N. DE GRANDE-DE KIMPE

Locally convex spaces for which $\Lambda(E) = \Lambda[E]$ and the Dvoretsky-Rogers theorem

Compositio Mathematica, tome 35, nº 2 (1977), p. 139-145
<http://www.numdam.org/item?id=CM_1977__35_2_139_0>

COMPOSITIO MATHEMATICA, Vol. 35, Fasc. 2, 1977, pag. 139–145
Noordhoff International Publishing
Printed in the Netherlands

LOCALLY CONVEX SPACES FOR WHICH $\Lambda(E) = \Lambda[E]$ AND THE DVORETSKY-ROGERS THEOREM

N. De Grande-De Kimpe

The classical Dvoretsky-Rogers theorem states that if E is a Banach space for which $\ell^1[E] = \ell^1(E)$, then E is finite dimensional. This property still holds for any ℓ^p $(1 < p < \infty)$ (see [5]).

Recently it has been shown (see [11]) that the result remains true when one replaces ℓ^1 by any non-nuclear perfect sequence space Λ, having the normal topology $n(\Lambda, \Lambda^x)$. (This situation does not contain the ℓ^p $(1 < p < \infty)$-case). The question whether the Dvoretsky-Rogers theorem holds for any perfect Banach sequence space Λ is still open. (A partial, positive answer to this problem, generalizing the ℓ^p-case for any p is given in this paper.) It seemed however more convenient to us to tackle the problem from the "locally-convex view-point".

The "locally-convex version" of the Dvoretsky-Rogers theorem was proved by A. Pietsch in [8] (for the definitions and notations see below):

If E is a locally convex space then the following are equivalent:

(i) E has property (B) and $\ell^1[E] = \ell^1(E)$

(ii) E'_β is nuclear (Remark that every Banach space E has the property (B)).

An inspection of the proof shows that Pietsch actually proves that (i) is equivalent to

(iii) For every $A \in \mathcal{B}_E$ there exists a $B \in \mathcal{B}_E$ $(A \subset B)$ such that the canonical mapping $\varphi_{AB} : E_A \to E_B$ is absolutely summing.

Since all the notions appearing in the ((i)$\Leftrightarrow$(iii))-version of the Dvoretsky-Rogers theorem have meaningfull generalizations when replacing ℓ^1 by any perfect Banach sequence space Λ one can ask if (i)$\Leftrightarrow$(iii) still holds in this generalized situation.

In this paper we show that the answer to this question is positive and we apply our result to some special cases.

§1. Preliminaries

All the classical notions and properties concerning locally convex spaces, as well as the elementary theory of sequence spaces, will be taken from [6]. They will be used without any further reference. The same will be done as far as nuclearity is concerned. Here we refer to [8].

We fix the following notations:

—If not specified E will denote a complete locally convex Hausdorff space with topological dual space E'.

—$\mathcal{P}_E$ is a fundamental system of semi-norms determining the topology of E.

—$\mathcal{U}_E$ is a fundamental system of barrelled neighbourhoods of the origin in E.

—$\mathcal{B}_E$ is a fundamental system of closed, convex, bounded subsets of E.

For $B \in \mathcal{B}_E$ we denote by E_B the Banach space $\bigcup_{n\in\mathbb{N}} n \cdot B$, normed by the gauge of B. This norm is denoted by $\|\cdot\|_B$.

—Λ is a perfect sequence space with α-dual space Λ^x. We assume that Λ is a Banach space for the strong topology $\beta(\Lambda, \Lambda^x)$. Elements of Λ are denoted by $\alpha = (\alpha_i)$.

—We consider the following generalized sequence spaces

$$\Lambda[E] = \{(x_i) | x_i \in E,\ i = 1, 2, \ldots \text{ and } \forall\, a \in E' : (\langle x_i, a\rangle)_i \in \Lambda\}$$

and

$$\Lambda(E) = \{(x_i) | x_i \in E,\ i = 1, 2, \ldots \text{ and } \forall\, p \in \mathcal{P}_E : (p(x_i))_i \in \Lambda\}$$

A locally convex Hausdorff topology on $\Lambda[E]$ is given by the semi-norms:

$$\sup_{a\in U^0} \|(\langle x_i, a\rangle)_i\|_\Lambda,\ (x_i) \in \Lambda[E],\ U \in \mathcal{U}_E.$$

A locally convex Hausdorff topology on $\Lambda(E)$ is given by the semi-norms

$$\|p(x_i)_i\|_\Lambda,\ (x_i) \in \Lambda(E),\ p \in \mathcal{P}_E.$$

For $\Lambda = \ell^1$ the above spaces are studied in [8]. In their general form the spaces $\Lambda[E]$ and $\Lambda(E)$ are studied in [7] and in [1] and [10] respectively. Obviously $\Lambda(E)$ is continuously embedded in $\Lambda[E]$.

—For a normal bounded subset R of Λ and $B \in \mathcal{B}_E$, define:

$$[R, B] = \{(x_i) | (x_i) \in \Lambda(E),\ x_i \in E_B,\ i = 1, 2, \ldots \text{ and } (\|x_i\|_B)_i \in R\}$$

—The space E is said to be fundamentally Λ-bounded if the collection of all $[R, B]$ forms a fundamental system of bounded subsets of $\Lambda(E)$. The notion of fundamentally Λ-bounded space has been introduced in [10]. A fundamentally ℓ^1-bounded space is exactly a space "having the property (B)" (see [8]).

—An operator (i.e. a continuous linear mapping) from a Banach space X to a Banach space Y is called Λ-summing if for every $(x_i) \in \Lambda[X]$ the sequence $(f(x_i))_i$ is an element of $\Lambda(Y)$. For $\Lambda = \ell^1$ (resp. $\Lambda = \ell^p$, $1 < p < \infty$) such an operator is called absolutely summing (resp. p-summing).

§2. A generalized Dvoretsky-Rogers theorem

LEMMA 1: *If D is a bounded subset of $\Lambda[E]$ then there exists $A \in \mathcal{B}_E$ such that $D \subset \Lambda[E_A]$ and*

$$\sup_{(x_i)\in D} \|(x_i)\|_{\Lambda[E_A]} \leq 1.$$

PROOF: For $U \in \mathcal{U}_E$ we put

$$\sigma_u = \sup_{(x_i)\in D} \sup_{a\in U^0} \|(\langle x_i, a\rangle)\|_\Lambda.$$

Then $\sigma_u < \infty$ since D is bounded in $\Lambda[E]$. If $A = \bigcap_{U\in\mathcal{U}_E} \sigma_u \cdot U$ then $A \in \mathcal{B}_E$. Take $(\lambda_i) \in \Lambda^x$ such that $\|(\lambda_i)\|_{\Lambda^x} \leq 1$. Then for $U \in \mathcal{U}_E$ and $a \in U$ we have:

$$\left|\left\langle \sum_i \lambda_i x_i, a\right\rangle\right| = \left|\sum_i \lambda_i \langle x_i, a\rangle\right| \leq \|(\lambda_i)\|_{\Lambda^x} \|(\langle x_i, a\rangle)\|_\Lambda \leq \sigma_u,$$

for all $(x_i) \in D$. Hence $\Sigma_i \lambda_i x_i \in \sigma_u \cdot U$ for each $U \in \mathcal{U}_E$. So $\Sigma_i \lambda_i x_i \in A$ (or $\|\Sigma_i \lambda_i x_i\|_A \leq 1$) for all $(\lambda_i) \in \Lambda^x$, $\|(\lambda_i)\|_{\Lambda^x} \leq 1$ and all $(x_i) \in D$. I.e.

$$\left|\left\langle \sum_i \lambda_i x_i, b\right\rangle\right| = \left|\sum_i \lambda_i \langle x_i, b\rangle\right| \leq 1$$

for all $(x_i) \in D$, all $(\lambda_i) \in \Lambda^x$, $\|(\lambda_i)\|_{\Lambda^x} \leq 1$ and all $b \in (E_A)'$ with $\|b\|_{(E_A)'} \leq 1$.

Since the unit ball in Λ^x is a normal subset of Λ^x we also have:

$$\sum_i |\lambda_i| |\langle x_i, b\rangle| \leq 1$$

under the same assumptions on (x_i), (λ_i) and b. Hence $(x_i) \in \Lambda[E_A]$ and $\|(x_i)\|_{\Lambda[E_A]} \leq 1$ for all $(x_i) \in D$.

LEMMA 2: *If $\Lambda(E) = \Lambda[E]$ and B is a bounded subset of $\Lambda[E]$, then B is also bounded in $\Lambda(E)$.*

PROOF: Remark that $B \subset \Lambda[E]$ is bounded if and only if for every $a \in E'$ the set

$$\{(\langle x_i, a\rangle)_i | (x_i) \in B\}$$

is bounded in Λ (see [7]). Also $B \subset \Lambda(E)$ is bounded if and only if for each $p \in \mathcal{P}_E$ the set

$$\{(p(x_i))_i | (x_i) \in B\}$$

is bounded in Λ (see [1]). Finally a subset A of Λ is bounded if and only if for each $\beta \in \Lambda^x$ the set

$$\left\{\sum_i |\alpha_i\beta_i| \,|\, \alpha \in A\right\}$$

is bounded in $\mathbb{R}$. Taking these facts in mind, the proof proceeds exactly as in [8] Theorem 2.1.2.

THEOREM: *The following are equivalent*

(i) *E is fundamentally Λ-bounded and $\Lambda(E) = \Lambda[E]$.*

(ii) *For every $A \in \mathcal{B}_E$ there exists a $B \in \mathcal{B}_E$ ($B \supset A$) such that the canonical injection $\varphi_{AB} : E_A \to E_B$ is Λ-summing.*

PROOF: (i) $\Rightarrow$ (ii): Take $A \in \mathcal{B}_E$ and put

$$D = \{(x_n) | (x_n) \in \Lambda[E_A],\ \|(x_n)\|_{\Lambda[E_A]} \leq 1\}$$

The continuous injection $i_A : E_A \to E$ extends canonically to a continuous injection $\bar{i}_A : \Lambda[E_A] \to \Lambda[E]$ ([2] Prop. 28). Hence D is bounded in $\Lambda[E]$.

Since $\Lambda[E] = \Lambda(E)$, D is also bounded in $\Lambda(E)$ (lemma 2). Since E is fundamentally Λ-bounded, there exists $B \in \mathcal{B}_E$ such that the set

$$\{(\|x_n\|_B)_n | (x_n) \in D\}$$

is bounded in Λ. In particular we have $D \subset \Lambda(E_B)$. So $(x_n) \in \Lambda[E_A]$ implies $(x_n) \in \Lambda(E_B)$ and (ii) is proved.

(ii) $\Rightarrow$ (i): Let D be a bounded subset of $\Lambda(E)$. Then D is bounded in $\Lambda[E]$ and, by lemma 1, D is bounded in $\Lambda[E_A]$ for some $A \in \mathcal{B}_E$. By (ii) there exists $B \in \mathcal{B}_E$ such that $\varphi_{AB} : E_A \to E_B$ is Λ-summing.

Then D is a bounded subset of $\Lambda(E_B)$ since the extended mapping $\varphi_{AB}:\Lambda[E_A]\to\Lambda(E_B)$ is continuous ([2] Prop. 28). So E is fundamentally Λ-bounded. For the second half of (i) suppose $(x_i)\in\Lambda[E]$. Then by lemma 1 there exists $A\in\mathscr{B}_E$ such that $(x_i)\in\Lambda[E_A]$. (Consider $\{(x_i)\}$ as a bounded subset of $\Lambda[E]$). By (ii) $(x_i)\in\Lambda(E_B)$ for some $B\in\mathscr{B}_E$. Finally the canonical injection $i_B:E_B\to E$ induces an injection $\overline{i_B}:\Lambda(E_B)\to\Lambda(E)$ and the conclusion follows.

§3. Examples and special cases

1. *Locally convex spaces for which* $\Lambda(E)=\Lambda[E]$.

Recall that a sequence (e_i) in E is called a Schauder basis for E if every $x\in E$ can be written uniquely as $x=\Sigma_i\,\alpha_i e_i$ and if the coefficient functionals $f_k:x\to\alpha_k$ are continuous. The basis (e_i) is strong if $\Sigma_i\,p(e_i)p_B(f_i)<\infty$ for every $p\in\mathscr{P}_E$ and every $B\in\mathscr{B}_E$ (p_B denotes the seminorm on E', $\beta(E',E)$ corresponding to B). For the connection between the existence of a strong basis with the nuclearity of the space, as well as for examples of spaces having a strong basis we refer to [3].

PROPOSITION 1: *If E has a strong basis (e_i,f_i) then $\Lambda(E)=\Lambda[E]$ whenever $\ell^1\subset\Lambda$.*

PROOF: Take $(y_n)\in\Lambda[E]$. Since E is semi-reflexive ([3] Prop. 4), its strong dual space E'_β is barrelled. Hence, by the Banach-Steinhaus theorem, the linear mapping

$$g:E'_\beta\to\Lambda:a\to(\langle y_n,a\rangle)_n$$

is continuous. I.e. $\exists\, B\in\mathscr{B}_E$, $\exists\, K>0$ such that

$$\|g(f_i)\|_\Lambda=\|(\langle y_n,f_i\rangle)_n\|_\Lambda\le Kp_B(f_i),\ i=1,2,\ldots.$$

For $p\in\mathscr{P}_E$ we then have:

$$p(y_n)=p\left(\sum_i\langle y_n,f_i\rangle e_i\right)\le\sum_i|\langle y_n,f_i\rangle|p(e_i)$$

So for $\beta\in\Lambda^x$ we obtain:

$$\sum_n|\beta_n|p(y_n)\le\sum_n\sum_i|\beta_n||\langle y_n,f_i\rangle|p(e_i)$$
$$\le\sum_i p(e_i)\|\beta\|_{\Lambda^x}\cdot\|(\langle y_n,f_i\rangle)\|_\Lambda$$

$$\leq \|\beta\|_{\Lambda^x} \cdot K \cdot \sum_i p(e_i) p_B(f_i) < \infty,$$

which implies that $(y_n) \in \Lambda(E)$.

2. *Spaces having property* (ii) *in the Theorem.*
For convenience we'll call them Λ-spaces.

PROPOSITION 2: *Under each of the following conditions E is a Λ-space*:
(a) *E'_β is nuclear*
(b) *E has a strong basis and is fundamentally Λ-bounded*
(c) *E is a Frechet space or a DF-space with a strong basis*
(d) *E is a Frechet space or a DF-space and $\Lambda(E) = \Lambda[E]$.*

PROOF:
(a) Every absolutely summing map between Banach spaces is Λ-summing (see [4]), then apply Pietsch's result mentioned in the introduction, and the theorem.
(b) From Prop. 1.
(c) Every Frechet space (and every *DF*-space) is fundamentally Λ-bounded (see [10]). Then apply (b)
(d) As in (c).

REMARK: Proposition 2(a) can also be interpreted as follows: If E has property (B) and $\ell^1(E) = \ell^1[E]$ then for every Λ, E is fundamentally Λ-bounded and $\Lambda(E) = \Lambda[E]$.

3. *The relation to nuclearity.*
If Λ is such that sufficiently many compositions of Λ-summing maps provide an absolutely summing map then every Λ-space E has a strong dual space E'_β which is nuclear. It is shown in [9] that this is the case whenever $\Lambda = \ell^p$ $(1 < p < \infty)$. We therefore obtain:

PROPOSITION 3: *The following are equivalent*:
(i) *E'_β is nuclear*
(ii) *E is fundamentally ℓ^p-bounded and $\ell^p(E) = \ell^p[E]$ (for some $1 \leq p < \infty$).*

COROLLARY: *IF E is a Frechet space or a DF-space then E (and E'_β) is nuclear if and only if $\ell^p(E) = \ell^p[E]$ for some p $(1 \leq p < \infty)$.*

(This result contains Grothendieck's result mentioned in the introduction).

REFERENCES

[1] N. DE GRANDE-DE KIMPE: Generalized sequence spaces. *Bull. Soc. Math. Belg. XXIII* (1971) 123–166.

[2] N. DE GRANGE-DE KIMPE: Operator theory for bornological spaces. *Bull. Soc. Math. Belg. XXVI*, (1974) 3–23.

[3] N. DE GRANGE-DE KIMPE: Criteria for nuclearity in terms of generalized sequence spaces. To appear in *Archiv der Mathematik*.

[4] ED. DUBINSKY and M. S. RAMANUJAN: Inclusion theorems for absolutely λ-summing maps. *Math. Ann. 192* (1971) 177–190.

[5] A. GROTHENDIECK: Sur certaines classes de suites dans les espaces de Banach et le théorème de Dvoretsky-Rogers. *Boletin Soc. Math. Sao Paulo, 8* (1956).

[6] G. KOTHE: *Topologische lineare Räume.* Springer Verlag (1960).

[7] A. PIETSCH: *Verallgemeinerte volkommene Folgenräume.* Akademie Verlag (1962).

[8] A. PIETSCH: *Nukleare lokalkonvexe Räume.* Akademie Verlag (1965).

[9] A. PIETSCH: Absolut p-summierende Abbildungen in normierte Räume. *Studia Math. 28* (1967) 333–353.

[10] R. C. ROSIER: Dual spaces of certain vector sequence spaces. *Pacific J. Math. 46(2)* (1973) 487–501.

[11] R. C. ROSIER: Vector sequence spaces and the Dvoretsky-Rogers theorem. (To appear).

(Oblatum 20–V–1976)

Vrije Universiteit Brussel
Departement voor Wiskunde
Pleinlaan 2, F7
1050 Brussel