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REPRESENTATIONS OF THE GROUP OF SMOOTH MAPPINGS OF A MANIFOLD X INTO A COMPACT LIE GROUP

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Abstract

Some important nonlocal representations of the group G^X consisting of C^∞-mappings of a Riemannian manifold X to a compact semisimple Lie group G are constructed. The irreducibility, as well as non-equivalence of the introduced representations corresponding to different Riemannian metrics are proved. The ring of representations is calculated.

Introduction

In this paper the unitary representations of the group G^X of smooth functions on a manifold X taking values in a compact Lie group G are being built.

According to the idea of a paper [20], one can construct irreducible nonlocal representations of the group of measurable G-valued functions on X, G being a Lie group, in the case when the unity representation is not an isolated point in the space of irreducible unitary representations. Such a construction for groups $G = SU(n, 1)$ and $G = SO(n, 1)$ is realized in [20] and [21]. The other examples of groups of this kind represent the groups of isometries of a Eucledean space, as well as solvable and nilpotent groups (cf. Araki [2], Streater [17], Guichardet [8], [9], Delorme [5], Parthasarathy-Schmidt [13] et al.).

In the case of compact Lie groups G the unity representation is an isolated point, and it is impossible to follow the scheme of [20]. If, however, one takes the group G^X of continuously differentiable mappings instead group of measurable mappings then it is possible to construct for this group a series of irreducible nonlocal representations. The idea of this construction is to consider at first the group $\theta^1(X; G)$ of smooth sections of the 1-jet fibre bundle $j^1(X; G) \to X$ (cf. §I); the initial group G^X is naturally imbedded in the $\theta^1(X; G)$ as a subgroup. As $\theta^1(X; G)$ may be regarded as a group of functions taking values in the skew-product $G \cdot (\underbrace{\mathfrak{G} \times \cdots \times \mathfrak{G}}_{\dim X})$, $\mathfrak{G}$ being the Lie algebra of the group G (cf. §I), and as the unity representation of this skew-product is not isolated, so it is possible, following the pattern of [20], to build nonlocal irreducible representation for $\theta^1(X; G)$ and to restrict it on the subgroup G^X. Here in this paper we study the representations of the group G^X obtained in this way.

The constructions of this kind of the representations of the group G^X have been found, after the appearance of [20], by the authors of the present paper and, independently, by Parthasarathy and Schmidt [14], R.S. Ismagilov [11], Albeverio and Hoeg-Krohn [1]. The irreducibility of these representations for $G = SU(2)$ in the case $\dim X \geq 5$ had been proved by R.S. Ismagilov [11].

As communicated to the authors A. Guichardet, P. Delorme had been studied the representations of the group G^X, where G is a compact Lie group. *It is being proved in the present paper the irreducibility of the representations for the group G^X, where G is any compact semisimple Lie group and* $\dim X \geq 2$.

The case $\dim X = 1$ remains open at the moment, the difficulty being connected with the more complicated, than for the spaces X of greater dimension, character of the restriction of the representation of the group G^X on the subgroup A^X, where $A \subset G$ is the Cartan subgroup.

We use in this paper, in a systematic way, the technique of Gaussian measures, which has been for the first time used in the study of nonlocal irreducible representations of current groups in [20], [21] and which has proved very useful for verifying the irreducibility and non-equivalence of functional group representations (see also [22]).

Here is the brief contents of the paper.

We consider, in §1, jet fibrations over X and the groups connected with them and we explain the main idea of the construction of the representation. We also introduce there the Maurer–Cartan cocycle, which plays a significant role in this construction. The §2 is of subsidiary character. There we give an account of what is connected with the construction $\mathrm{EXP}_\beta T$. This construction is more or less explicitly described in [2], [8], [13], but the usage of Gaussian measures, which began in [20], [21], allows to develop a systematic theory. It will be set forth in detail somewhere else. In §3 we construct the representations of the group G^X and formulate the principal results of the paper. The proofs of the main theorems are given in §§4 and 5.

In some papers of physical nature (see, for instance, [18], [3]) it was considered the so-called Sugawara algebra. The authors have noticed that the corresponding group is the central $\mathbb{R}^1$-extension of the group $\theta^1(X; G)$ (see above). (If we take $G = \mathbb{R}^1$, then we get the generalized functional Heisenberg group). We construct, in §6, nonlocal irreducible unitary representations for this group as well.

§1. The jets of smooth functions with values in a Lie group and Maurer–Cartan cocycle

Here we introduce the principal definitions concerning the group $C^\infty(X; G)$: the fibre bundle of k-jets, the group of sections of this fibre bundle, k-jet imbeddings etc. The description of the most important classes of representations of the group $C^\infty(X; G)$, both local and nonlocal, becomes more transparent if one passes to the group of sections of the k-jet fibre bundle (cf. section 3 of this paragraph). The most important for our purposes is the definition of the Maurer–Cartan cocycle, given in section 2.

1. *The group $G^X = C^\infty(X; G)$ and its jet extentions.* Let G be an arbitrary real Lie group, X – a real connected C^∞-manifold. Consider the set $C^\infty(X; G)$ of C^∞-mappings $\tilde{g}: X \to G$ such that $g(x) = 1$ outside of some compact set (depending of $\tilde{g}$). Let us supply the set $C^\infty(X; G)$ with the natural topology. The group operation in $C^\infty(X; G)$ is defined pointwise: $(g_1g_2)(x) = g_1(x)g_2(x)$. We shall denote the topological group $C^\infty(X; G)$ by G^X.

Let us define now the k-jet imbedding of the group G^X ($k = 0, 1, \ldots$). Recall that the k-jet of a mapping $X \to G$ in a point $x_0 \in X$ is, by definition, the class of smooth mappings $X \to G$, all of them taking the same value at x_0, and such that all corresponding partial derivatives of these mappings up to the k-th order, taken at x_0, coincide. One can define the k-jet space at a point $x_0 \in X$ as follows. Consider a subgroup $G^X_{x_0,k}$ of G^X, consisting of such functions $\tilde{g}: X \to G$ that $g(x_0) = 1$ and all partial derivatives of $\tilde{g}$ up to the k-th order are equal to zero in a point x_0. It is easy to verify that $G^X_{x_0,k}$ is a normal subgroup of the group G^X; the space of k-jets in a point x_0 is naturally identified with the factor group $G^X/G^X_{x_0,k}$.

It is clear that the factor-groups $G^X/G^X_{x,k}$ corresponding to different $x \in X$ are isomorphic. The group $G^X/G^X_{x,k}$ is, moreover, uniquely defined, up to isomorphism, by the group G, number k and dimension m of the manifold X. We shall denote this group by G^k_m and call it *the Leibnitz group of order k and degree m of the group G.*

In what follows we consider for the most part the case $k = 1$. Then $G^1_m \cong G \cdot (\mathfrak{G} \times \underbrace{\cdots}_{m} \times \mathfrak{G})$, with $\mathfrak{G}$ – the Lie algebra of G, i.e. G^1_m is a skew-product of the direct sum of m copies of Lie algebra $\mathfrak{G}$ and the group G; representation of the group G on $\mathfrak{G} \times \cdots \times \mathfrak{G}$ is adjoint representation. The group G^k_1 can easily be described as well (see [14]). In the general case the group G^k_m is a skew-product of G and a nilpotent group U^k_m, which space is a sum of several copies of the space $\mathfrak{G}$; the formula of the group rule in G^k_m for arbitrary k and m is actually rather complicated.

Let us define a k-jet fibre bundle $j^k(X; G) \to X$, which is a fibration over X with fibre corresponding to $x \in X$ consisting of all the k-jets in a point x. The structure of a fibre bundle is introduced in $j^k(X; G)$ in a natural way.[1]

The fibre bundle $j^k(X; G)$ gives reason to the following definition generalizing the definition of a vector bundle:

Definition: Let H be a connected Lie group. A smooth fibre bundle ξ with the standard fibre H and the structure group Aut H (= the group of all continuous group automorphisms of H) will be called a group bundle with a group H.

A group bundle with an additive group $\mathbb{R}^n$ (or $\mathbb{C}^n$) is a vector bundle in usual sense, that is why our notion presents a 'noncommutative' analogue of a vector bundle. The sections of a group bundle form a group, since each fibre has a group structure. A trivial group bundle with a group H and base X is a fibering $H \times X \to X$, and the group of all its smooth sections is the group H^X.

The jet fibre bundle $j^k(X; G) \to X$ defined above is a group bundle in the sense of our definition with the group G^k_m, $m = \dim X$. This bundle is not trivial unless the tangent bundle be so. Every fibre of this bundle over $x \in X$ is canonically provided with a structure of the group $G^X/G^X_{x,k} \cong G^k_m$.

Let us denote by $\theta^k(X; G)$ a space of all differentiable sections,

[1] It is a more traditional approach to consider $j^k(X; G)$ to be a fibre bundle over $X \times G$. The definition in the text is however, more suitable for us.

with compact support, of k-jet bundle $j^k(X; G) \to X$ supplied with usual topology. The space $\theta^k(X; G)$ is a group with regard to pointwise multiplication in the fibres.

Notice that there are defined natural group epimorphisms $\theta^{k+1}(X; G) \to \theta^k(X; G)$, $k = 0, 1, \ldots$; in particular, epimorphism $\theta^1(X; G) \to \theta^0(X; G) = G^X$.

DEFINITION: We call a k-jet imbedding a map

$$\mathfrak{J}^k : G^X \to \theta^k(X; G)$$

defined by: $(\mathfrak{J}^k \tilde{g})(x)$ is the k-jet of a function $\tilde{g}$ in a point $x \in X$.

The following is fairly evident: $\mathfrak{J}^k$ *is a group monomorphism.*

REMARK: All proposition of this section remains true if one substitutes the group G^X by the group of differentiable sections of any group bundle over X with the group G, $j^k(X; G)$ by the corresponding k-jet fibre bundle etc.

2. *The group $\theta^1(X; G)$ and Maurer–Cartan cocycle.* As we shall deal with the case $k = 1$ let us examine the group $\theta^1(X; G)$ in greater detail. For every $x \in X$ the elements of a factor group $G^X/G^X_{x,1}$ may be considered as the pairs $(g(x), a(x))$, where $g(x) \in G$ and $a(x)$ is a linear mapping of tangential spaces $T_xX \to T_{g(x)}G$, i.e. $a(x) \in \mathrm{Hom}\,(T_x, T_{g(x)})$. Consequently, one can put into correspondence to each element of the group $\theta^1(X; G)$ a map $TX \to TG$ of the tangential sheaves which is linear on the fibres. It is easy to see that this correspondence is an isomorphism of the group $\theta^1(X; G)$ and the group $(TG)^{TX}$ of all differentiable mappings $TX \to TG$ with compact support, which are linear on the fibres.

Further, the right trivialization of the tangential shief TG permits us to identify each tangential space T_gG with the space $T_eG \cong \mathfrak{G}$ and thus defines an isomorphism $TG \cong G \cdot \mathfrak{G}$ of the group TG and a semidirect product $G \cdot \mathfrak{G}$. Therefore the group $(TG)^{TX}$ is isomorphic to a skew-product of the group G^X and an additive group $\Omega^1(X; \mathfrak{G})$ of all differentiable mappings $TX \to \mathfrak{G}$ with compact support and linear on the fibres, i.e. a group of $\mathfrak{G}$-valued 1-forms on X.

Consequently, there are determined the canonical isomorphisms of the groups:

$$\theta^1(X; G) \cong (TG)^{TX} \cong G^X \cdot \Omega^1(X; \mathfrak{G}),$$

where $\Omega^1(X; \mathfrak{G})$ is the space of all differentiable $\mathfrak{G}$-valued 1-forms with compact support on X, and the action V of G^X on $\Omega^1(X;\mathfrak{G})$ is an adjoint action in every fibre:

$$(V(\tilde{g})\omega)(x) = \mathrm{Ad}\, g(x) \circ \omega(x).$$

REMARK: We have two different imbedding of G^X into $\theta^1(X;G)$: $\mathfrak{J}^1 : G^X \to \theta^1(X;G)$ and $G^X \to G^X \cdot 0 \subset G^X \cdot \Omega^1(X;\mathfrak{G})$.

Since G^X is acting on $\Omega^1(X;\mathfrak{G})$ and there is a natural epimorphism $\theta^1(X;G) \to G^X$, then it is defined in $\Omega^1(X;\mathfrak{G})$ also an action of the group $\theta^1(X;G)$.

Let us introduce the Maurer–Cartan 1-cocycle. Making use of the isomorphism $\theta^1(X;G) \to G^X \cdot \Omega^1(X;\mathfrak{G})$, define a 1-cocycle α of $\theta^1(X;G)$ taking values in $\Omega^1(X;\mathfrak{G})$ by the following formula. Given $\xi \in \theta^1(X;G)$, i.e. $\xi = (\tilde{g}, \omega)$, where $\tilde{g} \in G^X$, $\omega \in \Omega^1(X;\mathfrak{G})$, let

$$\alpha(\xi) = \omega.$$

DEFINITION: A restriction of the 1-cocycle α on the subgroup $\mathfrak{J}^1 G^X \cong G^X$ will be called *Maurer–Cartan cocycle* and denoted by β. Therefore

$$\beta(\tilde{g}) = \alpha(\mathfrak{J}^1 \tilde{g}).$$

For β to be a 1-cocycle means that for every $\tilde{g}_1, \tilde{g}_2 \in G^X$ it satisfies

$$\beta(\tilde{g}_1 \tilde{g}_2) = \beta \tilde{g}_1 + V(\tilde{g}_1) \beta \tilde{g}_2.$$

It is not hard to prove that the conception of the Maurer–Cartan cocycle is natural with respect to the diffeomorphisms of manifolds $X \to Y$.

Owing to the extreme importance of the Maurer–Cartan cocycle we shall give a direct definition of it. Denote by $d\tilde{g}$ a differential of a map $\tilde{g} : X \to G$,

$$d\tilde{g} : TX \to TG.$$

Let $R : TG \to \mathfrak{G}$ be the right trivialization. Then we have

$$\beta \tilde{g} = R \circ d\tilde{g}. \tag{1}$$

REMARK 1: Let $\sigma: TX \to TX$ be a map, preserving each fibre and linear on it. Then the map

$$\tilde{g} \to (\beta\tilde{g}) \circ \sigma \in \Omega^1(X; \mathfrak{G}), \quad \tilde{g} \in G^X \tag{2}$$

is a 1-cocycle to the group G^X again. The set of such cocycles forms a linear space. It is interesting to find out whether every 1-cocycle of G^X with values in $\Omega^1(X; \mathfrak{G})$ is cohomological to a cocycle of the form (2) with β the Maurer–Cartan cocycle.

REMARK 2: The group $G^1_m \cong G \cdot (\underbrace{\mathfrak{G} \times \cdots \times \mathfrak{G}}_{m})$ is represented with invariant scalar-product in the space $\underbrace{\mathfrak{G} \times \cdots \times \mathfrak{G}}_{m}$ and possesses a 1-cocycle with values in this space, defined by

$$\alpha_0(g, a) = a, \quad g \in G, \quad a \in \underbrace{\mathfrak{G} \times \cdots \times \mathfrak{G}}_{m}.$$

The 1-cocycle α of $\theta^1(X; G)$ introduced above can be defined locally in the following way. Assume that we have some local coordinates in X, and so the isomorphism $\operatorname{Hom}(T_x, \mathfrak{G}) \cong \underbrace{\mathfrak{G} \times \cdots \times \mathfrak{G}}_{m}$, $m = \dim X$ is defined. Thus the formula for the cocycle α may be written as follows:

$$\alpha(\tilde{g}, \omega)(x) = \alpha_0(g(x), \omega(x)).$$

Thus the existence of the Maurer–Cartan cocycle for a group G^X is connected with the existence of the 1-cocycle α_0 for G^1_m.

3. *Representations of the group* G^X. By using k-jet imbeddings $\mathfrak{J}^k: G^X \to \theta^k(X; G)$ one can construct various representations of G^X.

Consider at first arbitrary unitary representation π of the group G^k_m. Fix a point $x_0 \in X$ and define an isomorphism $G^X/G^X_{x_0,k} \cong G^k_m$. A representation of the group G^X can be defined by

$$T_\pi(\tilde{g}) = \pi((\mathfrak{J}^k\tilde{g})(x_0)). \tag{3}$$

The representation (3) is an analogue of the partial derivative in x_0. For instance, let $G = \mathbb{R}^1$, then $G^1_m = \mathbb{R}^1 \oplus \mathbb{R}^m$; if π is a one-dimensional representation of $G^1_m: (\xi_0, \xi_1, \ldots, \xi_m) \mapsto \exp(i \sum_{s=0}^m \alpha_s \xi_s)$, then $T_\pi(\tilde{g})$ is a one-dimensional representation $\tilde{g} \mapsto \exp[i(\alpha_0 g(x) + \sum_{s=1}^m \alpha_s (\partial g/\partial x^s))_{x=x_0}]$. For $G = SU(2)$, $k = 1$ and $m = 1$ such representations were described for the first time in [6]; they were constructed there directly by passing to a limit, like that used in the definition of a derivative.

The representations (3) are local ones since they depend on k-jets in a point $x_0 \in X$ only. Similarly, one can define representation of G^X depending on k-jets in a finite number of points.

More interesting representations of the group G^X arise due to the existence of nonlocal irreducible representations of the group $\theta^k(X; G)$, the restrictions of which to $\mathfrak{J}^k G^X$ remain irreducible.

In [20] there was given a construction of nonlocal representations for a current group, i.e. a group of functions on X taking values in some Lie group. For that construction to be applicable is necessary that the unity representation of the coefficient group be not isolated in the space of all its unitary representations, or, in more general terms,[1] that the first cohomology group with values in the space of some unitary representation of the coefficient group be non-trivial.

Since the local structure of the group $\theta^k(X; G)$ is similar to that of the group $C^\infty(X: G_m^k)$, $m = \dim X$, the construction of [20], [21] can be applied to $\theta^k(X; G)$ if the unity representation of G_m^k is not isolated. That is the case when G is a compact semisimple Lie group and $k = 1$, for G_m^1 then possesses an orthogonal representation in the space $\underbrace{\mathfrak{G} \times \cdots \times \mathfrak{G}}_{m}$ and, as we have seen, there exists a non-trivial 1-cocycle of the group G_m^1 with values in that space.

For the group G^X, with G a compact semisimple Lie group, there arises, consequently, a nonlocal unitary representation. We emphasize that though this representation is not local, thus resembling the representations constructed in [20], it does not admit any extention even to the group of continuous mappings $X \to G$, because its construction makes use of the 1-jet imbedding.

§2. Some subsidiaries preliminary facts

In the present paragraph we set forth some general definitions and statements, which will be used for the construction of representations for G^X, as well as in the proofs of the main theorems.

1. *The space* EXP $\bar{H}$. Let H be a real nuclear countably Hilbert-

[1] In all examples known to the authors the existence of a unitary representation of a Lie group with non-trivial first cohomology group accompanies the non-isolation of the unitary representation in the space of unitary representations. Moreover, such a representation admits a deformation which is at the same time a deformation of the unity representation. However, the necessity of such coincidence is not yet established.

norm space, and let $\langle , \rangle$ be some inner product in it. Denote by $\bar{H}$ a completion of H in the norm $\|h\| = \langle h, h \rangle^{1/2}$ and by H' a dual to H. Then there are natural embeddings $H \subset \bar{H} \subset H'$.

Let us define in H' a Gaussian measure μ with the zero mean and a correlation functional $B(h_1, h_2) = \langle h_1, h_2 \rangle$, by its Fourier transform

$$\int_{H'} e^{i\langle F,h \rangle} d\mu(F) = e^{-1/2\langle h,h \rangle} \quad (h \in H).$$

This measure μ will be called a *standard Gaussian measure* in H'.

Let us introduce a complex Hilbert space $L^2_\mu(H')$ of all square-integrable with respect to μ functionals on H'. The functional $\Omega \in L^2_\mu(H')$, identically equal to unity on H', will be called *the vacuum vector.*

We shall determine a natural isomorphism of the space $L^2_\mu(H')$ and another space which we call an exponential of H' and denote EXP $\bar{H}$.

Let $\bar{H}_{\mathrm{C}}$ be a complexification of $\bar{H}$ and $S^n\bar{H}_{\mathrm{C}}$ $(n = 1, 2, \ldots)$ the symmetrized tensor product of n copies of $\bar{H}_{\mathrm{C}}$. Let also $S^0\bar{H}_{\mathrm{C}} = \mathbb{C}$. Call an *exponential EXP $\bar{H}$ of a space $\bar{H}$, or Fock space corresponding to $\bar{H}$*, a complex Hilbert space

$$\mathrm{EXP}\, \bar{H} = \bigoplus_{n=0}^{\infty} S^n\bar{H}_{\mathrm{C}}.$$

To establish an isomorphism $\mathrm{EXP}\, \bar{H} \cong L^2_\mu(H')$ consider in the space EXP $\bar{H}$ the set of vectors

$$\exp h = 1 \oplus h \oplus \frac{1}{\sqrt{2!}} h \otimes h \oplus \frac{1}{\sqrt{3!}} h \otimes h \otimes h \oplus \cdots \quad (h \in H).$$

The set is known to be total in EXP $\bar{H}$ [8], i.e. its linear span is dense in EXP $\bar{H}$. Consider a mapping of the set $\{\exp h\}$ into the space $L^2_\mu(H')$:

$$(1) \qquad \exp h \mapsto e^{\|h\|^2/2} e^{i\langle \cdot, h \rangle} \in L^2_\mu(H').$$

One proves that this mapping conserves the inner product and so, by virtue of the totality of the sets $\{\exp h\}$ and $\{e^{i\langle \cdot, h \rangle}\}$ in EXP $\bar{H}$ and $L^2_\mu(H')$ correspondingly, the mapping (1) can be uniquely prolonged up to an isomorphism of the spaces:

$$\bigoplus_{n=0}^{\infty} S^n \bar{H}_C \cong L^2_\mu(H').$$

We call thus defined isomorphism *the canonical isomorphism* between the space $L^2_\mu(H')$ and its Fock model EXP $\bar{H} = \bigoplus_{n=0}^{\infty} S^n \bar{H}_C$. Notice that the canonical isomorphism puts in correspondence the vacuum vector $\Omega \in L^2_\mu(H')$ and the vector $1 = \exp 0$, as well as the subspace of deneralized Hermitian polynomials of degree n and the subspace $S^n \bar{H}_C$ in EXP $\bar{H}$ (see, for instance, [23], [12]).

REMARK: If there are two triples of spaces $H_1 \subset \bar{H}_1 \subset H'_1$, $H_2 \subset \bar{H}_2 \subset H'_2$ with $\bar{H}_1 = \bar{H}_2$, and μ_1, μ_2 are standard Gaussian measures in H'_1 and H'_2 correspondingly, then there exists a canonical isomorphism $L^2_{\mu_1}(H'_1) \cong L^2_{\mu_2}(H'_2)$. It arises from canonical isomorphisms between this space and the Fock space.

2. *The representation* $\mathrm{EXP}_\beta\, V$. Consider a topological group G and an orthogonal representation V of G in the space H. Let us extend the representation V to the space $H' \supset H$ dual to H, by

$$\langle V(g)F, h\rangle = \langle F, V^{-1}(g)h\rangle$$

for any $F \in H'$, $h \in H$.

Consider a 1-cocycle β of the group G with values in H, i.e. a continuous mapping $\beta : G \to H$ which for every $g_1, g_2 \in G$ satisfies

$$\beta(g_1 g_2) = \beta g_1 + V(g_1)\beta g_2.$$

Given a representation V and a 1-cocycle β, we shall construct a new representation, U, of the group G in the space $L^2_\mu(H') \cong \mathrm{EXP}\, \bar{H}$, by

$$(U(g)\Phi)(F) = e^{i\langle F, \beta g\rangle}\Phi(V^{-1}(g)F). \tag{2}$$

Call the representation U an *exponential* of the initial representation V of G (with respect to the 1-cocycle β) and denote it $\mathrm{EXP}_\beta\, V$. Now let us point out some simple properties of the representations $\mathrm{EXP}_\beta\, V$.

(1) Let the 1-cocycle $\beta, \beta' : G \to H$ be cohomological, i.e. there exists such a vector $h_0 \in H$ that $\beta' g - \beta g = V(g)h_0 - h_0$ for each $g \in G$. In this case the representations $U = \mathrm{EXP}_\beta\, V$ and $U' = \mathrm{EXP}_{\beta'}\, V$ are equivalent.

We can observe, indeed, that for every $g \in G$, $U'(g) = A_{h_0} U(g) A^{-1}_{h_0}$, where A_{h_0} is defined by $(A_{h_0}\Phi)(F) = e^{i\langle F, h_0\rangle}\Phi(F)$.

(2) $\mathrm{EXP}_{\beta_1} V_1 \otimes \mathrm{EXP}_{\beta_2} V_2 = \mathrm{EXP}_{\beta_1 \oplus \beta_2} (V_1 \oplus V_2)$.

(3) LEMMA 1. *Let V_C be the complexification of a representation V, $S^n V_C$ $(n = 1, 2, \ldots)$ a symmetrized tensor product of n copies of the representation V_C, $S^0 V_C$ a unity representation. If $\beta = 0$, then*

$$\mathrm{EXP}_\beta V \cong \bigoplus_{n=0}^{\infty} S^n V_C.$$

To prove the lemma it suffices to pass to the Fock model of the representation space.

Notice now that given a 1-cocycle $\beta : G \to H$ and an arbitrary bounded linear operator A in H which commutes with the operators of the representation V, the function

$$(A\beta)g = A(\beta g)$$

is a 1-cocycle of G taking values in H, too. Thus, with every 1-cocycle β one can connect a family of unitary representations of G in $L^2_\mu(H')$:

$$U_A = \mathrm{EXP}_{A\beta} V,$$

A being any bounded linear operator in H, commuting with the operators of the representation V.

LEMMA 2 (of a tensor product): *Let $\alpha = \binom{\alpha_{11}\,\alpha_{12}}{\alpha_{21},\alpha_{22}}$ be an arbitrary matrix of bounded linear operators $\alpha_{ij} : H \to H$, commuting with the operators of a representation V and satisfying the condition: $\alpha^*_{1i}\alpha_{1j} + \alpha^*_{2i}\alpha_{2j} = \delta_{ij}E$ $(i, j = 1, 2)$, E being the unity operator (i.e. $\alpha^*\alpha = 1$). Then*

$$U_{A_1} \otimes U_{A_2} \cong U_{\alpha_{11}A_1 + \alpha_{12}A_2} \otimes U_{\alpha_{21}A_1 + \alpha_{22}A_2}.$$

PROOF (cf. [22]): Define an operator in $L^2_\mu(H') \otimes L^2_\mu(H')$:

$$(R\Phi)(F_1, F_2) = \Phi(\alpha^*_{11}F_1 + \alpha^*_{21}F_2, \alpha^*_{12}F_1 + \alpha^*_{22}F_2).$$

The operator R is unitary since, as it is easy to see,

$$\int e^{\,i[\langle \alpha^*_{11}F_1 + \alpha^*_{21}F_2, f_1\rangle + \langle \alpha^*_{12}F_1 + \alpha^*_{22}F_2, f_2\rangle]} d\mu(F_1) d\mu(F_2)$$

$$= \int e^{\,i[\langle F_1, f_1\rangle + \langle F_2, f_2\rangle]} d\mu(F_1) d\mu(F_2).$$

For every $g \in G$,

$$R(U_{A_1}(g) \otimes U_{A_2}(g))R^{-1} = U_{\alpha_{11}A_1+\alpha_{12}A_2}(g) \otimes U_{\alpha_{21}A_1+\alpha_{22}A_2}(g),$$

whereof the statement of the lemma follows.

COROLLARY: *If A_1, A_2 are invertible operators in H, then*

$$U_{A_1} \otimes U_{A_2} \cong U_A \otimes U_0,$$

where $A = (A_1^* A_1 + A_2^* A_2)^{1/2}$, U_0-the representation, which corresponds to zero cocycle.

Indeed, put, for short, $B_1 = A_1^* A_1$, $B_2 = A_2^* A_2$ and notice that $B_1 - B_1 A^{-2} B_1 = B_2 - B_2 A^{-2} B_2$, this operator being self-adjoint and positively definite. Consider operators $\alpha_{11} = A^{-1} A_1^*$, $\alpha_{12} = A^{-1} A_2^*$,

$$\alpha_{21} = (B_1 - B_1 A^{-2} B_1)^{1/2} A_1^{-1}, \quad \alpha_{22} = -(B_2 - B_2 A^{-2} B_2)^{1/2} A_2^{-1}.$$

One easily proves that these operators satisfy the conditions of Lemma 2. On the other hand, $\alpha_{11}A_1 + \alpha_{12}A_2 = A$, $\alpha_{21}A_1 + \alpha_{22}A_2 = 0$, q.e.d.

3. *The definition of the representation for a group G with respect to a pair of* 1*-cocycles.* Let $U = \mathrm{EXP}_\beta\, V$ be the representation of G in the space $L^2_\mu(H')$ defined in the section 2. It is not difficult to verify that this representation is equivalent to the following representation U_λ in the space $L^2_\mu(H')$:

$$(U_\lambda(g)\Phi)(F) = e^{i\lambda[\langle F, \beta g\rangle + \mathrm{Im}\,\lambda \|\beta g\|^2]}\Phi(V^{-1}(g)F - 2\,\mathrm{Im}\,\lambda.\,\beta(g^{-1})),$$

λ being any complex number, $|\lambda| = 1$. Namely, $U_\lambda(g) = A_\lambda U(g) A_\lambda^{-1}$, where A_λ is the operator, uniquely defined by its action on the functionals $e^{i\langle\cdot, h\rangle}$:

$$A_\lambda : e^{i\langle\cdot, h\rangle} \mapsto e^{(\lambda^2-1)\|h\|^2/2} e^{i\lambda\langle\cdot, h\rangle}.$$

Given another 1-cocycle β' of the group G with values in H we define the operators $\tilde{U}_\lambda(g)$ in $L^2_\mu(H')$ by

$$(\tilde{U}_\lambda(g)\Phi)(F) = e^{i\langle F, \beta' g\rangle}(U_\lambda(g)\Phi)(F), \quad g \in G.$$

THEOREM: *The operators* $\tilde{U}_\lambda(g)$ are unitary and constitute a projective representation of the group G, namely

$$\tilde{U}_\lambda(g_1 g_2) = e^{2\,\mathrm{Im}\,\lambda \cdot \alpha(g_1,g_2)}\tilde{U}_\lambda(g_1)\tilde{U}_\lambda(g_2)$$

for every $g_1, g_2 \in G$, *where*

$$\alpha(g_1, g_2) = \langle \beta g_1^{-1}, \beta' g_2 \rangle.$$

The proof is immediate.

We want to define now an extension of the additive group $\mathbb{R}^+$ by the group G. For this purpose observe that the function α for every $g_1, g_2, g_3 \in G$ satisfies

$$\alpha(g_1, g_2) + \alpha(g_1 g_2, g_3) = \alpha(g_1, g_2 g_3) + \alpha(g_2, g_3).$$

Consequently, α is a 2-cocycle of the group G with values in $\mathbb{R}$ and therefore defines an extension $\tilde{G}$ of $\mathbb{R}^+$ by G. The elements of $\tilde{G}$ are the pairs (g, c), $g \in G$, $c \in \mathbb{R}^+$ with the following multiplication rule:

$$(g_1, c_1)(g_2, c_2) = (g_1 g_2, c_1 + c_2 - 2\,\mathrm{Im}\,\lambda\alpha(g_1, g_2)).$$

It corresponds to the projective representation of G in the space $L^2_\mu(H')$ defined above an (affine) unitary representation of the group $\tilde{G}$, given by

$$(\tilde{U}_\lambda(g, c)\Phi)(F) = e^{i[\lambda(\langle F,\beta g\rangle + \mathrm{Tm}\,\lambda\|\beta g\|^2) + (c + \langle F, \beta' g\rangle)]}\Phi(V^{-1}(g)F - 2\,\mathrm{Tm}\,\lambda \cdot \beta(g^{-1})).$$

4. *The singularity conditions for two measures.* Recall that two measures, μ and ν, in the space X are said to be equivalent if for every measurable set $A \subset X$ the conditions $\mu(A) = 0$ and $\nu(A) = 0$ hold simultaneously. The measures μ and ν are called mutually singular if there is a measurable set $A \subset X$ such that $\mu(A) = 0$, $\nu(X - A) = 0$.

The lemma which follows is well known in the theory of the Gaussian measure spaces.

LEMMA 3: *Let* μ *be the standard Gaussian measure in the space* H'. *The measures* μ *and* $\mu(\cdot + x)$ *are mutually singular if and only if* $x \notin \bar{H}$.

PROOF: We may assume that $\bar{H} = l^2$ and μ is a product measure, $\mu = m_1 \times \cdots \times m_n \times \cdots$ where $dm_i(t) = (2\pi)^{-1/2}e^{-t^2/2}dt$; $x = (x_1, \ldots, x_n, \ldots)$. The measure $\mu(\cdot + x)$ is a product measure either and therefore, by virtue of the zero-one law, the measures μ and $\mu(\cdot + x)$

are either mutually singular or equivalent. By virtue of Kakutani theorem the measures μ and $\mu(\cdot+x)$ are equivalent if and only if $\Pi_k \int \sqrt{dm_k \cdot dm_k(\cdot+x)} > 0$. The easy verification shows that this happens if and only if $\Sigma_k x_k^2 < \infty$, i.e. $x \in \bar{H}$.

LEMMA 4: *Let μ be the standard Gaussian measure in H' and ν a measure in H' satisfying $\nu(\bar{H}) = 0$. Then the measure μ is mutually singular with the convolution $\mu * \nu$ of the measures μ and ν.*

(A convolution of the measures μ and ν is defined by $(\mu * \nu)(\cdot) = \int \mu(\cdot - x) d\nu(x)$.)

PROOF: Assume that the measures μ and $\mu_1 = \mu * \nu$ are not mutually singular. Then $(d\mu_1/d\mu) = p > 0$ on a set of positive μ-measure. Since $\mu_1(\cdot) = \int \mu(\cdot - x) d\nu(x)$ then, by the Fubini theorem, $p(y) = \int [d\mu(y-x)/d\mu(y)] d\nu(x) > 0$ for y in the set mentioned above. Hence, there exists a set B, $\nu(B) > 0$, such that $[d\mu(\cdot - x)/d\mu] > 0$ for each $x \in B$. But then μ and $\mu(\cdot - x)$ are not mutually singular, and consequently, by Lemma 3, $x \in \bar{H}$, in which case $\nu(\bar{H}) > 0$. We came to a contradiction with the assumption.

5. *The spectral measures.* Let G be an abelian (not necessarily locally compact) topological group possessing a countable base of open sets, $\hat{G}$ the group of its continuous characters, U a unitary representation of G in the complex Hilbert space $\mathcal{H}$. By applying the spectral theorem (see, for example, [4]) to the C^*-algebra, generated by the operators of the representation U, we get the following decomposition: *There exists an isomorphism of the space $\mathcal{H}$ onto the direct integral of Hilbert spaces,*

$$T: \mathcal{H} \to \int_{\hat{G}}^{\oplus} \mathcal{H}_\chi d\mu(\chi),$$

with μ a Borel measure on $\hat{G}$, which transfers the operators $U(g)$, $g \in G$ into the operators

$$(TU(g)T^{-1}f)(\chi) = \chi(g) f(\chi). \tag{3}$$

The measure μ on $\hat{G}$ is defined by U uniquely up to equivalence and is called the *spectral measure* of the representation (3). The representation (3) of G in the space $\int_{\hat{G}}^{\oplus} \mathcal{H}_\chi d\mu(\chi)$ is called the *spectral decomposition* of the initial representation U. If $\dim \mathcal{H}_\chi = 1$ for a.e. χ it is said that the representation U has a simple spectre.

We give here some statements of the spectral measures.

(1) Two representations of G are disjoint if and only if their spectral measures are mutually singular.

(2) The spectral measure of the sum of two representations of G is equivalent to the sum of their spectral measures.

(3) The spectral measure of the tensor product of two representations of G is equivalent to the convolution of their spectral measures.

(4) The weakly closed algebra generated by the operators (3) in the space $\int_{\hat{G}}^{\oplus} \mathcal{H}_\chi d\mu(\chi)$ coincides with the algebra of the operators of multiplication by an arbitrary μ-measurable function $a(\chi)$: $f(\chi) \mapsto a(\chi)f(\chi)$ (i.e. this algebra is isomorphic to $L^\infty_\mu(\hat{G})$, see, for example, [4]).

In the §4 we shall make use of the following generalization of the statement 4):

Assume that a unitary representation of the group G can be decomposed into a direct integral of representations

$$U = \int_{\Xi}^{\oplus} U_\xi d\nu(\xi)$$

(Ξ is a measure space with the measure ν).

It means that U is equivalent to the representation in the direct integral of Hilbert spaces $\mathcal{H} = \int_{\Xi}^{\oplus} \mathcal{H}_\xi d\nu(\xi)$ given by

$$(U(g)f)(\xi) = U_\xi(g)f(\xi),$$

U_ξ being a representation of G in $\mathcal{H}_\xi$.

LEMMA 5: *If the representations U_{ξ_1}, U_{ξ_2}, are disjoint for almost every (with respect to ν) $\xi_1 \neq \xi_2$, then the weakly closed operator algebra generated by the operators $U(g)$, $g \in G$ contains the operators of multiplication by every bounded ν-measurable function $a(\xi): f(\xi) \mapsto a(\xi)f(\xi)$.*

COROLLARY: *Let a representation U of G be decomposed in a tensor product of two representations, $U = U' \otimes U''$, with the corresponding spectral measures μ' and μ''. If the measures $\mu'(\cdot + \chi_1)$ and $\mu'(\cdot + \chi_2)$ are mutually singular for almost every $\chi_1 \neq \chi_2$ (with respect to μ''), then the weakly closed operator algebra generated by the operators $U(g)$, $g \in G$, contains all operators $E \otimes U''(g)$, $g \in G$ (and, therefore, all operators $U'(g) \otimes E$).*

Indeed, let $U'' = \int_{\hat{G}}^{\oplus} U''_{\chi} d\mu''(\chi)$ be the spectral decomposition of the representation U''. Then $U = \int_{\hat{G}}^{\oplus} (U' \otimes U''_{\chi}) d\mu''(\chi)$. Since the spectral measure of the representation $U' \otimes U''_{\chi}$ is $\mu'(\cdot + \chi)$, then, by the conjecture made above, the representations $U' \otimes U''_{\chi_1}$ and $U' \otimes U''_{\chi_2}$ are disjoint for almost all (with respect to μ'') $\chi_1 \neq \chi_2$. It follows from the Lemma 5 that the weakly closed operator algebra generated by the operators $U(g)$, $g \in G$, contains the operators of multiplication by the functions $a_g(\chi) = \chi(g)$, i.e. the operators $E \otimes U''(g)$.

§3. Nonlocal representations of the Group G^X. The ring of representations

1. *Construction of the representations of the group G^X.* Let us begin to study of the representations of G^X which are connected with the Maurer–Cartan cocycle (see §1). *From now on we shall consider only those Lie groups G for which their Lie algebra possesses an inner product invariant under the adjoint action of the group G.* In particular, all compact and all abelian Lie groups satisfy this condition.

In order to construct a representation of G^X we assume that X has a structure of a Riemannian manifold. This structure induces an orthogonal structure in the tangent bundle TX, as well as a strictly positive smooth measure dx on X.

Let us introduce an inner product in the space $\Omega^1(X)$ of $\mathbb{R}$-valued 1-forms of the C^∞ class, with compact support by the formula

$$\langle \omega_1, \omega_2 \rangle = \int_X \langle \omega_1(x), \omega_2(x) \rangle_x dx,$$

where $\langle , \rangle_x$ is an inner product in the conjugate tangential space $T_x^* X$. Let us also fix an inner product in the Lie algebra $\mathfrak{G}$ of G which is invariant under the adjoint action of the group G.

Consider now the space $\Omega^1(X; \mathfrak{G}) = \Omega^1(X) \otimes \mathfrak{G}$ of differentiable $\mathfrak{G}$-valued 1-forms on X with compact support. The orthogonal structures in the spaces $\Omega^1(X)$ and $\mathfrak{G}$ introduced above induce the orthogonal structure in their tensor product $\Omega^1(X; \mathfrak{G})$. It is clear that the latter inner product in $\Omega^1(X; \mathfrak{G})$ is G^X-invariant.

Denote by H the pre-Hilbert space $\Omega^1(X; \mathfrak{G})$, by $\bar{H}$ its completion in the norm introduced in H, by $\mathcal{F}$ a space conjugate to H and by μ the standard Gaussian measure in $\mathcal{F}$.

Let us define, according to the general definition given in the section 2 of §2, the new unitary representation $U = \mathrm{EXP}_\beta V$ of the group G^X. For that purpose we shall extend the representation V of G^X from H to the space $\mathscr{F} \supset H$ by the formula

$$\langle V(\tilde{g})F, \omega\rangle = \langle F, V^{-1}(\tilde{g})\omega\rangle, \quad \omega \in H.$$

Let β be a Maurer–Cartan 1-cocycle, that is $\beta\tilde{g} = R \circ d\tilde{g}$ (see §1). Define a unitary representation U of the group G^X in the space $L^2_\mu(\mathscr{F})$ by

$$(U(\tilde{g})\Phi)(F) = e^{i\langle F, \beta\tilde{g}\rangle}\Phi(V^{-1}(\tilde{g})F).$$

Note that the representation U depends on the Riemann space structure in X.

We formulate now the main results of the paper.

THEOREM 1: *If G is a compact semisimple Lie group and* dim $X \geq 2$, *then the representation* $U = \mathrm{EXP}_\beta V$ *of the group G^X is irreducible.*

THEOREM 2: *Let G be a compact semisimple Lie group. Then the representations U of G^X corresponding to different Riemannian metrics on X are not equivalent.*

The proof of the theorems 1, 2 will be given in §5.[1] It rests upon the results of §4 where the restriction of the representation U of G^X to an abelian subgroup is studied. Note that the main results of §4 are valid for the case dim $X \geq 2$ only. The problem of irreducibility for the representation U in the case dim $X = 1$ still remains open.

REMARK 1: The representation U of G^X is a restriction to $\mathfrak{J}^1 G^X \cong G^X$ of the representation $\tilde{U}$ of the group $\theta^1(X; G) \cong G^X \cdot \Omega^1(X; \mathfrak{G})$ in the space $L^2_\mu(\mathscr{F})$ defined in the following way (see §1).

Let $\alpha \in \theta^1(X; G)$, i.e. $\alpha = (\tilde{g}, \omega)$, $\tilde{g} \in G^X$, $\omega \in \Omega^1(X; \mathfrak{G})$. Then

$$(\tilde{U}(\alpha)\phi)(F) = e^{i\langle F, \omega\rangle}\Phi(V^{-1}(\tilde{g})F).$$

The representation $\tilde{U}$ is an "integral of representations" in the sense of [20, 21]. Its irreducibility can be easily derived from the theorems of [20, 21]. Thus theorem 1 is asserting that the representation $\tilde{U}$ remains irreducible when restricted to the image of G^X.

[1] The notion of a spherical function of the representation U given in §5 allows one to prove Theorem 2 in a different way (independently of §4).

REMARK 2: The construction of the representation U of G^X presented above can be transferred in a natural way to the group $\theta(\xi)$ of all differentiable sections with compact support of an arbitrary fibre bundle ξ over X with a fibre G. One can easily see that the theorems 1 and 2 are true for groups $\theta(\xi)$ as well.

Indeed, there is in X an open dense submanifold $X_0 \subset X$ such that the fibration $\xi|_{X_0}$ is trivial and consequently $\theta(\xi|_{X_0}) \cong G^{X^0}$. When X is replaced by X_0, the space of the representation remains the same, and so the statements concerning the irreducibility and non-equivalence of the representations of the group $\theta(\xi)$ become analogous to those concerning representations of G^X, namely to the Theorems 1 and 2.

REMARK 3: Denote $W_2^1(X;G)$ the completion of G^X in the metric

$$d(\tilde{g}_1, \tilde{g}_2) = \langle \beta(\tilde{g}_1^{-1}\tilde{g}_2), \beta(\tilde{g}_1^{-1}\tilde{g}_2)\rangle^{1/2} + \langle \beta(\tilde{g}_1\tilde{g}_2^{-1}), \beta(\tilde{g}_1\tilde{g}_2^{-1})\rangle^{1/2}.$$

$W_2^1(X;G)$ is an analogue of Sobolev space $\mathring{W}_2^1(X)$.[1]

(For example, if $G = SU(n)$ and $\dim X = 1$, then $W_2^1(X;G) = \{g(\cdot) \mid \int_X (\sum_{i,k} |g'_{ik}(x)|^2) dx < \infty\}$.)

One sees from the formulae defining the representation U of G^X that this representation can be extended to a representation of the group $W_2^1(X;G)$.

REMARK 4: Representations $U = \mathrm{EXP}_\beta V$ of G^X induce the Hermitian representations of its Lie algebra $\mathfrak{G}^X$. The latter representations can be extended to the representations of the complexification $(\mathfrak{G}_\mathbb{C})^X$ of $\mathfrak{G}^X$. The explicit formulae for the operators of these (non-Hermitian) representations of the algebra $(\mathfrak{G}_\mathbb{C})^X$ can be easily put down as finite sums, if one uses the Fock model of the representation space. If, for example, $G = SO(n)$, then $\mathfrak{G}_\mathbb{C} = s\ell(n, \mathbb{C})$. Therefore, our representations give rise to (non-Hermitian) nonlocal representations of the current algebra $s\ell(n, \mathbb{C})$, which depend on 1-jets.

2. *Representations of the group G^X connected with subbundles of the tangent bundle TX.* Let us define now a more wide class of representations of G^X. Let E be an arbitrary differentiable subbundles of the tangent bundle TX. Consider the restrictions of 1-forms $\omega \in \Omega^1(X;\mathfrak{G})$

[1] This analogy is noticed also in [1], where the representation under consideration (defined independently and in a different way) is called the energy representation. There is in [1] useful realization in the space of functions on the trajectories of Group –Wiener process (for dim $X = 1$).

to the subbundle E. They form a linear space which we denote by H_E. In this space, as in the initial one, there is a naturally defined representation, V_E, of the group G^X. Let β be the Maurer–Cartan cocycle. Then for any $\tilde{g} \in G^X$ define $\beta_E \tilde{g}$ as a restriction of the mapping $\beta\tilde{g}: TX \to \mathfrak{G}$ to the subbundle E. It is clear that β_E is a 1-cocycle of G^X taking values in H_E.

Let there be a Riemannian manifold structure τ on X. This structure induces an inner product in the space H_E which is invariant under the representation V_E of G^X. Denote by $U_{E,\tau}$ an exponential of the representation V_E, which is connected with the cocycle β_E:

$$U_{E,\tau} = \mathrm{EXP}_{\beta_E} V_E.$$

The next theorem is a consequence the theorem 1.

THEOREM 3: *If G is a compact semisimple Lie group and* $\dim X \geq 2$ *then the representation $U_{E,\tau}$ of G^X is irreducible.*

To prove it let us decompose the fibre bundle TX into an orthogonal sum $TX = E \oplus E_\perp$ of the fibration E and its orthogonal complement $E_\perp$. Evidently,

$$\mathrm{EXP}_{\beta_E} V_E \otimes \mathrm{EXP}_{\beta_{E\perp}} V_{E_\perp} \cong \mathrm{EXP}_\beta V.$$

Therefore, the irreducibility of $U_{E,\tau} = \mathrm{EXP}_{\beta_E} V_E$ is an immediate consequence of that of the representation $\mathrm{EXP}_\beta V$.

THEOREM 4: *Let G be a compact semisimple Lie group. Then the representations U_{E_1,τ_1} and $U_{E_2\tau_2}$ of G^X are equivalent if and only if $E_1 = E_2$ and the inner products in the space $H_{E_1} = H_{E_2}$ induced by the Riemannian structures τ_1, τ_2 on X coincide.*

The proof of this theorem is similar to that of the theorem 2 (cf. §5).

3. *Decomposition of the tensor product of representations $U = \mathrm{EXP}_\beta V$.* The representations $U = \mathrm{EXP}_\beta V$ of G^X defined in the first section depend on the Riemannian space structure τ on X. To emphasize this circumstance we shall denote them by $U_\tau = \mathrm{EXP}_\beta (V, \tau)$. We may, on the contrary, consider the Riemannian metric as fixed, the parameter of a representation being the 1-cocycle of the group G^X originated from the Maurer–Cartan cocycle as it was explained in the Remark 2, section 2 of §1.

More exactly, let τ_0 be a fixed Riemannian metric on X, $(,)_x$ an inner product it induces in T_x, $x \in X$. Consider an arbitrary Riemannian metric τ on X and let $(,)_{\tau,x}$ stand for the inner product in T_x, $x \in X$ induced by τ. The latter inner product can be represented in the form

$$(\xi_1, \xi_2)_{\tau,x} = (\sigma_\tau(x)\xi_1, \sigma_\tau(x)\xi_2)_x,$$

$\sigma_\tau(x): T_x \to T_x$ being a self-adjoint positive linear operator. Observe that the function $x \mapsto \sigma_\tau(x)$ defines the Riemannian metric τ on X in a unique way.

Define the operator A_τ in the space $\Omega^1(X; \mathfrak{G})$ by

$$(A_\tau \omega)(x) = |\sigma_\tau(x)|^{1/2} \omega(x) \circ \sigma_\tau^{-1}(x),$$

where $|\sigma_\tau(x)| = \det \sigma_\tau(x)$. Evidently, A_τ commutes with the operators of the representation V. It follows that if β is the Maurer–Cartan cocycle, then the function

$$(A_\tau \beta)\tilde{g} = A_\tau(\beta \tilde{g}), \quad \tilde{g} \in G^X$$

is a 1-cocycle of G^X, too.

LEMMA 1: *There is an equivalence of the representations of* G^X:

$$\mathrm{EXP}_\beta\,(V, \tau) \cong \mathrm{EXP}_{A_\tau \beta}\,(V, \tau_0).$$

PROOF: Let μ_{τ_0}, μ_τ be the standard Gaussian measures in $\mathscr{F} = (\Omega^1(X; \mathfrak{G}))'$ which are induced by the Riemannian metrics τ_0, τ on X. It easily follows from the definition of the operator A_τ that the correlation functionals B_{τ_0}, B_τ of these measures are connected by

$$B_\tau(\omega_1, \omega_2) = B_{\tau_0}(A_\tau \omega_1, A_\tau \omega_2).$$

Evidently, the mapping $e^{i\langle \cdot, \omega\rangle} \mapsto e^{i\langle \cdot, A_\tau \omega\rangle}$ extends to the isomorphism of Hilbert spaces $L^2_{\mu_\tau}(\mathscr{F}) \to L^2_{\mu_{\tau_0}}(\mathscr{F})$ which transfers operators of the representation $\mathrm{EXP}_\beta\,(V, \tau)$ into operators of the representation $\mathrm{EXP}_{A_\tau \beta}\,(V, \tau_0)$.

In what follows we shall consider representations $\mathrm{EXP}_{A\beta}\,(V, \tau_0)$ with A an arbitrary self-adjoint positively definite linear operator in the space $\Omega^1(X; \mathfrak{G})$ commuting with the operators of the representation V of G^X. Using Lemma 2, §2, of tensor products, we obtain.

LEMMA 2:

$$\mathrm{EXP}_{A_1\beta}(V,\tau_0)\otimes \mathrm{EXP}_{A_2\beta}(V,\tau_0)\cong \mathrm{EXP}_{A\beta}(V,\tau_0)\otimes \mathrm{EXP}_0\cdot V,$$

where $A=(A_1^2+A_2^2)^{1/2}$, $\mathrm{EXP}_0\, V$ *is the representation corresponding the zero cocycle (it does not depend on the Riemannian space structure on* X*).*

The following theorem presents a decomposition of the tensor product of the representations which are examined and enables us to calculate the additive generators of the ring of representations.

THEOREM 5: *The representation* $\mathrm{EXP}_{A_1\beta}(V,\tau_0)\otimes \mathrm{EXP}_{A_2\beta}(V,\tau_0)$ *of the group* G^X *can be decomposed into a continual direct sum of the representations of the form*

$$\mathrm{EXP}_{A\beta}(V,\tau_0)\otimes V^{x_1}\otimes\cdots\otimes V^{x_n}\quad (n=0,1,\ldots),$$

where $A=(A_1^2+A_2^2)^{1/2}$ *and* V^{x_0} $(x_0\in X)$ *is a representation of* G^X *in the space* $\mathfrak{G}_{\mathrm{C}}$, *given by*

$$V^{x_0}(\tilde g)=\mathrm{Ad}\, g(x_0).$$

PROOF OF THEOREM 5: It is a consequence of Lemma 1 of §2 that $\mathrm{EXP}_0\, V=\bigoplus_{n=0}^{\infty} S^n V_{\mathrm{C}}$ where V_{C} is the complexification of the representation V, S^nV_{C} is a simmetrized tensor product of n copies of the representation V_{C}. The representation S^nV_{C} may, in its turn, be decomposed into a continual direct sum of the representations $V^{x_1}\otimes\cdots\otimes V^{x_n}$. Namely, S^nV_{C} is equivalent to finite multiple of a continual direct sum of the representations

$$\int_{\tilde X^n}^{\oplus} V^{x_1}\otimes\cdots\otimes V^{x_n}dx_1\ldots dx_n,$$

where the integral is taken over a domain $\tilde X^n\subset X^n$ which is fundamental with respect to the permutation group of $x_1,\ldots,x_n$. To complete the proof of the theorem one has to make use of Lemma 2.

COROLLARY: *Let* A *be a symmetric positively-definite linear operator in the space* $\Omega^1(X;\mathfrak{G})$, *commuting with the action of the group* G^X, *and* W *a local finitely-dimensional representation of* G^X. *Then the representations of the form* $\mathrm{EXP}_{A\beta}(V,\tau_0)\otimes W$ *are the additive generators in the ring of representations they generate.*

We shall point out that Theorem 5 is proved for a manifold X of any dimension and an arbitrary Lie group G satisfying the conditions given in the beginning of the Section 1.

REMARK 1: If G is a compact semisimple Lie group, $\dim X \geq 2$ and $x_1, \ldots, x_n$ are different points of X, then the representation $\mathrm{EXP}_{A_\beta}(V, \tau_0) \otimes V^{x_1} \otimes \cdots \otimes V^{x_n}$ of G^X is irreducible. The proof of this statement can be given along the same lines as for Theorem 1.

REMARK 2: Lemma 2 and Theorem 5 can be easily formulated in terms of Riemannian metrics on X.

LEMMA 2′: *If* $\dim X \neq 2$, *then*

$$\mathrm{EXP}_\beta(V, \tau_1) \otimes \mathrm{EXP}_\beta(V, \tau_2) \cong \mathrm{EXP}_\beta(V, \tau) \otimes \mathrm{EXP}_0\, V,$$

where τ *is a Riemannian metric uniquely defined by the equation*

$$|\sigma_\tau(x)|\sigma_\tau^{-2}(x) = |\sigma_{\tau_1}(x)|\sigma_{\tau_1}^{-2}(x) + |\sigma_{\tau_2}(x)|\sigma_{\tau_2}^{-2}(x); \tag{1}$$

for the definition of $\sigma_\tau(x)$ see p. 20.[1]

THEOREM 5′: *If* $\dim X \neq 2$, *then the representation* $\mathrm{EXP}_\beta(V, \tau_1) \otimes \mathrm{EXP}_\beta(V, \tau_2)$ *can be decomposed into a continual direct sum of the representations of the form* $\mathrm{EXP}_\beta(V, \tau) \otimes V^{x_1} \otimes \cdots \otimes V^{x_n}$ $(n = 0, 1, \ldots)$ *the Riemannian metric* τ *being defined as in the Lemma* 2.

The similar statements take place for a tensor product $U_{E,\tau_1} \otimes U_{E,\tau_2}$ of the representations described in section 2, where E is an arbitrary subbundle of the tangent bundle TX.

Let us formulate now some statements concerning tensor products $U_{E_1,\tau_1} \otimes U_{E_2,\tau_2}$ with $E_1 \neq E_2$.

If $E_1 \cap E_2 = 0$, then there exists such a Riemannian space structure τ on X that $U_{E_1,\tau_1} \otimes U_{E_2,\tau_2} \cong U_{E_1+E_2,\tau}$. This Riemannian structure is defined by the conditions: (a) E_1 and E_2 are mutually orthogonal in the metric on TX induced by τ, (b) the Riemannian structures τ, τ_i induce the same inner product in the space H_{E_i} $(i = 1, 2)$ (see section 2).

[1] The equation (1) can fail to have a solution $\sigma_\tau(x)$ in the case $\dim X = 2$.

If $E_1 \cap E_2 = E \neq 0$ and E is also a subbundle of the tangent bundle, then let $E_i^\perp \subset E_i$ designate the orthogonal complementation to E in E_i with respect to the metric induced by τ_i $(i = 1, 2)$. Then we have an isomorphism $U_{E_i,\tau_i} \cong U_{E_i^\perp,\tau_i} \otimes U_{E,\tau_i}$.

Consequently,

$$U_{E_1,\tau_1} \otimes U_{E_2,\tau_2} \cong (U_{E_1^\perp,\tau_i} \otimes U_{E_2^\perp,\tau_2}) \otimes (U_{E,\tau_1} \otimes U_{E,\tau_2}).$$

This reduces the problem of decomposition of the tensor product $U_{E_1,\tau_1} \otimes U_{E_2,\tau_2}$ to the cases considered above.

§4. Restriction of the representation U of the group G^X to an abelian subgroup

Let G be a compact semisimple Lie group, X a connected open manifold of the class C^∞, $U = \mathrm{EXP}_\beta V$ the representation of the group G^X in the space $L^2_\mu(\mathscr{F})$ constructed in section 1 of §2.

Let $\mathfrak{A}$ be an arbitrary Cartan subalgebra of the algebra $\mathfrak{G}$, $A \subset G$ – the Cartan subgroup corresponding to $\mathfrak{A}$. Let us put $\mathfrak{A}^X$ for the additive group of differentiable C^∞ mappings $a: X \to \mathfrak{A}$ with a compact support and exp for the exponential mapping $\mathfrak{A}^X \to A^X$.

Define the representation W of the group $\mathfrak{A}^X$ in the space $L^2_\mu(\mathscr{F})$ by

$$W(a) = U(\exp a), \quad a \in \mathfrak{A}^X.$$

Since $\beta(\exp a) = da$, the operators $W(a)$ have the following form:

$$(W(a)\Phi)(F) = e^{i\langle F, da\rangle}\Phi(V^{-1}(\exp a)F). \tag{1}$$

We proceed now to the calculation of the spectral measure of the representation W. Notice that the character group $(\mathfrak{A}^X)^\wedge$ of $\mathfrak{A}^X$ is isomorphic to $(\mathfrak{A}^X)' = (C^\infty(X))' \otimes \mathfrak{G}$ (the isomorphism is given by the correspondence $F \mapsto \chi_F(\cdot) = e^{i\langle F,\cdot\rangle}$). The spectral measure can be therefore considered as a measure in the space $(\mathfrak{A}^X)'$ conjugate to $\mathfrak{A}^X$.

Let $\mathfrak{m}$ denote the orthogonal complementation in $\mathfrak{G}$ to $\mathfrak{A}: \mathfrak{G} = \mathfrak{A} \oplus \mathfrak{m}$. Let $\mathscr{F}_\mathfrak{A} \subset \mathscr{F}$, $\mathscr{F}_\mathfrak{m} \subset \mathscr{F}$ be the subspaces of, correspondingly, $\mathfrak{A}$-valued and $\mathfrak{m}$-valued generalized 1-forms on X, and $\mu_\mathfrak{A}$, $\mu_\mathfrak{m}$ be the standard Gaussian measures in $\mathscr{F}_\mathfrak{A}$ and $\mathscr{F}_\mathfrak{m}$. It is clear that $\mathscr{F} = \mathscr{F}_\mathfrak{A} \oplus \mathscr{F}_\mathfrak{m}$, $\mu = \mu_\mathfrak{A} \times \mu_\mathfrak{m}$ and

$$L^2_\mu(\mathscr{F}) = L^2_{\mu_\mathfrak{A}}(\mathscr{F}_\mathfrak{A}) \otimes L^2_{\mu_\mathfrak{m}}(\mathscr{F}_\mathfrak{m}).$$

LEMMA 1: *The representation W of the group $\mathfrak{A}^X$ can be decomposed into the tensor product $W = W_{\mathfrak{A}} \otimes W_{\mathfrak{m}}$ of the representations in the spaces $L^2_{\mu_{\mathfrak{A}}}(\mathscr{F}_{\mathfrak{A}})$ and $L^2_{\mu_{\mathfrak{m}}}(\mathscr{F}_{\mathfrak{m}})$ which are defined by*

$$(W_{\mathfrak{A}}(a)\Phi(F) = e^{i\langle F, da\rangle}\Phi(F), \tag{2}$$

$$(W_{\mathfrak{m}}(a)\Phi)(F) = \Phi(V^{-1}(\exp a)F). \tag{3}$$

This lemma is a straight consequence of (1), if one notes that $V(\exp a)$ is acting trivially on $\mathscr{F}_{\mathfrak{A}}$.

COROLLARY: *The spectral measure of the representation W is equivalent to the convolution of the spectral measures of $W_{\mathfrak{A}}$ and $W_{\mathfrak{m}}$.*

LEMMA 2: *The spectral measure of the representation $W_{\mathfrak{A}}$ is equivalent to thie Gaussian measure on $(\mathfrak{A}^X)'$ with the zero mean and the correlation functional*

$$B(a_1, a_2) = \langle da_1, da_2\rangle, \quad a_1, a_2 \in \mathfrak{A}^X \tag{4}$$

(the angular brackets denote the inner product in $\Omega^1(X; \mathfrak{A}) \subset \Omega^1(X; \mathfrak{G})$).

PROOF: Consider the differentiation operator

$$d: \mathfrak{A}^X \to \Omega^1(X; \mathfrak{A}).$$

Evidently, its kernel is zero.[1] Let

$$d^*: (\Omega^1(X; \mathfrak{A}))' = \mathscr{F}_{\mathfrak{A}} \to (\mathfrak{A}^X)'$$

be the mapping conjugate to d. It follows from the formula (2) for operators $W_{\mathfrak{A}}(a)$ that the image $d^*\mu_{\mathfrak{A}}$ of $\mu_{\mathfrak{A}}$ is the spectral measure of $W_{\mathfrak{A}}$. It is well known that a linear transformation transfers a Gaussian measure into a Gaussian one, the correlation functionals of these measures being connected by the formula

$$B_{d^*\mu_{\mathfrak{A}}}(a_1, a_2) = B_{\mu_{\mathfrak{A}}}(da_1, da_2).$$

Consequently, $B_{d^*\mu_{\mathfrak{A}}}(a_1, a_2) = \langle da_1, da_2\rangle$. The lemma is proved.

[1] Since X is open, $\mathfrak{A}^X$ does not contain constants.

Our aim now is to find the spectral measure of the representation $W_{\mathfrak{m}}$. Let H_{C} be the complexification of the space $\Omega^1(X, \mathfrak{m})$, $V_{\mathfrak{m}}$ – the representation of $\mathfrak{A}^X$ in H_{C} given by

$$(V_{\mathfrak{m}}(a)\omega)(x) = \mathrm{Ad}\,(\exp a(x)) \circ \omega(x).$$

Denote by $S^n V_{\mathfrak{m}}$ the symmetrized tensor product of n copies of the representation $V_{\mathfrak{m}}$, $n = 1, 2, \ldots$, and by $S^0 V_{\mathfrak{m}}$ the unity representation.

LEMMA 3: $W_{\mathfrak{m}} \cong \bigoplus_{n=0}^{\infty} S^n V_{\mathfrak{m}}$.

This lemma follows immediately from Lemma 1, §2.

COROLLARY: $W \cong W_{\mathfrak{A}} \oplus (\bigoplus_{n=1}^{\infty} (W_{\mathfrak{A}} \otimes S^n V_{\mathfrak{m}}))$.

Define for any root α of the algebra $\mathfrak{G}$ (with respect to $\mathfrak{A}$) and any $x_0 \in X$ a distribution $\varphi_{x_0}^{\alpha} \in (\mathfrak{A}^X)'$ by

$$\langle \varphi_{x_0}^{\alpha}, a \rangle = \alpha(a(x_0)). \tag{5}$$

LEMMA 4: *The spectral measure ν_n of the representation $S^n V_{\mathfrak{m}}$ is concentrated on the subset of distributions of the form $\varphi_{x_1}^{\alpha_1} + \cdots + \varphi_{x_n}^{\alpha_n}$. Moreover, on each subset $\{\varphi_{x_1}^{\alpha_1} + \cdots + \varphi_{x_n}^{\alpha_n} | x_1, \ldots, x_n \in X\}$ where $\alpha_1, \ldots, \alpha_n$ are fixed, the measure ν_n is equivalent to the measure $dx_1 \ldots dx_n$.*

PROOF: It suffices to check the statement for the case $n = 1$. Let $\mathfrak{m}_{\mathrm{C}}$ denote the complexification of $\mathfrak{m}$ and $\mathfrak{G}^{\alpha} \subset \mathfrak{m}_{\mathrm{C}}$ the rooted subspace corresponding to the root α. Consider the subspaces $H_{\mathrm{C}}^{\alpha} = \Omega^1(X) \otimes \mathfrak{G}^{\alpha}$ of the space $H_{\mathrm{C}} = \Omega^1(X) \otimes \mathfrak{m}_{\mathrm{C}}$. They are orthogonal for different α, $\mathfrak{A}^X$-invariant and

$$H_{\mathrm{C}} = \bigoplus_{\alpha \in \Delta} H_{\mathrm{C}}^{\alpha}$$

(Δ is the set of all roots).

The representation operators $V_{\mathfrak{m}}$ are given on each subspace H_{C}^{α} by

$$(V_{\mathfrak{m}}(a)\omega)(x) = e^{i\alpha(a(x))}\omega(x).$$

It is clear that the spectral measure of the representation $V_{\mathfrak{m}}$ in the subspace H_{C}^{α} is concentrated on the subset of the distributions φ_x^{α},

$x \in X$ defined by (5), this measure being equivalent to dx under the identification $\varphi_x^\alpha \mapsto x \in X$. It follows that the spectral measure of the representation $V_{\mathfrak{m}}$ in the whole space H_C is concentrated on the set $\{\varphi_x^\alpha \mid \alpha \in \Delta, x \in X\}$ and equivalent (under the identification $\varphi_x^\alpha \mapsto (x, \alpha) \in X \times \Delta$) to the product of the measure dx on X and a uniform measure on Δ. Lemma is proved.

COROLLARY: *The Riemannian structure on the manifold X changed, the spectral measure of the representation $W_{\mathfrak{m}}$ becomes equivalent to the former measure.*

LEMMA 5: *Let $\varphi \in (\mathfrak{A}^X)'$ be a distribution,*

$$\varphi = \lambda_1 \varphi_{x_1}^{\alpha_1} + \cdots + \lambda_n \varphi_{x_n}^{\alpha_n}, \ \lambda_k \in \mathbb{R}, \ \varphi \neq 0, \tag{6}$$

where $\varphi_{x_k}^{\alpha_k}$ are given by (5). *If* $\dim X \geq 2$, *then φ is not an element of the completion $\overline{\mathfrak{A}^X}$ of the space $\mathfrak{A}^X \subset (\mathfrak{A}^X)'$ in the norm $\|a\| = \langle da, da \rangle^{1/2}$.*

PROOF: As the distributions φ_x^α are local with respect to x, it suffices to prove that for every $\alpha \in \Delta$, $x_0 \in X$ and a neighbourhood X_0 of x_0 the distribution $\varphi = \varphi_x^\alpha$ is not an element of the completion of the space $\mathfrak{A}^{X_0} = C^\infty(X_0) \otimes \mathfrak{A}$.

Let X_0 be a sufficiently small neighbourhood of $x_0 \in X$; $x^1, \ldots, x^n$ local coordinates in X_0; $e_1, \ldots, e_\tau$ and orthonormal basis in $\mathfrak{A}$. We do not lose in generality if assume that X_0 is the open unit ball with centre x_0. Let us represent the elements $a \in \mathfrak{A}^{X_0}$ in the form

$$a = \sum_{k=1}^{\tau} a_k e_k, \quad a_k \in C^\infty(X_0).$$

In the chosen coordinates $\|a\|$ and $\langle \varphi_{x_0}^\alpha, a \rangle$ are expressed as follows:

$$\|a\|^2 = \sum_{k=1}^{\tau} \int_{X_0} \left(\sum_{i,j=1}^{m} \tau_{ij}(x) \frac{\partial a_k}{\partial x^i} \frac{\partial a_k}{\partial x^j} \right) dx, \tag{7}$$

where τ_{ij} is the metric tensor in X_0;

$$\langle \varphi_{x_0}^\alpha, a \rangle = \sum_{k=1}^{\tau} \alpha(e_k) a_k(0). \tag{8}$$

We observe that $\alpha(e_k) \neq 0$ for at least one k.

Suppose now that $\varphi^{\alpha}_{x_0}$ is an element of the completion $\mathfrak{A}^{X_0}$ in the norm $\|.\|$. Then, by virtue of (7) and (8), there exist such functions $\omega_1, \ldots, \omega_m$ on X_0, that

$$\int\limits_{X_0}\left(\sum_{i,j=1}^{m} \tau_{ij}(x)\omega_i(x)\omega_j(x)\right)dx < \infty \tag{9}$$

and at the same time

$$\int\limits_{X_0}\left(\sum_{i,j=1}^{m} \tau_{ij}(x)\frac{\partial f}{\partial x^i}\,\omega_j\right)dx = f(0) \tag{10}$$

for any $f \in C^{\infty}(X_0)$. (In other words, delta function $\delta(x)$ is an element of the completion of the space $C^{\infty}(X_0)$ in the 'energy norm' $\|f\|^2 = \int_{X_0}[\Sigma_{i,j}\,\tau_{ij}(\partial f/\partial x^i)(\partial f/\partial x^j)]dx$. It is known (see, for example, [16]) that it is impossible, if dim $X \geq 2$.[1] q.e.d.

COROLLARY 1: *Let μ be the spectral measure of the representation $W_{\mathfrak{A}}$ of $\mathfrak{A}^X$. If* dim $X \geq 2$ *then for every distributions $\varphi_1 \neq \varphi_2$ of the form* (6) *the measures $\mu(\cdot + \varphi_1)$ and $\mu(\cdot + \varphi_2)$ are mutually singular.*

It follows from Lemma 2, indeed, that μ is a Gaussian measure in $(\mathfrak{A}^X)'$ with the correlation functional $B(a_1, a_2) = \langle da_1, da_2\rangle$. As we have seen, $\varphi_1 - \varphi_2$ is not an element of the completion $\overline{\mathfrak{A}}^X$ of $\mathfrak{A}^X$ in the norm $\|a\| = \langle da, da\rangle^{1/2}$. Therefore, the statement is an immediate consequence of Lemma 3, §2.

COROLLARY 2: *The spectral measure μ of the representation $W_{\mathfrak{A}}$ and the spectral measure $\mu * \nu_n$ of the representation $W_{\mathfrak{A}} \otimes S^n V_m$* $(n = 1, 2, \ldots)$ *are mutually singular.*

Indeed, in view of Lemma 4 and 5, $\nu_n(\overline{\mathfrak{A}^X}) = 0$, *and the statement is an immediate consequence of Lemma* 4, §2.

LEMMA 6: *Let $\mathfrak{U}_{\mathfrak{A}}$ be a weakly closed operator algebra generated by operators of the representation W of the group $\mathfrak{A}^X : (W(a)\Phi)(F) = e^{i\langle F, da\rangle}\Phi(V^{-1}(\exp a)F)$. If* $\dim X \geq 2$, *then the algebra $\mathfrak{U}_{\mathfrak{A}}$ contains every shift operator $\Phi(F) \mapsto \Phi(V^{-1}(\exp a)F)$, $a \in \mathfrak{A}^X$, and consequently, every operator $\Phi(F) \mapsto e^{i\langle F, da\rangle}\Phi(F)$, $a \in \mathfrak{A}^X$.*

PROOF: Let μ, ν be the spectral measures of the representations

[1] If dim $X \geq 2$, $W_2^1(X) \not\subset C(X)$.

$W_{\mathfrak{A}}$, $W_{\mathfrak{m}}$ of $\mathfrak{A}^X$ composing the tensor product $W = W_{\mathfrak{A}} \otimes W_{\mathfrak{m}}$ introduced in Lemma 1. It follows from Lemma 4 and Lemma 5 (Corollary 1) that the measures $\mu(\cdot + \varphi_1)$ and $\mu(\cdot + \varphi_2)$ are mutually singular for almost every (with respect to measure ν) pair of functions $\varphi_1 \neq \varphi_2$ from $(\mathfrak{A}^X)'$. Hence, by Lemma 5, §2 (Corollary), the weakly closed operator algebra generated by the operators $W(a)$, $a \in \mathfrak{A}^X$ contains operators $E \otimes W_{\mathfrak{m}}(a)$, $a \in \mathfrak{A}^X$ (E stands for unity operator). Finally, observe that the operator $E \otimes W_{\mathfrak{m}}(a)$ is the shift operator $\Phi(F) \mapsto \Phi(V^{-1})(\exp a)F)$.

LEMMA 7: *Let τ_1, τ_2 be two different Riemannian metrics on X and W^1, W^2 the corresponding representations given by* (1) *of the group $\mathfrak{A}^X$. Then the spectral measures of W^1, W^2 are mutually singular.*

We shall reduce now Lemma 7 to a simpler proposition. Let us start with the remark that if $Y \subset X$ is an arbitrary neighbourhood where the metrics τ_1 and τ_2 do not coincide, it suffices to prove lemma for the restrictions of the representations W^1, W^2 on a subgroup $\mathfrak{A}^Y \subset \mathfrak{A}^X$. Therefore, one can without loss of generality assume that X is a unit ball.

We decompose now the representation W^i ($i = 1, 2$) into a tensor product: $W^i = W^i_{\mathfrak{A}} \otimes W^i_{\mathfrak{m}}$ (Lemma 1), and let μ^i, ν^i be the spectral measures of the representations $W^i_{\mathfrak{A}}$ and $W^i_{\mathfrak{m}}$ correspondingly. Recall that, by Lemma 2, μ^i is a Gaussian measure in $(\mathfrak{A}^X)'$ with the zero mean and the correlation functional $B^i(a, a) = \langle da, da \rangle_i$, where $\langle , \rangle_i$ is an inner product in $\Omega^1(X; \mathfrak{A})$ induced in X by τ_i ($i = 1, 2$).

It is true that if the measures μ^1, μ^2 are mutually singular, the same holds for the spectral measures $\mu^1 * \nu^1$ and $\mu^2 * \nu^2$ of the representations W^1, W^2. It is known, on the one hand, that if two Gaussian measures μ^1, μ^2 are mutually singular, then so are any shifts $\mu^1(\cdot + \varphi_1)$, $\mu^2(\cdot + \varphi_2)$ of these measures (see, for example, [15], pp. 117–118). On the other hand, in view of Lemma 4 (Corollary), the measures ν^1 and ν^2 are equivalent. Hence, the measures $\mu^1 * \nu^1$ and $\mu^2 * \nu^2$ are mutually singular. Therefore, to prove the Lemma we need to show the mutual singularity of μ^1 and μ^2.

At the end, let us remark that $\mathfrak{A}^X = \mathbb{R}^X_1 \oplus \cdots \oplus \mathbb{R}^X_r$, $\mathbb{R}_i = \mathbb{R}$ ($r = \dim \mathfrak{A}$), and the representations $W^i_{\mathfrak{A}}$ of the group $\mathfrak{A}^X$ are tensor products of the representations of the groups $\mathbb{R}^X_i$. Thus, it suffices to prove the singularity for spectral measures of the representations of each of the latter subgroup $\mathbb{R}^X_i$.

Consequently, making use of the explicit expressions (7) for the correlation functionals $\langle da, da \rangle_i$ we have reduced the proof of Lemma 7 to the proof of the following statement.

PROPOSITION: *Let X be the open unity ball in $\mathbb{R}^m$ and $\tau^1_{kl}(x)$, $\tau^2_{kl}(x)$ $(k, l = 1, \ldots, m)$ be differentiable functions on X with the matrices $\tau^1(x) = \|\tau^1_{kl}(x)\|$, $\tau^2(x) = \|\tau^2_{kl}(x)\|$ positively definite in every $x \in X$. Let*

$$B^i(f, f) = \int_X \left(\sum_{k,l=1}^m \tau^i_{kl}(x) \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j} \right) dx, \quad f \in C^\infty(X) \quad (i = 1, 2).$$

Suppose $\tau^1(x) \neq \tau^2(x)$. Then the Gaussian measures in $(C^\infty(X))'$ with the zero mean and the correlation functionals B^1 and B^2 are mutually singular.

PROOF: Consider the operators

$$B^i = -\sum_{k,l=1}^m \frac{\partial}{\partial x^l} \left(\tau^i_{kl} \frac{\partial}{\partial x^k} \right) \quad (i = 1, 2). \tag{11}$$

in the a space $C^\infty(X)$ of differentiable functions on X with compact support. Note that the inner products $B^1(f, g) = \langle B^1 f, g \rangle$ and $B^2(f, g) = \langle B^2 f, g \rangle$ are mutually equivalent in the space $C^\infty(X)$ and also equivalent to any inner product defined by an elliptic operator of the form (11), and by the operator $\Delta = \Sigma_{k=1}^m \partial^2/(\partial x^k)^2$, in particular. The completion of $C^\infty(X)$ in the inner product defined by Δ is the Sobolev space $\mathring{W}^1_2$.

By theorem of Feldman, the Gaussian measures μ^1, μ^2 are equivalent if and only if the operator $B^1 - B^2$ is of Hilbert–Schmidt class with respect to the inner product determined by each of the forms $B^1(f, g)$, $B^2(f, g)$.[1] It is equivalent thing to say that $B^1 - B^2$ is a Hilbert–Schmidt operator in $\mathring{W}^1_2$. But since $B^1 - B^2$ is a differentiation operator of the form (11) again, it is possible in the case $B^1 = B^2$ only. The Proposition is proved.

REMARK: All results of this paragraph can also be formulated without considerable change for the representations $U_{E,\tau}$ of the group G^X determined by arbitrary subbundles E of the tangent bundle TX.

[1] Formulated in a different way (see [15], p. 130, Theorem 4), the Feldman's theorem says that the measures μ^1 and μ^2 are equivalent if and only if $B^1 - B^2 = (B^2)^{1/2} \Gamma (B^2)^{1/2}$ with Γ a Hilbert–Schmidt operator.

The analogue of Lemma 7 for these representations is the following lemma. Let

$$W_{E_i,\tau_i}(a) = U_{E_i,\tau_i}(\exp a),\ a \in \mathfrak{A}^X \quad (i = 1, 2).$$

LEMMA 8: *The spectral measures of the representations W_{E_1,τ_1} and W_{E_2,τ_2} of the group $\mathfrak{A}^X$ are equivalent if and only if $E_1 = E_2$ and the inner products in the space $H_{E_1} = H_{E_2}$ induced by the Riemannian metrics τ_1, τ_2, coincide.*

(The definitions of the space H_E and the representation $U_{E,\tau}$ see in section 2 of §3).

§5. Proof of the main theorems

1. To begin with, notice that it suffices to prove theorems 1 and 2 under the assumption that X is an open manifold diffeomorphic to $\mathbb{R}^m$. Indeed, every smooth connected manifold X, $\dim X = m$, contains an open submanifold Y, which is everywhere dense in X and diffeomorphic to $\mathbb{R}^m$. It is evident that the set $\mathcal{F}_Y = (\Omega^1(Y; \mathfrak{G}))'$, is a subset of full Gaussian measure μ in $\mathcal{F} = (\Omega^1(X; \mathfrak{G}))'$. Hence, we have the coincidence $L^2_\mu(\mathcal{F}_Y) = L^2_\mu(\mathcal{F})$ and thus the assertions of Theorems 1 and 2 about representations of G^X reduce to those of representations of G^Y.

Let, therefore, G be a compact semisimple Lie group, X an open connected manifold, and let $U = \mathrm{EXP}_\beta V$ be the unitary representation of G^X in the space $L^2_\mu(\mathcal{F})$ defined in section 1 of §3:

$$(U(\tilde{g})\Phi)(F) = e^{i\langle F, \beta\tilde{g}\rangle}\Phi(V^{-1}(\tilde{g})F).$$

LEMMA 1: *If $\dim X \geq 2$, then the cyclic subspace $H \subset L^2_\mu(\mathcal{F})$ of the group G^X generated by the vacuum vector $\Omega \in L^2_\mu(\mathcal{F})$ is irreducible.*

PROOF: Let $\mathfrak{A} \subset \mathfrak{G}$ be an arbitrary Cartan subalgebra, $\mathfrak{m}$ its orthogonal complementation in $\mathfrak{G}$. Let us decompose the space $L^2_\mu(\mathcal{F})$ into the tensor product $L^2_\mu(\mathcal{F}) = L^2_{\mu_\mathfrak{A}}(\mathcal{F}_\mathfrak{A}) \otimes L^2_{\mu_\mathfrak{m}}(\mathcal{F}_\mathfrak{m})$ according to Lemma 1, §4. Let $H_\mathfrak{A}$ denote the space of all functionals $\Phi \in L^2_\mu(\mathcal{F})$ such that $\Phi(\cdot + F_\mathfrak{m}) = \Phi(\cdot)$ for every $F_\mathfrak{m} \in \mathcal{F}_\mathfrak{m}$. It is also evident that $H_\mathfrak{A}$ is invariant under the action of operators $W(a) = U(\exp a)$, $a \in \mathfrak{A}^X$, and the restriction of the representation W of $\mathfrak{A}^X$ to the subspace $H_\mathfrak{A}$ is equivalent to the representation $W_\mathfrak{A}$ (see Lemma 1, §4).

It follows from Lemma 3, §4 (Corollary) and Lemma 5, §4 (Corollary 2) that the restrictions of the representation W of the group $\mathfrak{A}^X$ to the subspace $H_{\mathfrak{A}}$ and its orthogonal complement are disjoint. Consequently, every bounded linear operator C in $L^2_\mu(\mathscr{F})$ which commutes with the operators $U(\tilde{g})$, $\tilde{g} \in G^X$ and, therefore, with the operators $W(a)$, $a \in \mathfrak{A}^X$ leaves the subspace $H_{\mathfrak{A}}$ invariant.

Let now $\mathfrak{A}_1$ be another Cartan subalgebra in $\mathfrak{G}$, such that $\mathfrak{A} \cap \mathfrak{A}_1 = 0$ (it certainly exists), let $\mathfrak{m}_1$ be the orthogonal complementation to $\mathfrak{A}_1$ in $\mathfrak{G}$, $H_{\mathfrak{A}_1} \subset L^2_\mu(\mathscr{F})$ a subspace corresponding to $\mathfrak{A}_1$. We shall verify that $H_{\mathfrak{A}} \cap H_{\mathfrak{A}_1} = \{c\Omega\}$. Indeed, if $\mathfrak{A} \cap \mathfrak{A}_1 = 0$, then $\mathfrak{m} + \mathfrak{m}_1 = \mathfrak{G}$ and $\mathscr{F}_{\mathfrak{m}} + \mathscr{F}_{\mathfrak{m}_1} = \mathscr{F}$. Hence, if $\Phi \in H_{\mathfrak{A}} \cap H_{\mathfrak{A}_1}$, that is $\Phi(\cdot + F_{\mathfrak{m}}) = \Phi(\cdot + F_{\mathfrak{m}_1}) = \Phi(\cdot)$ for every $F_{\mathfrak{m}} \in \mathscr{F}_{\mathfrak{m}}$, $F_{\mathfrak{m}_1} \in \mathscr{F}_{\mathfrak{m}_1}$, then $\Phi(\cdot + F) = \Phi(\cdot)$ for every $F \in \mathscr{F}$ and, consequently, $\Phi = \text{const}$.

Let C be an arbitrary bounded linear operator in $L^2_\mu(\mathscr{F})$ which commutes with the operators $U(\tilde{g})$, $\tilde{g} \in G^X$. Since C, as was demonstrated above, leaves both subspaces $H_{\mathfrak{A}}, H_{\mathfrak{A}_1}$ invariant and since $H_{\mathfrak{A}} \cap H_{\mathfrak{A}_1} = \{c\Omega\}$, then $C\Omega = c\Omega$. The statement of the Lemma follows.

LEMMA 2. *If* $\dim X \geq 2$, *then the weakly closed operator algebra* $\mathfrak{U}$ *generated by the operators* $U(\tilde{g})$, $\tilde{g} \in G^X$ *in the space* $L^2_\mu(\mathscr{F})$, *contains the operators of multiplication by the functionals of the form*

$$F \mapsto e^{i\langle F, \Sigma_{k=1}^n V(\tilde{g}_k) du_k\rangle}, \tilde{g}_k \in G^X, u_k \in \mathfrak{A}^X, n = 1, 2, \ldots.$$

PROOF: In view of Lemma 6, §4, $\mathfrak{U}$ contains the operators of multiplication by $e^{i\langle \cdot, da\rangle}$, $a \in \mathfrak{A}^X$, where $\mathfrak{A}$ is an arbitrary Cartan subalgebra in $\mathfrak{A}$. Consequently, $\mathfrak{U}$ also contains every operator A_u of multiplication by $e^{i\langle \cdot, du\rangle}$, $u \in \mathfrak{G}^X$. It is evident that the operator of multiplication by $e^{i\langle F, \Sigma_{k=1}^n V(\tilde{g}_k) du_k\rangle}$ equals to the product of the operators $U(\tilde{g}_k) A_{u_k} U^{-1}(\tilde{g}_k)$ and, therefore, is also an element of $\mathfrak{U}$.

LEMMA 3. *The set* $M = \{\Sigma_{k=1}^n V(\tilde{g}_k) du_k | \tilde{g}_k \in G^X, \ u_k \in \mathfrak{G}^X, \ n = 1, 2, \ldots\}$ *is dense in the space* $\Omega^1(X; \mathfrak{G})$ *of* 1-*forms in the norm introduced there.*

PROOF: The representation V of G^X in the space $\Omega^1(X; \mathfrak{G})$ is a continual direct sum $V = \int_X^{\oplus} \tilde{V}^x dx$, $\tilde{V}^x$ being a representation in the space H^x of 1-forms in $x \in X$ taking their values in $\mathfrak{G}$.

It is evident that these representations $\tilde{V}^x$ are pairwise disjoint. In the meantime one proves that the component of the set M in every space $H^x \cong \bigoplus^m \mathfrak{G}$ ($m = \dim X$) coincides with H^x. It follows that M is dense in $\Omega^1(X; \mathfrak{G})$.

REMARK: It is a consequence of Lemma 3 that the set $\{\beta\tilde{g} | \tilde{g} \in G^X\}$ is a total one in $\Omega^1(X; \mathfrak{G})$ with respect to the norm introduced there. This fact can be, however, established directly.

PROOF OF THEOREM I. Let $\mathscr{H} \subset L^2_\mu(\mathscr{F})$ be the cyclic subspace for the group G^X which is generated by the vacuum vector Ω. In view of Lemma I, $\mathscr{H}$ is irreducible. By Lemma 2, $\mathscr{H}$ contains all the functionals $e^{i\langle\cdot,\omega\rangle}$, $\omega \in M$, where $M = \{\Sigma_{k=1}^n V(\tilde{g}_k) du_k | \tilde{g}_k \in G^X,\ u_k \in \mathfrak{G}^X, n = 1, 2, \ldots\}$. Using Lemma 3, M is dense in $\Omega^1(X; \mathfrak{G})$ and, therefore, the functionals $e^{i\langle\cdot,\omega\rangle}$, $\omega \in M$ form the set total in $L^2_\mu(\mathscr{F})$. Consequently, $\mathscr{H} = L^2_\mu(\mathscr{F})$ and Theorem I is proved.

PROOF OF THEOREM 2. Let τ_1, τ_2 be different Riemannian metrics on X, U^1 and U^2 the corresponding representations of the group G^X. Consider the representations of the group $\mathfrak{A}^X$, where $\mathfrak{A}$ a Cartan subalgebra of $\mathfrak{G}$: $W^i(a) = U^i(\exp a)$, $i = 1, 2$. In view of Lemma 7, §4, the representations W^1 and W^2 are disjoint and, consequently, the same is true for the representations U^1 and U^2 of G^X. The Theorem is proved.

Along these lines Theorem 4 of the equivalence conditions for the representations U_{E_1,τ_1}, U_{E_2,τ_2} of G^X can be deduced from Lemma 8, §4.

2. *Spherical function of the representation* $U = \mathrm{EXP}_\beta\, V$. We show here that *if G is a compact semisimple Lie group and* $\dim X \geq 2$, *then the vacuum vector Ω is invariantly defined in the space of representation* $U = \mathrm{EXP}_\beta\, V$ *of* G^X.

Let us, indeed, consider the subspace $H_{\mathfrak{A}} \subset L^2_\mu(\mathscr{F})$ introduced in the proof of Lemma I. We have established above that (a) $H_{\mathfrak{A}}$ is invariant with respect to the representation $W(a)$ of the group $\mathfrak{A}^X$; (b) the spectral measure of the representation $W(a)$ in the subspace $H_{\mathfrak{A}}$ is a Gaussian measure in $(\mathfrak{A}^X)'$ with the zero mean; (c) the restrictions $W(a)$ to the subspace $H_{\mathfrak{A}}$ and its orthogonal complementation are mutually disjoint. On the contrary, it easily follows from the results of §4 that the subspace $H_{\mathfrak{A}} \subset L^2_\mu(\mathscr{F})$ is determined in invariant way by the conditions (a), (b) and (c).

Since the intersection of the subspaces $H_{\mathfrak{A}_i}$, where $\mathfrak{A}_i$ runs over different Cartan subalgebras in $\mathfrak{G}$, contains the multiples of Ω only, then the vector Ω is also invariantly determined, up to a multiplier, in the space of representation.

We shall call *the spherical function of the representation* $U = \mathrm{EXP}_\beta V$ the following function on G^X:

$$\psi(\tilde{g}) = \langle U(\tilde{g})\Omega, \Omega\rangle.$$

It follows from the definition of operators $U(\tilde{g})$ that

$$\psi(\tilde{g}) = \exp\left(-\tfrac{1}{2}\|\beta\tilde{g}\|^2\right).$$

The representation U is uniquely determined by its sperical function and it is a consequence of the invariant definition of the vacuum vector that equivalent representations U determine equal spherical functions ψ. One easily deduces from this fact the statement of Theorem 2 in the case $\dim X \geq 2$.

§6. Extension of the group $\theta^1(X; G)$ of section of 1-jet fibre bundle and its representations

In the papers of physical character [18], [3] and others the so-called Sugawara algebra, accompanying a Lie algebra, is considered. It turns out that Sugawara algebra is the Lie algebra of a certain infinite-dimensional group which can be described in terms of the present paper. Let us give the precise definitions.

Let X be a Riemannian manifold, G a real Lie group such that its Lie algebra $\mathfrak{G}$ possesses an inner product invariant under the adjoint action of the group G. Let us consider the group

$$\theta^1(X; G) = G^X \cdot \Omega^1(X; \mathfrak{G})$$

of all differentiable sections with compact support of the 1-jet fibre bundle $j^1(X; G) \to X$ (see §I). The group $\theta^1(X; G)$ is acting in the space $\Omega^1(X; \mathfrak{G})$ of differentiable $\mathfrak{G}$-valued 1-forms on X with compact support.

Let us introduce two cocycles of the group $\theta^1(X; G)$ taking values in $\Omega^1(X; \mathfrak{G})$:

$$\alpha(\tilde{g}, \tilde{a}) = \tilde{a},$$
$$\tilde{\beta}(\tilde{g}, \tilde{a}) = \beta\tilde{g} \quad (\tilde{g} \in G^X, \tilde{a} \in \Omega^1(X; \mathfrak{G})),$$

where β is the Maurer–Cartan cocycle.

Define for any elements $f_1 = (\tilde{g}_1, \tilde{a}_1)$ and $f_2 = (\tilde{g}_2, \tilde{a}_2)$ of $\theta^1(X; G)$:

$$\gamma(f_1, f_2)(x) = \langle(\tilde{\beta} f_1^{-1})(x), (\alpha f_2)(x)\rangle_x,$$
$$\gamma_0(f_1, f_2) = \langle\tilde{\beta} f_1^{-1}, \alpha f_2\rangle = \int_X \gamma(f_1, f_2)(x)dx.$$

It is clear that for every $f_1, f_2, f_3 \in \theta^1(X; G)$ the following is true:

$$\gamma(f_1, f_2) + \gamma(f_1 f_2, f_3) = \gamma(f_1, f_2 f_3) + \gamma(f_2, f_3).$$

Therefore, γ is a 2-cocycle of the group $\theta^1(X; G)$ with values in $\mathbb{R}^X$. Similarly, γ_0 is a 2-cocycle of $\theta^1(X; G)$ with values in $\mathbb{R}$.

Let us verify the non-triviality of 2-cocycles γ and γ_0. Suppose, for example, that the 2-cocycle γ is trivial. It means that there exists a mapping $c: \theta^1(X; G) \to \mathbb{R}^X$ such that

$$\gamma(f_1, f_2) = c(f_1 f_2) - c(f_1) - c(f_2) \tag{1}$$

for every $f_1, f_2 \in \theta^1(X; G)$. Let $\mathfrak{A} \subset \mathfrak{G}$ be an arbitrary abelian subalgebra, $A \subset G$ a corresponding abelian subgroup. Consider the abelian subgroup $\theta^1(X; A) \subset \theta^1(X; G)$. It follows from (1) that $\gamma(f_1, f_2) = \gamma(f_2, f_1)$ for every $f_1, f_2 \in \theta^1(X; A)$. In the meantime it is clear that $\gamma(f_1, f_2) \neq \gamma(f_2, f_1)$ on $\theta^1(X; A)$. Therefore, $\gamma \not\sim 0$. One proves the non-triviality of the cocycle γ_0 likewise.

The cocycles γ and γ_0 define nontrivial extensions of the additive groups $\mathbb{R}^X$ and $\mathbb{R}$ by means of the group $\theta^1(X; G)$. Let $S(X; G)$ and $S^0(X; G)$, correspondingly, denote these extensions. Thus, the elements of the group $S(X; G)$ are the pairs (f, c), $f \in \theta^1(X; G)$, $c \in \mathbb{R}^X$ with the multiplication rule

$$(f_1, c_1)(f_2, c_2) = (f_1 f_2, c_1 + c_2 + \gamma(f_1, f_2));$$

the elements of the group $S^0(X; G)$ are the pairs (f, c), $f \in \theta^1(X; G)$, $c \in \mathbb{R}$ with the multiplication rule

$$(f_1, c_1)(f_2, c_2) = (f_1 f_2, c_1 + c_2 + \gamma_0(f_1, f_2)).$$

Note that the group $S^0(X; G)$ is isomorphic to the factor group of $S(X; G)$ with the subgroup of elements type $(1, c)$, $c \in \mathbb{R}^X$ with $\int_X c(x)dx = 0$.

To indicate the connection of the groups constructed above with Sugawara algebra, let us suppose G a compact Lie group. Then Lie algebra of the group $S^0(\mathbb{R}^4; G \times G)$ is Sugawara algebra, its generators being described in [3].

Let us construct now representations of the groups $S(X; G)$ and $S^0(X; G)$. Let $\mathscr{F} = (\Omega^1(X; \mathfrak{G}))'$, and μ be the standard Gaussian measure on $\mathscr{F}$. According to Sect. 3, §2 one defines a unitary representation of the group $S^0(X; G)$ in the space $L^2_\mu(\mathscr{F})$ for a pair of cocycles $\alpha, \tilde{\beta}$ of the group $\theta^1(X; G)$ by the following:

$$(U(f, c)\Phi)(F) = e^{is(c+\langle F, \alpha f\rangle)+\frac{1}{2}\langle F, \tilde{\beta} f\rangle - \frac{1}{4}\|\tilde{\beta} f\|^2}\Phi(V^{-1}(\tilde{g})F + \tilde{\beta} f^{-1}), \tag{2}$$

where $f = (\tilde{g}, \tilde{a}) \in \theta^1(X;G)$ and $s \neq 0$ is a real parameter. If G is a compact semisimple Lie group and $\dim X \geq 2$, then, making use of Theorem I, one proves that the representation (2) is irreducible.

As the group $S^0(X; G)$ is isomorphic to the factor group of $S(X; G)$, the representation (2) of the group trivially extends to a representation of the group $S(X; G)$, the result being the representation of $S(X; G)$ in the space $L^2_\mu(\mathscr{F})$ given by

$$\begin{aligned}(\tilde{U}(f, c)\Phi)(F) = {} & e^{is(\int_X c(x)dx + \langle F, \alpha f\rangle)+\frac{1}{2}\langle F, \tilde{\beta} f\rangle - \frac{1}{4}\|\tilde{\beta} f\|^2} \\ & \times \Phi(V^{-1}(\tilde{g})F + \tilde{\beta} f^{-1}).\end{aligned} \tag{3}$$

EXAMPLE: $X = \mathbb{R}$, $G = \mathbb{R}^+$. In this case the elements of the group $S(X; G)$ are the triples $(g(x), a(x), c(x))$, $x \in \mathbb{R}$ with the multiplication rule:

$$(g_1, a_1, c_1)(g_2, a_2, a_2) = \left(g_1 + g_2, a_1 + a_2, c_1 + c_2 + \frac{dg_1}{dx} a_2\right).$$

The factor group of $S(X; G)$ with subgroup G_0 of elements $(g, 0, 0)$ with $g = \text{const.}$, is isomorphic to the functional Heisenberg group with δ'-commutation law. The representation (3) of $S(X; G)$ is trivial on G_0 and therefore it defines the representation of this Heisenberg group.

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