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THE NUMBER OF CRITICAL POINTS IN MORSE APPROXIMATIONS

Henry King*

Given a real valued smooth function on a manifold one can ask how many critical points nearby morse function can have. Usually the topology of the situation will make it necessary to have some critical points, but one can ask if it is always possible to smoothly approximate by a morse function with the least number of singularities topologically possible (i.e. the least number of singularities in close continuous approximations). The theorem below shows that this is not always possible for C^2 or better approximations and counter examples are numerous.

Denote $eS^{n-1} = \{x \in R^n \,\big|\, |x| = e\}$ and $eB^n = \{x \in R^n \,\big|\, |x| \leq e\}$. ∂ will mean boundary and $\approx$ will denote homeomorphism.

DEFINITION: If $f: M \to R$ and $g: N \to R$, then $f \oplus g: M \times N \to R$ is the map $f \oplus g(x, y) = f(x) + g(y)$.

DEFINITION: If $f: M \to R$ is C^k, then $u_k(f)$ is the least number such that there exist morse functions arbitrarily close to f in $C^k(M, R)$ with $u_k(f)$ critical points. Here $C^k(M, R)$ is endowed with the Whitney C^k topology.

PROPOSITION 1: *Suppose* $f: (R^n, 0) \to (R, 0)$ *has no singularities except* 0 *and* $h: R^k \to R$ *is given by* $h(x_1, \ldots, x_k) = \Sigma_{i=0}^{k} \pm x_i^2$. *Then* $u_k(f \oplus h) = u_k(f)$ *if* $k \geq 2$.

PROOF: If g is C^k close to f, then $g \oplus h$ is C^k close to $f \oplus h$ so $u_k(f \oplus h) \leq u_k(f)$.

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Conversely, if g is C^k close enough to $f \oplus h$, then by a parameterized Morse lemma there is a C^{k-1} diffeomorphism p C^{k-1} close to the identity so $gp(x, y) = f'(x) + h(y)$ where f' is C^k close to f. Then g has as many critical points as f' does, so $u_k(f \oplus h) \geq u_k(f)$.

DEFINITION: If $f : (R^n, 0) \to (R, 0)$ is a polynomial, and 0 is an isolated singularity of f, then $N(f) = eS^{n-1} \cap f^{-1}((-\infty, 0])$ for sufficiently small e. In [1] it was shown that this is independent of small e.

PROPOSITION 2: *If $f : (R^n, 0) \to (R, 0)$ is a polynomial with 0 an isolated singularity of f and $u_1(f) = 0$, then $N(f)$ is contractible.*

PROOF: It follows from [1] that for small e and d,

$$(eB^n, eS^{n-1} \cap f^{-1}((-\infty, 0])) \approx (eB^n \cap f^{-1}(2dB^1), eB^n \cap f^{-1}(-2d)).$$

Pick e and d so the above holds and also so f has no singularities in $eB^n - 0$ and $2dB^1$ contains no critical values of f restricted to eS^{n-1}. Pick c in $(0, e)$ so $f(cB^n) \subset dB^1$. Pick a g C^1 close to f with no critical points in cB^n. We may assume by altering g if necessary that $g(x) = f(x)$ if $|x| \geq c$. Then the gradient of g (slightly modified near eS^{n-1} to be tangent to eS^{n-1}) is a nonzero vector field on $eB^n \cap f^{-1}(2dB^1)$ whose flow gives a diffeomorphism from $eB^n \cap f^{-1}(2dB^1)$ to $(eB^n \cap f^{-1}(-2d)) \times [0, 1]$. But $eB^n \cap f^{-1}(2dB^1) \approx eB^n$ and $eB^n \cap f^{-1}(-2d) \approx N(f)$, so $N(f)$ is contractible.

PROPOSITION 3: *If $p : (R^n, 0) \to (R, 0)$ is a polynomial with 0 the only critical point of p and if $N(p)$ is contractible and $n \geq 6$, then $u_0(p) = 0$.*

PROOF: Pick any $e > 0$. Pick $a > 0$ and b in $(0, e/2)$ so $N(p) \approx aS^{n-1} \cap p^{-1}((-\infty, 0])$ and $(aB^n, aS^{n-1} \cap p^{-1}((-\infty, 0]) \approx (aB^n \cap p^{-1}(bB^1), aB^n \cap p^{-1}(-b))$. Denote $W = aB^n \cap p^{-1}(bB^1)$. Approximate p by a morse function f so that $p(x) = f(x)$ for x near ∂W. By the relative h-cobordism theorem we may cancel all f's handles and produce a $g : W \to bB^1$ without critical points so $g(x) = f(x)$ for x near ∂W. Then the function

$$h(x) = \begin{cases} g(x) & \text{if } x \text{ is in } W \\ p(x) & \text{otherwise} \end{cases}$$

is e close to p in the C^0 topology, so $u_0(p) = 0$.

PROPOSITION 4: *$N(f \oplus t^2) \approx N(f) \times [0, 1]$ and $N(f \oplus -t^2) \approx eB^n \times S^0/(x, 1) = (x, -1)$ if $x \in eS^{n-1} \cap f^{-1}((-\infty, 0])$ for e sufficiently small.*

PROOF: It follows from [1] or [2], Lemma 3.6 that if e is small there is a diffeomorphism $c : eS^{n-1} \times (0, 1] \to eB^n - 0$ so that $|c(x, t)| = te$ for all x and t and so $fc(x, t) = 0$ iff $f(x) = 0$ and so $(\partial/\partial t)|fc(x, t)| > 0$ for all t if $f(x) \neq 0$. Define a homeomorphism $h : eS^n \to eS^n \subset R \times R^n$ by

$$h(t, x) = \begin{cases} (t, 0) & \text{for } t = \pm e \text{ and } x = 0 \\ (t, c(ex/|x|, (\sqrt{e^2 - t^2})/e)) & \text{otherwise} \end{cases}$$

then there is a function $b : eS^{n-1} \to [0, e)$ so $b^{-1}(0) = eS^{n-1} \cap f^{-1}(0)$ and $h^{-1}(f \oplus t^2)^{-1}((-\infty, 0]) = \{(t, x) \in eS^n | f(ex/|x|) \leq 0 \text{ and } |t| \leq b(ex/|x|)\}$ and so $h^{-1}(f \oplus - t^2)^{-1}((-\infty, 0]) = \{(t, x) | x = 0 \text{ or } f(ex/|x|) \leq 0 \text{ or } |t| \geq b(ex/|x|)\}$. The result follows.

PROPOSITION 5: *Suppose $M \subset RP^n$ is a codimension 1 compact smooth submanifold of real projective n space. Then M is ϵ-isotopic to a real non-singular projective variety.*

PROOF: See [3] and note that his trick works for this case also. Unless a component of M doesn't separate RP^n, it will be necessary to define ϕ so $\phi(x) = \phi(-x)$ (rather than $\phi(x) = -\phi(-x)$) and likewise $\Phi = (\Phi(x) + \Phi(-x))/2$. The isotopy on S^n can be canonical, hence equivariant. See also [7].

THEOREM: *For every $n \geq 6$ there is a weighted homogeneous polynomial $q : (R^n, 0) \to (R, 0)$ of finite codimension so that $0 = u_1(q) < u_2(q)$. If $n \geq 7$ we may even require that $N(q)$ is a disc so that q is topologically equivalent to a function without critical points and yet is not C^2 close to any function without critical points.*

PROOF: It follows from [4] that there is a compact n-2 dimensional manifold U in $RP^{n-2} - RP^{n-3}$ so that U has the homology of a point but U is not simply connected. Take the connected sum of ∂U and RP^{n-3} in RP^{n-2} by removing a disc from each of ∂U and RP^{n-3} and attaching a tube $S^{n-4} \times [0, 1]$. Call M the resulting submanifold of RP^{n-2}. By Proposition 5 there is a homogeneous polynomial $h : R^{n-1} \to R$ so $h^{-1}(0) \cap S^{n-2}$ is isotopic to $p^{-1}(M)$ where $p : S^{n-2} \to RP^{n-2}$ is the usual projection and so 0 is an isolated critical point of h. Milnor has pointed out to me that since homogeneous polynomials of finite codimension are dense in all homogeneous polynomials (c.f. [5]) we may assume h has finite codimension. Notice that $N(h)$ has the homology of a point but is not simply connected. (To see this, note that by Lefschetz duality, ∂U has the homology of a sphere. But then $\partial N(h) \approx \partial U \# \partial U$ has the homology of a sphere also so by Alexander duality, $S^{n-2}-h^{-1}(0)$ has the homology of two points, hence $N(h)$ has the homology of a point. $N(h)$ is not simply connected because it has the homotopy type of the wedge of U with something else.)

Now Proposition 4 says that $N(h\oplus -t^2)$ is the union of two discs along $N(h)$ so Van Kampen's theorem implies $N(h\oplus -t^2)$ is simply connected and the Mayer-Vietoris sequence implies $N(h\oplus -t^2)$ has the homology of a point, hence $N(h\oplus -t^2)$ is contractible. But then Proposition 3 implies $u_0(h\oplus -t^2)=0$. But $u_2(h)=u_2(h\oplus -t^2)$ by Proposition 1 and $u_2(h)\geq u_1(h)>0$ by Proposition 2. So $0=u_0(h\oplus -t^2)< u_2(h\oplus -t^2)$.

$N(h\oplus -t^2)$ might not be a disc since its boundary might not be simply connected. But by Proposition 4, $\partial N(h\oplus s^2-t^2)$ is the double of $N(h\oplus -t^2)$ which is simply connected, so $N(h\oplus s^2-t^2)$ is a disc. Since $N(f)$ determines the local topological type of f (see [2], [6] or do as an exercise in the weighted homogeneous case), $h\oplus s^2-t^2$ is topologically equivalent to a projection.

To get $u_1(h\oplus -t^2)=0$, pick any function $g: R^n \to R$ without critical points so that $g(x)=(h\oplus -t^2)(x)$ if $|x|\geq 1$. Let d be the degree of h. Define $g_t(x,y)=(g(t^2x, t^dy))/t^{2d}$ all (x,y) in $R^{n-1}\times R$. Then for larger and larger t, g_t is a closer and closer C^1 approximation to $h\oplus -t^2$.

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