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CURVATURE ON ALGEBRAIC PLANE CURVES I

Linda Ness

Introduction

We will consider holomorphic curves C in $\mathbb{C}^2$; these are the zero sets of holomorphic functions defined on an open connected subset of $\mathbb{C}^2$. For such a curve

$$C^* = C - \{\text{singular points}\}$$

is a Riemann surface. We endow it with the metric induced by the Fubini-Study metric on $\mathbb{C}^2$; with respect to the usual coordinates of $\mathbb{C}^2$ this is the metric given by

$$(1) \quad \frac{2}{1+|y|^2+|z|^2}((1+|z|^2)dyd\bar{y} - \bar{y}z\, dyd\bar{z} - y\bar{z}\, dzd\bar{y} + (1+|y|^2)dzd\bar{z})$$

Hence we may view C^* as a real Riemannian surface. In this paper we study the Gaussian curvature of C^*.

Both Part I and Part II are portions of the author's thesis. The author would like to thank her advisor David Mumford for his many helpful insights and his encouragement.

The main result in section 1 is a formula for the Gaussian curvature K in terms of the defining equation of the curve and the affine hessian of the defining equation. A corollary gives a curvature formula for algebraic curves in terms of the homogeneous polynomial defining the curve, its Hessian, and the homogeneous coordinates of the point. The formulas show that $K \leq 2$ and that K attains the maximum value 2 precisely at the flexes (if C contains no line components and is not a conic).

In section 2 we consider an analytic family of curves

$$C_t: f(y, z) = t: 0 \leq |t| < \epsilon$$

where C_0 is a singular curve. Let $P \in C_0$ be a singular point of order m and let B be a ball centered at P which is so small that $B \cap C_0 = \bigcup_{i=1}^{r} C_i$ where C_i is a holomorphic curve, $B \cap C_i \cap C_j = \{P\}$ for $i \neq j$, and where r is the number of branches of C_0 at P. In section 2 we study the curvature on $B \cap C_t$.

Before stating the main result of this section we define the *affine hessian* of a holomorphic function $f: \mathbb{C}^2 \to \mathbb{C}$ by

$$H(f) = 2f_{zy}f_yf_z - f_{yy}f_z^2 - f_{zz}f_y^2 \tag{2}$$

Then $Q \in C_t$ is a flex $\langle = \rangle$ C_t is nonsingular at Q and $Q \in (H(f) = 0) \cap C_t$. In the case that P is an ordinary singularity of C_0 we will assume B is so small that in B $H(f)$ may be factored as

$$H(f) = \left(\prod_{i=1}^{m} h_i\right) \cdot g$$

where h_i and g are holomorphic, $h_i = 0$ is tangent to C_i, $g = 0$ has a singularity order of $2m - 4$ at P, and where the tangents to $g = 0$ at P are distinct from the tangents to C_0. [3]. If $g = 0$, and hence $H(f) = 0$, in fact has an ordinary singularity at P we will assume that B is so small that in B g may be factored as

$$g = \prod_{j=1}^{2m-4} g_j$$

where $g_j: \mathbb{C} \to \mathbb{C}$ is holomorphic.

We now state the main result of section 2:

THEOREM 2: *Suppose C_0 has an ordinary singularity at P of order m. For ϵ sufficiently small, positive, depending on C_0 and B, the following statements are true for C_t, $|t| < \epsilon$.*

(i) *Let l be any line through P not tangent to $H(f) = 0$. Let $P_t \in C_t \cap B \cap l$. Then there are m such points P_t counted with multiplicity and $K(P_t) \to -\infty$ as $t \to 0$.*

(ii) *$B \cap C_t$ contains at least $3m(m-1)$ flexes counted with multiplicity. More precisely for each $i = 1, \ldots, m$*

$$B \cap C_t \cap (h_i = 0)$$

contains at least $m+1$ flexes counted with multiplicity and it contains more $\langle = \rangle$ P is a flex of $h_i = 0$. If $g = 0$ has an ordinary singularity at P then for $j = 1, \dots, 2m-4$

$$B \cap C_t \cap (g_j = 0)$$

contains exactly m flexes counted with multiplicity.

0

We first recall a few facts about the Fubini-Study metric. Geometrically the Fubini-Study metric is the metric induced on $\mathbb{C}^2$ by stereographic projection of the unit sphere $S^4 \subset \mathbb{R}^5$, with the usual metric, from $(0,0,0,1)$ onto $\mathbb{R}^4$ viewed as $\mathbb{C}^2$.

The Fubini-Study metric on $\mathbb{CP}^2$ is defined on each of the three usual affine coordinate patches of $\mathbb{CP}^2$ by formula (1). It is the unique Kahler metric with constant sectional curvature equal to 2 which is invariant under the automorphisms of $\mathbb{CP}^2$ induced by elements of $U(3, \mathbb{C})$ [2]. This invariance under unitary changes of homogeneous coordinates can be seen from the following observation. If $\Pi\colon \mathbb{C}^3 - \{0\} \to \mathbb{CP}^2$ is the natural projection which takes a line through the origin in $\mathbb{C}^3$ to the corresponding point in $\mathbb{CP}^2$ and if Φ denotes the 1–1 form associated to the Fubini-Study metric, then

$$\Pi^*\Phi = \partial\bar{\partial} \log(|x|^2 + |y|^2 + |z|^2)$$

where x, y, and z denote the usual coordinates on $\mathbb{C}^3$.

1

Hence forth we will assume that all curves (holomorphic or algebraic are endowed with the metric induced by the Fubini-Study metric (on $\mathbb{C}^2$ or $\mathbb{CP}^2$).

THEOREM 1: *Let $C \subset \mathbb{C}^2$ be a holomorphic curve defined by $f(y, z) = 0$. The Gaussian curvature at a nonsingular point of C is given by the formula*

$$K(y, z) = 2 - \frac{(1 + |y|^2 + |z|^2)^3 |H(f)|^2}{(|f_y|^2 + |f_z|^2 + |yf_y + zf_z|^2)^3} \tag{3}$$

COROLLARY: *Let $C \subset \mathbb{CP}^2$ be an algebraic curve defined by $F(x, y, z) = 0$, where $F(x, y, z)$ is a homogeneous polynomial of degree d. The Gaussian curvature at a nonsingular point P of C is given by the formula*

$$K(P) = 2 - \frac{\|P\|^6 |\text{Hessian } F|^2}{(d-1)^4 \|\text{grad } F\|^6} \tag{4}$$

where the Hessian and the gradient are evaluated at the homogeneous coordinates of P and where $\| \ \|$ indicates the usual norm in $\mathbb{C}^3$.

PROOF OF THE THEOREM: It suffices to consider the case $f_y(P) \neq 0$. Locally, then, C may be viewed as the graph of a holomorphic function $y(z)$. The Gaussian curvature for a metric $2hdzd\bar{z}$ is given by $(-1/h)(\partial^2 \log h/\partial z \partial \bar{z})$. In our case we may write $h = B/A^2$ where $A = 1 + |y|^2 + |z|^2$ and $B = A - |\partial A/\partial z|^2$.

Note that for any positive real-valued function $P: \mathbb{C} \to \mathbb{C}$.

$$* \qquad \frac{\partial^2 \log P}{\partial z \partial \bar{z}} = \frac{\partial^2 P}{\partial z \partial \bar{z}} - \frac{1}{P^2} \left| \frac{\partial P}{\partial z} \right|^2$$

Applying $*$ to A and B and simplifying using the formula

$$(f_y)^3 \frac{\partial^2 y}{\partial z^2} = H(f)$$

gives (3).

PROOF OF THE COROLLARY: We may assume without loss of generality that P is in the affine subset of $\mathbb{CP}^2$ where $x \neq 0$. Then locally C is defined by $f(y, z) = F(1, y, z) = 0$. The corollary follows by applying Euler's formula to the denominator and the following formula, which is an exercise in [1] to the numerator

$$\text{Hessian } F(1, y, z) = (d-1)^2 H(f)$$

From the theorem we can conclude that the points of maximum curvature are projective invariants. Clearly $K \leq 2$ and $K \equiv 2$ on lines. If P is a point on an analytic curve $C: f = 0$ which contains no line components

$$K(P) = 2 \langle = \rangle P \in f = 0 \cap H(f) = 0 \text{ and } P \text{ is nonsingular}$$
$$\langle = \rangle P \text{ is a flex on } C$$

The flexes are precisely the nonsingular points where the tangents have higher order contact. Hence they are preserved by maps of C induced by projective transformations of $\mathbb{C}^2$ (or $\mathbb{C}^3$ in the algebraic case). In general Gaussian curvature is invariant just under isometries, which in the case of the Fubini-Study metric are maps of C induced by unitary transformations of $\mathbb{C}^2$(or $\mathbb{C}^3$). Intuitively the curve is "flatter" at the flexes so $K(\text{flex}) = 2$ is not surprising.

If C is an irreducible algebraic curve of degree $d > 1$ Bezout's theorem implies that there are $\leq 3d(d-2)$ flexes counted with multiplicity, with equality if and only if C is nonsingular. Nonsingular conics, then, are the only irreducible algebraic curves where the maximum curvature is <2.

We now apply the theorem to cubics. Up to unitary transformation every irreducible cubic C may be defined by an affine equation of the form

$$(y - (\alpha z + \beta))^2 - g(z) = 0 \tag{5}$$

where g is a cubic polynomial. The line "at ∞" for a cubic C defined by (5) contains exactly one point of C, a flex, so the curvature "at ∞" is 2. The affine branch points occur on the line $y = \alpha z + \beta$ at the points where $g(z) = 0$. C is singular if and only if g has multiple roots, i.e. if and only if the discriminant of g is 0. If C is singular, the singular points occur among the branch points.

PROPOSITION: *Suppose C is an irreducible nonsingular cubic defined by an equation of the form (5), where $g(z) = c(z - b_1)(z - b_2)(z - b_3)$, $c \neq 0$. Let B_i be the branch point $(\alpha b_i + \beta, b_i)$. Then*

$$\prod_{i=1}^{3} (2 - K(B_i)) = \frac{M}{|\text{disc } g|^2}$$

where M depends on b_i and $M \geq 64$, with equality $\langle = \rangle$ the branch points all lie on the z-axis.

PROOF: Apply formula (3)

2

We follow the notation of the Introduction and give the proof of Theorem 2.

PROOF OF (i): Recall the following classical [3]

LEMMA 1: *If $f(y, z) = 0$ is an analytic curve with a singularity of order m at P, $H(f) = 0$ has a singularity of order $\geq 3m - 4$ at P; the inequality is strict if and only if all the tangent lines to $f = 0$ at P coincide.*

Since curvature is invariant under unitary transformations we may assume that $P = (0, 0)$ and that $y = 0$ is tangent to $H(f) = 0$ at P. Theorem 1 and Lemma 1 imply that it is sufficient to notice:

$$\lim_{y\to 0} \frac{|y|^2 |H(f)|^2}{(|f_y|^2 + |f_z|^2 + |yf_y + zf_z|^2)^3} \Bigg|_{(y, \alpha y)} = C_\alpha \tag{6}$$

where C_α is continuous in α and vanishes at α only if $z = \alpha y$ is tangent to $H(f) = 0$.

PROOF OF (ii): In B f may be factored as $f = \Pi_{i=1}^m f_i$ where f_i is analytic and C_i: $f_i = 0$. It is sufficient to prove

$$*I(P, f_1 \cap H(f) = 3(m-1) + I(P, f_1 \cap H(f_1))$$
$$**I(P, h_1 \cap f) = m + 1 + I(P, f_1 \cap H(f_1))$$

and in the case that $g = 0$ has an ordinary singularity at P

$$***I(P, g_i \cap f) = m$$

***holds since $g = 0$ and $f = 0$ have no common tangents when the singularity of f at P is ordinary.

LEMMA 3: *If p and q are analytic functions $\mathbb{C}^2 \to \mathbb{C}$, then*

$$H(pq) = p^3 H(q) + q^3 H(p) + pqr \tag{7}$$

where r: $\mathbb{C}^2 \to \mathbb{C}$ is analytic.

PROOF OF LEMMA 3: Straightforward using the defining formula of the affine hessian (2).

To prove * we apply the lemma letting $p = f_1$, $q = \Pi_{i=2}^m f_i$. Next

$$I(P, h_1 \cap f) = \sum_{i=1}^m I((P, h_1 \cap f_i) = m - 1 + I(P, h_1 \cap f_1)$$

$$I(P, h_1 \cap f_1) = I(P, H(f) \cap f_1) - I\left(P, f_1 \cap g \cdot \prod_{j=2}^{m} h_j\right)$$

Since the tangents are distinct $I(P, f_1 \cap g \cdot \Pi_{j=2}^{m} h_j) = 3m - 5$. Finally we apply * and obtain **.

COROLLARY 1: *Part* (i) *is true in the more general case that C_0 has a singularity of order m at P with at least two distinct tangents.*

PROOF: Clear from the proof of (i).

COROLLARY 2: *Let $l_1 \ldots l_n$ denote the distinct tangent lines of $H(f) = 0$ at P. Let D_i denote a small solid cone with vertex at P and axis l_i. Let $\tilde{C}_t = C_t \cap B - D_1 - \cdots - D_r$. Given $M > 0$, there exists $\epsilon > 0$ such that if $|t| < \epsilon$, the flexes of $C_t \cap B$ are in the interiors of the cones D_i, $i = 1, \ldots, n$ and*

$$\max_{\tilde{C}_t} K < -M > 0$$

PROOF: It is sufficient to note that C_α in formula (6) is bounded on $B - D_1 - \cdots - D_r$.

From Corollary 2 we obtain a picture of $C_t \cap B$ when C_0 has an ordinary singularity at P. The flexes collapse toward the tangent lines to the hessian curve which include the tangent lines to C_0 at P, and leave regions of large negative curvature in the regions between the tangents to the hessian. We can visualize nipples of positive curvature crowding toward the tangent lines to the hessian leaving more and more extreme saddles between the tangent lines.

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