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## BORDISM WITH CODIMENSION-ONE SINGULARITIES

S. Buoncristiano and C. De Concini

### 0. Introduction

Throughout the first part of this introduction we shall place ourselves in the differentiable category.

Thus ‘manifold’ will mean differentiable manifold.

Let  $l$  be an integer  $\geq 2$  and  $G$  a group of permutations on  $l$ -letters. Define a  $G$ -manifold of dimension  $n$  to be a pair  $M, S_M$ , where  $M - S_M$  is an oriented manifold of dimension  $n$ ,  $S_M$  is a (possibly empty) oriented manifold of dimension  $n - 1$  and a typical neighbourhood of  $S_M$  in  $M$  is the mapping cylinder on an  $l$ -sheeted covering of  $S_M$  having  $G$  as a structure group.

There is a bordism group  $\hat{\Omega}_n^G$  of  $G$ -manifolds of dimension  $n$ , constructed in the usual way.

The calculation of this group is an interesting but difficult problem. However one can try to compute various bordism groups obtained from  $\hat{\Omega}_n^G$  by imposing restrictive conditions on the covering  $\pi$ .

For instance, suppose we require that  $\pi$  be a trivial covering, i.e.  $G = (1)$ . Then the concept of  $G$ -manifold coincides with that of  $Z_l$ -manifold introduced by Sullivan in [4]. The bordism group of  $Z_l$ -manifolds can be proved to be equal to the bordism group of oriented manifolds with coefficients  $Z_l$ , denoted  $\Omega_n$  (point,  $Z_l$ ) (see [1] for details and generalisations). Therefore in this case the computation is possible using the universal-coefficient formula.

One other particular case of interest is obtained by requiring that  $\pi$  be a 2-sheeted covering,  $G = Z_2$ . The associated bordism group is easily proved to be isomorphic to the usual bordism group of unoriented manifolds.

In §4 we give a complete computation of  $\hat{\Omega}_n^G$  in the case where  $l$  is an odd prime  $p$  and  $\pi$  is a principal covering. For this group we continue to use the notation  $\hat{\Omega}_n^{Z_p}$ . We show that  $\hat{\Omega}_n^{Z_p}$  is isomorphic to the reduced bordism group of  $Z_p$ -equivariant oriented manifolds of

dimension  $n + 1$ . The proof, which works in a more general context, is based on transversality constructions which take place in a classifying space for  $Z_p$ .

The concept of  $G$ -manifold makes equally sense if we do not start from differentiable manifolds but from the cycles of any connected generalised homology theory (see [1])  $U_*(-)$ , so that one can form an associated bordism group  $\hat{U}_n^G$ . Moreover one can construct bordism groups  $\{\hat{U}_n^G(X)\}$  ( $X$  a topological space) using maps of  $G$ -manifolds into  $X$  in the usual way.

In §1 we establish a long exact sequence which generalizes in a natural way the usual Bockstein sequence for  $Z_l$  coefficients.

In §2 we take into consideration the bordism group, still denoted  $\hat{U}_n^G(X)$ , of those  $G$ -manifolds in  $X$  for which  $\pi$  is principal. Under the assumption that  $U_n(X)$  is free of  $l$ -torsion we establish a short exact sequence relating  $\hat{U}_n^G(X)$  to the bordism group of  $G$ -invariant manifolds of dimension  $n - 1$ . This exact sequence, together with the results of §4, yields another short exact sequence (theorem 4.5) which is the  $Z_p$ -analogue ( $p$  odd prime) of a short exact sequence established by Wall ([6]) in the  $Z_2$ -case.

Finally, in §5 we set  $l = p$  odd prime and prove that the theory  $\hat{\Omega}_n^{S_p}(-)$  does not represent homology with  $Z_p$ -coefficients. Here  $S_p$  is the full symmetric group on  $p$  letters. Note that, if  $l = 2$ , then  $\hat{\Omega}_n^{S_2}(-)$  is just unoriented differentiable bordism and therefore a famous theorem of Thom's ([5]) ensures that  $\hat{\Omega}_n^{S_2}(-)$  represents  $Z_2$ -homology. The fact that  $\hat{\Omega}_n^{S_p}(-)$  does not map onto  $Z_p$ -homology is interesting because in [3] Rourke has proved that the multiplicative closure of  $\hat{\Omega}_n^{S_p}(-)$  does represent  $Z_p$ -homology.

## 1. A Long Exact Sequence

Let  $U$  be an oriented geometric homology theory. The cycles of  $U$ , endowed with a well-defined orientation will be called  $U$ -manifolds. Let  $G$  be a group of permutations on  $l$  letters.

A covering  $\pi: \bar{M} \rightarrow M$  of order  $l$  is a  $(U, G)$ -covering of dimension  $n$  if

- (1)  $\bar{M}$  and  $M$  are  $U$ -manifolds of dimension  $n$  (possibly with boundary) and  $\pi$  is orientation preserving
- (2)  $G$  is a structural group for  $\pi$ .

$\pi|: \partial\bar{M} \rightarrow \partial M$  is the *boundary* of the  $(U, G)$ -covering  $\pi$ , written  $\partial\pi$ ; it is clearly a  $(U, G)$ -covering of dimension  $n - 1$ . Then there is an obvious notion of *bordism* of  $(U, G)$ -coverings and the bordism

classes can be added by disjoint union. Thus, for each integer  $n \geq 0$ , we obtain an abelian group  $U_n^G$ , the *group of bordism classes of  $(U, G)$ -coverings of dimension  $n$* .

We now give the basic definition of this paper.

A  $\hat{U}^G$ -manifold of dimension  $n$  is a pair  $P, S_P$  such that:

- (1)  $P - S_P$  is a (possibly non-compact)  $U$ -manifold of dimension  $n$
- (2)  $S_P$  is a (possibly empty) compact  $U$ -manifold of dimension  $n - 1$
- (3) there exists a homeomorphism

$$h: (N, S_P) \rightarrow (Z_n, S_P)$$

where:

$N$  is a neighbourhood of  $S_P$  in  $P$

$\pi: \bar{S}_P \rightarrow S_P$  is a  $(U, G)$ -covering

$Z_n$  is the mapping cylinder of  $\pi$ .

Moreover we require that  $h|_{S_P}$  is the identity and  $h|_{N - S_P} \rightarrow Z_n - S_P$  is an isomorphism of  $U$ -manifolds.

*Note:*  $S_P$  stands for ‘singularity of  $P$ ’. Of course if  $S_P = \emptyset$ , then  $N = Z_n = \emptyset$ .

There is an obvious notion of boundary of a  $\hat{U}^G$ -manifold of dimension  $n$ , the boundary being itself a  $\hat{U}^G$ -manifold of dimension  $n - 1$ . Therefore the *group*,  $\hat{U}_n^G$ , of bordism classes of  $\hat{U}^G$ -manifolds of dimension  $n$  is formed in the usual way.

For the sake of simplicity and when no confusion is possible we shall make use of the following notation and terminology: ‘manifold’ will stand for  $U$ -manifold, ‘covering’ for  $(U, G)$ -covering without boundary, ‘relative covering’ for covering with boundary; we shall write  $\hat{U}$  instead of  $\hat{U}^G$  and refer to a  $\hat{U}$ -manifold also as a ‘manifold with singularities’. Finally  $[ ]$  will denote ‘bordism class’ in the appropriate category  $(U, U^G$  or  $\hat{U})$ .

Now let  $X$  be a topological space. Groups  $U_n^G(X)$  and  $\hat{U}_n(X)$  can be defined in the usual way by mapping coverings and  $\hat{U}$ -manifolds of dimension  $n$  respectively into  $X$ . Therefore the elements of  $U_n^G(X)$  are bordism classes  $[\pi, \bar{f}, f]$  where  $\pi: \bar{M} \rightarrow M$  is a covering of dimension  $n$ ,  $f$  is a continuous map and  $\bar{f} = f\pi$ . The elements of  $\hat{U}_n(X)$  are bordism classes  $[P, g]$  where  $P$  is a  $\hat{U}$ -manifold of dimension  $n$  and  $g$  is a continuous map  $P \rightarrow X$ . In the following, for the sake of clarity of exposition and when no confusion is possible, we shall omit to mention the various maps into  $X$ .

We want to set up homomorphisms connecting the groups  $\{U_n(X)\}$ ,  $\{U_n^G(X)\}$  and  $\{\hat{U}_n(X)\}$ .

The *singularity map*  $\sigma: \hat{U}_n(X) \rightarrow U_{n-1}^G(X)$  assigns to each  $\hat{U}$ -manifold  $P$ ,  $S_P$  the covering  $\pi: \bar{S}_P \rightarrow S_P$  mentioned in the definition of  $P$ ;  $\pi$  is mapped into  $X$  by restricting the given map  $P \rightarrow X$ . It is immediately seen that this correspondence yields a well-defined homomorphism of abelian groups.

The *forgetful map*  $\phi: U_n^G(X) \rightarrow U_n(X)$  is defined by  $\phi(\pi) = \bar{M}$  for each covering  $\pi: \bar{M} \rightarrow M$  mapped into  $X$ . The *natural map*  $\psi: U_n(X) \rightarrow \hat{U}_n(X)$  is defined to be the identity on representatives.

The *augmentation*  $\epsilon_n: U_n^G(X) \rightarrow U_n(X)$  is defined by  $\epsilon_n[\pi, \bar{f}, f] = [M, f]$  for  $\pi: \bar{M} \rightarrow M$ .

We set  $\bar{U}_n^G(X) = \text{Ker } \epsilon_n$ .

1.1 THEOREM: *The long sequence*

$$\cdots \rightarrow U_n(X) \xrightarrow{\psi} \hat{U}_n(X) \xrightarrow{\sigma} U_{n-1}^G(X) \xrightarrow{\phi} U_{n-1}(X) \rightarrow \cdots \quad (\text{L})$$

is exact.

PROOF: As the space  $X$  does not play any essential role in the proof, we shall assume, for the sake of simplicity,  $X = \text{point}$ .

- (1)  $\sigma\psi = 0$ . This is obvious since, for each manifold  $M$ ,  $\psi(M)$  does not have any singularities.
- (2)  $\phi\sigma = 0$ . If  $P, S_P$  is a  $\hat{U}$ -manifold, then  $\phi\sigma(P)$  bounds a manifold isomorphic to the closure of  $P - N$  (see the definition of  $\hat{U}$ -manifold).
- (3)  $\psi\phi = 0$ . Let  $\pi: \bar{M} \rightarrow M$  be a covering. Then  $\psi\phi(\pi)$  is the boundary of the  $\hat{U}$ -manifold  $Z_\pi, M$ .
- (4)  $\text{Ker } \sigma \subseteq \text{Im } \psi$ . Let  $[P] \in \text{Ker } \sigma$ . Then the covering  $\pi: \bar{S}_P \rightarrow S_P$  is a boundary, i.e. there exists a relative covering  $\pi': \bar{W} \rightarrow W$  such that  $\partial\pi' = \pi$ . Form  $P \times [0, 1]$  and glue the mapping cylinder  $Z_{\pi'}$ , to  $P \times 1$  by means of the isomorphism  $h$  given in the definition of  $\hat{U}$ -manifold. The resulting object provides a bordism between  $P \cong P \cong 0$  and a  $\hat{U}$ -manifold without singularities are required.
- (5)  $\text{Ker } \phi \subseteq \text{Im } \sigma$ . If  $[\pi: \bar{M} \rightarrow M]$  is in  $\text{Ker } \phi$ , then there exists a manifold  $Q$  with  $\bar{M} = \partial Q$ . The  $\hat{U}$ -manifold  $P$  obtained by glueing  $Q$  to  $Z_\pi$  along  $\bar{M}$  is such that  $\sigma(P) = \pi$ .
- (6)  $\text{Ker } \psi \subseteq \text{Im } \phi$ . If  $[V] \in \text{Ker } \psi$ , there exists a manifold with singularities  $P$  such that  $\partial P \cong V$ . Then the manifold  $Q = \text{closure of } P - N$  provides a bordism between  $V$  and a covering.

Thus exactness is proved  $\square$

Now suppose that  $G$  is the trivial group 1: a  $(U, 1)$ -covering is, up to

isomorphism, a product  $M \times \underline{l} \xrightarrow{\text{projection}} M$  with  $\underline{l} =$  discrete set  $\{1, 2, \dots, l\}$ . Then the augmentation  $\epsilon_n: U_n^1(X) \rightarrow U_n(X)$ , which associates to each trivial covering its base, defines an isomorphism of abelian groups. Moreover, in the case  $G = 1$ , a manifold with singularities is nothing else than a ‘manifold with  $Z_l$ -coefficients’ as defined in [4] and [1] and the group  $\hat{U}_n^1(X)$  coincides, by definition, with  $U_n(X; Z_l)$ .

The following proposition, the proof of which is trivial, explains how the sequence (L) may be viewed as a generalisation of the Bockstein sequence of  $U_n(X; Z_l)$  (see [4] and [1]).

1.2 PROPOSITION: *There is an isomorphism of long exact sequences given by:*

$$\begin{array}{ccccccccccc}
 (L) \cdots & \rightarrow & U_n(X) & \xrightarrow{\psi} & \hat{U}_n(X) & \xrightarrow{\sigma} & U_{n-1}^1(X) & \xrightarrow{\phi} & U_{n-1}(X) & \rightarrow & \cdots \\
 & & \downarrow \text{id} & & \downarrow \text{id} & & \downarrow \epsilon_{n-1} & & \downarrow \text{id} & & \\
 \text{Bockstein } \cdots & \rightarrow & U_n(X) & \xrightarrow{\psi'} & U_n(X; Z_l) & \xrightarrow{\sigma'} & U_{n-1}(X) & \xrightarrow{l_*} & U_{n-1}(X) & \rightarrow & \cdots
 \end{array}$$

where  $\psi'$  is the natural map,  $\sigma'$  is restriction to singularities and  $l_*$  is ‘multiplication by  $l$ ’  $\square$

## 2. A Short Exact Sequence

From now on we shall restrict our attention to those  $(U, G)$ -coverings  $\pi: \bar{M} \rightarrow M$  which satisfy a further condition, namely:

(3)  $\pi$  is a principal covering (i.e. a fibre bundle with fibre  $G$  and action given by multiplication).

Of course in this case  $l = |G| =$  order of  $G$ .

The definitions of the bordism groups of principal  $(U, G)$ -coverings and the associated bordism groups of manifolds with singularities are formulated in exactly the same way as their analogues in the unrestricted case and we continue to use the notation of §1.

Clearly propositions 1.1 and 1.2 are not affected by condition (3).

Let  $BG$  be a classifying space for the group  $G$ . The following theorem is important because it interprets the group  $U_n^G$  as the group  $U_n(BG)$ . Its proof, in the case where  $X =$  point and  $U$  is oriented differentiable bordism, can be found in [2]. The extension to the general case is trivial.

2.1 THEOREM: *Let  $[\pi, \bar{f}, f] \in U_n^G(X)$  and  $\varphi: M \rightarrow BG$  be a classify-*

ing map for  $\pi$ . The correspondence  $[\pi, \bar{f}, f] \rightsquigarrow [M, g]$ , where  $g: M \rightarrow X \times BG$  is given by  $g(z) = (f(z), \varphi(z))$ , defines an isomorphism between  $U_n^G(X)$  and  $U_n(X \times BG)$   $\square$

It follows from 2.1 that commutativity holds in the diagram

$$\begin{array}{ccc} U_n^G(X) & & \\ \parallel & \searrow^{\epsilon_n} & \\ U_n(X \times BG) & \rightarrow & \bar{U}_n(X) \end{array}$$

where the vertical arrow is the isomorphism of 2.1 and the horizontal arrow is induced by the projection  $X \times BG \rightarrow X$ . From this commutativity and from exactness, applied to the pair  $(X \times BG, X)$ , it follows that  $\bar{U}_n^G(X)$  can be identified with  $U_n(X \times BG, X)$ .

From now on  $X$  will be assumed to be a CW complex.

NOTATION: For each  $n$ ,  $U_n$  will stand for  $U_n$  (point).

2.2 PROPOSITION: For each element  $x \in \bar{U}_n^G(X)$  there exists a power  $l^!$  with  $l^!x = 0$  (recall that  $l = |G|$ ).

PROOF: Consider the spectral sequence  $\{E_{p,q}^r\}$  associated with  $U_n(X \times BG, X)$ . In this spectral sequence  $E_{p,q}^2 = H_p(X \times BG, X; U_q)$ , where  $H(-)$  denotes singular homology, and there is a filtration

$$0 \subseteq J_{0,n} \subseteq J_{1,n-1} \subseteq \cdots \subseteq J_{p,q} \subseteq \cdots \subseteq J_{n,0} = \bar{U}_n^G(X).$$

By the universal-coefficient theorem

$$\begin{aligned} H_p(X \times BG, X; U_q) &\cong H_p(X \times BG, X) \otimes U_q \oplus \\ &+ \text{Tor}(H_{p-1}(X \times BG, X), U_q). \end{aligned}$$

By the Kunneth formula

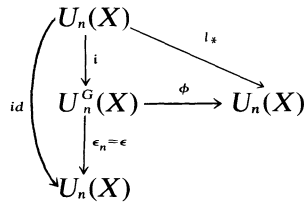
$$H_p(X \times BG, X) \cong \sum_{i=1}^p H_{p-i}(X) \otimes H_i(BG) \oplus \sum_{i=0}^{p-1} \text{Tor}(H_{p-i-1}(X), H_i(BG)).$$

Now it is known that, if  $z \in \check{H}(BG, \mathcal{A})$ ,  $\mathcal{A}$  any abelian group, then  $lz = 0$ . From this and the Kunneth formula follows that each element  $y \in H_p(X \times BG, X)$  is such that  $ly = 0$ . Similarly for  $H_{p-1}(X \times BG, X)$ . Therefore the universal-coefficient formula above ensures that for every element  $u \in H_p(X \times BG, X, U_q)$  one has  $lu = 0$ . We deduce that

$|E_{p,q}^2|$  divides some power of  $l$ . Because  $E_{p,q}^{r+1} = H(E_{p,q}^r)$  we obtain inductively that  $|E_{p,q}^\infty|$  divides a power of  $l$ . From the relationship  $E_{p,q}^\infty \cong J_{p,q}/J_{p-1,q+1}$  we deduce that also  $|J_{p,q}|$  divides a power of  $l$ . Therefore induction on  $p$  brings us to the conclusion that  $|\bar{U}_n^G(X)|$  is a divisor of some power  $l^t$ . In particular, for each  $x \in \bar{U}_n^G(X)$  one has  $l^t x = 0$ , which proves the theorem  $\square$

2.3 THEOREM: *If  $U_n(X)$  has no  $l$ -torsion, we have  $\bar{U}_n^G(X) = \text{Ker } \phi$ , where  $\phi: U_n^G(X) \rightarrow U_n(X)$  is the forgetful map.*

PROOF: Consider the commutative diagram



where  $i[M] = [G \times M \xrightarrow{\text{projection}} M]$ .

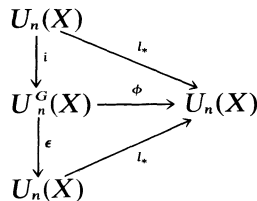
We must show that  $\text{Ker } \phi = \text{Ker } \epsilon$ .

$\text{Ker } \epsilon \subseteq \text{Ker } \phi$ . Let  $x \in \text{Ker } \epsilon$ . By theorem 2.2 there exists  $t > 0$  with  $l^t x = 0$ . Since  $U_n(X)$  has no  $l$ -torsion,  $\phi(x) = 0$  necessarily.

$\text{Ker } \phi \subseteq \text{Ker } \epsilon$ . Let  $x \in \text{Ker } \phi$ . Because  $\epsilon i = \text{id}$ ,  $x$  can be written uniquely as  $y + z$  with  $y = i(y')$ ,  $y' \in U_n(X)$ , and  $z \in \text{Ker } \epsilon$ . We want to show that  $y = 0$ . Suppose  $y \neq 0$ ; then  $y' \neq 0$  and  $\phi(y) = \phi i(y') = l_*(y') \neq 0$  because  $l_*$  is injective by the hypothesis that  $U_n(X)$  is free of  $l$ -torsion. On the other hand  $z \in \text{Ker } \epsilon \subseteq \text{Ker } \phi$  implies  $0 = \phi(x) = \phi(y + z) = \phi(y) + \phi(z) = \phi(y)$  which is a contradiction. It follows that  $y = 0$  and the theorem is proved  $\square$

2.4 THEOREM: *If  $U_n(X)$  has no  $l$ -torsion, one has  $[\bar{M}, \bar{f}] = l[M, f]$ , as elements of  $U_n(X)$ , for each  $[\pi, \bar{f}, f] \in U_n^G(X)$ .*

PROOF: Consider the diagram





In order to prove the theorem we need to show that the lower triangle is commutative.

Let  $y \in U_n^G(X)$ ;  $y$  can be written uniquely as  $u + v$  with  $u = i(u')$ ,  $u' \in U_n(X)$  and  $v \in \text{Ker } \epsilon$ . Then we have

$$l_*\epsilon(y) = l_*\epsilon(u + v) = l_*\epsilon(u) = l_*\epsilon i(u') = l_*u'.$$

On the other hand,  $v \in \text{Ker } \phi$  by theorem 2.3 and therefore

$$\phi(y) = \phi(u) + \phi(v) = \phi(u) = \phi i(u') = l_*u'.$$

Thus we have proved that  $\phi(y) = l_*\epsilon(y)$  as required  $\square$

Note that the proof of 2.4 uses only the inclusion  $\text{ker } \epsilon \subseteq \text{Ker } \phi$ .

We now come to the main theorem of this section.

**2.5 THEOREM:** *If  $U_n(X)$  has no  $l$ -torsion, there is a short exact sequence:*

$$0 \rightarrow U_n(X; Z_l) \xrightarrow{j} \hat{U}_n(X) \xrightarrow{\sigma} \bar{U}_{n-1}^G(X) \rightarrow 0$$

where  $j$  is the identity on representatives.

**PROOF:** Consider the map of exact sequences

$$\begin{array}{ccccccccccc} \cdots & \rightarrow & U_n(X) & \xrightarrow{l_*} & U_n(X) & \xrightarrow{\psi'} & U_n(X; Z_l) & \xrightarrow{\sigma'} & U_{n-1}(X) & \xrightarrow{l_*} & U_{n-1}(X) & \xrightarrow{\psi'} & \cdots \\ & & \downarrow i & & \parallel id & & \downarrow j & & \downarrow i & & \parallel id & & \\ \cdots & \rightarrow & U_n^G(X) & \xrightarrow{\phi} & U_n(X) & \xrightarrow{\psi} & \hat{U}_n(X) & \xrightarrow{\sigma} & U_{n-1}^G(X) & \xrightarrow{\phi} & U_{n-1}(X) & \rightarrow & \cdots \end{array}$$

By 2.3 we have  $\text{Im } \sigma = \text{Ker } \phi = \bar{U}_{n-1}^G(X)$ .

Therefore, in order to establish the theorem, we only need to prove that  $j$  is an isomorphism of  $U_n(X; Z_l)$  onto  $\text{Ker } \sigma$ .

(1)  $j(U_n(X; Z_l)) \subseteq \text{Ker } \sigma$ . Because  $l_*$  is injective,  $\psi'$  is onto and therefore, given  $x \in U_n(X; Z_l)$ , there exists  $y \in U_n(X)$  with  $\psi'(y) = x$ . It follows that

$$\sigma j(x) = \sigma j \psi'(y) = \psi \sigma(y) = 0 \quad \text{so that } j(x) \in \text{Ker } \sigma.$$

(2)  $\text{Ker } \sigma \subseteq j(U_n(X; Z_l))$ . Given  $x \in \text{Ker } \sigma$ , there exists  $y \in U_n(X)$  with  $\psi(y) = x$ ; then  $x' = \psi'(y)$  is such that  $j(x') = x$ .

(3)  $j$  is injective. Let  $x \in \text{Ker } j$ . Choose  $y \in U_n(X)$  such that  $\psi'(y) = x$ . Because  $\psi(y) = 0$  there exists  $z \in U_n^G(X)$  with  $\phi(z) = y$ . On the other hand theorem 2.4 implies that  $\text{Im } \phi = \text{Im } l_*$ ; therefore there

exists  $w \in U_n(X)$  such that  $l_*(w) = y$ . But then

$$x = \psi'(y) = \psi'l_*(w) = 0$$

and everything is proved  $\square$

2.6 COROLLARY (Generalised Rohlin sequence): *If  $U_n(X)$  has no  $l$ -torsion, there is an exact sequence*

$$U_n(X) \xrightarrow{l^*} U_n(X) \xrightarrow{\psi} \hat{U}_n(X).$$

PROOF: The result follows from theorem 1.1, the Bockstein sequence, theorem 2.5 and the commutative diagram

$$\begin{array}{ccccc} U_n(X) & \xrightarrow{l^*} & U_n(X) & \xrightarrow{\psi} & \hat{U}_n(X) \\ & & \searrow \psi' & & \nearrow j \\ & & & & U_n(X; Z_1). \end{array}$$

$\square$

2.7 COROLLARY: *Suppose that the theory  $U(-)$  is ‘oriented differentiable bordism  $\Omega(-)$ ’;  $l = |G|$  is an odd number,  $X$  is a space whose integral homology is finitely generated and has no odd torsion. Then there is an exact sequence*

$$0 \rightarrow \Omega_n(X; Z_l) \xrightarrow{j} \hat{\Omega}_n(X) \xrightarrow{\sigma} \bar{\Omega}_{n-1}^G(X) \rightarrow 0$$

and an exact Rohlin sequence

$$\Omega_n(X) \xrightarrow{l^*} \Omega_n(X) \xrightarrow{\psi} \hat{\Omega}_n(X).$$

PROOF: By Conner and Floyd [2] (15.2)

$$\Omega_n(X) \cong \sum_{p+q=n} H_p(X; \Omega_q).$$

Therefore, since  $\Omega_q$  has no odd torsion,  $\Omega_n(X)$  has no  $l$ -torsion. The result follows from 2.5 and 2.6  $\square$

Before ending this section we give a proposition containing two sufficient conditions for a space  $X$  to have an  $l$ -torsion-free  $U_n(X)$ , assuming that  $U_n(\text{point})$  is  $l$ -torsion-free.

2.8 PROPOSITION: *Suppose  $U_n$  is free of  $l$ -torsion.*

- (1) If  $\hat{H}(X; Z)$  is a torsion group and the torsion is prime to  $l$ , then  $U_n(X)$  is also such a group.
- (2) If  $H(X; Z)$  has no torsion and the natural map  $e: U_n(X) \rightarrow H(X; U_0)$  is onto, then  $U_n(X)$  has no  $l$ -torsion.

PROOF: Consider the spectral sequence

$$E_{p,q}^2 = H_p(X; U_q) \Rightarrow U_{p+q}(X)$$

In the case (1) the universal-coefficient formula implies that  $E_{p,q}^2$  is a torsion group, the torsion being prime to  $l$ . Therefore the  $E^\infty$ -term and the limit are also such groups.

In case (2) the universal-coefficient formula implies that  $E_{p,q}^2 = H_p(X) \otimes U_q$ . This, together with the hypothesis on  $e$ , implies by a standard argument that the spectral sequence collapses.

The result then follows  $\square$

### 3. Brief Review of Geometric Cohomology Theory

In this section we briefly recall the theory of mock bundles developed in [1].

These objects give a geometric picture, based essentially on transversality constructions, of the cohomological theory  $U^*(-)$  dual to the theory  $U_*(-)$  which we are considering. We shall make much use of the language of mock bundles in the following section.

Here we simply outline the main features of mock bundles and the reader is referred to [1] for the details.

Let  $K$  be an oriented ball complex.

A  $(U, q)$ -mock bundle  $\xi^q/K$  with base  $K$  and total space  $E_\xi$  consists of a  $PL$  projection  $p_\xi: E_\xi \rightarrow |K|$  such that, for each  $\sigma \in K$ ,  $p_\xi^{-1}(\sigma)$  is a  $U$ -manifold of dimension  $q + \dim \sigma$  with boundary  $p^{-1}(\dot{\sigma})$ ;  $\xi(\sigma) = p^{-1}(\sigma)$  is the *block* over  $\sigma$ . We also require  $\epsilon(\xi(\tau), \xi(\sigma)) = \epsilon(\tau, \sigma)$  where  $\tau$  is a codimension-one face of  $\sigma$  and  $\epsilon(-, -)$  denotes the usual incidence number.

If  $U(-)$  is not an oriented theory (i.e. the cycles of  $U$  are not oriented) then the reference to the incidence numbers is to be omitted.

The empty set is regarded as a manifold of any dimension; thus  $q$  may be negative.

Mock bundles  $\xi, \eta/K$  are *isomorphic*, written  $\xi \cong \eta$ , if there exists a

PL isomorphism  $h: E_\xi \rightarrow E_\eta$  which restricts to an isomorphism of  $U$ -manifolds  $h: \xi(\sigma) \rightarrow \eta(\sigma)$  for each  $\sigma \in K$ .

Given  $\xi/K$  and  $L \subset K$ , the restriction  $\xi|L$  is defined by  $E(\xi|L) = p_\xi^{-1}(L)$  and  $p_{(\xi|L)} = p_\xi|E(\xi|L)$ .

A  $(U, q)$ -mock bundle over  $(K, L)$  is a  $(U, q)$ -mock bundle over  $K$  which is empty over  $L$ .

$\xi_0/(K, L)$  is cobordant to  $\xi_1/(K, L)$  if there exists a  $(U, q)$ -mock bundle  $\eta/(K \times I, L \times I)$  such that  $\eta|K \times \{0\} \cong \xi_0$  and  $\eta|K \times \{1\}$  is obtained from  $\xi_1$  by reversing the orientation in each block and each ball of the base. If  $U(-)$  is not oriented,  $\eta|K \times \{1\} = \xi_1$ . Cobordism is an equivalence relation and  $[\xi]$  stands for "equivalence class of  $\xi$ ".

Define  $T^q(K, L)$  as the set of cobordism classes of  $(U, q)$ -mock bundles over  $K, L$ . It is easy to see  $T^q(K, L)$  is an abelian group under disjoint union. The zero-element is the empty mock bundle.

**3.1 THEOREM:** *Let  $M$  be a PL manifold of dimension  $m$ , which is oriented with respect to the theory  $U(-)$ ;  $\xi/K$  a  $(U, q)$ -mock bundle over a triangulation of  $M$ . Then  $E_\xi$  is a  $U$ -manifold of dimension  $(m + q)$  with boundary  $p_\xi^{-1}(\partial M)$ .*

The proof of this theorem can be found in [1].  $\square$

In the following, for the sake of simplicity, we shall be deliberately careless about orientation in most of the definitions.

Let  $\xi^a/K'$  be a  $(U, q)$ -mock bundle and  $K'$  a subdivision of  $K$ . Then, by 3.1,  $p_\xi: E_\xi \rightarrow |K|$  is a  $(U, q)$ -mock bundle called the *amalgamation* of  $\xi$  and written  $\text{am}(\xi)$ .

One has an abelian-group homomorphism

$$\text{am}: T^q(K') \rightarrow T^q(K).$$

**3.2 PROPOSITION:** *Let  $f: |K| \rightarrow |L|$  be a simplicial map and  $E_\xi \xrightarrow{p_\xi} |L|$  a  $(U, q)$ -mock bundle. Then the pull-back  $f^*(\xi)$*

$$\begin{array}{ccc} E(f^*(\xi)) & \xrightarrow{\quad\quad} & E_\xi \\ \downarrow \text{dashed} & & \downarrow p_\xi \\ |K| & \xrightarrow{\quad f \quad} & |L| \end{array}$$

*is also a  $(U, q)$ -mock bundle.*

**3.3 SUBDIVISION THEOREM:** *Given  $\xi/K$  and  $K'$  a subdivision of  $K$ , there exists  $\xi'/K'$  such that  $\text{am}(\xi') \cong \xi$ .*

The proof can be found in [1], II, §2  $\square$

There is also a relative version of the subdivision theorem, which gives a homomorphism

$$sd: T^q(K) \rightarrow T^q(K').$$

### 3.4 THEOREM:

- (a) *am and sd are inverse isomorphisms*
- (b) *the isomorphism-type of  $T^q(K)$  depends only on the polyhedron  $|K|$ .  $\square$*

The extension of the above theorems to the case of pairs  $K, L$  is straightforward.

We now look at the functoriality of  $T^q(-)$ .

Let  $f: P \rightarrow Q$  be a continuous map. We want to define  $f^*: T^q(Q) \rightarrow T^q(P)$ . Using the  $PL$  approximation theorem we assume that  $f$  is a  $PL$  map (up to a homotopy). Let  $K, L$  be triangulations of  $P, Q$  respectively with  $f: |K| \rightarrow |L|$  simplicial and let  $\xi/L$  be a  $(U, q)$ -mock bundle. We set  $f^*[\xi] = [f^*(\xi)]$  (see 3.2). It is not difficult to check that  $f^*$  is well-defined, functorial, and depends only on the homotopy class of  $f$ .

Also maps between pairs of polyhedra induce well-defined homomorphisms in the same fashion so that we obtain a homotopy functor  $T^q(-; -)$  defined on the category of pairs of polyhedra.

Let  $\xi^q/K$  and  $\eta^r/E_\xi$  be  $(U, q)$ -mock bundles such that the blocks of  $\xi$  are  $PL$  manifolds oriented with respect to the theory  $U$ . Subdivide  $\eta$  so as to obtain that the blocks of  $\xi$  are subcomplexes. Then, given  $\sigma \in K$ , we have that  $E(\eta|_{\xi(\sigma)})$  is a  $U$ -manifold by 3.1. Then there is a  $(U, q+r)$ -mock bundle

$$E_\eta \xrightarrow{p_\xi \cdot p_\eta} |K|$$

This gives a *transgression-homomorphism*

$$(p_\xi)_! T^r(E_\xi) \rightarrow T^{q+r}(|K|).$$

Let  $M^m$  be a closed  $PL$  manifold oriented with respect to the theory  $U(-)$ ;  $\xi^q/M$  a  $(U, q)$ -mock bundle.

Then  $p_\xi: E_\xi \rightarrow M$  represents an element of  $U_{m+q}(M)$  by 3.1. Thus we have a function

$$D: T^q(M) \rightarrow T_{m+q}(M)$$

$$D[\xi] = [p_\xi].$$

**3.5 POINCARÉ DUALITY THEOREM:** *D is an isomorphism.*

For the proof see [1], II, §3.

**3.6 ALEXANDER–WHITEHEAD DUALITY THEOREM:** *Let  $M^m$  be a PL manifold oriented with respect to the theory  $U$ ;  $X \subset M$  a subpolyhedron such that  $M$  is obtained from  $X$  by attaching one  $m$ -cell  $e^m$ . Then there is an isomorphism*

$$\varphi: T^q(X) \cong T_{m+q}(M, \hat{e})$$

where  $\hat{e}$  is the center of  $e^m$ .

**SKETCH OF PROOF:** We shall only define the homomorphism  $\varphi$ . Let  $R \subset M$  be the closure of the complement of an oriented  $m$ -disk  $B$  centered at  $\hat{e}$ . We define  $\varphi$  to be the composition

$$\begin{array}{ccccc}
 T^q(X) & \xleftarrow{i_*} & T^q(R) & \xrightarrow{a} & T_{m+q}(R, \partial B) \\
 & & & & \downarrow i_* \\
 & & & & T_{m+q}(M, B) \\
 & \searrow \varphi & & & \Downarrow \text{homotopy} \\
 & & & & T_{m+q}(M, \hat{e})
 \end{array}$$

where the  $i$ 's are the injections and  $a$  is the map that regards a  $(U, q)$ -mock bundle on  $R$  as a relative  $U$ -manifold in  $R$ ,  $\partial B$ .

Observe that  $a$  is the relative version of the Poincaré-duality map. It is easy to check that  $\varphi$  is an isomorphism  $\square$

**TERMINOLOGY:** In the following section we shall refer to  $\varphi$  as the process of ‘expanding’ a mock bundle on  $X$  over the regular neighbourhood  $R$ .

Finally, the duality isomorphism allows us to state the following

**3.7 THEOREM:** *The functor  $\{T^q(-)\}_q$  is naturally equivalent to  $\{U^{-q}(-)\}_q$  where  $U^*(-)$  is the cohomological theory dual to  $U_*(-)$   $\square$*

#### 4. The $Z_p$ -case

In this section we examine closely the theory of  $\hat{U}^{z_p}$ -manifolds,  $p$  an odd prime.

We recall that a classifying space for  $Z_p$  may be constructed as

follows. Given an integer  $A > 0$ , consider the sphere

$$S^{2A-1} = \{(z_0, \dots, z_{A-1}) \in \mathbb{C}^A \mid \sum |z_i|^2 = 1\}$$

where the  $z_i$ 's are complex coordinates, and define  $T: S^{2A-1} \rightarrow S^{2A-1}$  by

$$T(z_0, \dots, z_{A-1}) = (e^{2\pi i/p} z_0, \dots, e^{2\pi i/p} z_{A-1})$$

Then  $T$  determines a free action of  $Z_p$  on  $S^{2A-1}$ ; the orbit-space of this action is the lens space of dimension  $2A - 1$  and is denoted by  $L_{2A-1} = S^{2A-1}/T$ .

The orbit of  $(z_0, \dots, z_{A-1})$  will be written  $[z_0, \dots, z_{A-1}]$ .

Now  $Z_p$  acts on the union  $E(Z_p) = \bigcup_{A \geq 1} S^{2A-1}$  by means of  $T$ ; this action of  $T$  on  $E(Z_p)$  makes  $E(Z_p)$  a universal space for  $Z_p$  and  $BZ_p = E(Z_p)/T$  a classifying space for  $Z_p$ . Moreover  $BZ_p$  is a CW complex whose  $2A - 1$  skeleton is the lens space  $L_{2A-1}$ .

Let  $Q_{2A}$  be the CW complex obtained from  $L_{2A-1}$  by attaching one  $2A$ -cell  $e^{2A}$  via the projection-map  $q: S^{2A-1} \rightarrow L_{2A-1}$ . It is known that  $L_{2A+1}$  is obtained from  $Q_{2A}$  by adjoining one  $(2A + 1)$ -cell  $e^{2A+1}$ .

From now on we shall make the assumption that the lens spaces  $L_{2A-1}$  are endowed with a fixed orientation with respect to the theory  $U$ .

**4.1 THEOREM:** *Fix integers  $n$ ,  $A \geq 0$  with  $n < 2A$ . There is an isomorphism between  $\hat{U}_n = \hat{U}_n(\text{point})$  and  $U^{2A-n}(Q_{2A})$ .*

**PROOF:** Let  $P, S_p$  be a  $\hat{U}$ -manifold of dimension  $n$ . Let  $D^{2A} \subset e^{2A}$  be a disk centered at the origin of  $e^{2A}$ . Then  $Q_{2A} - \mathring{D}^{2A}$  ( $\mathring{D}$  = interior of  $D$ ) can be identified to the mapping cylinder of the projection  $q: S^{2A-1} \rightarrow L_{2A-1}$ . Therefore, since  $n < 2A$ , there is a (classifying) map

$$\gamma: (N, S_p) \rightarrow (Q_{2A} - \mathring{D}^{2A}, L_{2A-1})$$

preserving the mapping-cylinder structures on  $N$  and  $Q_{2A} - \mathring{D}^{2A}$ . Here  $N$  is the (regular) neighbourhood of  $S_p$  mentioned in the definition of  $P$ .

Now  $\gamma$  can be extended to a map of  $P - N$  into  $\mathring{D}^{2A}$  using the connectivity of  $S^{2A-1}$ .

This construction gives a map  $\theta: P, S_p \rightarrow Q_{2A}, L_{2A-1}$  which is unique up to homotopy. Moreover, by choosing suitable ball decompositions of  $P$  and  $Q_{2A}$ ,  $\theta$  becomes an  $(n - 2A)$ -mock bundle on  $Q_{2A}$ .

It is not difficult to check that the correspondence  $[P] \rightsquigarrow [\theta]$  defines a

homomorphism

$$\alpha: \hat{U}_n \rightarrow U^{2A-n}(Q_{2A}).$$

That  $\alpha$  is an isomorphism follows immediately from the fact that, because  $L_{2A-1}$  is oriented, an  $(n-2a)$ -mock bundle on  $Q_{2A}$  can be amalgamated to give a  $\hat{U}$ -manifold of dimension  $n$ .  $\square$

4.2 THEOREM: *There is an isomorphism  $\lambda$  between  $\hat{U}_n$  and the reduced bordism group  $\tilde{U}_{n+1}(BZ_p)$ .*

PROOF: Let  $A$  be such that  $n < 2A$ . By 4.1  $\hat{U}_n$  can be identified with  $U^{2A-n}(Q_{2A})$ . Now we apply the duality theorem 3.6 with  $M = L_{2A+1}$ ,  $X = Q_{2A}$ ,  $R = 'L_{2A+1}$  minus an oriented disk centered at the origin of  $e^{2A+1}'$  and we obtain an isomorphism  $\lambda_{2A+1}: \hat{U}_n \rightarrow \tilde{U}_{n+1}(L_{2A+1})$ . Because  $n < 2A$  the natural inclusion  $j^{2A+1}: L_{2A+1} \subset BZ_p$  induces an isomorphism  $j_*^{2A+1}: \tilde{U}_{n+1}(L_{2A+1}) \rightarrow \tilde{U}_{n+1}(BZ_p)$ .

We now prove that the composition  $j_*^{2A+1}\lambda_A$  does not depend on  $A$ . In order to clarify the geometry of the following argument we first remark that the proof of 3.6 shows that the isomorphism  $\lambda_{2A+1}: U_n \rightarrow \tilde{U}_{n+1}(L_{2A+1})$  is given by expanding a  $\hat{U}$ -manifold, thought of as a mock bundle on  $Q_{2A}$ , over the regular neighbourhood  $R$ . The inverse isomorphism  $\lambda_{2A+1}^{-1}$  is given by first regarding an  $(n+1)$ -manifold  $V$  in  $L_{2A+1}$  as a mock bundle on  $L_{2A+1}$  (this is essentially Poincaré Duality) and then taking its restriction to  $Q_{2A}$  (this is transversality with respect to  $Q_{2A}$ ).

In  $S^{2A+1} = \{(z_0, \dots, z_A) \mid \sum |z_i|^2 = 1\}$  consider the subspace  $\bar{Q}_{2A} = \{(z_0, \dots, z_{A-1}, t e^{2\pi i k/p}) \mid t \in R \text{ and } |z_0|^2 + \dots + |z_{A-1}|^2 + t^2 = 1; k = 0, \dots, p-1\}$ .

Clearly  $\bar{Q}_{2A}$  is invariant under the  $Z_p$ -action and its orbit space is  $Q_{2A}$ . Let  $\mu_{2A}: Q_{2A} \subset L_{2A+1}$  be the inclusion induced by  $\bar{Q}_{2A} \subset S^{2A+1}$ .

Let  $\mu_{2A+1}: L_{2A+1} \subset Q_{2A+2}$  be the inclusion

$$[z_0, \dots, z_A] \rightsquigarrow [z_0, \dots, z_A, 0] \in Q_{2A+2}.$$

Let  $i_{2A}: Q_{2A} \hookrightarrow Q_{2A+2}$  be the inclusion

$$[z_0, \dots, z_{A-1}, t e^{2\pi i k/p}] \rightsquigarrow [0, z_0, \dots, z_{A-1}, t e^{2\pi i k/p}].$$

It is easy to see that  $i_{2A}$  can be regarded as the projection of a mock bundle.

Finally, let  $\nu^{2A+1}: L_{2A+1} \hookrightarrow L_{2A+3}$  be the inclusion  $[z_0, \dots, z_A] \rightsquigarrow$



$[z_0, \dots, z_A, 0]$  and  $\bar{\nu}^{2A+1}: L_{2A+1} \hookrightarrow L_{2A+3}$  the inclusion

$$[z_0, \dots, z_A] \rightsquigarrow [0, z_0, \dots, z_A].$$

Now consider the commutative diagram

$$\begin{array}{ccccc} Q_{2A} & \xrightarrow{\mu_{2A}^*} & L_{2A+1} & & \\ & & \downarrow \mu_{2A+1} & \searrow \bar{\nu}^{2A+1} & \\ & & Q_{2A+2} & \xrightarrow{\mu_{2A+2}^*} & L_{2A+3} & \xrightarrow{\mu_{2A+3}^*} & BZ_p \\ & & & & \uparrow j^{2A+1} & & \end{array}$$

Let  $[P] \in \hat{U}_n$  and  $[\theta]$  be the corresponding element of  $U^{2A-n}(Q_{2A})$  (theorem 4.1). From definitions and commutativity we have

$$j_*^{2A+1} \lambda_{2A+1}[P] = j_*^{2A+3} \nu_*^{2A+1} D\mu_{2A}^*[\theta]$$

where  $D$  stands for Poincaré dual.

Consider also the commutative diagram

$$\begin{array}{ccccccc} Q_{2A} & \xrightarrow{i_{2A}^*} & Q_{2A+2} & \xrightarrow{\mu_{2A+2}^*} & L_{2A+3} & \xrightarrow{j^{2A+3}} & BZ_p \\ & & & & \uparrow \bar{\nu}^{2A+1} & & \\ & & & & L_{2A+1} & & \end{array}$$

We have

$$\begin{aligned} j_*^{2A+3} \lambda_{2A+3}[P] &= j_*^{2A+3} D\mu_{2A+2}^*(i_{2A})_*[\theta] \\ &= j_*^{2A+3} \bar{\nu}_*^{2A+1} D\mu_{2A}^*[\theta]. \end{aligned}$$

Because  $\nu^{2A+1}$  and  $\bar{\nu}^{2A+1}$  are clearly homotopic maps, we have proved that

$$j_*^{2A+1} \lambda_{2A+1} = j_*^{2A+3} \lambda_{2A+3}.$$

Thus we obtain a well-defined isomorphism  $\lambda: \hat{U}_n \rightarrow \tilde{U}_{n+1}(BZ_p)$  for each  $n$   $\square$

*Note added in proof.* If the theory  $U$  is multiplicative then  $\lambda$  extends to an isomorphism of  $U_*$  modules  $\lambda_*: \hat{U}_* \cong \tilde{U}_*(BZ_p)$  (here  $U_* = \sum_n U_n$ ;  $\hat{U}_* = \sum_n \hat{U}_n$ ;  $\tilde{U}_*(BZ_p) = \sum_n \tilde{U}_n(BZ_p)$ ).

The above theorem reduces the calculation of  $\hat{U}_n^{Z_p}$  to that of  $\tilde{U}_{n+1}(BZ_p)$ . For instance, in the case where  $U(-)$  is oriented differentiable bordism  $\Omega(-)$  this calculation has been performed by Conner and Floyd and we refer the reader to [2] (36.5) for a complete statement of the results.

Theorem 4.2 deals with the case  $X = \text{point}$ .

However one can prove the corresponding result for a general CW complex  $X$  (theorem 4.4. below) using the direct construction of  $\lambda$  outlined in the proof of 4.2.

We leave the details to the reader.

**4.4 THEOREM:** *There is an isomorphism*

$$\lambda_X: \hat{U}_n(X) \rightarrow U_{n+1}(BZ_p \times X, X)$$

which reduces to  $\lambda$  for  $X = \text{point}$ .  $\square$

**4.5 THEOREM:** *Suppose  $U_n(X)$  has no  $p$ -torsion for each  $n$ . Let  $\hat{S}: \hat{U}_n(X) \rightarrow \hat{U}_{n-2}(X)$  be the composition*

$$\hat{U}_n(X) \xrightarrow{\sigma} \bar{U}_{n-1}^{Z_p}(X) \cong U_{n-1}(BZ_p \times X, X) \xrightarrow{\lambda_X^{-1}} \hat{U}_{n-2}(X).$$

Then the sequence

$$0 \rightarrow U_n(X; Z_p) \xrightarrow{j} U_n(X) \xrightarrow{\hat{S}} \hat{U}_{n-2}(X) \rightarrow 0$$

is exact.

**PROOF:** The theorem follows immediately from 2.5 and 4.4.  $\square$

## 5. Representing homology cycles

Let  $X$  be a topological space and  $\varphi: \hat{U}_{*}^{Z_p}(X) \rightarrow H_*(X; Z_p)$  be the edge-homomorphism of the spectral sequence associated with the theory  $\hat{U}^{Z_p}(-)$ .

Recall that  $\varphi$  is obtained by assigning to a  $\hat{U}^{Z_p}$ -manifold  $M$  the chain formed by the sum of the top dimensional simplexes of any triangulation of  $M$ .

Consider the square

$$\begin{array}{ccc} \hat{U}^{Z_p}(X) & \xrightarrow{\sigma} & U^{Z_p}(X) \\ \downarrow \varphi & & \downarrow \gamma \\ H(X; Z_p) & \xrightarrow{\beta} & H(X) \end{array}$$

where  $\gamma$  is the composition of the augmentation  $\epsilon: U^{Z_p}(X) \rightarrow U(X)$  and the usual Thom map. The map  $\beta$  is homology Bockstein.

The following lemma is elementary and we leave the proof to the reader.

5.1 LEMMA: *The above square is commutative*  $\square$

Now we are ready to prove the following

5.2 THEOREM: *If  $\gamma$  is not onto for  $X = \underbrace{BZ_p \times \cdots \times BZ_p}_n$  then also  $\varphi$  is not onto (for the same  $X$ ).*

PROOF: Let us suppose that  $\varphi$  is onto. Then, since each element of  $H^*(X)$  has order  $p$  we obtain, by the exactness of the Bockstein sequence, that  $\beta$  is onto; by lemma 5.1  $\gamma$  has to be onto while we have assumed  $\gamma$  not onto.

The theorem is proved  $\square$

If we take  $U(-)$  to be oriented differentiable bordism  $\Omega(-)$  then the Thom example  $H\mathbb{Z}(BZ_p \times BZ_p)$  (see [5]) together with our theorem shows that  $\hat{\Omega}^{Z_p}$  does not represent mod  $p$  homology.

REMARK: The same argument applies also in the stronger case in which, instead of the theory  $\hat{U}^{Z_p}(-)$ , we consider the theory where we allow  $(U, S_p)$  coverings of order  $p$ ,  $S_p$  being the full symmetric group of permutations on  $p$  letters.

#### REFERENCES

- [1] BUONCRISTIANO, S., C. P. ROURKE and B. J. SANDERSON: A Geometric Approach to Homology Theory. London Mathematical Society Lecture Note Series (to appear).
- [2] CONNER, P. E. and E. E. FLOYD: *Differentiable Periodic Maps*. Springer-Verlag 1964.
- [3] ROURKE, C. P.: Representing Homology Classes. *Bull. London Math. Soc.* 5 (1973) 257-260.
- [4] SULLIVAN, D.: *Geometric Topology Notes*. Princeton MIT 1970.
- [5] THOM, R.: Quelques propriétés globales des variétés différentiables. *Comment. Math. Helv.* 28 (1954) 17-86.
- [6] WALL, C. T. C.: Cobordism exact sequences for differential and combinatorial manifolds. *Ann. of Math.* 77 (1963) 1-15.

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