u}$$

is absolutely convergent.

3. Adelic Integrals

3.1: The first major step in our investigation of the Fourier coefficients $a_s(\lambda)$ of E_s, $s \in S(\boldsymbol{Q})$, is to show that $a_s(\lambda)$ is cyclotomic if the character λ is generic. Our method is reduction to the case $s \equiv 1$, where the work of Tsao [36] and Baily [7] applies. The main problem is that the double cosets $P(\boldsymbol{Q})xK^+$, on which s is constant, are not restricted direct products. Our solution is to show first that s is invariant by $G(\boldsymbol{Q}_p)$ for all but finitely many primes p, i.e., $s(x)$ depends only on finitely many of the components x_p. Then we find an open subgroup H of $P(\boldsymbol{A})$ such that

(1) H is the restricted direct product of groups $H_v \subset P(\boldsymbol{Q}_v)$;
(2) $H_p = P(\boldsymbol{Q}_p)$ for almost all finite primes, and $H_\infty = P(\boldsymbol{R})^o$;
(3) the map $u \mapsto s(wu)$ is invariant under H acting as affine transformations as in 1.5.1.

Finally, we show that the integrals of $u \mapsto e(-\lambda o)c(wu)\epsilon(-\lambda u)$ over the orbits of H have cyclotomic values and that almost all of them vanish.

3.2: To show that $s(x)$ does not depend on the component x_p, i.e., s is $G(\boldsymbol{Q}_p)$-invariant, the crux is to show that s is invariant on the left under $T(\boldsymbol{Q}_p)$. For this we prove two easy lemmas.

3.2.1. Lemma: *If $x \in G(\boldsymbol{A})$ and $\Delta = P(\boldsymbol{Q})xK^+$, then $U(\boldsymbol{A})\Delta = \Delta$.*

Let $H = x \cdot K^+ \cdot x^{-1}$, so $\Delta = P(\boldsymbol{Q})H \cdot x^{-1}$. Then strong approximation shows that $U(\boldsymbol{A}) = U(\boldsymbol{Q})(U \cap H)$. Thus, $U(\boldsymbol{A})P(\boldsymbol{Q})H = P(\boldsymbol{Q})H$, and it follows that $U(\boldsymbol{A})\Delta = \Delta$.

3.2.2. Lemma: *If H is an open subgroup of $Z_{\boldsymbol{\Theta}}(\boldsymbol{A})$ and contains $\mathcal{D}Z_{\boldsymbol{\Theta}}(\boldsymbol{R})^o$, then $Z_{\boldsymbol{\Theta}}(\boldsymbol{Q})H$ is a normal open subgroup of $Z_{\boldsymbol{\Theta}}(\boldsymbol{A})$.*

It suffices to show that the derived group of $Z_{\boldsymbol{\Theta}}(\boldsymbol{A})$ is contained in $Z_{\boldsymbol{\Theta}}(\boldsymbol{Q})H$. Let Y be the simply connected cover of $\mathcal{D}Z_{\boldsymbol{\Theta}}$, so that $Y(\boldsymbol{R})$ is a connected Lie group. Then the required fact follows from real approximation for $Z_{\boldsymbol{\Theta}}$, [20; 0.4], and from strong approximation for Y combined with the proof of [27; Satz 6.1]. For a similar argument, see 5.5.2. Note that $Z_{\boldsymbol{\Theta}}$ is never of type E_8, so the facts we need about strong approximation were established by Kneser. For a proof of strong approximation independent of classification, see [31].

3.2.3. Lemma: *For a double coset $\Delta = P(\boldsymbol{Q})xK^+$ with $x \in G(\boldsymbol{A})$, let N_Δ be the normalizer of Δ in $P(\boldsymbol{A})$ for left multiplication. Then N_Δ is an open subgroup of $P(\boldsymbol{A})$ and contains $U(\boldsymbol{A})$.*

Proof: Again, let $H = x \cdot K^+ \cdot x^{-1}$. It suffices to show that $(Z_{\boldsymbol{\Theta}} \cap H)U(\boldsymbol{A})P(\boldsymbol{Q})H = P(\boldsymbol{Q})H$. We have

$$(Z_{\boldsymbol{\Theta}} \cap H)U(\boldsymbol{A})P(\boldsymbol{Q})H = (Z_{\boldsymbol{\Theta}} \cap H)U(\boldsymbol{A})Z_{\boldsymbol{\Theta}}(\boldsymbol{Q})H = U(\boldsymbol{A})Z_{\boldsymbol{\Theta}}(\boldsymbol{Q})H$$

by Lemma 3.2.2. By Lemma 3.2.1, $U(\boldsymbol{A})Z_{\boldsymbol{\Theta}}(\boldsymbol{Q})H = P(\boldsymbol{Q})H$.

3.2.4. THEOREM: *For all but finitely many primes p, the group $G(\boldsymbol{Q}_p)$ leaves each $s \in S(C)$ invariant under left multiplication.*

It suffices to show, for each of the finitely many double cosets $\Delta = P(\boldsymbol{Q})xK^+$ with $x \in G(\boldsymbol{A})$, that $G(\boldsymbol{Q}_p) \cdot \Delta = \Delta$ for all but finitely many p.

Clearly, $N_\Delta \cap T$ is an open subgroup of $T(\boldsymbol{A})$ and contains $T(\boldsymbol{Q})$. Since T splits over a finite extension of $\boldsymbol{Q}$, it follows from the finiteness of class number, [41; IV, §4], that $T(\boldsymbol{Q}_p) \subset N_\Delta$ for almost all p. Fix such a prime p.

If α is a root of T in Z_Θ and if V_α is its root-group, then $V_\alpha \cap N_\Delta$ is an open subgroup of $V_\alpha(\boldsymbol{A})$. We identify V_α with its Lie algebra via the exponential map. Then the action of $\mathrm{Int}(t)$ is identified with scalar multiplication by $\alpha(t) \in \boldsymbol{Q}_p^\times$ for each $t \in T(\boldsymbol{Q}_p)$. Since the image of $T(\boldsymbol{Q}_p)$ under α contains elements of arbitrarily large p-adic order, the union over all $t \in T(\boldsymbol{Q}_p)$ of $\mathrm{Int}(t)(V_\alpha(\boldsymbol{Q}_p) \cap N_\Delta)$ must exhaust $V_\alpha(\boldsymbol{Q}_p)$. Thus, $V_\alpha(\boldsymbol{Q}_p) \subset N_\Delta$ for each root α. It follows that $Z_\Theta(\boldsymbol{Q}_p) \subset N_\Delta$ for almost all rational primes p.

Combined with Lemma 3.2.1, this shows that $P(\boldsymbol{Q}_p) \cdot \Delta = \Delta$ for almost all p. On the other hand, $G(\boldsymbol{Z}_p)xK^+ = xK^+$ for almost all p. Thus, for almost all rational primes p,

$$\Delta = P(\boldsymbol{Q})xK^+ = P(\boldsymbol{Q})P(\boldsymbol{Q}_p)xK^+ = P(\boldsymbol{Q})P(\boldsymbol{Q}_p)G(\boldsymbol{Z}_p)xK^+ = G(\boldsymbol{Q}_p) \cdot \Delta$$

since $P(\boldsymbol{Q}_p)G(\boldsymbol{Z}_p) = G(\boldsymbol{Q}_p)$.

3.3: The primes p for which $G(\boldsymbol{Q}_p) \cdot \Delta = \Delta$ for each double coset $\Delta = P(\boldsymbol{Q})xK^+$, $x \in G(\boldsymbol{A})$, we call *good primes*. The remaining finite collection of primes we call *bad primes*. Let $H_\infty = P(\boldsymbol{R})^o$, $H_p = P(\boldsymbol{Q}_p)$ for each good prime $p < \infty$ and $H_p = (Z_\Theta \cap x \cdot K^+ \cdot x^{-1})U(\boldsymbol{Z}_p)$ for each bad prime. The restricted direct product of the H_v's with respect to the subgroups $H_v \cap P(\boldsymbol{Z}_p)$ is then an open subgroup of $P(\boldsymbol{A})$. In 1.5.1 we defined an algebraic action of P acting as affine linear transformations on U. Under this action the orbits of H on $U(\boldsymbol{A})$ are open. For each orbit $H \cdot u$, we choose a representative x with $x_v = 0$ for all good places v (including $v = \infty$).

According to 2.8.1 and 2.5 (2)

$$a_s(\lambda) = e(-\lambda o) \int_{U(\boldsymbol{A})} c(wu)\epsilon(-\lambda u)\, du. \tag{1}$$

The integral is absolutely convergent and breaks up as a sum of absolutely convergent integrals over the orbits $H \cdot x$. Now,

$$(2) \qquad \int_{H \cdot x} c(wu)\epsilon(-\lambda u)\,\mathrm{d}u = s(wx) \prod_{v \leq \infty} \int_{H_v \cdot x_v} \phi(wu)\epsilon_v(-\lambda u) d_v u$$

and for good places v the integral is

$$(3) \qquad \mathscr{I}_v = \int_{U(\mathbf{Q}_v)} \phi_v(wu)\epsilon_v(-\lambda u) d_v u.$$

Tsao [36] and Baily [7] have proved that, for all *finite* primes p, $\mathscr{I}_p$ is rational, and also, that $e(-\lambda o)$ times the product of $\mathscr{I}_v$ over *all* places v is rational. Therefore, $e(-\lambda o)\Pi_{v \,\mathrm{good}} I_v$ is also rational. If p is a bad prime, then $u \mapsto \phi_p(wu)$ is constant on $H_p \cdot x_p$, so

$$(4) \qquad \int_{H_p \cdot x_p} \phi_p(wu)\epsilon_p(-\lambda u) d_p u = \phi_p(wx_p) \int_{H_p \cdot x_p} \epsilon_p(-\lambda u) d_p u.$$

Since ϵ_p is invariant under $\mathbf{Z}_p$, we see at once that the right side of (4) lies in $\mathbf{Q}_{ab}$. Thus,

$$(5) \qquad e(-\lambda o) \int_{H \cdot x} c(wu)\epsilon(-\lambda u) du \in \mathbf{Q}_{ab}.$$

3.4: It remains to show that almost all the orbital integrals 3.3 (5) vanish. For this it will suffice to show that, for each bad prime p, the integral

$$(1) \qquad \int_{H_p \cdot x_p} \epsilon_p(-\lambda u) d_p u$$

fails to vanish for only finitely many orbits $H_p \cdot x_p$. We fix such a bad prime p. Now the problem is purely local, so we suppress the subscript p wherever possible.

Define a p-adic lattice $\Lambda = K_p \cap U(\mathbf{Z}_p)$ and let the height of an orbit be defined as follows:

$$\mathrm{height}(H \cdot x) = \log_p([\mathbf{Z}_p \cdot x : \mathbf{Z}_p \cdot x \cap \Lambda]).$$

The orbits are open, so only finitely many have given height; thus, it is enough to show that, whenever height$(H \cdot x)$ exceeds some bound,

$$\int_{H \cdot x} \epsilon(-\lambda u) du = 0.$$

The existence of such a bound will follow from Lemma 1.6.1 applied to $H' = Z_{\Theta} \cap H$. First, however, we must embed Z_{Θ} as an algebraic $\boldsymbol{Q}_p$-subgroup of some special linear group.

Recall that Z_{Θ} is embedded in $GL(U)$ by Int_G. We define an embedding α of $GL(U)$ into $SL(U \oplus U)$ by letting $g \in GL(U)$ take $(u, v) \in U \oplus U$ to $(g(u), g^{-1}(v))$. The composition $\beta = \alpha \circ \mathrm{Int}_G$ then embeds Z_{Θ} as a $\boldsymbol{Q}_p$-subgroup of $SL(U \oplus U)$, and β is a submersion from Z_{Θ} to its image. Thus, viewing Z_{Θ} as a $\boldsymbol{Q}$-subgroup of $SL(U \oplus U)$, we see that $H' = H_p \cap Z_{\Theta}$ is an open compact subgroup of $Z_{\Theta}(\boldsymbol{Q}_p)$.

We view the endomorphism ring $\mathrm{End}(U \oplus U)$ as an algebraic algebra E, with p-adic points $E(\boldsymbol{Q}_p)$ containing a p-adic lattice Ω, defined to be the stabilizer in E of the p-adic lattice $\Lambda \oplus \Lambda$. Then $H' \subset \Omega$. For $u \in U$, define a linear map $f_u : E \to U \oplus U$ by letting $f_u(T) = T(u, 0)$ for each $T \in E$. If $u \in \Lambda$, then f_u maps Ω into Λ and, for each positive integer ν, induces a map

$$\psi_\nu : \Omega / p^\nu \Omega \to \Lambda / p^\nu \Lambda. \tag{2}$$

Extend λ to a linear functional L on $U \oplus U$ by letting $L(u, v) = \lambda(u)$ and let $\tau_u(T) = -L(T(u))$ for each $u \in U$ and $T \in E$. Note that if $z \in Z_{\Theta}$, then $\tau_u(z) = -\lambda(z(u))$. If $\mathrm{height}(H \cdot x) = m$, let $y = p^m x$. Choose ν large enough so that $H' + p^\nu \Omega = H'$. Then

$$\int_{H' \cdot x} \epsilon(-\lambda u) du = q \sum_u \epsilon(-p^{-m} \lambda u), \quad (u \in H' \cdot y (\mathrm{mod}\, p^\nu \Lambda)),$$

with some $q \in \boldsymbol{Q}^{\times}$. If the kernel of Ψ_ν has order t, then it follows from (2) that

$$\int_{H' \cdot x} \epsilon(-\lambda u) du = \frac{q}{t} \sum_q \epsilon(p^{-m} \tau_u(g)), \quad (g \in H' (\mathrm{mod}\, p^\nu \Omega)), \tag{3}$$

To use Lemma 1.6.1 we must have estimates of $m(w)$ and $\kappa(w)$ for $w = (u, 0) \in U \oplus U$. This will follow from the next lemma and some calculations of Tsao. Write $m(u) = m(w)$, $\kappa(u) = \kappa(w)$.

Recall that $u \in \Lambda$ is *primitive* if $u \notin p\Lambda$. By standard facts about ground field restriction, it follows from [36; Lemma 9.6] that there is a positive integer ν such that, for each primitive vector $u \in \Lambda$ and each sufficiently large positive integer m, there exists $g \in H'$ satisfying: (i) $g \equiv 1_E (\mathrm{mod}\, p^m \Omega)$, and (ii) $\tau_u(g) \not\equiv \tau_u(1_E) (\mathrm{mod}\, p^{m+\nu})$. One can even take $g \in H' \cap \mathfrak{D}G$. Given a primitive vector $u \in \Lambda$, let

$\nu(u)$ be the smallest positive integer ν such that, for each sufficiently large $m \in \boldsymbol{Z}(>0)$, conditions (i) and (ii) hold for some $g \in H'$. From the proof of Lemma 9.6 of [36], it is clear that $\nu(u)$ depends on u only modulo $p^k \Lambda$ for some small fixed integer k.

3.5. LEMMA: *With notation as in* 1.6 *and* 3.4, $m_a(u) = \nu(a \cdot u) - 1$, *(*$a \in H'$, *$u$ primitive in* Λ*).*

Since $\tau_{a \cdot u}(g) = -\lambda(g(a \cdot u)) = \tau_u(g \cdot a)$ for each $g \in H'$, we may choose $g \in H'$ such that $g \equiv 1_E (\operatorname{mod} p^m \Omega)$ but $\tau_u(g \cdot a) \not\equiv \tau_u(a)$ $(\operatorname{mod} p^{m+\nu(a \cdot u)})$. Note that $g \cdot a \in H'$ and that $g \cdot a = a + p^m y$ for some $y \in \Omega$ because Ω is H'-stable. By [3; Cor. 2] there is a constant $\mu \in \boldsymbol{Z}(\geq 0)$ independent of m and such that, for some $z \in \Omega$, $b = p^m y + p^{2m-\mu} z$ lies in $\mathfrak{p}(a)$, cf. 1.6. Clearly, then

$$\tau_u(b) \equiv \tau_u(p^m y) (\operatorname{mod} p^{2m-\mu}), \tag{1}$$

$$\tau_u(a + p^m y) \not\equiv \tau_u(a) (\operatorname{mod} p^{m+\nu(a \cdot u)}). \tag{2}$$

If m is sufficiently large, then $b \in p^m \Omega$ but $\tau_u(b) \not\equiv 0$ $(\operatorname{mod} p^{m+\nu(a \cdot u)})$. In particular, $\tau_u(p^{-m} b) \not\equiv 0$ $(\operatorname{mod} p^{\nu(a \cdot u)})$, so $m_a(u) < \nu(a \cdot u)$, cf. 1.6 (1).

Conversely, if $\nu = m_a(u) + 1$, then we can find $y \in \mathfrak{p}(a) \cap \Omega$ such that $\tau_u(y) \not\equiv 0$ $(\operatorname{mod} p^\nu)$, so $\tau_u(p^m y) \not\equiv 0$ $(\operatorname{mod} p^{\nu+m})$. By [3; Cor. 2] there exists $z \in \Omega$ such that

$$b = a + p^m y + p^{2m-\mu} z \in Z_{\boldsymbol{\Theta}} \cap \Omega.$$

If m is large enough and if $x \in Z_{\boldsymbol{\Theta}}(\boldsymbol{Q}_p)$, then $x \equiv a (\operatorname{mod} p^m)$ implies $x \in H'$. Thus, we may assume that $b \in H'$. Now $a^{-1} \in H' \subset \Omega$, so if $g = ba^{-1}$, then $g \equiv 1_E (\operatorname{mod} p^m \Omega)$ but $\tau_{a \cdot u}(g) \not\equiv \tau_{a \cdot u}(1_E) (\operatorname{mod} p^{m+\nu})$. Consequently, $m_a(u) + 1 \geq \nu(a \cdot u)$. Since $m_a(u) < \nu(a \cdot u)$, we have $m_a(u) = \nu(a \cdot u) - 1$, as required.

3.6. LEMMA: *There is a constant m_p such that if* height$(H_p \cdot x) >$ m_p, *then*

$$\int_{H_p \cdot x} \epsilon_p(-\lambda u) d_p u = 0.$$

We need to use Lemma 1.6.1. Since $\nu(u)$ depends on primitive $u \in \Lambda$ only modulo $p^k \Lambda$ for some integer k, we see that the map

$a \mapsto m_a(u) = \nu(a \cdot u) - 1$ is constant on the cosets of the open subgroup

$$H'(k) = \{g \in H' : g \equiv 1_E (\mathrm{mod}\, p^k \Omega)\}$$

of H'. With κ as in 1.6, then $\kappa(u) \leq k$ for each primitive $u \in \Lambda$. Moreover, $m(u)$ exists and is bounded so we can apply Lemma 1.6.1 to finish the proof.

3.6.1. COROLLARY: *There is a finite set B of non-archimedean rational primes such that, given a generic character $\lambda \in L(n)$, there exist an open compact subset $Y = Y(\lambda)$ of U_B and a rational number $q = q(\lambda)$ satisfying*

$$(1) \qquad a_s(\lambda) = q \int_Y s(wu)\phi(wu)\epsilon(-\lambda u) d_B u,$$

for each $s \in S$. Moreover, we may assume that Y is normalized by $P(\hat{\boldsymbol{Z}}) \cap P_B$ and that Y has rational volume.

Let B equal the set of bad primes p, i.e., p such that some $s \in S$ is not invariant by $G(\boldsymbol{Q}_p)$. Let $H_B = H \cap P_B$ and let Y be the union of orbits $H_B \cdot x$ in U_B with height$(H_p \cdot x_p) \leq m(p)$ for each $p \in B$. Then Y has the required properties.

3.6.2. COROLLARY: *For each $s \in S(\boldsymbol{Q})$ and each generic character $\lambda \in L(n)$, the Fourier coefficient $a_s(\lambda) \in \boldsymbol{Q}_{ab}$.*

The integrand in 3.6.1 (1) is locally constant and its values lie in $\boldsymbol{Q}_{ab}$. Thus, for some positive integer k, the compact open set Y is partitioned by a finite family of cosets $u_i + p^k \Lambda$ on which the integrand $u \mapsto c(wu)\epsilon(-\lambda u)$ is constant. The corollary follows.

§4. Cyclotomy of the Fourier coefficients

4.0: This paragraph is devoted to proving that all the Fourier coefficients of E_s lie in the field $\boldsymbol{Q}_{ab}$ whenever $s \in S(\boldsymbol{Q})$. We have shown already that $a_s(\lambda) \in \boldsymbol{Q}_{ab}$ for each generic character λ. The main idea is to reduce to this case by using the natural fibration of D over its boundary components. More precisely, we show that if λ does not have maximal rank, then $a_s(\lambda)$ appears as a generic coefficient in the

Fourier expansion of an Eisenstein series induced from E_s on one of the rational boundary components of D, say D_*.

Since the rational boundary components of D_* are naturally identified with rational boundary components of D, we can restrict our attention to D_* of rank $n-1$. Namely, we can proceed inductively if we establish that $a_s(\lambda) = a_{s_*}(\lambda_*)$, the Fourier coefficient of an Eisenstein series E_{s_*} on D_* attached to a character λ_* with the same rank as λ. This method is easier to describe than direct descent to an arbitrary rational boundary component but amounts to the same thing.

4.1. A STANDARD BOUNDARY COMPONENT: We now construct the standard rational boundary component of rational rank $n-1$. Recall that there is a basis of strongly orthogonal $\boldsymbol{Q}$-roots, $\{2\alpha_1, \ldots, 2\alpha_n\}$, such that the simple roots of T in P are $\alpha_1 - \alpha_2, \ldots, \alpha_{n-1} - \alpha_n, 2\alpha_n$. Let $U_{\max}$ be the unipotent subgroup of P generated by the root groups of T and P. Let $T^{\#}$ be the largest subtorus of T on which all the simple roots of T in P, except for $\alpha_1 - \alpha_2$, are trivial, let G_* be the centralizer of $T^{\#}$ in G, and let $P^{\#} = G_* U_{\max}$ be the standard maximal parabolic $\boldsymbol{Q}$-subgroup attached to $\alpha_1 - \alpha_2$. Then G_* is a Levi component of $P^{\#}$, and with respect to T, the Dynkin diagram of G_* has type C_{n-1}; it is obtained from that of G by deleting the vertex corresponding to $\alpha_1 - \alpha_2$.

The boundary component corresponding to $P^{\#}$ is the symmetric space D_* attached to G_*, so it is a rational tube domain of rank $n-1$ over $\boldsymbol{Q}$. Its boundary point at infinity corresponds to the maximal parabolic $\boldsymbol{Q}$-subgroup $P_* = P \cap G_*$ of G_*. Let U_* be the unipotent radical of P_*. We note that U_* is the Jordan subalgebra U_{n-1} of U introduced in 2.6; see [14; §4].

Each integral automorphic form F on D, e.g., $F = E_s$, induces an automorphic form $\mathrm{Ind}_* F$ on D_* via extension to the boundary as in [9, §8]. By the argument of [6; 7.2], if F has Fourier expansion

$$F(z) = \sum_{\lambda \in L} a(\lambda) e(\lambda z), \tag{1}$$

then, for $z \in D_*$,

$$\mathrm{Ind}_* F(z) = \sum_{\lambda \in L_*} a(\lambda) e(\lambda z), \tag{2}$$

where $L_* = \{\lambda \in L : \lambda(\perp U_{n-1}) = 0\}$. It will suffice to show that $\mathrm{Ind}_* E_s$ is an Eisenstein series on D_*.

4.2: There is a canonical fibration $\pi_*: D \to D_*$ defined in [9; 1.7]. Recall that j is commensurable with the functional determinant; hence, by [9; 1.11], the function $z \mapsto j(g, z)$ is constant along the fibres of π_* whenever $g \in PP^\# \cap G(\boldsymbol{R})^o$. Furthermore, if $g \notin PP^\#$, then $j(g, z)$ tends to zero as z tends to the boundary of D in a suitable Siegel set, see [9; 7.7(ii)]. Since $(PP^\#)(\boldsymbol{Q}) = P(\boldsymbol{Q})P^\#(\boldsymbol{Q})$, we have for $g \in G(\boldsymbol{R})^o$ and $z = g(o)$

$$(1) \qquad (\mathrm{Ind}_* \, E_s) \circ \pi_*(z) = \sum_\gamma s(\gamma) N(\gamma) j(\gamma, z),$$

where the sum ranges over γ in $(P(\boldsymbol{Q}) \cap P^\#(\boldsymbol{Q}))\backslash P^\#(\boldsymbol{Q})$.

Since we identify G_* with $U^\#\backslash P^\#$ and we have $G_*(\boldsymbol{Q}) = U^\#(\boldsymbol{Q})\backslash P^\#(\boldsymbol{Q})$, it follows that the sum on the right side of (1) can be taken over all γ in $P_*(\boldsymbol{Q})\backslash G_*(\boldsymbol{Q})$. For $\gamma \in G_*$, however, $j(\gamma, z)$ depends only on $\pi_*(z)$, and we write $j(\gamma, z) = j'(\gamma, \pi_*(z))$. For $z_* \in D_*$ we can find $g \in G_*(\boldsymbol{R})^o$ such that $z_* = g(\pi_*(o))$, hence

$$(2) \qquad \mathrm{Ind}_* \, E_s(z_*) = \sum_\gamma s(\gamma) N(\gamma) j'(\gamma, z_*), \quad (\gamma \in P_*(\boldsymbol{Q})\backslash G_*(\boldsymbol{Q})).$$

4.3: Let $\rho_* = \rho \mid P_*$ and $K_* = K \cap G_*$. Then K_* is a compact open subgroup of $G_*(\boldsymbol{Z})$, and, according to Bruhat-Tits theory, $G_*(\boldsymbol{Z}_p)$ is a special maximal compact subgroup of $G(\boldsymbol{Q}_p)$ for each finite prime p.

To show that the right side of 4.2 (2) is an Eisenstein series attached to $\mathfrak{A}_* = (G_*, T, P_*, \rho_*, K_*)$ it suffices to check that j' is the automorphy factor attached to $\mathfrak{A}_*$.

4.3.1: Recall that there is a rational map $\alpha: G \times U \to U \times Z_\Theta$ defined by setting $\alpha(g, x) = (A_g(x), \mathfrak{z}(g, x))$ if $gx \in A_g(x) \cdot \mathfrak{z}(g, x) \cdot U^-$; see 1.5. The automorphy factor j is defined by $j(g, x) = \rho(\mathfrak{z}(g, x))$. Similarly, one defines a rational map $\alpha_*: G_* \times U_* \to U_* \times (Z_\Theta \cap G_*)$, and α_* is clearly the restriction of α to $G_* \times U_*$. The restriction of $j(g, x)$ to $G_* \times U$ is thus the automorphy factor j_* attached to $\mathfrak{A}_*$. To see that $j' = j_*$ we must verify that $j(g, \pi_*(x)) = j(g, x)$ for a Zariski-dense set of pairs $(g, x) \in G_* \times U$. For this, it suffices to see that π_* is the restriction to D of the projection $U \to U_{n-1}$ arising from the Peirce decomposition with respect to the idempotent e_1, i.e., π_* is the projection onto U_{n-1} with kernel $\perp U_{n-1}$. Indeed, then $j(g, x) = j(g, x')$ whenever $x - x' \in \perp U_{n-1}$ because j is algebraic.

4.3.2: To verify the above identification is routine in principle but complicated in practice. We begin by decomposing each of the

$\boldsymbol{Q}$-idempotents e_i as a sum of $\boldsymbol{R}$-idempotents e_{ij} to get a maximal set of orthogonal $\boldsymbol{R}$-idempotents. As in 1.5.4, this gives rise to a maximal $\boldsymbol{R}$-split torus ${}_{\boldsymbol{R}}T$, which contains T. The root system of ${}_{\boldsymbol{R}}T$ in G can then be identified with the root system ${}_{\boldsymbol{R}}\Phi(a, g)$ considered by Baily and Borel in [9; 1.2], where $\mathfrak{g} = \mathrm{Lie}(G(\boldsymbol{C}))$, $\mathfrak{a} = \mathrm{Lie}({}_{\boldsymbol{R}}T(\boldsymbol{C}))$. In particular, we may assume that the idempotents $e_{\mu\nu}$ correspond in some order to the root vectors X_i of [9; 1.1, p. 448]. Let $\Delta = \{X_1, \ldots, X_t\}$ be the set of all the X_i's where t is the rank of G as a semisimple group over $\boldsymbol{R}$. To each $X = X_i \in \Delta$ one attaches an element $c_X = \exp((\pi/4)(E_{-i} - E_i))$, as in [9; 1.1], and to each subset $\sigma \subset \Delta$ one attaches the product c_σ of those c_X's such that $X \in \sigma$. Let $b = \{e_{\mu\nu} : \mu = n-1\}$, and let a be the complement of b in Δ. The standard boundary components correspond to certain subsets $\sigma \subseteq \Delta$, and to each boundary component $\mathscr{F}_\sigma$ is attached an unbounded realization $r_\sigma : D \to \mathfrak{p}^+$ obtained from Harish-Chandra's realization $r_\emptyset$ by using the partial Cayley transform c_σ. Here $\mathfrak{p}^+$ is the Lie algebra of a vector subgroup P^+ of $G(\boldsymbol{C})$; P^+ is the unipotent radical of a parabolic subgroup $K_{\boldsymbol{C}}P^+$.

According to [9; 1.7], $\mathrm{Int}(c_\Delta)P(\boldsymbol{R})^o \subset K_{\boldsymbol{C}}P^+$, hence $P^+ \subset \mathrm{Int}(c_\Delta)U(\boldsymbol{C})$. Since D embeds as an open subset of $\mathfrak{p}^+$ and also as an open subset of $U(\boldsymbol{C})$, dimension considerations show that $P^+ = \mathrm{Int}(c_\Delta)U(\boldsymbol{C})$. Our identification of root systems allows us to identify $\mathrm{Ad}(c_\Delta)^{-1}\mathfrak{p}_b^+$ with $\mathrm{Lie}(U_{n-1}(\boldsymbol{C}))$ and $\mathrm{Ad}(c_\Delta)^{-1}\mathfrak{q}_b$ with $\mathrm{Lie}(\perp U_{n-1}(\boldsymbol{C}))$.

4.3.3: To complete the argument, it remains to pass from the realization r_b, used by Baily and Borel in [9; 1.7] in defining the canonical fibration π_*, to the tube realization r_Δ. The main point is to characterize $\mathfrak{q}_b$. Let A_b be the $\boldsymbol{R}$-subtorus of $\exp(\mathfrak{a})$ corresponding to the Lie subalgebra of $\mathfrak{a}$ generated by root vectors in b. Then $\mathfrak{q}_b$ is spanned by root vectors on which $\mathrm{Int}(c_\Delta)A_b$ acts non-trivially. Therefore, there is a one-parameter subgroup γ in A_b such that, for $v \in \mathfrak{q}_b$,

$$\gamma(t)v \to 0 \quad \text{as } t \to 0. \tag{1}$$

In fact, (1) characterizes $\mathfrak{q}_b$ as a subspace of $\mathfrak{p}^+$.

For each subset σ of Δ, let $S_\sigma = r_\sigma(\Delta)$ and let $\bar{S}_\sigma$ be the closure of S_σ in the natural topology of $\mathfrak{p}^+$. Let $\nu : S_b \to S_\Delta$ be the map such that $r_\Delta = \nu \circ r_b$. Then ν has meromorphic extension ν^+ to $\mathfrak{p}^+$, and the restriction of ν^+ to $\mathfrak{p}_b^+$ gives the Cayley transform from the bounded realization of the standard boundary component D_* to the tube realization of D_*. In particular, $\nu^+(0)$ is well-defined. We must show that $\nu(\mathfrak{q}_b \cap S_b) \subseteq \mathfrak{q}_b + \nu(0)$.

The natural action of $g \in A_b$ on S_Δ takes $x \in S_\Delta$ to $\mathrm{Ad}(c_\Delta \cdot g \cdot c_\Delta^{-1})x$. Since c_a centralizes A_b, the action of $g \in A_b$ on S_b, which takes $x \in S_b$ to $\mathrm{Ad}(c_b \cdot g \cdot c_b^{-1})x$, is linear and coincides with the action of A_b on S_Δ. In particular, A_b fixes $0 \in \mathfrak{p}^+$, and if $x \in \mathfrak{q}_b$, then

$$\gamma(t)(\nu(x) - \nu(0)) = \nu(\gamma(t)x) - \nu(0) \to 0$$

at $t \to 0$. Therefore, $\nu(x) - \nu(0)$ lies in $\mathfrak{q}_b$ whenever x does, so $\nu(\mathfrak{q}_b \cap S_b) \subseteq \mathfrak{q}_b + \nu(0)$ as required.

We have now shown that $j(g, \pi_*(x)) = j(g, x)$ for $g \in G_*$ and $x \in U$; therefore, j_* is the automorphy factor attached to $\mathfrak{A}_* = (G_*, T_*, P_*, \rho_*, K_*)$.

4.4: If $\lambda \in L$ and $\lambda(\perp U_{n-1}) = 0$, then $a_s(\lambda)$ appears in the Fourier expansion of $\mathrm{Ind}_*(E_s)$ on G_* by 4.1 (2). Furthermore, we have just verified that $\mathrm{Ind}_*(E_s) = {}_{\mathfrak{A}}E_s$. In particular, if $rk_{\boldsymbol{Q}}(\lambda) = n-1$ and $\lambda(\perp U_{n-1}) = 0$, then λ has maximal rank as a character on U_*, and this implies that $a_s(\lambda) \in \boldsymbol{Q}_{ab}$. Moreover, by applying 3.6.1 to $\mathfrak{A}_*$ in place of $\mathfrak{A}$, one has an expression for $a_s(\lambda)$ as a rational number times a certain finite exponential sum. We now extend this result to all characters $\lambda \in L$. We begin by observing that the Eisenstein series ${}_{\mathfrak{A}}E_s$ does not depend on the choice of torus T, nor even on the choice of Levi component Z_Θ, because the construction of the automorphy factor j did not use T. That is, if $\mathfrak{A}' = (G, T', P, \rho, K)$, then ${}_{\mathfrak{A}'}E_s = {}_{\mathfrak{A}}E_s$ for each $s \in S_{\mathfrak{A}} = S_{\mathfrak{A}'}$.

Now suppose that λ has rank k. Recall that this means that $(\lambda \circ \mathrm{Int}(z))(\perp U_k) = 0$ for some $z \in Z_\Theta$, and $\lambda \circ \mathrm{Int}(z)|U_k$ is generic. By changing T to $\mathrm{Int}(z^{-1})T$ we may therefore assume that $\lambda(\perp U_k) = 0$ and that $\lambda \mid U_k$ is generic. Let $G_n = G$ and successively define $G_j = (G_{j+1})_*$ for $j = n-1, n-2, \ldots, 1, 0$. Then $a_s(\lambda)$ appears as a generic coefficient in the Fourier expansion of an Eisenstein series ${}_{\mathfrak{A}(k)}E_s$ on G_k, where $\mathfrak{A}(k) = (G_k, T, P \cap G_k, \rho \mid P \cap G_k, K \cap G_k)$. One can apply 3.6.1 to $\mathfrak{A}(k)$. In particular we have shown that all the Fourier coefficients of E_s lie in $\boldsymbol{Q}_{ab}$. We summarize what we have proved in the following lemma.

4.5. Lemma: *For each $\lambda \in L$ there is a $\boldsymbol{Q}$-subspace U_* of U, an element $w \in G(\boldsymbol{Q})$, a finite set of non-archimedean rational primes B and an open compact subset Y of $U_{*B} = \prod_{p \in B} U_*(\boldsymbol{Q}_p)$ such that*

$$a_s(\lambda) = \int_Y s(wu)\phi(wu)\epsilon(-\lambda u)d_*u.$$

The Haar measure d_*u on U_{*B} is normalized so that the volume of $U_{*B} \cap U(\hat{\boldsymbol{Z}})$ is rational.

4.6. REMARK: It is convenient to call any field containing all the Fourier coefficients of an automorphic form F *a field of definition for* F.

§5. Galois action

5.1: Our next goal is to show that the Eisenstein series ${}_{\mathfrak{A}}E_s$ is defined over a cyclotomic number field, which can be chosen independently of $s \in S(\boldsymbol{Q})$ and of the character ρ. To do this we study the effect of replacing each Fourier coefficient with its conjugate by a fixed element τ in the Galois group $\mathfrak{G}$ of the maximal abelian extension $\boldsymbol{Q}_{ab}$ over $\boldsymbol{Q}$. The main result is that this galois action replaces an Eisenstein series with respect to K by an Eisenstein series with respect to another compact open subgroup of $G(\boldsymbol{A}_{na})$. Information on the field of definition then follows because the isotropy group of each Eisenstein series is open in $\mathfrak{G}$.

We identify $\mathfrak{G}$ with $GL(1, \hat{\boldsymbol{Z}})$ as usual and write the action of $\mathfrak{G}$ on $\boldsymbol{Q}_{ab}$ exponentially so that a^τ is the image of $a \in \boldsymbol{Q}_{ab}$ under the action of $\tau \in \mathfrak{G}$. For $\tau \in \mathfrak{G}$ and $x \in \boldsymbol{A}$ let $x \mapsto \tau \cdot x$ denote multiplication by the idèle τ. Note that $\epsilon(x) \in \boldsymbol{Q}_{ab}$ and that $\epsilon(x)^\tau = \epsilon(\tau \cdot x)$. By Lemma 1.4.1 there is a one-parameter $\boldsymbol{Q}$-subgroup $\mu : GL(1) \rightarrow \mathrm{Int}_G(T_\Theta)$ such that $\mu(\tau)$ maps each $u \in U(\boldsymbol{A})$ to its scalar multiple $\tau \cdot u$, $\tau \in \mathfrak{G} = GL(1, \hat{\boldsymbol{Z}})$. In particular, if $\lambda \in L$, then $\epsilon(-\lambda u)^\tau = \epsilon(-\lambda \circ \mu(\tau)(u))$.

5.2: Fix $\mathfrak{A} = (G, T, P, \rho, K)$ and suppose that $\tau \in \mathfrak{G}$ and $s \in S_{\mathfrak{A}}$. To emphasize the rôle of $\mathfrak{A}$ we write ${}_{\mathfrak{A}}E_s$ in place of E_s. Let $\mu(\tau)G$ be the algebraic group G endowed with co-ordinates such that $(\mu(\tau)G)(\hat{\boldsymbol{Z}}) = \mu(\tau)(G(\hat{\boldsymbol{Z}}))$, and let $\tau \cdot A = (\mu(\tau)G, T, P, \rho, \mu(\tau)K)$. For any function f on $G(\boldsymbol{A})$ define τ^*f by

$$\tau^*f(x) = f(w \cdot \mu(\tau^{-1})(w^{-1}x)), \quad (x \in G(\boldsymbol{A})).$$

We shall prove that τ, acting on Fourier coefficients, takes ${}_{\mathfrak{A}}E_s$ to ${}_{\tau\cdot\mathfrak{A}}E_{\tau^*s}$. First we need two lemmas.

5.2.1. LEMMA: *For any commutative ring R and any $\tau \in \mathfrak{G}$,*

$$\tau^*(S_{\mathfrak{A}}(R)) = S_{\tau\cdot\mathfrak{A}}(R).$$

More precisely, if $x \in G(\boldsymbol{A})$, we claim that

$$(1) \qquad \mu(\tau)(w^{-1}P(\boldsymbol{Q})xK^+) = w^{-1}P(\boldsymbol{Q})w \cdot \mu(\tau)(w^{-1}x) \cdot \mu(\tau)K^+.$$

Indeed, by Lemma 3.2.1,

$$w^{-1}P(\boldsymbol{Q})xK^+ = U^-(A)Z_{\boldsymbol{\Theta}}(\boldsymbol{Q})w^{-1}K^+$$

and this is mapped by $\mu(\tau)$ to

$$U^-(\boldsymbol{A})Z_{\boldsymbol{\Theta}}(\boldsymbol{Q})\mu(\tau)(w^{-1}x) \cdot \mu(\tau)K^+ = w^{-1}P(\boldsymbol{Q})w \cdot \mu(\tau)(w^{-1}x) \cdot \mu(\tau)K^+.$$

5.2.2. LEMMA: $\tau^*(N_{\mathfrak{A}}) = N_{\tau \cdot \mathfrak{A}}$.

Since $\mathrm{Int}(w)$ acts as inversion on $T_{\boldsymbol{\Theta}} \cap \mathscr{D}G$, the center of $\mathscr{D}G$ meets $T_{\boldsymbol{\Theta}}$ in a subgroup of order ≤ 2. Therefore, $\mu(\tau)^{-2} = \mathrm{Int}_G(t)$ for some $t \in T_{\boldsymbol{\Theta}}(\boldsymbol{A}_{na})$. Clearly, t lies in the greatest compact subgroup of $T_{\boldsymbol{\Theta}}(\boldsymbol{A}_{na})$, so $|\rho(t)|_{\boldsymbol{A}} = 1$. Calculating locally, we see that if $g \in G(\boldsymbol{A})$ then

$$w \cdot \mu(\tau)^{-1}(g) = t \cdot \mu(\tau)^{-1}(wg).$$

Thus, $\tau^* f(g) = f(t \cdot \mu(\tau)^{-1}(g))$. If $g \in G(\boldsymbol{A}_{na})$, then $g = b\omega$ with $b \in P(\boldsymbol{A})$ and $\omega \in \mu(\tau)G(\hat{\boldsymbol{Z}})$. A brief calculation shows that

$$\tau^* N_{\mathfrak{A}}(b\omega) = |\rho(b)|_{\boldsymbol{A}} = N_{\tau \cdot \mathfrak{A}}(b\omega),$$

as required.

5.2.3. THEOREM: *Suppose that $\tau \in \mathfrak{G}$ and that the Eisenstein series ${}_{\mathfrak{A}}E_s$ has Fourier expansion*

$${}_{\mathfrak{A}}E_s(x) = \sum_{\lambda \in L} a_s(\lambda) \cdot \boldsymbol{e}(\lambda x), \quad (x \in D).$$

Then $a_s(\lambda) \in Q_{ab}$ for each $\lambda \in L$ and

$${}_{\tau \cdot \mathfrak{A}}E_{\tau^* s}(x) = \sum_{\lambda \in L} a_s(\lambda)^\tau \cdot \boldsymbol{e}(\lambda x), \quad (x \in D).$$

That is, ${}_{\tau \cdot \mathfrak{A}}a_{\tau^* s}(\lambda) = ({}_{\mathfrak{A}}a_s(\lambda))^\tau$ for each $\lambda \in L$.

Replacing $\mathfrak{A}$ by (G, T^z, P, ρ, K) for suitable $z \in Z_{\boldsymbol{\Theta}}(\boldsymbol{Q})$, we may assume that $\lambda(\perp U_i) = 0$ and $rk_{\boldsymbol{Q}}(\lambda) = i$ for some $i \leq n$. Furthermore, the action of τ^* commutes with restriction from $G(\boldsymbol{A})$ to $G_i(\boldsymbol{A})$, so replacing $\mathfrak{A}$ by $\mathfrak{A}(i) = (G_i, T_i, P_i, \rho_i, K_i)$, as in 4.4, and s by its restriction to $G_i(\boldsymbol{A})$, we may assume that λ is generic.

Let $c = s\phi$ as in 2.3. There is a finite set B of non-archimedean places of $\boldsymbol{Q}$ such that

$$(1) \qquad a_s(\lambda) = q \int_Y c(wu)\epsilon(-\lambda u) d_B u$$

for some rational number $q = q(\lambda)$ and some compact open subset $Y = Y(\lambda) \subset U_B$; see 4.5. We can take Y to be normalized by $\mu(\tau)$ as in 3.6.1. Since the integrand is locally constant, the integral in (1) reduces to a finite sum. Thus,

$$(2) \qquad a_s(\lambda)^\tau = q \int_Y c(wu)\epsilon(-\lambda u)^\tau d_B u.$$

A change of variable replaces the integrand of (2) by

$$c(w \cdot \mu(\tau)^{-1}(u))\epsilon(-\lambda u) = \tau^* s(wu)\tau^*\phi(wu)\epsilon(-\lambda u),$$

so the theorem follows from Lemma 5.2.2, i.e.,

$$(3) \qquad a_s(\lambda) = q \int_Y \tau^* s(wu)\phi_{\tau\cdot\mathfrak{A}}(wu)\epsilon(-\lambda u) d_B u.$$

5.3. THEOREM: *If $s \in S$, then its isotropy group $\mathfrak{G}_s$ in $\mathfrak{G}$ is open and each τ in $\mathfrak{G}_s \cap \mathcal{N}_{\mathfrak{G}}(G(\hat{\boldsymbol{Z}}))$ fixes the Fourier coefficients of E_s. In particular, if $s \in S(\boldsymbol{Q})$, then the Eisenstein series E_s is defined over the finite cyclotomic extension of $\boldsymbol{Q}$ corresponding to $\mathfrak{G}_s \cap \mathcal{N}_{\mathfrak{G}}(G(\hat{\boldsymbol{Z}}))$.*

Clearly, the normalizer of $G(\hat{\boldsymbol{Z}})$ in $\mathfrak{G}$ is open. Thus, in view of 5.2.2, the only point that needs to be checked is that $\mathfrak{G}_s$ is open. Since the set of cusps is finite, it suffices to show that the condition, on $\tau \in \mathfrak{G}$,

$$\mu(\tau)(w^{-1}P(\boldsymbol{Q})xK^+) = w^{-1}P(\boldsymbol{Q})xK^+$$

is open for each $x \in G(\boldsymbol{A})$. This follows at once from 5.2.1 (1) because the condition $\mu(\tau)K = K$ is open, K being both compact and open.

5.3.1. COROLLARY: *There is a root of unity Y, which depends only on K, not on ρ, such that each Eisenstein series E_s is defined over the number field generated by Y and by the finite set of values $s(G(\boldsymbol{A}))$.*

Indeed, the character ρ appears nowhere in the description of $\mathfrak{G}_s$. If $s \in S(C)$, then $s = \Sigma_i \alpha_i s_i$ for some $\alpha_i \in C$, where the s_i's give the basis of $S(\boldsymbol{Q})$ corresponding to the point-cusps; $\alpha_i = s(x)$ for some $x \in G(\boldsymbol{A})$, and $E_s = \Sigma_i \alpha_i E_{s_i}$.

5.3.2. REMARK: The value $\Phi_a(f)$ of an automorphic form f at a cusp $\Delta(a) = P(\boldsymbol{Q})aK^+$ is defined in [9; §8], for instance. By a result of Baily [4; (9) on p. 146] and by 2.4 (5), we have

$$\Phi_a(E_s) = N(a)s(a).$$

In the statement of Corollary 5.3.1, since $N(a) \in \boldsymbol{Q}$, we may replace the values of s by the values of E_s at the various cusps.

5.4: We now give some concrete examples in which we can specify the order of Y in Theorem 5.3. The key point is to find criteria for $\tau \in \mathfrak{G}$ to fix a given element $s \in S_{\mathfrak{A}}$. It is convenient here to make two additional assumptions about $\mathfrak{A}$:

(A6) *The character* μ factors through $T_{\boldsymbol{\Theta}}$;

(A00) *Co-ordinates on G are chosen so that $T_{\boldsymbol{\Theta}}(\hat{\boldsymbol{Z}})$ is the greatest compact subgroup of $T_{\boldsymbol{\Theta}}(\boldsymbol{A}_{na})$*; see 1.3.0.

We can then identify $\mathfrak{G}$ with a subgroup of $T_{\boldsymbol{\Theta}}(\hat{\boldsymbol{Z}})$. Condition (A6) holds whenever G is an adjoint group; however, it is more convenient to deal with examples of the following type. Let G be a group satisfying assumptions (A0) to (A5) of 1.2 plus assumption (A00) above. Then $GL(1)$ acts on G via an isomorphism $\mu : GL(1) \to \mathrm{Int}_G(T_{\boldsymbol{\Theta}})$, and we can use this action to form the semi-direct product $G^* = GL(1) \times G$. Withe obvious choice of co-ordinates, G^* satisfies (A0) and (A00). Extend ρ to a character of $P^* = GL(1) \times P$ by letting $\rho^*(t, b) = t \cdot \rho(b)$. Let $T^* = GL(1) \times T$ and let $K^* = \mathcal{N} \times K$, where $\mathcal{N}$ is the normalizer of K in $\mathfrak{G} = GL(1, \hat{\boldsymbol{Z}})$. Then $\mathfrak{A}^* = (G^*, T^*, P^*, \rho^*, K^*)$ satisfies (A1) to (A6) as well as (A0) and (A00). For example, if $G = SL(2)$, then $G^* = GL(2)$, while if $G = \mathrm{Sp}(2n)$, then $G^* = \mathrm{Gp}(2n)$, as in [32]. This gives examples of $\mathfrak{A}$, as in 1.1, but for $\mathrm{Gp}(2n)$ instead of $\mathrm{Sp}(2n)$. We shall refer to these examples of $\mathfrak{A}$ as *standard*.

5.4.1: View $\mathfrak{G}$ as embedded in $T_{\boldsymbol{\Theta}}$. A brief calculation shows that the double coset $P(\boldsymbol{Q})xK^+$ is fixed by τ in the sense of 5.2, i.e., $P(\boldsymbol{Q})w \cdot \tau w^{-1}xK^+\tau^{-1} = P(\boldsymbol{Q})xK^+$, if both

1(a) *τ normalizes K*;

1(b) *the commutator $\tau(w^{-1}x)^{-1} \cdot \tau^{-1} \cdot (w^{-1}x)$ lies in K.*

In particular, if K is normal in $G(\hat{Z})$ and if s is the characteristic function of $P(Q)wK^{+}$, then the corresponding Eisenstein series has rational Fourier coefficients. This occurs, for instance, if $G = GL(2)$ and $\mathfrak{A}$ is standard with $K = K(m)$ for a positive integer m; cf. [22; (4) on p. 48] and [29; IV – 30, Prop. 17] with $d = 0$. Moreover, this shows that whenever the corresponding arithmetic group Γ has only two cusps, their Eisenstein series must both be defined over Q since the sum of the Eisenstein series certainly is by 5.3.2. For example, this occurs if $G = GL(2)$ and $K = K_0(p)$ with p a prime number; see [23; Satz 11].

Since $\tau w \tau w^{-1}$ lies in the center of G, a brief calculation shows that 1(a) and 1(b) both follow from

$$(2) \qquad \tau \in K \cap xKx^{-1}.$$

In particular, if s is the characteristic functions of $P(Q)xK^{+}$, then $\tau^{*}s = s$ whenever τ satisfies 2).

5.4.2: Suppose that the derived group of G is simply connected. Then, by 2.2.1 we can choose representatives for the double cosets $P(Q)\backslash G(Q)K^{+}/K^{+}$ from the set $T_{\Theta}(A)G(\hat{Z})$. Suppose also that K is normal in $G(\hat{Z})$. If $\tau \in T_{\Theta} \cap K$, then it follows from 5.4.1(2) that $\tau^{*}s = s$ for each $s \in S_{\mathfrak{A}}(Q)$. One can use what has been shown to treat the following explicit examples.

Firstly, if G is itself semi-simple and simply connected and if $K = G(\hat{Z})$, then the Eisenstein series attached to the various cusps are defined over Q. Their sum is E_s with $s \equiv 1$ and was treated by Tsao [36] and by Baily [7].

Secondly, one can handle principal congruence subgroups of any Hilbert modular group. Suppose that G is obtained from $GL(2)$ by restricting from a totally real ground field k down to Q. Let $\mathfrak{o}$ denote the maximal order of k, let $\mathfrak{y}$ be an integral ideal of k with index $N(\mathfrak{y})$ in $\mathfrak{o}$, and let $\Gamma(\mathfrak{y})$ denote the subgroup of $G(Z) = GL(2, \mathfrak{o})$ consisting of matrices congruent to the identity modulo $\mathfrak{y}$. Then one checks readily that for each cusp, the corresponding Eisenstein series is defined over the field generated by the $N(\mathfrak{y})$-th roots of unity. This result was first proved by H. Klingen in [25]. Our techniques also yield analogous results for principal congruence subgroups of Hilbert-Siegel modular groups.

5.5. In this section we assume that G normalizes $G(\hat{Z})$: Even so, the Eisenstein series attached to individual cusps are usually not defined over $\boldsymbol{Q}$. However, we shall prove that both $\mathfrak{E}$ and ${}^o\mathfrak{E}$ have bases of Eisenstein series defined over $\boldsymbol{Q}$. In other words, the spaces $\mathfrak{E}$ and ${}^o\mathfrak{E}$ are defined over $\boldsymbol{Q}$.

For any $\boldsymbol{Q}_{ab}$ subspace V of S let $\mathfrak{E}_V$ denote the space of Eisenstein series E_s with $s \in V$. Note that if $V = {}^oS$, then $\mathfrak{E}_V$ is just ${}^o\mathfrak{E}$, cf. 2.2.2.

5.5.1. LEMMA: *If V is a $\mathfrak{G}$-invariant $\boldsymbol{Q}_{ab}$-subspace of S, then $\mathfrak{E}_V$ has a basis of Eisenstein series defined over $\boldsymbol{Q}$.*

The action of $\mathfrak{G}$ on S was chosen so that, for λ any rational character on V defined over $\boldsymbol{Q}$, the linear map $s \mapsto a_s(\lambda)$ is $\mathfrak{G}$-equivariant. By Theorem 5.3 it therefore suffices to show that V is generated over $\boldsymbol{Q}_{ab}$ by the $\boldsymbol{Q}$-subspace $V^{\mathfrak{G}}$ of all $\mathfrak{G}$-invariant elements of V. Note that every character χ on the compact group $\mathfrak{G}$ satisfies $\chi(\sigma)^\tau = \chi(\sigma\tau)$ for $\sigma, \tau \in \mathfrak{G}$. It follows that for each such χ, the map that takes $s \in V$ to

$$(^*s)^\wedge(\chi) = \int_{\mathfrak{G}} \chi(\tau)\tau^*s \, d\tau$$

is a projection onto $V^{\mathfrak{G}}$. Note that $(^*s)^\wedge$ is the Fourier transform of the function $^*s : \tau \mapsto \tau^*s$ from $\mathfrak{G}$ to S. By Fourier inversion, it is clear that $V^{\mathfrak{G}}$ must generate V over $\boldsymbol{Q}_{ab}$, as required. Note also that the integral in (1) is actually just a finite sum.

5.5.2. LEMMA: *For each $\tau \in \mathfrak{G}$, the $\boldsymbol{Q}_{ab}$-subspace ${}^oS$ of S is invariant under the endomorphism τ^* of S.*

Recall that $\mathfrak{G}$ is identified with a subgroup of $(\operatorname{Int} G)(\boldsymbol{A})$ and that ${}^oS(\boldsymbol{Q})$ is spanned by $P(\boldsymbol{Q})\backslash G(\boldsymbol{Q})K^+/K^+$. Since K^+ contains $\mathscr{C}_G(\boldsymbol{A}) = \mathscr{C}_{G(\boldsymbol{A})}$ and since $w\tau(w^{-1}G(\boldsymbol{Q})K^+) = G(\boldsymbol{Q})K^+$ if $\tau(G(\boldsymbol{Q})K^+) = G(\boldsymbol{Q})K^+$, it suffices to show that $\operatorname{Int}_G(G(\boldsymbol{Q})K^+)$ is a normal subgroup of $(\operatorname{Int} G)(\boldsymbol{A})$. The following proof is just a slight variation on Kneser's proof of Satz 6.1 in [27].

Real approximation shows that $G(\boldsymbol{Q})G(\boldsymbol{R})$ is contained in $G(\boldsymbol{Q})K^+$; see [20; 0.4]. Also the complement of $G(\boldsymbol{Q})K^+$, as a union of cosets of K^+, is clearly open; hence $\mathscr{C}\ell(G(\boldsymbol{Q})G(\boldsymbol{R}))$, the closure of $G(\boldsymbol{Q})G(\boldsymbol{R})$ in $G(\boldsymbol{A})$, is contained in $G(\boldsymbol{Q})K^+$. Let F be the derived subgroup of G, and let $f: E \to F$ be a simply connected cover of F. Then using strong approximation for E, we have

$$f(E(\boldsymbol{A})) = f(\mathscr{C}\ell(E(\boldsymbol{Q})E(\boldsymbol{R}))) \subseteq \mathscr{C}\ell(G(\boldsymbol{Q})G(\boldsymbol{R})) \subseteq G(\boldsymbol{Q})K^+.$$

Thus,

$$\mathrm{Int}_G \circ f(E(A)) \subseteq \mathrm{Int}_G(G(\boldsymbol{Q})K^+).$$

However, $\mathrm{Int}_G \circ f$ is the simply connected cover of Int G, so Kneser's proof of Hilfsatz 6.2 in [27] shows that the derived group of $(\mathrm{Int}\, G)(A)$ is contained in $\mathrm{Int}_G(G(\boldsymbol{Q})K^+)$, hence the normality of $\mathrm{Int}_G(G(\boldsymbol{Q})K^+)$, as required.

5.5.3. THEOREM: *Suppose that $\mathfrak{G}$ normalizes $G(\hat{\boldsymbol{Z}})$. Then each of the spaces E and $^o\mathfrak{E}$ has a basis of Eisenstein series defined over $\boldsymbol{Q}$.*

This follows at once from 5.5.1 and 5.5.2.

5.6: Let H denote the group of all holomorphic isometries of the domain D; it is a Lie group though possibly not connected. We wish to extend Theorem 5.3 to certain arithmetic subgroups of H, cf. [9; 3.3(i)]. For this we need some terminology and facts.

Fix (G, P) as in 1.2 and assume that G has trivial center. Then H^o is isomorphic to G^∞. We fix an isomorphism and henceforth identify H^o with G^∞. Let $H^o(\boldsymbol{Q}) = G(\boldsymbol{Q}) \cap H^o$ and let $H(\boldsymbol{Q}) = \mathcal{N}_H(H^o(\boldsymbol{Q}))$. Let $B^o = P \cap H^o$, $B = \mathcal{N}_H(B^o)$, $B(\boldsymbol{Q}) = B \cap H(\boldsymbol{Q})$ and $B^o(\boldsymbol{Q}) = B \cap H^o(\boldsymbol{Q})$. Then $H(\boldsymbol{Q}) = B(\boldsymbol{Q}) \cdot H^o(\boldsymbol{Q})$, cf. [4; p. 145]. The reader should keep in mind that, despite the notation here, H is *not* an algebraic group.

Suppose that Γ is an arithmetic subgroup of H. Then $\Gamma \subset \mathrm{H}(\boldsymbol{Q})$, cf. [4; §4], and there is a finite subset X of $H^o(\boldsymbol{Q})$ such that

$$H(\boldsymbol{Q}) = \coprod_{a \in X} B^0(\boldsymbol{Q}) a \Gamma. \tag{1}$$

If $g \in H$ and $z \in D$, let $J(g, z)$ be the jacobian determinant of g at z. There is then an integer $d = d(\Gamma)$ divisible by the order of B/B^o and such that, for $a \in X$,

$$J(b, z)^d = 1, \quad (b \in a\Gamma a^{-1} \cap B,\ z \in D),$$

cf. [8; pp. 239–240]. Let $\Gamma_a = \Gamma \cap a^{-1}Ba$.

If Δ is any arithmetic subgroup of Γ, then there is a finite subset $Y \subset \Gamma$ such that $\Gamma = \coprod_{\omega \in Y} \Gamma_a \cdot \omega \cdot \Delta$. For each $\omega \in Y$ we can find $b = b(\omega) \in B(\boldsymbol{Q})$ such that $b\omega \in H^0(\boldsymbol{Q})$. For each $g \in H$ let $g(\omega) = bg\omega$. Then $a(\omega) \in H^0(\boldsymbol{Q})$ for each $a \in X$ because $bH^0(\boldsymbol{Q})\omega = H^0(\boldsymbol{Q})$.

Given $\gamma \in \Gamma$ we choose $\omega \in Y$, $\nu \in \Gamma_a$ and $\gamma' \in \Delta$ such that $\gamma = \nu\omega\gamma'$. Then $a\gamma = {}^a\nu \cdot b^{-1} \cdot a(\omega)\gamma'$. If $z \in D$, then $J({}^a\nu, z)^d = 1$ and $J(b^{-1}, z)$ is independent of z. Let $c(\omega) = J(b^{-1}, z)$. Then $J(a\gamma, z)^d = c(\omega)^d J(a(\omega)\gamma', z)^d$. It is easy to verify that $\Gamma_a \cdot \omega \cdot \gamma' \mapsto \Delta_{a(\omega)}\gamma'$ gives a one-to-one correspondence between right cosets of Γ_a in $\Gamma_a\omega\Delta$ and right cosets of $\Delta_{a(\omega)}$ in Δ. If ℓ is an integer divisible by d, then it follows that least formally that, for $z \in D$,

$$(2) \qquad \sum_{\gamma \in \Gamma_a \backslash \Gamma} J(a\gamma, z)^\ell = \sum_{\omega \in Y} c(\omega)^\ell \sum_{\gamma'} J(a(\omega)\gamma', z)^\ell, \quad (\gamma' \in \Delta_{a(\omega)} \backslash \Delta).$$

Now let $\Delta = \Gamma \cap H^o$. If ℓ is a sufficiently large positive integer, then a criterion of Godement, cf. [8; Chap. 11, sec. 2], show that each inner sum on the right side of (2) converges normally to a Δ-automorphic form ${}^\Delta\mathscr{E}_{a(\omega),\ell}$. Thus, the sum on the left side of (2) also converges normally, this time to a Γ-automorphic form ${}^\Gamma\mathscr{E}_{a\ell}$.

5.7. Definition: If Γ is an arithmetic subgroup of H such that $\Gamma \cap H^o$ is a congruence subgroup of G, then Γ is called a *congruence subgroup* of H.

5.9. Theorem: *Given a congruence subgroup Γ of H there exists a root of unity $Y = Y(\Gamma)$ and integers ℓ_0 and $d(\Gamma)$ such that, for each even integer $\ell > \ell_0$ and divisible by $d(\Gamma)$, the Eisenstein series ${}^\Gamma\mathscr{E}_{a\ell}$ is defined over the field $\boldsymbol{Q}(Y)$.*

From [9; Theorem 1.11] it follows that $b \mapsto J(b, *)$, $(b \in B^o)$, is the restriction to B^o of a rational character ρ_0 on P. Choose ℓ_0 such that, if $\ell > \ell_0$ and if $d(\Gamma)$ divides ℓ, then 5.6 (2) converges to a Γ-automorphic form ${}^\Gamma\mathscr{E}_{a\ell}$ for each $a \in X \subset H^o(\boldsymbol{Q})$). Fix such an ℓ and let $\rho = (\rho_0)^{-\ell}$, let $\Delta = \Gamma \cap H^o$, and choose K as in 1.2 such that $\Delta = G(\boldsymbol{Q}) \cap K^+$. Then $\mathfrak{A} = (G, T, P, \rho, K)$ satisfies (A0) to (A5) of 1.2, and $j_{\mathfrak{A}} = J^{-\ell}$, cf. [7; §1]. From 5.6 (2) we have

$$(1) \qquad {}^\Gamma\mathscr{E}_{a\ell} = \sum_{\omega \in Y} c(\omega)^\ell \cdot {}^\Delta\mathscr{E}_{a(\omega)\ell},$$

where Y is a finite subset of Γ. Note that $c(\omega)^\ell = J(b^{-1}, *)^\ell = J(b^{-\ell}, *)$ where $b = b(\omega) \in B(\boldsymbol{Q})$. Since the order of B/B^o divides ℓ, $b^{-\ell} \in P(\boldsymbol{Q})$, hence $c(\omega)^\ell = \rho_0(b^{-\ell}) \in \boldsymbol{Q}$.

Fix $\omega \in Y$ and let s be the characteristic function of the double coset $P(\boldsymbol{Q}) \cdot a(\omega) \cdot K^+$. Then ${}^\Delta\mathscr{E}_{a\ell} = N(a(\omega))_{\mathfrak{A}} E_s$ and $N(a(\omega)) \in \boldsymbol{Q}$, cf.

2.3. Furthermore, ${}_{\mathfrak{A}}E_s$ is defined over some cyclotomic field $\boldsymbol{Q}(Y)$ which, by Corollary 5.3.1, is independent of ℓ. The theorem now follows.

§6. Arithmetic quotients

6.1: We now turn to the problem of determining a field of definition for the Satake compactification $\mathfrak{V}(\Gamma)$ of the arithmetic quotient $\Gamma\backslash D$, where Γ is a congruence subgroup of $H = \mathrm{Hol}(D)$, cf. 5.6. We shall present a partial solution by providing a condition on Γ that is necessary and sufficient for certain polynomials in the Eisenstein series ${}^{\Gamma}\mathscr{E}_{a\ell}$ to generate the field k_Γ of all Γ-automorphic functions. As an application of the results of §5, we show that if Γ satisfies this condition (specified in 6.3) then $\mathfrak{V}(\Gamma)$ is defined over some cyclotomic field $\boldsymbol{Q}(Y)$. In fact, the important point is to check that a variety or morphism is defined over $\boldsymbol{Q}_{ab}$ since it then follows immediately that it is defined over $\boldsymbol{Q}(Y)$ for some root of unity Y. In some cases one can give upper bounds on the degree of Y.

In this paragraph $\boldsymbol{P}^r$ will denote projective space of dimension r. Let $d = d(\Gamma)$ and ℓ_0 both be as in Theorem 5.9, and let $j = J^{-1}$, where J is the jacobian determinant. Suppose that $\psi_1, \ldots, \psi_r$ are Γ-automorphic forms of weights $w_1, \ldots, w_r$, respectively, with respect to j. Then the monomial $\psi_1 \ldots \psi_r$ is said to have *weight* $w_1 + \cdots + w_r$. If $f(\psi_1, \ldots, \psi_r)$ is a polynomial in the ψ_i's then it is said to be *isobaric of weight* w if all its terms are monomials of weight w. We call a quotient of two isobaric polynomials of the same weight an *isobaric rational function.*

6.2: Let $\mathfrak{V}^* = \mathfrak{V}(\Gamma)$ be the Satake compactification, as in [9; 10.4], of $\Gamma\backslash D$. Then the sheaf $\mathscr{S}$ of germs of holomorphic functions on $\Gamma\backslash D$ extends to an ample sheaf $\mathscr{S}^*$ on $\mathfrak{V}^*$, and by [8; 8.5] each Eisenstein series (in the notation of §5)

$${}^{\Gamma}\mathscr{E}_{a\ell}(x) = \sum_{\gamma} j(a\gamma, x)^{\ell}, \quad (\gamma \in \Gamma_a\backslash\Gamma,\ x \in D)$$

extends to a section ${}^{\Gamma}\mathscr{E}^*_{a\ell}$ of $\mathscr{S}^*$. For each positive integer w let ${}^{\Gamma}I(w)$ be the module of all isobaric polynomials of weight w in the sections ${}^{\Gamma}\mathscr{E}^*_{a\ell}$, where d divides ℓ, $\ell > \ell_0$ and a runs through a finite set $X = X(\Gamma)$ of representatives in $H^o(\boldsymbol{Q})$ for the double cosets $B^o(\boldsymbol{Q})\backslash H(\boldsymbol{Q})/\Gamma$, cf. 5.6.

For infinitely many positive integers m, the sections of $\mathcal{S}^*$ belonging to ${}^{\Gamma}I(m)$ have no common zeros, [4; Proposition 1]. For each such m, choose a basis $\{\beta_1, \ldots, \beta_{r(m)}\}$ of monomials in ${}^{\Gamma}I(m)$. For such a basis there is a map ${}^{\Gamma}\Psi_m : \mathfrak{V}^* \to \boldsymbol{P}^{r(m)-1}$ that takes $z \in \mathfrak{V}^*$ to the point in $\boldsymbol{P}^{r(m)-1}$ with homogeneous co-ordinates $[b_1(z) : \cdots : \beta_{r(m)}(z)]$. Each ${}^{\Gamma}\Psi_m$ is a morphism of complex spaces, so its image $\mathfrak{W}_m$ is a projective variety. The function field of $\mathfrak{V}^*$ is then a finite algebraic extension of the function field of $\mathfrak{W}_m$. Let d_m be the degree of this extension. Then d_m is monotone decreasing as m tends to infinity, so d_m reaches a stationary value, which we denote $d_\infty(\Gamma)$. Let $q : D \to \Gamma \backslash D$ be the quotient map.

6.3: Definition: Let Γ be an arithmetic subgroup of H. We call a Γ-conjugacy class of subgroups $H(\boldsymbol{Q})$-conjugate to B a *cusp of type B* for Γ, or sometimes, a *point-cusp* for Γ. If Γ' is another arithmetic subgroup of H that has precisely the same point-cusps as Γ, then we say that Γ and Γ' are *B-isocuspidal subgroups.* In case no arithmetic subgroup Δ of $H(\boldsymbol{Q})$ properly containing Γ is B-isocuspidal with Γ, then we say that Γ is *B-saturated* or, simply, Γ is *saturated.*

6.3.1. Remarks: (1) The point-cusps of Γ correspond to the zero-dimensional rational boundary components adherent to a given fundamental domain for Γ in D.

(2) Γ and Γ' are B-isocuspidal precisely when the double-coset decompositions $B(\boldsymbol{Q})\backslash H(\boldsymbol{Q})/\Gamma$ and $B(\boldsymbol{Q})\backslash H(\boldsymbol{Q})/\Delta$ coincide. That is, $B(\boldsymbol{Q})a\Gamma = B(\boldsymbol{Q})a\Delta$ for each $a \in H(\boldsymbol{Q})$.

(3) It is easily seen that $d_\infty(\Gamma) = 1$ if and only if the field of Γ-automorphic functions is generated by isobaric polynomials in the Eisenstein series. We next show that if two arithmetic groups $\Gamma \subset \Gamma'$ are isocuspidal, then they have the same Eisenstein series, and finally, that $d_\infty(\Gamma) = 1$ precisely when Γ is saturated.

6.4. Lemma: *Suppose that $\tau \in H$ and that Γ is an arithmetic subgroup of H. Let $\Delta = \langle \tau, \Gamma \rangle$ be the subgroup of H generated by τ and Γ. If Δ is arithmetic and isocuspidal with Γ, then ${}^{\Gamma}\Psi_m \circ q \circ \tau = {}^{\Gamma}\Psi_m \circ q$ for each sufficiently large integer m divisible by the constant $d(\Gamma)$.*

It is sufficient to prove that for each $a \in H(\boldsymbol{Q})$ and $z \in D$ we have

$$(1) \qquad \sum_{\gamma \in \Gamma_a \backslash \Gamma} j(a\gamma, z)^{\ell} = \sum_{\gamma \in \Delta_a \backslash \Delta} j(a\gamma, z)^{\ell}$$

for each ℓ divisible by $d(\Gamma)$. However, $\Gamma_a \backslash \Gamma$ is in one-to-one correspondence with $B(\mathbf{Q})\backslash B(\mathbf{Q})a\Gamma$ and $\Delta_a\backslash\Delta$ is in one-to-one correspondence with $B(\mathbf{Q})\backslash B(\mathbf{Q})a\Delta$. Since $B(\mathbf{Q})a\Gamma = B(\mathbf{Q})a\Delta$, we have (1), and the lemma is proved.

6.5. LEMMA: *We use the notation of* 6.2. *Suppose that Δ is an arithmetic subgroup of H containing Γ, that $\tau \in H$ and that ${}^{\Gamma}\Psi_m \circ q \circ \tau = {}^{\Gamma}\Psi_m \circ q$ for each sufficiently large integer m divisible by the constant $d(\Gamma)$. Let Δ^* be the subgroup of H generated by τ and Δ. Then Δ^* is arithmetic and is B-isocuspidal with Δ.*

Suppose that Δ^* does *not* have properly discontinuous action on D. Then some orbit $\mathcal{Q}$ of Δ^* has an accumulation point, which lies in the interior of some fundamental domain F for Γ. Therefore, for some m there are infinitely many distinct points in $\mathcal{Q} \cap F$ all mapped to a single point of $\mathfrak{W}_m$ by ${}^{\Gamma}\Psi_m \circ q$. This, however, contradicts the finiteness of the fibres of ${}^{\Gamma}\Psi_m * q$; see [2; Lemma 2].

Therefore, the action of Δ^* on D must be properly discontinuous. It follows that Δ^* is discrete and since $\Delta\backslash H$ has finite invariant volume, Δ is of finite index in Δ^*. Hence, Δ^* is arithmetic and must lie in $H(\mathbf{Q})$; see [4; p. 144].

Let ℓ be an even integer greater than ℓ_0 and divisible by both $d(\Gamma)$ and $d(\Delta^*)$. Let

$$\omega_\ell(z) = \sum_{\gamma} j(\gamma, z)^\ell, \quad (\gamma \in (\Delta^* \cap B)\backslash\Delta^*,\ z \in D).$$

By 5.6 (2) the series above converges normally to a Δ^*-automorphic form lying in ${}^{\Gamma}I(\ell)$. Since ${}^{\Gamma}\Psi_\ell \circ q \circ \tau = {}^{\Gamma}\Psi_\ell \circ q$, it follows that for each $a \in H(\mathbf{Q})$, ${}^{\Gamma}\mathscr{E}_{a\ell}/\omega_\ell$ is a Δ^*-automorphic function; hence, each ${}^{\Gamma}\mathscr{E}_{a\ell}$ is actually an automorphic form with respect to Δ^*, not just Γ.

Suppose that Δ^* is not B-isocuspidal with Δ. Then there exist two zero-dimensional rational boundary components $\mathscr{F}_1$ and $\mathscr{F}_2$ that are Δ^*-equivalent but not Δ-equivalent. *A fortiori* $\mathscr{F}_1$ and $\mathscr{F}_2$ are not Γ-equivalent. Every Δ^*-automorphic form that vanishes at $\mathscr{F}_1$ must also vanish at $\mathscr{F}_2$. However, $\mathscr{F}_2$ is the boundary component normalized by $\mathrm{Int}(a)B$ for some $a \in X$, hence ${}^{\Gamma}\mathscr{E}_{a\ell}$ vanishes at $\mathscr{F}_1$ but not at $\mathscr{F}_2$; see [4; §5 (1)]. Since ${}^{\Gamma}\mathscr{E}_{a\ell}$ is Δ^*-automorphic, this contradicts the assumption that Δ^* is not B-isocuspidal with Δ and completes the proof.

6.5.1. COROLLARY: *Suppose $\Gamma \subset \Delta$ are arithmetic subgroups of H and $\tau \in H$. If $\Gamma^* = \langle\Gamma, \tau\rangle$ is arithmetic and isocuspidal with Γ, then $\Delta^* = \langle\Delta, \tau\rangle$ is arithmetic and isocuspidal with Δ.*

This follows immediately from 6.4 and 6.5. *G.* Prasad has given an algebraic proof of the corresponding statement with H replaced by an algebraic $\boldsymbol{Q}$-group G and B replaced by any parabolic $\boldsymbol{Q}$-subgroup P of G.

6.5.2. COROLLARY: *If Γ and Γ' are saturated arithmetic subgroups of H, then so is $\Gamma \cap \Gamma'$.*

Suppose $\Delta = \Gamma \cap \Gamma'$ is not saturated. Then Δ is properly contained in an arithmetic group $\Delta^* = \langle \Delta, \tau \rangle$ that is isocuspidal with Δ. It follows from 6.5.1 that $\langle \Gamma, \tau \rangle$ and $\langle \Gamma', \tau \rangle$ are both arithmetic and that they are isocuspidal, respectively, with Γ and with Γ'. Since Γ and Γ' are both saturated, it follows that $\tau \in \Gamma \cap \Gamma'$, contradiction.

6.5.3. REMARK: Corollary 6.5.2 was suggested by L.-C. Tsao.

6.6. THEOREM: *Suppose that Γ is an arithmetic subgroup of H. Then the following are equivalent*:

(i) *There is a saturated arithmetic subgroup Γ' of H containing Γ and such that if Δ is a subgroup of Γ' properly containing Γ, then*

$$B(\boldsymbol{Q})\backslash H(\boldsymbol{Q})/\Delta \neq B(\boldsymbol{Q})\backslash H(\boldsymbol{Q})/\Gamma;$$

(ii) $d_\infty(\Gamma) = 1$;

(iii) *Γ is saturated.*

It is obvious that (iii) implies (i).

(i) implies (ii): Suppose that $d_\infty(\Gamma) \neq 1$. Then Lemmas 3 and 4 of [4] show that there exists $\tau \in H$ such that ${}^{\Gamma}\Psi_m \circ q \circ \tau = {}^{\Gamma}\Psi_m \circ q$ for every sufficiently large integer m divisible by $d(\Gamma)$ but $\tau \notin \Gamma$. Let Δ be the subgroup of H generated by Γ and by τ. Then Lemma 6.5 shows that $B(\boldsymbol{Q})\backslash H(\boldsymbol{Q})/\Delta = B(\boldsymbol{Q})\backslash H(\boldsymbol{Q})/\Gamma$ and $\tau \in \Gamma'$. Thus (i) implies (ii).

(ii) *implies* (iii): Suppose that Γ is not saturated, i.e., Γ is properly contained in an arithmetic subgroup Δ of H such that Γ and Δ are B-isocuspidal. It follows that Γ and Δ have precisely the same Eisenstein series. Thus ${}^{\Gamma}I(m) = {}^{\Delta}I(m)$, and we may assume that ${}^{\Gamma}\Psi_m = {}^{\Delta}\Psi_m$ for each pertinant m, cf. 6.2. Suppose that $F(\Delta)$ is a fundamental domain for Δ and that $\sigma_1, \ldots, \sigma_n$ form a complete set of left-coset representatives for Γ in Δ. Then $\bigcup_{i=1}^{n} \sigma_i F(\Delta)$ is a fundamental domain for Γ, so $d_\infty(\Gamma) \geq [\Delta : \Gamma] > 1$, which shows that (ii) implies (iii). The theorem is proved.

6.7. Definition: If for each even integer $\ell > \ell_0$ and divisible by $d(\Gamma)$ the space generated by all ${}^{\Gamma}\mathscr{E}_{a\ell}$, where $a \in H(\boldsymbol{Q})$, has a basis of forms defined over a field F, then we say that F is a *field of definition for* Γ.

6.8. Theorem: *Suppose that* Γ *is a saturated arithmetic subgroup of* H *and that* F *is a field of definition for* Γ. *Then the Satake compactification* $\mathfrak{V}(\Gamma)$ *has a projective model* $\mathfrak{W}(\Gamma)$ *defined over* F. *In particular, if* Γ *is also a congruence subgroup, then* $\mathfrak{W}(\Gamma)$ *is defined over the cyclotomic field* $\boldsymbol{Q}(Y)$ *for some root of unity* Y

In view of Theorem 5 it suffices to construct a projective model $\mathfrak{W}(\Gamma)$ of $\mathfrak{V}(\Gamma)$ defined over F. Since Γ is saturated, Theorem 6.5 shows that $d_\infty(\Gamma) = 1$, i.e., we can find an even integer $m > \ell_0$, divisible by $d(\Gamma)$ and such that ${}^{\Gamma}\Psi_m : \mathfrak{V}(\Gamma) \to \boldsymbol{P}^{r-1}$ has degree 1 for every choice of basis in ${}^{\Gamma}I(m)$. In other words, ${}^{\Gamma}\Psi_m$ is a birational morphism onto its image.

Choose a basis B of ${}^{\Gamma}I(m)$ such that each $\beta \in B$ is defined over F. Since $\mathfrak{V}(\Gamma)$ is a normal variety [9; 10.11], Zariski's Main Theorem shows that ${}^{\Gamma}\Psi_m : \mathfrak{V}(\Gamma) \to \mathfrak{W}_m$ is a normalization. It follows that the graded $\boldsymbol{C}$-algebra $\mathscr{A}$ of integral Γ-automorphic forms (with respect to j) is integral over the $\boldsymbol{C}$-algebra generated by B.

By [5] the algebra $\mathscr{A}$ is generated over $\boldsymbol{C}$ by a finite number of automorphic forms $\phi_1, \ldots, \phi_s$ each defined over F. In view of [9; 10.11], we can then choose a basis $\psi_1, \ldots, \psi_t$ of the space of integral Γ-automorphic forms defined over F, of some suitably large weight ℓ, such that the map Ψ^* that takes $z \in \mathfrak{V}(\Gamma)$ to the point of $\boldsymbol{P}^{t-1}$ with homogeneous co-ordinates $[\psi_1(z) : \cdots : \psi_t(z)]$ is embedding of $\mathfrak{V}(\Gamma)$ into $\boldsymbol{P}^{t-1}$. The module of relations among any finite set of automorphic forms defined over F is easily seen to be defined over F as well. Let $\mathfrak{W}(\Gamma)$ be the image of $\mathfrak{V}(\Gamma)$ under the embedding Ψ^*. Then $\mathfrak{W}(\Gamma)$ is clearly defined over F.

6.9: Let Γ be an arithmetic subgroup of H and suppose $\tau \in H$. Let $\Delta = \mathrm{Int}(\tau) \cdot \Gamma$. Then the isometry $x \mapsto \tau^{-1} \cdot x$ of D induces a map τ_0 from $\Delta \backslash D$ to $\Gamma \backslash D$. The next lemma shows that, if $\tau \in H(\boldsymbol{Q})$, then Δ is arithmetic. It follows then that τ_0 extends to a map $\tau^* : \mathfrak{W}(\Delta) \to \mathfrak{W}(\Gamma)$ for each $\tau \in H(\boldsymbol{Q})$. Notice that the lemma is not entirely trivial because the group H is not an algebraic group; see 5.6.

6.9.1: Lemma: *If* $\tau \in H(\boldsymbol{Q})$, *then* $\mathrm{Int}(\tau)\Gamma$ *is an arithmetic subgroup of* H.

Since this is a question of commensurability, we may assume that $\Gamma \subset H^o(\mathbf{Q})$. Choose a $\boldsymbol{Q}$-group G such that $H^o(\boldsymbol{Q}) = G(\boldsymbol{Q}) \cap G(\boldsymbol{R})^o$ and a faithful finite-dimensional matrix representation ρ of G. Let A be the matrix ring generated by $\rho(\Gamma)$. Then the action of H on $A \otimes_{\boldsymbol{Z}} \boldsymbol{R}$ by conjugation defines an embedding $r: H \to GL(A, \boldsymbol{R})$. Since $G(\boldsymbol{Q}) \subset A \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$, cf. [4; §4], and since τ normalizes $G(\boldsymbol{Q}) \cap G(\boldsymbol{R})^o$, which clearly generates the algebra $A \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$, it follows that $r(\tau) \in GL(A, \boldsymbol{Q})$. Now $r(\Gamma)$ is an arithmetic subgroup of $r(G)$, and $r(\mathrm{Int}(\tau) \cdot \Gamma) = \mathrm{Int}(r(\tau)) \cdot r(\Gamma)$ is arithmetic in $r(G)$; therefore, $\mathrm{Int}(\tau) \cdot \Gamma$ is arithmetic in G (and in H), as required.

6.9.2. THEOREM: *Suppose that Γ is saturated and that $\tau \in H(\boldsymbol{Q})$. Then $\Delta = \mathrm{Int}(\tau)\Gamma$ is saturated and, if F is a field of definition for all the Eisenstein series ${}^{\Gamma}\mathscr{E}_{a\ell}$ and ${}^{\Delta}\mathscr{E}_{a\ell}$ ($a \in H(\boldsymbol{Q})$, $\ell < \ell_0$ and divisible by $d(\Gamma)$), then $\tau^*: \mathfrak{W}(\Delta) \to \mathfrak{W}(\Gamma)$ is defined over F.*

It is clear that Δ is saturated, so it suffices to show that, for each rational function ϕ on $\mathfrak{W}(\Gamma)$ defined over F, the function $L(\tau)\phi: x \mapsto \phi(\tau^{-1} \cdot x)$ on $\mathfrak{W}(\Delta)$ is also defined over F. For some positive integer d, the field of rational functions defined over F on $\mathfrak{W}(\Gamma)$ is generated over F by the automorphic functions

$$\phi = {}^{\Gamma}\mathscr{E}_{a,\ell d}/({}^{\Gamma}\mathscr{E}_{b,d})^{\ell} \quad (a, b \in H(\boldsymbol{Q});\ \ell \in \boldsymbol{Z}(>0)).$$

It therefore suffices to check that $L(\tau)\phi$ is defined over F for each such ϕ. We have

$$L(\tau) \cdot {}^{\Gamma}\mathscr{E}_{a,\ell d}(x) = j(\tau^{-1}, x)^{-\ell d} \cdot {}^{\Delta}\mathscr{E}_{a',\ell d}(x), \quad (a' = a\tau^{-1});$$

thus,

$$L(\tau)\phi = {}^{\Delta}\mathscr{E}_{a',\ell d}/({}^{\Delta}\mathscr{E}_{b',d})^{\ell}, \quad (b' = a'\tau^{-1}).$$

The theorem now follows because ${}^{\Delta}\mathscr{E}_{a',\ell d}$ and ${}^{\Delta}\mathscr{E}_{b',d}$ are both defined over F.

6.9.3. DEFINITION: Let Γ be an arithmetic subgroup of H. If Γ contains a normal subgroup Γ' that is a saturated congruence subgroup of H, then Γ is said to be *ample*.

6.9.4. COROLLARY: *Let Γ be an ample arithmetic subgroup of $H(\boldsymbol{Q})$, and let $\tau \in H(\boldsymbol{Q})$. Then $\Delta = \mathrm{Int}(\tau)\Gamma$ is ample and $\tau^*: \mathfrak{W}(\Delta) \to$*

$\mathfrak{W}(\Gamma)$ is defined over $\boldsymbol{Q}_{ab}$. In particular, if τ normalizes Γ, then τ^* is an automorphism of $\mathfrak{W}(\Gamma)$ defined over $\boldsymbol{Q}_{ab}$.

It is clear that Δ is ample, and from 6.9.2 we have $\mathfrak{W}(\Gamma')$ defined over $\boldsymbol{Q}_{ab}$. There is a natural finite and surjective morphism μ from $\mathfrak{W}' = \mathfrak{W}(\Gamma')$ to $\mathfrak{W} = \mathfrak{W}(\Gamma)$ and the fibres of μ are orbits of the finite group Γ/Γ' whose elements act as automorphisms of $\mathfrak{W}(\Gamma')$ defined over $\boldsymbol{Q}_{ab}$. The homogeneous co-ordinate ring $P[\mathfrak{W}]$ can be identified with the $\boldsymbol{C}$-algebra of Γ-automorphic forms whose weights are divisible by some suitable integer, and similarly for $P[\mathfrak{W}']$. We may identify $P[\mathfrak{W}]$ with the ring of invariants of Γ/Γ' in $P[\mathfrak{W}']$. By classical invariant theory, $\mathfrak{W}$ is therefore defined over $\boldsymbol{Q}_{ab}$, and it is obvious that τ^* is also defined over $\boldsymbol{Q}_{ab}$.

6.9.5. Remark: The proof of 6.9.4 shows that $\mathfrak{W}(\Gamma)$ (*resp.* τ^*) is defined over any field of definition for Γ' (*resp.* Γ' and $\mathrm{Int}(\tau)\Gamma'$).

6.10. Subvarieties: Suppose that Γ is ample. We now show that $\boldsymbol{Q}_{ab}$ is a field of definition for certain subvarieties of $\mathfrak{W}(\Gamma)$, for example, $\Gamma\backslash D$ and the various cusps. Since there are only finitely many $G(\boldsymbol{Q})$-orbits among the parabolic $\boldsymbol{Q}$-subgroups and since there are only finitely many cusps for Γ, in view of Theorem 6.9.2 it suffices to show the following lemma.

6.10.1. Lemma: *Let Q be a standard maximal parabolic $\boldsymbol{Q}$-subgroup for the order chosen on $\Sigma(T, G)$. Let k be a field of definition for $\mathfrak{W}(\Gamma)$. Then the image in $\mathfrak{W}(\Gamma)$ of the boundary component $D(Q)$ is an irreducible algebraic subvariety $\mathfrak{W}_Q$ defined over k.*

Firstly observe that if F is a Γ-automorphic form on D, then there is a positive integer $n = n(Q)$ such that the form induced by F on the boundary component $D(Q)$ has Fourier coefficients corresponding to characters λ of U that vanish on U_n. This follows easily from the discussion in §4.1.

Secondly, note that we may view the ring of Γ-automorphic forms of non-negative weights divisible by some suitable integer $d = d(\Gamma)$ as the homogeneous co-ordinate ring $P[\mathfrak{W}]$ of the variety $\mathfrak{W} = \mathfrak{W}(\Gamma)$. Each rational character λ on U then gives a k-linear form on the space of automorphic forms, namely, $F \mapsto \langle\lambda, F\rangle$, where $\langle\lambda, F\rangle$ denotes the Fourier coefficient (of the form F) indexed by λ.

The family of k-linear forms on $P[\mathfrak{W}]$ just described, call it L, separates homogeneous co-ordinates on $\mathfrak{W}$. Moreover, a co-ordinate F is k-rational if and only if $\langle\lambda, F\rangle \in k$ for each $\lambda \in L$.

Let L_Q be the subset of L consisting of all characters vanishing on U_n. Then the set

$$I = \{f \in P[\mathfrak{W}] : \langle \lambda, f \rangle = 0 \text{ for each } \lambda \in L_Q\}$$

is the homogeneous ideal defining the image $\mathfrak{W}_Q$ of $D(Q)$ in $\mathfrak{W}(\Gamma)$. Clearly, I is defined over k as a vector subspace of $P[\mathfrak{W}]$ and hence as a homogeneous ideal. Indeed, dimension considerations show that $\mathfrak{W}_Q$ is defined by the vanishing of a finite family of k-rational co-ordinates.

6.10.2. THEOREM: *Suppose that Γ is an ample arithmetic subgroup of $G(\boldsymbol{R})^o$. If Q' is any parabolic $\boldsymbol{Q}$-subgroup of G, then the image in $\mathfrak{W} = \mathfrak{W}(\Gamma)$ of the boundary component $D(Q')$ is an irreducible algebraic subvariety $\mathfrak{W}_{Q'}$ defined over a cyclotomic field $\boldsymbol{Q}(Y)$.*

Let Γ' be a saturated normal congruence subgroup of Γ. Choose an element $\tau \in H(\boldsymbol{Q})$ such that $\tau^{-1}Q'\tau = Q$ is a standard parabolic $\boldsymbol{Q}$-subgroup of G with respect to our ordering of $\Sigma(T, G)$. Then $\Delta' = \mathrm{Int}(\tau)\Gamma'$ is saturated, $\Delta = \mathrm{Int}(\tau)\Gamma$ is ample and there is a cyclotomic field $k = \boldsymbol{Q}(Y)$ such that all the Eisenstein series ${}^{\Gamma'}\mathscr{E}_{a\ell}$ and ${}^{\Delta'}\mathscr{E}_{a\ell}$ (with $a \in H(\boldsymbol{Q})$, $\ell > \ell_0$ and divisible by $d(\Gamma)$) are defined over k. By Theorem 6.9.2 and Remark 6.9.5, then the isomorphism $\tau^* : \mathfrak{W}(\Delta) \to \mathfrak{W}(\Gamma)$ is defined over k. The variety $\mathfrak{W}(\Delta)_Q$ is defined over k, hence its image $\mathfrak{W}(\Gamma)_{Q'} = \mathfrak{W}_{Q'}$ is defined over k as well.

6.10.3. COROLLARY: *If Γ is ample then there is a cyclotomic field of definition for the Shimura variety $\Gamma\backslash D$ considered as an open subvariety of $\mathfrak{W}(\Gamma)$.*

This is clear because the complement in $\mathfrak{W}(\Gamma)$ of $\Gamma\backslash D$ is the union of all cusps.

6.11. EXAMPLES: (1) Hecke-type congruence subgroups of Siegel modular groups are always saturated. Indeed, suppose that $G = \mathrm{Gp}(2m)$ as in §5.4 and make the standard choice for $\mathfrak{A}$. Let N be any non-zero integer and let x be the block-matrix $\left(\begin{smallmatrix} I & 0 \\ 0 & N \cdot I \end{smallmatrix}\right) \in G(\boldsymbol{Q})$, where I is the m-rowed identity matrix. Since $G(\boldsymbol{Z})$ is known to be maximal arithmetic in $G(\boldsymbol{R})$, both $G(\boldsymbol{Z})$ and $xG(\boldsymbol{Z})x^{-1}$ are saturated. By 6.5.2, if we let the Hecke congruence subgroup of level N be

$$\Gamma_0(N) = G(\boldsymbol{Z}) \cap xG(\boldsymbol{Z})x^{-1},$$

then $\Gamma_0(N)$ is a saturated subgroup of $G(\boldsymbol{R})$. Since $G(\boldsymbol{Z})$ and its conjugate are congruence subgroups, so is $\Gamma_0(N)$. This gives one family of examples where the theorems of §5 and §6 apply. In particular, the Satake compactification $\mathfrak{V}(\Gamma_0(N))$ carries a natural $\boldsymbol{Q}_{ab}$-structure. Similar examples can be constructed for other groups by using Lemma 1.4.1.

(2) In [19], Christian determined the principal congruence subgroups Γ of $\mathrm{Sp}(2m, \boldsymbol{Z})$ for which the Eisenstein series ${}^{\Gamma}\mathscr{E}_{a\ell}$ generate the field of modular functions for Γ; thus, he determined which Γ are saturated. His result is that all the principal congruence subgroups are saturated in case the rank m is odd; however, in case m is even, the only principal congruence subgroups that are saturated are those of level 1, 2, 4, p^a or $2p^a$, where p^a is a power of an odd prime. Tsao has recovered Christian's results by some brief calculations. Moreover, when the principal congruence subgroup $\Gamma(q)$ of level q is not saturated, Tsao has calculated the smallest saturated group $\overline{\Gamma(q)}$ containing $\Gamma(q)$; $\overline{\Gamma(q)}$ is the saturation of $\Gamma(q)$. It exists by Corollary 6.5.2.

Tsao has also investigated saturation for principal congruence subgroups of Baily's modular group, which acts on the exceptional tube domain of complex dimension 27, and principal congruence subgroups acting on type A domains. In the former case $\Gamma(q)$ is always saturated, while in the latter case, as in the case $G = \mathrm{Sp}(4n)$, the saturations $\overline{\Gamma(\mathrm{q})}$ do not form a cofinal family of arithmetic subgroups even when the groups $\Gamma(q)$ do. In the latter case, the saturated congruence subgroups do not form a cofinal family of arithmetic subgroups.

(3) In [38], Tsao has determined which normal congruence subgroups of $SL(2, \boldsymbol{Z})$ are saturated.

REFERENCES

[1] A. ASH et al: *Smooth Compactification of Locally Symmetric Varieties*, Math. Sci. Press, Brookline, Massachusetts, 1976.

[2] W.L. BAILY, JR: On the theory of θ-functions, the moduli of Abelian varieties, and the moduli of curves, *Ann. of Math. 75* (1962) 342–381.

[3] W.L. BAILY JR: On Hensel's lemma and exponential sums, *Global Analysis: Papers in Honor of K. Kodaira* (D.C. Spencer and S. Iyanga, ed.), Princeton University Press, Princeton, 1969, 85–100.

[4] W.L. BAILY, JR: Eisenstein series on tube domains, *Problems in Analysis: A symposium in Honor of Salomon Bochner*, R.C. Gunning, ed., Princeton University Press, Princeton, 1970, 139–156.

[5] W.L. BAILY, JR: Automorphic forms with integral Fourier coefficients, Several Complex Variables I—Maryland 1970, *Lecture Notes in Mathematics 155*, Springer-Verlag, Berlin - Heidelberg - New York, 1970, 1–8.

[6] W.L. BAILY, JR: An exceptional arithmetic group and its Eisenstein series, *Ann. of Math. 91* (1970) 512–549.

[7] W.L. BAILY, JR: On the Fourier coefficients of certain Eisenstein series on the adèle group, *Number Theory, Algebraic Geometry and Commutative Algebra: in honor of Y. Akizuki*, Kinokuniya, Tokyo, 1973.

[8] W.L. BAILY, JR: *Introductory Lectures on Automorphic Forms*, Publications of the Mathematical Society of Japan 12, Iwanami Shoten, Publishers, and Princeton University Press, Tokyo, 1973.

[9] W.L. BAILY, JR. and A. BOREL: Compactification of arithmetic quotients of bounded symmetric domains, *Ann. of Math. 84* (1966) 442–528.

[10] A. BOREL: Some finiteness properties of adèle groups over number fields, *Publ. Math. I.H.E.S. 16* (1963) 101–126.

[11] A. BOREL: Introduction to Automorphic Forms, in Algebraic Groups and Discontinuous Subgroups (A. Borel and G.D. Mostow, ed.): *Proc. Symp. Pure Math. IX*, Amer. Math. Soc., Providence, R. I., 1966, 199–210.

[12] A. BOREL: *Linear Algebraic Groups* (Notes by H. Bass), Benjamin Lecture Notes Series, New York, 1969.

[13] A. BOREL: personal communication.

[14] A. BOREL and J. TITS: Groupes réductifs, *Publ. Math. I.H.E.S. 27* (1965) 55–151.

[15] A. BOREL, Compléments à l'article "Groupes réductifs," *Publ. Math. IHES. 41* (1972) 253–276.

[16] F. BRUHAT and J. TITS: Groupes réductifs sur un corps local: I. Données radicielles valuées, *Publ. Math. I.H.E.S. 41* (1972) 5–252.

[17] C. CHEVALLEY: *Séminaire C. Chevalley* 1956–1958: *Classification des groupes de Lie algébriques*, mimeographed notes, École Normal Supérieure, Paris, 1958.

[18] U. CHRISTIAN: Zur Theorie der Hilbert-Siegelschen Modulfunktionen, *Math. Ann. 152* (1963) 275–341.

[19] U. CHRISTIAN, Zur Theorie der Hilbert-Siegelschen Modulfunktionen über den Eisensteinreihen, *Abhandl. math. Sem. Hamburg Univ. 27* (1964).

[20] P. DELIGNE: Travaux de Shimura, Séminaire Bourbaki, 1970/71, exposé 389, in *Lecture Notes in Mathematics 244*, Springer-Verlag, Berlin-Heidelberg-New York, 1971.

[21] R. GODEMENT: *Introduction à la théorie de Langlands*, Séminaire Bourbaki, 1966/67, exposé 321, W.A. Benjaminin, New York, and Amsterdam, 1968.

[22] R. GUNNING: *Lectures on Modular Forms*, Princeton University Press, Princeton, 1962.

[23] E. HECKE: *Analytische Arithmetik der positiven quadratischen Formen*, Kgl. Danske Vid. Selsk. Math. - fys. Medd. XVII, *12* (1940), (Math. Werke; 789–918).

[24] S. HELGASON: *Differential Geometry and Symmetric Spaces*, Academic Press, New York, 1962.

[25] H. KLINGEN: Über den arithmetischen Charakter der Fourier koeffizienten von Modulformen, *Math. Ann. 147* (1962) 176–188.

[26] H.D. KLOOSTERMAN: Theorie der Eisensteinschen Reihen von mehreren Veränderlichen, *Abhandl. math. Sem. Hamburg Univ. 6* (1927) 163–188.

[27] M. KNESER: Starke Approximation in Algebraischen Gruppen. I, *J. f. reine angew. Math. 218* (1965) 190–203.

[28] A. KORANYI and J. WOLF: Realization of Hermitian Symmetric Spaces as Generalized Half-Planes, *Ann. of Math. 81* (1965) 265–288.

[29] A. OGG: *Modular Forms and Dirichlet Series*, W.A. Benjamin, New York and Amsterdam, 1969.

[30] H. PETERSSON: Über den Körper der Fourierkoeffizienten der von Hecke untersuchten Eisensteinreihen, *Abhandl. math. Sem. Hamburg Univ. 16* (1949) 101–113.

[31] G. PRASAD: Strong approximation for semi-simple groups over function fields. *Ann. of Math. 105* (1977) 553–572.

[32] G. SHIMURA: On canonical models of arithmetic quotients of bounded symmetric domains, I, II, *Ann. of Math 91* (1970) 144–222; *92* (1970) 528–549.

[33] G. SHIMURA: On the Fourier coefficients of modular forms of several variables, *Nachr. Akad. Wiss.* Göttingen, 1975, Nr. *17*.

[34] C.L. SIEGEL: Über die analytische Theorie der quadratischen Formen, *Ann. of Math 36* (1935) 527–606 (Ges. Abhandl. I, 326–405).

[35] T.A. SPRINGER: *Jordan Algebras and Algebraic Groups*, Springer-Verlag, New York-Heidelberg-Berlin, 1973.

[36] L.-C. TSAO: The rationality of the Fourier coefficients of certain Eisenstein series on tube domains, *Compositio Math. 32* (1976) 225–291.

[37] L.-C. TSAO: *On saturated subgroups*, preprint.

[38] L.-C. TSAO: *Saturated normal congruence subgroups of the modular group*, preprint.

[39] E.B. VINBERG: Homogeneous cones, *Soviet Math. 1* (1960) 787–793 (translated from Doklady Akad. Nauk SSSR *133* (1960)).

[40] A. WEIL: *Adèles and algebraic groups* (Notes by M. Demazure and T. Ono), Institute for Advanced Study, Princeton, 1961.

[41] A. WEIL: *Basic Number Theory*. Springer-Verlag, Berlin-Heidelberg-New York, 1967.

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