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THE SEMI-M PROPERTY FOR NORMED RIESZ SPACES

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1. Introduction

It is well-known that if (Δ, Γ, μ) is a σ-finite measure space and if $1 \leq p < \infty$, then the Banach dual L_p^* of the Banach space $L_p = L_p(\Delta, \mu)$ can be identified with $L_q = L_q(\Delta, \mu)$, where $p^{-1} + q^{-1} = 1$. For $p = \infty$ the situation is different; the space L_1 is a linear subspace of L_∞^*, and only in a very trivial situation (the finite-dimensional case) we have $L_1 = L_\infty^*$. Restricting ourselves to the real case, the Banach dual L_∞^* is a (real) Riesz space, i.e., a vector lattice, and L_1 is now a band in L_∞^*. The disjoint complement (i.e., the set of all elements in L_∞^* disjoint to all elements in L_1) is also a band in L_∞^*, called the band of singular linear functionals on L_∞. It is evident that for any pair F_1, F_2 of positive elements in L_∞^* we have $\|F_1 + F_2\| = \|F_1\| + \|F_2\|$. This is due to the fact that L_∞^* is an abstract L-space.

More generally, let Φ and Ψ be a pair of conjugate and continuous Orlicz functions (also called Young functions). It is well-known that if Φ does not increase too fast, then the Banach dual L_Φ^* of the Orlicz space L_Φ can be identified with the Orlicz space L_Ψ. This holds in particular if Φ satisfies the so-called Δ_2-condition (i.e., there exists a constant $M > 0$ such that $\Phi(2u) \leq M\Phi(u)$ for all $u \geq 0$). However, if Φ increases too fast (e.g. $\Phi(u) = e^u - 1$), then L_Ψ is a proper linear subspace of L_Φ^*. More precisely, L_Ψ is a band in the (real) Riesz space L_Φ^*. The disjoint complement of L_Ψ in L_Φ^* is again a band in L_Φ^*, called the band of the (bounded) singular linear functionals on L_Φ. It was proved by T. Ando ([1], 1960) that this subspace of all singular linear functionals on L_Φ is an L-space, i.e., if S_1, S_2 are positive singular linear functionals on L_Φ, then $\|S_1 + S_2\| = \|S_1\| + \|S_2\|$. The proof was extended to the case of discontinuous Orlicz functions by M. M. Rao

([9], 1968). His definition of singular functionals, however, did not cover all possible cases. The general situation for Orlicz spaces was discussed by the present author ([3], 1975).

Orlicz spaces are special examples of normed Köthe spaces. It may be asked, therefore, for which normed Köthe spaces we have the above-mentioned triangle equality for positive singular bounded linear functionals. More generally, we may ask for which normed Riesz spaces (i.e., normed vector lattices) we have the triangle equality for positive singular bounded linear functionals.

Let L_ρ be a (real) normed Riesz space with Riesz norm ρ (i.e., ρ is a norm such that $\rho(f) = \rho(|f|)$ for all $f \in L_\rho$ and $0 \le u \le v$ implies $\rho(u) \le \rho(v)$). The notation $f_n \downarrow f_0$ means that the sequence $\{f_n: n = 1, 2, \ldots\}$ in L_ρ is decreasing and $\inf f_n = f_0$; the notation $f_n \uparrow f_0$ in L_ρ is defined similarly. The Banach dual L_ρ^* of L_ρ is also a Riesz space; the positive bounded linear functionals on L_ρ form the positive cone of L_ρ^*. The element $F \in L_\rho^*$ is called an integral if $|F|(f_n) \downarrow 0$ holds for any sequence $f_n \downarrow 0$ in L_ρ. It is well-known that the set of all integrals is a band in L_ρ^*, which we shall denote by $\overset{*}{L}_{\rho,c}$. If L_ρ is the space $L_\infty = L_\infty(\Delta, \mu)$, then $\overset{*}{L}_{\rho,c}$ is exactly the space $L_1(\Delta, \mu)$. If L_ρ is the Orlicz space L_Φ, then $\overset{*}{L}_{\rho,c}$ is exactly the space L_Ψ, where Ψ is the conjugate Orlicz function of Φ. We return to the general case. The set $\overset{*}{L}_{\rho,s}$ of all elements disjoint to $\overset{*}{L}_{\rho,c}$ is also a band in L_ρ^*, and we have $L_\rho^* = \overset{*}{L}_{\rho,c} \oplus \overset{*}{L}_{\rho,s}$. The elements in $\overset{*}{L}_{\rho,s}$ are called the singular bounded linear functionals on L_ρ. The problem is now to find sufficient and (or) necessary conditions for $\overset{*}{L}_{\rho,s}$ to be an abstract L-space. For the formulation of a satisfactory answer, we present the following definition.

DEFINITION 1.1. *The normed Riesz space L_ρ is called a semi-M-space if L_ρ satisfies the following condition: If u_1 and u_2 are positive elements in L_ρ such that $\rho(u_1) = \rho(u_2) = 1$ and if* $\sup(u_1, u_2) \ge v_n \downarrow 0$, *then* $\lim \rho(v_n) \le 1$.

It is evident that any M-space in the sense of Kakutani is semi-M. Another special case of a semi-M-space arises whenever ρ is an absolutely continuous norm, i.e. whenever $u_n \downarrow 0$ in L_ρ implies $\rho(u_n) \downarrow 0$ (as in L_p spaces for $1 \le p < \infty$ and in Orlicz spaces L_Φ if Φ satisfies the Δ_2-condition).

Our main result is that the space L_ρ is a semi-M-space if and only if $\overset{*}{L}_{\rho,s}$ is an abstract L-space. Since all Orlicz spaces are semi-M-spaces this takes care of the Orlicz spaces. In section 3 we derive another characterization of semi-M-spaces if L_ρ has the principal projection property. We recall that a band A in the Riesz space L is called a projection band if L is the direct sum of A and the disjoint complement A^d of A. The space L is then said to have the principal projection property (from now on abbreviated as p.p.p.) whenever every principal band in L is a projection band. Every normed Köthe space is Dedekind complete which implies the p.p.p., and so the results of section 3 can be applied in the theory of normed Köthe spaces.

There is still another class of spaces L_ρ for which a useful characterization of the semi-M property can be given. This is done in section 4. We recall several facts.

The element $f_0 \in L_\rho$ is said to be of absolutely continuous norm if it follows from $|f_0| \geq u_n \downarrow 0$ that $\rho(u_n) \downarrow 0$. The set L_ρ^a of all elements of absolutely continuous norm is a norm closed order ideal in L_ρ. By way of example, if the measure space is the real line with Lebesgue measure, then $L_\rho^a = L_\rho$ if L_ρ is an L_p space ($1 \leq p < \infty$), but $L_\rho^a = \{0\}$ if L_ρ is the space L_∞. If L_ρ is the sequence space l_∞, then L_ρ^a is the subspace (c_0) of all null sequences. The ideal L_ρ^a in L_ρ and the band $L_{\rho,s}^*$ in L_ρ^* are related. It is known that L_ρ^a is the inverse annihilator ${}^\perp(L_{\rho,s}^*)$ of $L_{\rho,s}^*$. We recall that for any subset B of L_ρ^* the inverse annihilator ${}^\perp B$ of B is defined by

$$ {}^\perp B = \{f : f \in L_\rho, F(f) = 0 \text{ for all } F \in B\}. $$

Similarly the annihilator $A^\perp$ of any $A \subset L_\rho$ is defined by

$$ A^\perp = \{F : F \in L_\rho^*, F(f) = 0 \text{ for all } f \in A\}. $$

It follows from $L_\rho^a = {}^\perp\{L_{\rho,s}^*\}$ that $(L_\rho^a)^\perp \supset L_{\rho,s}^*$, and the last inclusion may be a strict inclusion. Our result is now that in spaces L_ρ satisfying $(L_\rho^a)^\perp = L_{\rho,s}^*$, L_ρ is a semi-M-space if and only if L_ρ/L_ρ^a is an abstract M-space.

Having proved the general theorem (sections 2–6), we study the semi-M property for an important class of normed Köthe spaces, the rearrangement invariant Köthe spaces (sections 7–9). Using the results of this paper it can be proved that whenever L_ρ is a normed Köthe space having the semi-M property, then $L_{\rho,s}^*$ is Riesz isomorphic and isometric to a band in $L_{\infty,s}^*$ (L_∞ defined on the same measure space as L_ρ). This generalizes results of T. Ando ([1], 1960) and M. M. Rao ([9], 1968). To avoid unduly lengthening of this paper we shall report on this elsewhere ([4]).

2. The main theorem

Unless stated otherwise L_ρ will denote a normed real Riesz space. If $F \in L_\rho^*$ is positive (notation $F \geq \Theta$, where Θ is the null functional), it is well-known that the integral component of F is equal to F_L, where

$$F_L(u) = \inf \{\lim F(u_n): 0 \leq u_n \uparrow u\}$$

for all $u \in L_\rho^+$. From this expression we derive a necessary and sufficient condition for $\Theta \leq F \in L_\rho^*$ to be singular.

LEMMA 2.1. *Let $\Theta \leq S \in L_\rho^*$. Then $S \in L_{\rho,s}^*$ holds if and only if there exists for every $u \in L_\rho^+$ and for every $\epsilon > 0$ a sequence $\{u_{n,\epsilon}: n = 1, 2, \dots\} \subset L_\rho^+$ such that $u_{n,\epsilon} \uparrow u$ and $S(u_{n,\epsilon}) < \epsilon$ for all n.*

PROOF: (i) Assume that $S \in L_{\rho,s}^*$, and let $u \in L_\rho^+$ and $\epsilon > 0$ be given. Then $S_L(u) = 0$. Hence

$$\inf \{\lim S(u_n): 0 \leq u_n \uparrow u\} = S_L(u) = 0.$$

This implies the existence of a sequence $\{u_{n,\epsilon}: n = 1, 2, \dots\}$ as indicated in the lemma.

(ii) Now assume that for each $u \in L_\rho^+$ and for each $\epsilon > 0$ a sequence $\{u_{n,\epsilon}: n = 1, 2, \dots\}$ as indicated in the lemma exists. It is obvious then that $S_L(u) < \epsilon$. This holds for every $\epsilon > 0$, so $S_L(u) = 0$. This holds for every $u \in L_\rho^+$, so $S \in L_{\rho,s}^*$.

We shall denote by ρ^* the norm in the Banach dual L_ρ^* of L_ρ. Next, we state and prove our first main result.

THEOREM 2.2. *L_ρ is a semi-M-space if and only if $L_{\rho,s}^*$ is an abstract L-space.*

PROOF: (i) Assume that L_ρ is a semi-M-space. Let $S_1, S_2 \in L_{\rho,s}^*$, $S_1, S_2 \geq \Theta$ and $\epsilon > 0$ be given. Then there exist elements $u_1, u_2 \in L_\rho^+$ such that $\rho(u_1) = \rho(u_2) = 1$ and

$$S_i(u_i) > \rho^*(S_i) - \tfrac{1}{2}\epsilon \quad (i = 1, 2).$$

Setting $u = \sup(u_1, u_2)$ and $S = S_1 + S_2$ it follows that $\Theta \leq S \in L_{\rho,s}^*$ and that $u \geq 0$. Hence, in view of Lemma 2.1, there exists a sequence

$\{u_{n,\epsilon}: n = 1, 2, \ldots\} \subset L_\rho^+$ such that $u_{n,\epsilon} \uparrow u$ and $S(u_{n,\epsilon}) < \epsilon$ for all n. Defining $v_n = u - u_{n,\epsilon}$ for all n, the sequence $\{v_n: n = 1, 2, \ldots\}$ satisfies $u \geq v_n \downarrow 0$. Hence $\lim \rho(v_n) \leq 1$ as $n \to \infty$, since L_ρ is semi-M by hypothesis. Thus there exists a number n_0 such that $\rho(v_n) < 1 + \epsilon$ for all $n \geq n_0$. This implies that

$$\begin{aligned}\rho^*(S_1 + S_2) = \rho^*(S) &\geq S(v_n/(1+\epsilon)) = S((u - u_{n,\epsilon})/(1+\epsilon)) \\ &> S(u/(1+\epsilon)) - \epsilon/(1+\epsilon) \geq S_1(u_1/(1+\epsilon)) + S_2(u_2/(1+\epsilon)) - \epsilon/(1+\epsilon) \\ &> (1+\epsilon)^{-1}\{\rho^*(S_1) + \rho^*(S_2) - 2\epsilon\},\end{aligned}$$

provided $h \geq n_0$. Since the above inequality holds for all $\epsilon > 0$, we obtain

$$\rho^*(S_1 + S_2) \geq \rho^*(S_1) + \rho^*(S_2).$$

The inverse inequality is obvious, so $\rho^*(S_1) + \rho^*(S_2) = \rho^*(S_1 + S_2)$. This shows that $L_{\rho,s}^*$ is an abstract L-space.

(ii) For the converse direction, assume that L_ρ is not a semi-M-space. Then there exist $u_1, u_2 \in L_\rho^+$, $\rho(u_1) = \rho(u_2) = 1$ and there exists a sequence $\{v_n: n = 1, 2, \ldots\} \subset L_\rho$ such that $\sup(u_1, u_2) \geq v_n \downarrow 0$ and

$$\lim \rho(v_n) = \alpha > 1 \text{ as } n \to \infty.$$

Now there is for all n $(= 1, 2, \ldots)$ a functional $F_n \in L_\rho^*$ satisfying

$$\rho^*(F_n) = 1,\ F_n \geq \Theta,\ F_n(v_n) = \rho(v_n).$$

Since the unit ball in L_ρ^* is weak star compact, the sequence $\{F_n: n = 1, 2, \ldots\}$ has a weak star cluster point F_0. It is obvious that $F_0 \geq \Theta$ and $\rho^*(F_0) \leq 1$. Furthermore, it is obvious that

$$0 \leq \lim F_0(v_n) \leq \lim \rho(v_n) = \alpha \text{ as } n \to \infty.$$

Next, fix n and $\epsilon > 0$, and consider the weak star open neighbourhood

$$U = \{F: |F(v_n) - F_0(v_n)| < \epsilon\}$$

of F_0. Then U contains infinitely many F_n's. Since $m \geq n$ implies

$$F_m(v_n) \geq F_m(v_m) = \rho(v_m) \geq \alpha,$$

it follows that $\rho(v_m) \leq F_0(v_n) + \epsilon$ for those $m \geq n$ such that $F_m \in U$.

Thus $\lim \rho(v_n) \leq \lim F_0(v_n)$ as $n \to \infty$. Therefore $\lim F_0(v_n) = \alpha$ as $n \to \infty$. Next, if $F_0 = S + I$, $\Theta \leq S \in L^*_{\rho,s}$, $\Theta \leq I \in L^*_{\rho,c}$, then $\lim I(v_n) = 0$, so $\lim S(v_n) = \alpha > 1$. Especially it follows that $S(\sup(u_1, u_2)) \geq \alpha > 1$. Let now S_1 be defined by

$$S_1(u) = \sup\{S(\inf(u, n(u_1 - u_2)^+)): n = 1, 2, \ldots.\}$$

for all $u \in L^+_\rho$, and $S_1(f) = S_1(f^+) - S_1(f^-)$ for arbitrary $f \in L_\rho$. Then $S_1 \in L^*_\rho$, $\Theta \leq S_1 \leq S$, so $S_1 \in L^*_{\rho,s}$. Moreover, $S_1((u_1 - u_2)^+) = S((u_1 - u_2)^+)$ and $S_1((u_1 - u_2)^-) = 0$. Set $S_2 = S - S_1$. Then

$$\begin{aligned}\rho^*(S_1) + \rho^*(S_2) &\geq S_1(u_1) + S_2(u_2) = S_1(u_1) + S(u_2) - S_1(u_2)\\ &= S(u_2) + S_1(u_1 - u_2) = S(u_2) + S((u_1 - u_2)^+)\\ &= S(\sup(u_1, u_2)) \geq \alpha > 1 \geq \rho^*(S).\end{aligned}$$

So $L^*_{\rho,s}$ is not an abstract L-space.

REMARK: The proof of part (ii) is due to D. H. Fremlin. Originally we proved this part if L_ρ has the principal projection property (see section 3). Fremlin kindly gave permission to publish his proof here.

For applications it is often useful to have another representation of semi-M-spaces. This representation is given in the following lemma.

LEMMA 2.3: *The following conditions are equivalent.*

(a) *L_ρ is a semi-M-space.*

(b) *There exists a function M_ρ: $L^+_\rho \to \mathbb{R}^+$ (where $+\infty$ is allowed), such that*

  (i) *$\rho(u) \leq 1 + M_\rho(u)$ for all $u \in L^+_\rho$,*

  (ii) *$u_1, u_2 \in L^+_\rho$, $\rho(u_1) = \rho(u_2) = 1$ and $u = \sup(u_1, u_2)$ implies $M_\rho(v_n) \to 0$ for any sequence $\{v_n: n = 1, 2, \ldots\} \subset L^+_\rho$ satisfying $u \geq v_n \downarrow 0$.*

PROOF: (a) $\Rightarrow$ (b). Define $M_\rho(u) = \{\rho(u) - 1\}^+$ for all $u \in L^+_\rho$. It is easily verified that M_ρ satisfies the conditions (i) and (ii) of condition (b).

(b) $\Rightarrow$ (a). Obvious.

Using this lemma it is easy to show that all Orlicz spaces are semi-M-spaces.

Example 2.4. The Orlicz spaces $L_\Phi[\|\cdot\|_\Phi]$ *and* $L_\Phi[N_\Phi]$

Let (Δ, Γ, μ) be a σ-finite measure space, and let Φ be any (not necessarily continuous) Orlicz function on $[0, \infty)$. By Ψ we denote the complementary (or conjugate) Orlicz function of Φ (for more details and further references see [3], [11]). For all $f \in M$ (the set of all measurable functions on Δ), we define

$$M_\Phi(f) = \int_\Delta \Phi(|f|)d\mu;\ M_\Psi(f) = \int_\Delta \Psi(|f|)d\mu;$$
$$L_{M\Phi} = \{f \in M;\ M_\Phi(f) < \infty\};$$
$$N_\Phi(f) = \inf\{k > 0;\ M_\Phi(k^{-1}f) \leq 1\};$$
$$\|f\|_\Phi = \sup\left\{\int |fg|d\mu;\ M_\Psi(g) \leq 1\right\};$$
$$L_\Phi[N_\Phi] = \{f \in M;\ N_\Phi(f) < \infty\};$$
$$L_\Phi[\|\cdot\|_\Phi] = \{f \in M;\ \|f\|_\Phi < \infty\}.$$

It is well-known that $L_\Phi[N_\Phi]$ and $L_\Phi[\|\cdot\|_\Phi]$ are the same, when regarded as point sets. We denote this set by L_Φ. Furthermore, identifying μ-almost everywhere equal functions in M, it follows that $L_\Phi[N_\Phi]$ and $L_\Phi[\|\cdot\|_\Phi]$ are normed Riesz spaces, with Riesz norms N_Φ and $\|\cdot\|_\Phi$ respectively. The set $L_{M\Phi}$ is in general not a linear space but in [3], lemma 6.2 it is proved that $L_{M\Phi}$ is a lattice. We shall show now that $L_\Phi[N_\Phi]$ and $L_\Phi[\|\cdot\|_\Phi]$ are semi-M-spaces. To this end, note that

$$N_\Phi(f) \leq \|f\|_\Phi \leq 1 + M_\Phi(f)$$

for all $f \in L_\Phi$ (cf. [11] p. 79, th. 1). Furthermore, if $f_1, f_2 \in L_\Phi^+$ satisfy $\|f_1\|_\Phi = \|f_2\|_\Phi = 1$ (or $N_\Phi(f_1) = N_\Phi(f_2) = 1$), then $f_1, f_2 \in L_{M\Phi}$. Hence $f = \sup(f_1, f_2)$ satisfies $f \in L_{M\Phi}$. Let now $\{g_n\colon n = 1, 2, \ldots\}$ be a sequence of functions in L_Φ^+ such that $f \geq g_n \downarrow 0$. Then, according to the theorem on dominated convergence of integrals, $M_\Phi(g_n) \downarrow 0$. Thus M_Φ satisfies all conditions of Lemma 2.3 (ii) and therefore $L_\Phi[N_\Phi]$ and $L_\Phi[\|\cdot\|_\Phi]$ are semi-M-spaces.

REMARK: In view of th. 2.2 and ex. 2.4 it follows that $L^*_{\Phi,s}[N_\Phi]$ and $L^*_{\Phi,s}[\|\cdot\|_\Phi]$ are abstract L-spaces. Hence th. 2.2 is a generalization of ([3], th. 6.4). The open question, stated in ([3], remark (i), p. 62) is now also solved, i.e., it is true that $L^*_{\Phi,s}[\|\cdot\|_\Phi]$ is an abstract L-space.

More examples will be presented in section 6. We finally note that it can be proved similarly as above, that not only Orlicz spaces, but more generally even the modulared spaces in the sense of Nakano are semi-M-spaces.

3. The case that L_ρ has the principal projection property

Throughout this section L_ρ will again be a normed Riesz space. Let Π be an exhausting sequence in L_ρ, i.e. $\Pi = \{K_1, K_2, \ldots\}$ where all K_n are bands in L_ρ such that $K_1 \subset K_2 \subset \ldots$, and such that the band generated by the set of all K_n is L_ρ itself (note that such a sequence always exists, for instance $K_n = L_\rho$ for all n is such a sequence). In this situation we shall denote by I_Π the norm closure of the order ideal generated by the set of all K_n. In general I_Π will not be the whole of L_ρ.

DEFINITION 3.1: *The exhausting sequence Π will be called an exhausting projection sequence (abbreviated as e.p. sequence) if every band $K_n \in \Pi$ is a projection band (so $L_\rho = K_n \oplus K_n^d$ for all n, where K_n^d denotes the disjoint complement of K_n).*

LEMMA 3.2: *Let Π be an e.p. sequence and let $S \in L_\rho^*$ satisfy $S \geq \Theta$ and $S(u) = 0$ for all $u \in I_\Pi$. Then $S \in L_{\rho,s}^*$ (so $\{I_\Pi\}^\perp$ is a band in $L_{\rho,s}^*$ in view of* [6], *th.* 21.1(*i*)).

PROOF: Let $g \in L_\rho^+$ be given. Since $\Pi = (K_1, K_2, \ldots)$ is an e.p. sequence it follows that g has a unique decomposition $g = g_n + g_n'$, where $g_n \in K_n$ and $g_n' \in K_n^d$ with $g_n, g_n' \geq 0$ for all n. Also, since Π is exhausting, we have $g_n \uparrow g$. By assumption $S(g_n) = 0$ for all n. Thus, according to lemma 2.1, it follows that $S \in L_{\rho,s}^*$.

If L_ρ has the p.p.p., then there exist many non-trivial e.p. sequences. This will become clear from the next lemma.

LEMMA 3.3. *Assume that L_ρ has the p.p.p., and let $\{f_n\colon n = 1, 2, \ldots\} \subset L_\rho^+$ satisfy $f_n \downarrow 0$. Then there exists for every $\epsilon > 0$ an e.p. sequence $\Pi_\epsilon = \{K_1, K_2, \ldots\}$ such that $\rho(f_{nn}) \leq \epsilon$ for all n, where f_{nn} denotes the component of f_n in the band K_n.*

PROOF: Let $\epsilon > 0$ be given. Without loss of generality we may assume that $\rho(f_1) > 0$. Setting

$$g = \epsilon\{\rho(f_1)\}^{-1} f_1;\ v_n = \sup\{g - f_n, 0\} = (g - f_n)^+,$$

it follows that $0 \leq v_n \uparrow g$. Hence $\{[v_1], [v_2], \ldots\}$, where $[v_n]$ stands for the principal band generated by v_n ($n = 1, 2, \ldots$), is an exhausting sequence in the band $[g] = [f_1]$. Thus, since L_ρ has the p.p.p., it follows that $\{[v_1], [v_2], \ldots\}$ is an e.p. sequence in $[g]$. Now note that $[g]^d = [f_1]^d$ is a projection band in L_ρ (again since L_ρ has the p.p.p.). Hence, setting

$$K_n = [v_n] \oplus [f_1]^d$$

for $n = 1, 2, \ldots$, $\Pi_\epsilon = \{K_1, K_2, \ldots\}$ is an e.p. sequence in L_ρ. Let f_{nn} be the component f_n in K_n. We have $0 \leq f_{nn} \leq g$ for all n. To prove this, first note that $f_{nn} \in [v_n]$, since $f_{nn} \leq f_n \leq f_1$, so the component of f_{nn} in $[f_1]^d$ is zero. Thus

$$f_{nn} = \sup \{\inf (f_n, mv_n) \colon m = 1, 2, \ldots\}$$

for all n. Therefore it suffices to prove that $\inf (f_n, mv_n) \leq g$ for all m (n fixed). To this end; note that $h \in L_\rho$, $f \in L_\rho^+$ implies $(h - f)^- \geq h^-$, which implies

$$(mg - mf_n - f_n)^- \geq (mg - mf_n)^- = m(g - f_n)^- \geq (g - f_n)^-.$$

Hence

$$\begin{aligned} \inf (0, mg - mf_n - f_n) = -(mg - mf_n - f_n)^- &\leq -(g - f_n)^- \\ &\leq (g - f_n)^+ - (g - f_n)^- = g - f_n, \end{aligned}$$

and so

$$\inf (f_n, mg - mf_n) \leq g.$$

Now note that

$$\begin{aligned} \inf (f_n, mv) &= \inf (f_n, m \sup ((g - f_n), 0)) \\ &= \sup \{(\inf (f_n, m(g - f_n)), (\inf (f_n, 0))\} = \sup \{0, \inf(f_n, mg - mf_n)\} \leq g. \end{aligned}$$

By what was proved above. Thus $0 \leq f_{nn} \leq g$ for all n. It follows that $\rho(f_{nn}) \leq \rho(g) \leq \epsilon$, which is the desired result.

Once more assume that Π is an exhausting sequence in L_ρ, and let the order ideal I_Π be as described in the beginning of this section. Furthermore, let d_Π be the Riesz norm in the factor space L_ρ / I_Π (in the sequel we shall consider d_Π also as a Riesz semi-norm on L_ρ). In this situation it is possible to give a useful characterization for semi-M-spaces, provided L_ρ has the principal projection property. First we prove the following lemma.

LEMMA 3.4: *Let $f \in L_\rho^+$ be given, and let $\Pi = \{K_1, K_2, \ldots\}$ be any exhausting projection sequence. Then $d_\Pi(f) = \lim_{n\to\infty} \rho(f'_{\Pi n})$, where $f'_{\Pi n}$ denotes the component of f in the band K_n^d $(n = 1, 2, \ldots)$.*

PROOF; For all n, let $f_{\Pi n}$ be the component of f in K_n. Since obviously $f_{\Pi n} \in I_\Pi$ for all n, we have $d_\Pi(f) \le \rho(f - f_{\Pi n}) = \rho(f'_{\Pi n})$. Hence

$$d_\Pi(f) \le \lim_{n\to\infty} \rho(f'_{\Pi n}).$$

For the converse inequality let $\epsilon > 0$ be given. Then there exists an element $g' \in I_\Pi$ such that $d_\Pi(f) > \rho(f - g') - \frac{1}{2}\epsilon$. Setting

$$g = \sup\{\inf(f, g'), 0\},$$

it follows that $0 \le g \le f$ and $g \in I_\Pi$. Moreover, $0 \le f - g \le |f - g'|$. Hence

$$\rho(f - g) \le \rho(f - g') < d_\Pi(f) + \tfrac{1}{2}\epsilon.$$

Next, the definition of I_Π implies the existence of a positive integer n_0 and an element $h \in K_{n_0}$ such that $\rho(h - g) < \frac{1}{2}\epsilon$. Similarly as above it can be shown that h can be chosen such that $0 \le h \le g$. Since Π is exhausting it follows that $h \in K_n$ for all $n \ge n_0$. Thus $0 \le h \le g \le f$, so $0 \le h \le f_{\Pi n}$ for all $n \ge n_0$. It follows that

$$d_\Pi(f) > \rho(f - g) - \tfrac{1}{2}\epsilon > \rho(f - h) - \epsilon \ge \rho(f - f_{\Pi n}) - \epsilon = \rho(f'_{\Pi n}) - \epsilon$$

for all $n \ge n_0$. This holds for all $\epsilon > 0$, so

$$d_\Pi(f) \ge \lim_{n\to\infty} \rho(f'_{\Pi n}),$$

which completes the proof.

The next theorem is our second main result.

THEOREM 3.5: *Let L_ρ have the principal projection property. Then L_ρ is a semi-M-space if and only if L_ρ/I_Π is an abstract M-space for any exhausting projection sequence Π.*

PROOF: (i) Assume that L_ρ is a semi-M-space, and let Π be an e.p.

sequence. In view of lemma 3.2, $I_{\Pi}^{\perp}$ is a band in $L_{\rho,s}^{*}$. Moreover, in view of theorem 2.2 $L_{\rho,s}^{*}$ is an abstract L-space. Hence, $I_{\Pi}^{\perp}$ and therefore $(L_\rho/I_\Pi)^*$ are also abstract L-spaces. It follows that $(L_\rho/I_\Pi)^{**}$ is an abstract M-space. Since L_ρ/I_Π can be considered as a Riesz subspace of $(L_\rho/I_\Pi)^{**}$, the space L_ρ/I_Π is also an abstract M-space.

(ii) Assume that L_ρ/I_Π is an abstract M-space for any e.p. sequence Π. To show that L_ρ is a semi-M-space, let $f_1, f_2 \in L_\rho^+$ be given such that $\rho(f_1) = \rho(f_2) = 1$, and set $f = \sup(f_1, f_2)$. Moreover, let $\{g_n: n = 1, 2, \ldots\} \subset L_\rho^+$ be given such that $f \geq g_n \downarrow 0$. To show that $\lim \rho(g_n) \leq 1$, let $\epsilon > 0$ be given. According to lemma 3.3 there exists an e.p. sequence Π_ϵ such that $\rho(g_{nn}) \leq \epsilon$ for all n (where g_{nn} is defined similarly as f_{nn} in lemma 3.3). Now we have

$$d_{\Pi_\epsilon}(f) = \max(d_{\Pi_\epsilon}(f_1), d_{\Pi_\epsilon}(f_2)) \leq \max(\rho(f_1), \rho(f_2)) = 1,$$

since L_ρ/I_{Π_ϵ} is an abstract M-space. It follows that

$$\begin{aligned}\lim \rho(g_n) &= \lim \rho(g_{nn} + g'_{nn}) \leq \lim \rho(g_{nn}) + \lim \rho(g'_{nn}) \\ &\leq \epsilon + \lim \rho(f'_{\Pi n}) = d_{\Pi_\epsilon}(f) + \epsilon \leq 1 + \epsilon\end{aligned}$$

in view of lemma 3.4. This holds for all $\epsilon > 0$, so L_ρ is a semi-M-space.

One might ask whether there exist normed Riesz spaces possessing the semi-M property but not the principal projection property. As follows from the next example such spaces do indeed exist.

Example 3.6. *The space* $C(X)$

Let X be any compact topological space and $C(X)$ the space of all real-valued continuous functions on X. We define $f \leq g$ if $f(x) \leq g(x)$ for all $x \in X$ ($f, g \in C(X)$). It is well-known that $C(X)$, partially ordered in this manner, is a Riesz space. In addition, if we define $\|f\| = \sup\{|f(x)|; x \in X\}$, then $C(X)$ becomes a normed Riesz space with Riesz norm $\|\cdot\|$. Since $C(X)$ is an abstract M-space, it follows that $C(X)$ is also a semi-M-space, but it is well-known that $C(X)$ does not have the p.p.p. in general. For example if $X = [0, 1]$ with the ordinary topology, then $C(X)$ does not have the p.p.p.

4. The case that $(L_\rho^a)^\perp = L_{\rho,s}^*$

Once more, let L_ρ be a normed (real) Riesz space. Furthermore, let L_ρ^a be the norm closed order ideal of L_ρ consisting of all elements having an absolutely continuous norm. We recall that L_ρ^a and $L_{\rho,s}^*$ satisfy $L_\rho^a = {}^\perp\{L_{\rho,s}^*\}$.

THEOREM 4.1. *If L_ρ satisfies $\{L_\rho^a\}^\perp = L_{\rho,s}^*$, then L_ρ is a semi-M-space if and only if L_ρ/L_ρ^a is an abstract M-space.*

PROOF: This is an immediate consequence of theorem 2.2. Indeed, note that

$$(L_\rho/L_\rho^a)^* = \{L_\rho^a\}^\perp = L_{\rho,s}^*$$

in the present case, so L_ρ is a semi-M-space if and only if $(L_\rho/L_\rho^a)^*$ is an abstract L-space. By the same arguments as used in theorem 3.5 it follows that $(L_\rho/L_\rho^a)^*$ is an abstract L-space if and only if L_ρ/L_ρ^a is an abstract M-space.

REMARK: It can be proved that if L_ρ/L_ρ^a is an abstract M-space, then L_ρ is a semi-M-space. We leave the rather technical but straightforward proof to the reader.

5. Normed Köthe spaces

In the next section we investigate whether the semi-M property is satisfied in some well-known Riesz spaces. All these spaces will be normed Köthe spaces. For that reason we recall in the present section some facts from the theory of normed Köthe spaces. For the general theory we refer to ([12], Ch. 15).

From now on (Δ, Γ, μ) will always be a σ-finite measure space. By M we denote the set of all real-valued measurable functions on Δ (the values $+\infty$ and $-\infty$ are allowed for functions in M). The μ-almost everywhere equal functions in M are identified in the usual way. By ρ we shall denote a function norm on M^+, so ρ satisfies: (i) $0 \leq \rho(u) \leq \infty$ for all $u \in M^+$, $\rho(u) = 0$ if and only if $u = 0$ μ-almost everywhere, (ii) $\rho(au) = a\rho(u)$ for all $a \geq 0$ and for all $u \in M^+$, (iii) $\rho(u+v) \leq \rho(u) + \rho(v)$ for all $u, v \in M^+$, (iv) $0 \leq u \leq v$, $u, v \in M^+$ implies $\rho(u) \leq \rho(v)$. The function norm ρ is extended to the whole of M by setting $\rho(f) = \rho(|f|)$ for all $f \in M$. The set L_ρ is defined by

$$L_\rho = \{f \in M : \rho(f) < \infty\}.$$

It is obvious that L_ρ, defined in this way, is a normed Riesz space with

Riesz norm ρ (this is due to the fact that μ-a.e. equal functions are identified and to the fact that $\rho(f) < \infty$ implies $|f(x)| < \infty$ μ-a.e. on Δ). Furthermore, it is well-known that the space L_ρ is Dedekind complete (cf. [8], ex. 23.3 (iv)). Hence, L_ρ has the principal projection property. From the preceding sections we derive immediately the following theorem.

THEOREM 5.1: *Let L_ρ be a normed Köthe space. Then the following conditions are equivalent:*
(a) *L_ρ is a semi-M-space.*
(b) *If $u_1, u_2 \in L_\rho^+$ with $\rho(u_1) = \rho(u_2) = 1$, and if $\{\Delta_n : n = 1, 2, \ldots\}$ is a sequence of μ-measurable sets such that $\Delta_n \downarrow \phi$ and $u = \sup(u_1, u_2)$, then $\lim \rho(uX_{\Delta_n}) \leq 1$ where X_{Δ_n} denotes the characteristic function of Δ_n.*
(c) *$L_{\rho,s}^*$ is an abstract L-space.*
If, in addition, L_ρ satisfies $\{L_\rho^a\}^\perp = L_{\rho,s}^*$, then (a), (b) *and* (c) *are also equivalent to*
(d) *L_ρ/L_ρ^a is an abstract M-space.*

In the sequel the order ideal L_ρ^b of L_ρ will be of great use. This order ideal is the norm closed ideal spanned by all essentially bounded functions f in L_ρ such that $\mu(\text{supp}(f)) < \infty$, where $\text{supp}(f) = \{x \in \Delta;\ f(x) \neq 0\}$. We note that supp (f) is not uniquely determined since we are working with equivalence classes of functions. However if $f_1 = f_2$ μ-a.e. on Δ, then supp (f_1) and supp (f_2) differ at most a μ-null set. In [3] lemma 2.1 it was shown that $L_\rho^a \subset L_\rho^b$ does always hold, but the equality $L_\rho^a = L_\rho^b$ can also occur.

LEMMA 5.2: *If the measure space (Δ, Γ, μ) is purely atomic, all atoms having equal measure, then $L_\rho^a = L_\rho^b$.*

PROOF: We have to show that any essentially bounded function f having a support A of finite measure is a member of L_ρ^a. This is obvious since $A \supset \Delta_n \downarrow \phi$, where $A, \Delta_n \in \Gamma (n = 1, 2, \ldots)$ and $\mu(A) < \infty$ implies $\mu(\Delta_n) = 0$ for $n \geq N_0$ (N_0 some positive constant) in the present case.

We finally note that if $L_\rho^a = L_\rho^b$, then $\{L_\rho^a\}^\perp = L_{\rho,s}^*$ (cf. [3], th. 2.2). However, $\{L_\rho^a\}^\perp = L_{\rho,s}^*$ does not always imply $L_\rho^a = L_\rho^b$. This will follow from ex. 6.2 below.

6. Examples

Throughout this section (Δ, Γ, μ) and M will be as defined in section 5, unless stated otherwise. Note first that the Orlicz spaces L_Φ are normed Köthe spaces with the semi-M property (see ex. 2.4).

Example 6.1. *The space of Gould and its associate space*
Assume that (Δ, Γ, μ) is atomless and $\mu(\Delta) = \infty$. Setting

$$\rho(f) = \sup\left\{\int_E |f| d\mu : \mu(E) = 1, E \in \Gamma\right\}$$

for all $f \in M^+$, it follows that ρ is a function norm (see [7], ex. 1.2). The space L_ρ is called the space of Gould. It is well-known that the first associate norm ρ' of ρ satisfies $\rho'(f) = \sup\{\|f\|_1, \|f\|_\infty\}$ for all $f \in M$, where $\|\cdot\|_1$ and $\|\cdot\|_\infty$ denote the familiar L_1 and L_∞ norms respectively (cf. [7], th. 3.5 or [12], exercise 71.3). Defining the Orlicz functions Φ_1, Φ_2 and Φ by

$$\Phi_1(x) = x \text{ for all } x \geq 0,$$

$$\Phi_2(x) = 0 \text{ for } 0 \leq x \leq 1;\ \Phi_2(x) = \infty \text{ for } x > 1,$$

$$\Phi(x) = x \text{ for } 0 \leq x \leq 1;\ \Phi(x) = \infty \text{ for } x > 1,$$

it follows that $\Phi(x) = \max\{\Phi_1(x), \Phi_2(x)\}$ for all $x \geq 0$. Moreover,

$$N_{\Phi_1}(f) = \|f\|_1 \text{ and } N_{\Phi_2}(f) = \|f\|_\infty$$

for all $f \in M$. Hence, according to ([3], th. 5.4), we have

$$N_\Phi(f) = \max\{N_{\Phi_1}(f), N_{\Phi_2}(f)\} = \rho'(f)$$

for all $f \in M$. This implies that $L'_\rho = L_\Phi[N_\Phi]$. Let Ψ be the complementary Orlicz function of Φ, so

$$\Psi(x) = 0 \text{ for } 0 \leq x \leq 1;\ \Psi(x) = x - 1 \text{ for } x \geq 1$$

It follows from the general Orlicz space theory that the first associate space of $L_\Phi[N_\Phi]$ (which equals L'_ρ) is $L_\Psi[\|\cdot\|_\Psi]$. Hence $L_\rho = L_\Psi[\|\cdot\|_\Psi]$. Thus, in view of example 2.4, L_ρ and L'_ρ are semi-M-spaces, and hence $L^*_{\rho,s}$ and $L^*_{\rho',s}$ are abstract L-spaces.

Example 6.2. *The space of Korenblyum, Kreĭn and Levin*

In this example, let Δ be the closed interval $(x: 0 \leq x \leq 1)$ and let μ be Lebesgue measure. Defining

$$\rho(f) = \sup\left(h^{-1}\int_0^h |f|d\mu : 0 < h \leq 1\right),$$

for all $f \in M^+$, it follows that ρ is a function norm. The space L_ρ is called the space of Korenblyum, Kreĭn and Levin (cf. [7], ex. 1.3). We note that $L_\rho^b = L_\rho$ and that $L_\rho^a \neq L_\rho$. The proof of the first equality is routine and left to the reader, for $L_\rho^a \neq L_\rho$ we refer to ([7], th. 5.2). Also it follows from ([7], th. 5.3) combined with ([12], th. 72.6 and th. 72.7) that $\{L_\rho^a\}^\perp = L_{\rho,s}^*$. We shall prove now that L_ρ is not a semi-M-space. To this end, divide $[\frac{1}{2}, 1]$ into 10 equal interval $E_{1,1}, \ldots, E_{1,10}$ (from the left to the right), then divide $[\frac{1}{4}, \frac{1}{2}]$ into 100 equal intervals $E_{2,1}, \ldots, E_{2,100}$, and so on. Generally, divide $[2^{-n}, 2^{-n+1}]$ into 10^n equal intervals $E_{n,1}, \ldots, E_{n,10^n}$. Let

$$E_1 = \cup\{E_{i,j}; j \text{ odd}\}; E_2 = \cup\{E_{i,j}; j \text{ even}\}.$$

Note that $\mu(E_1) = \mu(E_2) = \frac{1}{2}$, and that for any $h > 0$ we have $\mu(E_i \cap [0, h]) < \frac{2}{3}h (i = 1, 2)$.

Defining $u_i = \frac{3}{2}X_{E_i}$ $(i = 1, 2)$ it follows that $\rho(u_i) \leq 1$ for $i = 1, 2$. Note that $u = \sup(u_1, u_2)$ is identically equal to $\frac{3}{2}$. Set $v_n = uX_{F_n}$ $(n = 1, 2, \ldots)$, where $F_n = [0, n^{-1}]$. It is routine to prove that $\rho(u) = \rho(v_n) = \frac{3}{2}$ for all n. Since $u \geq v_n \downarrow 0$, it follows that L_ρ is not a semi-M-space. In view of th. 5.1 we can, therefore, conclude that L_ρ/L_ρ^a is not an abstract M-space and also that $L_{\rho,s}^*$ is not an abstract L-space.

In the sections 8 and 9 we shall investigate two other important classes of normed Köthe spaces (the Lorentz space M_Λ and L_Λ). Before doing so we introduce the rearrangement invariant Köthe spaces.

7. Rearrangement invariant Köthe spaces

Most of the important normed Köthe spaces are examples of the so-called rearrangement invariant Köthe spaces. In this section we derive some general properties of spaces of this type. We first recall some definitions.

DEFINITION 7.1: *Let* $f \in M$. *Then*

(i) $\lambda_f(y) = \mu\{x \in \Delta\colon |f(x)| > y\}$ *for all* $y \geq 0$.

(ii) $f^*(x) = \inf\{y\colon \lambda_f(y) \leq x\}$ *for all* $x \geq 0$, *where* $\inf\{\phi\} = +\infty$.

The function f^*, *defined on* $\mathbb{R}^+$, *is called the non-increasing rearrangement of* $|f|$.

We note that the value $+\infty$ is allowed for f^*. Furthermore it is obvious that f^* is a non-increasing function on $\mathbb{R}^+$. We present some examples.

(a) Let (Δ, Γ, μ) be an arbitrary measure space, and let $A \in \Gamma$, where $\mu(A) < \infty$ as well as $\mu(A) = \infty$ is allowed. Setting $f = X_A$, it follows that $\lambda_f(y) = \mu(A)$ for $0 \leq y < 1$; $\lambda_f(y) = 0$ for $y \geq 1$. Hence, for $\mu(A) < \infty$ we have $f^*(x) = 1$ for $0 \leq x \leq \mu(A)$ and $f^*(x) = 0$ for $x > \mu(A)$, and for $\mu(A) = \infty$ we have $f^*(x) = 1$ for all x.

(b) Let $f(x) = tg\, x$ for $x \in \mathbb{R}^+$, $x \neq \frac{1}{2}\pi + k\pi$ and $f(x) = 0$ for $x = \frac{1}{2}\pi + k\pi$ $(k = 1, 2, \ldots)$. Then $\lambda_f(y) = \infty$ for all $y \geq 0$ and hence $f^* = \infty$ for all $x \geq 0$.

(c) Let $f(x) = x(x+1)^{-1}$ on $\mathbb{R}^+$. Then $\lambda_f(y) = \infty$ for $0 \leq y < 1$; $\lambda_f(y) = 0$ for $y \geq 1$. Hence $f^*(x) = 1$ for all $x \in \mathbb{R}^+$.

DEFINITION 7.2: *A function norm* ρ *is said to be rearrangement invariant (abbreviated as r.i.) whenever it follows from* $f, g \in M$ *and* $f^* \leq g^*$ *on* $\mathbb{R}^+$ *that* $\rho(f) \leq \rho(g)$.

Let ρ be a r.i. function norm. We prove first that for $f \in L_\rho^+$ we have $f^*(x) < \infty$ for all $x > 0$. For this purpose let $A_n = \{x \in \Delta\colon f(x) > n\}$ for $n = 1, 2, \ldots$. If $\mu(A_n) = \infty$ for all n, then $\lambda_f(y) = \infty$ for all y, so $f^*(x) = \infty$ for all x. The function $f_0 = \infty \cdot X_\Delta$ also satisfies $f_0^*(x) = \infty$ for all x, so $f^* = f_0^*$. This implies $\rho(f) = \rho(f_0) = \infty$, contradicting $f \in L_\rho$. Hence we must have $\mu(A_n)$ finite from some index on, so $\mu(A_n) \downarrow \mu(A)$ for $A = \cap A_n$. If $\mu(A) = \alpha > 0$, then $f \geq nX_A$ for $n = 1, 2, \ldots$, so $\rho(f) \geq n\rho(X_A)$ for all n. But $\mu(A) = \alpha > 0$ implies $\rho(X_A) > 0$, so $\rho(f) = \infty$, which again contradicts $f \in L_\rho$. Hence we have $\mu(A_n) \downarrow 0$, i.e., $\lambda_f(n) \downarrow 0$ as $n \to \infty$. This implies $\lambda_f(y) \downarrow 0$ as $y \to \infty$, and so

$$f^*(x) = \inf\{y\colon \lambda_f(y) \leq x\}$$

is finite for every $x > 0$.

Examples of r.i. function norms are the L_p-norms $(1 \leq p \leq \infty)$ and more generally the Orlicz norms N_Φ and $\|\cdot\|_\Phi$, defined in example 2.4. An example of a function norm which fails to be r.i. is the norm of

Korenblyum, Krein and Levin, defined in example 6.2. For more details on r.i. norms we refer to [5].

Let now ρ be a non-trivial r.i. function norm, and let L_ρ, L_ρ^a and L_ρ^b be as defined in section 5. Note that ρ is saturated and so $X_A \in L_\rho^b \subset L_\rho$ for any $A \in \Gamma$ satisfying $0 < \mu(A) < \infty$. Also note that f, $g \in M$, $f^* = g^*$ on $\mathbb{R}^+$ implies $\rho(f) = \rho(g)$.

In the remaining part of this section we shall assume (Δ, Γ, μ) to be an atomless measure space.

LEMMA 7.3: (i) *If* $X_A \in L_\rho^a$ *for some* $A \in \Gamma$, $0 < \mu(A) < \infty$, *then* $X_B \in L_\rho^a$ *for all* $B \in \Gamma$ *satisfying* $\mu(B) < \infty$.

(ii) *We have either* $L_\rho^a = \{0\}$ *or* $L_\rho^a = L_\rho^b$.

PROOF: (i) Let $B \in \Gamma$ satisfy $\mu(B) < \infty$ and let $\{B_n: n = 1, 2, \ldots\} \subset \Gamma$ be such that $B \supset B_n \downarrow \phi$. Then $\mu(B_n) \downarrow 0$, so we may assume that $\mu(B_n) \leq \mu(A)$ holds for all n. Since (Δ, Γ, μ) is atomless, there exists a sequence $\{A_n: n = 1, 2, \ldots\}$ in Γ such that $\mu(A_n) = \mu(B_n)$ for all n, and $A \supset A_n \downarrow \phi$. Then $X_{A_n}^* = X_{B_n}^*$ for all n, and on account of $X_A \in L_\rho^a$ it follows that

$$\rho(X_{B_n}) = \rho(X_{A_n}) \downarrow 0 \text{ as } n \to \infty,$$

so $X_B \in L_\rho^a$.

(ii) Obvious from part (i).

LEMMA 7.4: (i) *If* $L_\rho^a = \{0\}$, *there exists a constant* $c > 0$ *such that* $\rho(X_B) \geq c$ *for all* $B \in \Gamma$ *satisfying* $\mu(B) > 0$.

(ii) *If* $L_\rho^a = \{0\}$, then $c\|f\|_\infty \leq \rho(f)$ *for all* $f \in M$ *(c is the constant of* (i)). *Hence* $L_\rho^a = \{0\}$ *implies* $L_\rho \subset L_\infty$.

PROOF: (i) Let $A \in \Gamma$ satisfy $0 < \mu(A) < \infty$. Then $X_A \notin L_\rho^a$, so there exists a constant $c > 0$ and a sequence $\{A_n: n = 1, 2, \ldots\}$ in Γ such that $A \supset A_n \downarrow \phi$ and $\rho(X_{A_n}) \geq c$ for all n. Note that $\mu(A) < \infty$ implies $\mu(A_n) \downarrow 0$. Now, let $B \in \Gamma$ satisfy $\mu(B) > 0$, so $\mu(B) \geq \mu(A_{n_0})$ for some n_0. This implies $X_B^* \geq X_{A_{n_0}}^*$, so $\rho(X_B) \geq \rho(X_{A_{n_0}}) \geq c$, which is the desired result.

(ii) Let $f \in M$ be given. If $\|f\|_\infty = 0$ there is nothing to prove, so assume that $\|f\|_\infty > 0$. Take α such that $0 < \alpha < \|f\|_\infty$. There exists a set $A \in \Gamma$ such that $\mu(A) > 0$ and $|f| \geq \alpha X_A$. Hence

$$\rho(f) \geq \rho(f X_A) \geq \alpha \rho(X_A) \geq \alpha c.$$

This holds for all $\alpha < \|f\|_\infty$, so $\rho(f) \geq c\|f\|_\infty$.

8. The Lorentz space M_Λ

In the next section we shall investigate the semi-M property for r.i. function norms. Before doing so we present a class of r.i. spaces from which many counterexamples can be derived.

Let (Δ, Γ, μ) be an arbitrary σ-finite measure space. Furthermore, let Λ be a real-valued function on $[0, \infty)$ satisfying:

(i) $\Lambda(0) = 0$, $\Lambda(x) > 0$ if $x > 0$, Λ is right-continuous at zero,

(ii) Λ is non-decreasing and concave on $[0, \infty)$.

It follows that Λ is continuous on the whole of $[0, \infty)$. Furthermore, note that it is not excluded that Λ is bounded on $[0, \infty)$. Setting now

$$\|f\|_M = \sup \{\{\Lambda(\mu(E))\}^{-1} \int_E |f| d\mu : E \in \Gamma, 0 < \mu(E) < \infty\}$$

for all $f \in M$, it follows that $\|\cdot\|_M$ is a function norm on M. For details we refer to [2] or [10]. The normed Köthe space generated by $\|\cdot\|_M$ will be denoted by M_Λ. In ([10], th. 3.3) it is proved that

$$\|f\|_M = \sup \left\{(\Lambda(x))^{-1} \int_0^x f^*(t) dt : x > 0\right\}$$

for all $f \in M$, and it is obvious therefore that $\|\cdot\|_M$ is a r.i. function norm. Now, let M_Λ^a be the order ideal in M_Λ consisting of all functions having an absolutely continuous norm, and let M_Λ^b be the norm closed order ideal spanned by all essentially bounded functions in M_Λ having a support of finite measure.

Lemma 8.1: *Assume that (Δ, Γ, μ) contains an atomless set of positive measure. Then $M_\Lambda^a = M_\Lambda^b$ if and only if $\lim x/\Lambda(x) = 0$ as $x \downarrow 0$ (or equivalently $\Lambda'(0) = \infty$, where Λ' denotes the right derivative of Λ).*

Proof: (a) Assume that $\lim x/\Lambda(x) = 0$ as $x \downarrow 0$ and let $A \in \Gamma$, $0 < \mu(A) < \infty$ be given. It suffices to show that $X_A \in M_\Lambda^a$. To this end let $\{A_n : n = 1, 2, \ldots\}$ in Γ satisfy $A \supset A_n \downarrow \phi$. Then

$$\|X_{A_n}\|_M = \sup \left\{(\Lambda(x))^{-1} \int_0^x X_{A_n}^*(t) dt : x > 0\right\}$$

$$= \mu(A_n)/\Lambda(\mu(A_n)),$$

so $\|X_{A_n}\|_M \to 0$ as $A_n \downarrow \phi$, since $\mu(A_n) \downarrow 0$. Thus $M_\Lambda^a = M_\Lambda^b$.

(b) Assume that $\lim x/\Lambda(x) = c > 0$ as $x \downarrow 0$. Let $A \in \Gamma$ be an atomless set such that $0 < \mu(A) < \infty$. Then there exists a sequence $\{A_n: n = 1, 2, \ldots\} \subset \Gamma$ such that $A \supset A_n \downarrow \phi$ and $\mu(A_n) > 0$ for all n. Similarly as in part (a) it follows now that $\|X_{A_n}\|_M \geq c > 0$ for all n. Hence $X_A \notin M_\Lambda^a$, so $M_\Lambda^a \neq M_\Lambda^b$. This concludes the proof.

We present three examples of normed Köthe spaces L_ρ having a r.i. function norm ρ but failing to have the semi-M property (and thus $L_{\rho,s}^*$ is not an abstract L-space in these cases).

EXAMPLE 8.2: Let $\Delta = (0, 2]$ be provided with Lebesgue measure, and let

$$\Lambda(x) = x^p, 0 < p < 1.$$

Furthermore, define

$$\begin{aligned} &f_1(x) = px^{p-1} = \Lambda'(x) \text{ for } x \in (0, 1]; f_1(x) = 0 \text{ for } x \in (1, 2], \\ &f_2(x) = p(x-1)^{p-1} = \Lambda'(x-1) \text{ for } x \in (1, 2]; f_2(x) = 0 \text{ for } x \in (0, 1], \\ &f(x) = \max\{f_1(x), f_2(x)\} \text{ for all } x \in (0, 2]. \end{aligned}$$

It is easily seen that $\|f_1\|_M = \|f_2\|_M = 1$. Next, define

$$\Delta_n = (0, 2^{-n}) \cup (1, 1 + 2^{-n})$$

for $n = 1, 2, \ldots$, and set $g_n = fX_{\Delta_n}$ for all n. It follows that $f \geq g_n \downarrow 0$. Now, in view of

$$\int_{\Delta_n} g_n(x)dx = 2\Lambda(2^{-n})$$

for all n, we have $\|g_n\|_M \geq 2\Lambda(2^{-n})/\Lambda(2^{-n+1}) = 2^{1-p}$. Hence

$$\lim \|g_n\|_M \geq 2^{1-p} > 1,$$

and so M_Λ is not a semi-M-space in this case. We finally note that due to lemma 8.1 $M_\Lambda^a = M_\Lambda^b$ holds in the present case.

EXAMPLE 8.3: Let $\Delta = \mathbb{R}$ be provided with Lebesgue measure, and let

$$\Lambda(x) = \log(x+1) \text{ for all } x \geq 0.$$

Furthermore, let

$$f_1(x) = (x+1)^{-1} = \Lambda'(x) \text{ for all } x \geq 0;\ f_1(x) = 0 \text{ for } x < 0,$$
$$f_2(x) = (1-x)^{-1} = \Lambda'(-x) \text{ for all } x \leq 0;\ f_2(x) = 0 \text{ for } x > 0,$$
$$f(x) = \max\{f_1(x), f_2(x)\} \text{ on } \mathbb{R}.$$

Then $\|f_1\|_M = \|f_2\|_M = 1$. Next, let $\Delta_n = (-\infty, -n) \cup (n, \infty)$ for $n = 1, 2, \ldots$, and set $g_n = fX_{\Delta_n}$ for all n. Again $f \geq g_n \downarrow 0$ holds. Computing the norm of g_n, we obtain $\|g_n\|_M = 2$ for all n. Hence, M_Λ is not a semi-M-space in the present case. We note that $M_\Lambda^a = \{0\}$ in the present case (according to lemma 8.1 and lemma 7.3 (ii)).

EXAMPLE 8.4: Let $\Delta = \mathbb{Z}$, the set of all integers, and let $\mu(n) = 1$ for all $n \in \mathbb{Z}$. Thus (Δ, Γ, μ) is a purely atomic measure space. Furthermore, let $\Lambda(x) = \log(x+1)$ as in the preceding example. Similarly as in the preceding example it can be shown that M_Λ is not a semi-M-space, now using the functions

$$f_1(n) = (n+1)^{-1} \text{ for } n = 1, 2, \ldots;\ f_1(n) = 0 \text{ for } n = 0, -1, -2, \ldots,$$
$$f_2(n) = (1-n)^{-1} \text{ for } n = -1, -2, \ldots;\ f_2(n) = 0 \text{ for } n = 0, 1, 2, \ldots,$$
$$f = \sup(f_1, f_2).$$

It follows that $M_\Lambda^a \neq M_\Lambda$ and hence $M_\Lambda^b \neq M_\Lambda$ in the present case (since $M_\Lambda^a = M_\Lambda^b$ in view of lemma 5.2). Furthermore, we note for use in the sequel that $X_\Delta \in M_\Lambda$ does not hold in the present case.

9. The semi-M property for rearrangement invariant Köthe spaces

As observed in the preceding section not every Köthe space having a r.i. norm is a semi-M-space. In this section it will be shown that under certain conditions on the measure space (Δ, Γ, μ) and on the order ideals L_ρ^a and L_ρ^b the space L_ρ is a semi-M-space.

First we consider the case $L_\rho^a = \{0\}$. Using lemma 5.2 it follows easily that (Δ, Γ, μ) does not contain any atoms in this case (in fact, the characteristic function of an atom is always a member of L_ρ^a).

THEOREM 9.1: *Let ρ be a r.i. function norm.*

(i) *If $L_\rho^a = \{0\}$ and $\mu(\Delta) < \infty$, then L_ρ is a semi-M-space.*

(ii) *If (Δ, Γ, μ) is atomless and $L_\rho^b = L_\rho$, then L_ρ is a semi-M-space.*

PROOF: (i) Let $f_1, f_2 \in L_\rho^+$ be given such that $\rho(f_1) = \rho(f_2) = 1$ and let $f = \sup(f_1, f_2)$. Since $L_\rho^a = \{0\}$ we have $L_\rho \subset L_\infty$ (cf. lemma 7.4 (ii)), so $f_1^*(0)$, $f_2^*(0)$ and $f^*(0)$ exist as finite positive numbers. In fact, note that any $g \in L_\rho$ satisfies $g^*(0) = \|g\|_\infty$. Hence, since $\|f\|_\infty = \|f_1\|_\infty$ or $\|f\|_\infty = \|f_2\|_\infty$ must hold it follows that $f^*(0) = f_1^*(0)$ or $f^*(0) = f_2^*(0)$. Assume that $f^*(0) = f_1^*(0)$ holds, and let ϵ satisfy $0 < \epsilon < 1$. Since f_1^* is right-continuous at zero there exists $\delta > 0$ such that

$$|f_1^*(0) - f_1^*(x)| = f_1^*(0) - f_1^*(x) < \epsilon f^*(0)$$

for all $x \in [0, \delta]$, so for these values of x we have

$$f^*(x)/f_1^*(x) \leq f^*(0)/f_1^*(\delta) \leq f^*(0)/(f^*(0) - \epsilon f^*(0)) = (1-\epsilon)^{-1}.$$

Hence

$$(1-\epsilon)f^* \leq f_1^* \leq f^*$$

on $[0, \delta]$ ($f_1^* \leq f^*$ is obvious since $f_1 \leq f$ on Δ). Now, let $\{\Delta_n : n = 1, 2, \ldots\}$ in Γ satisfy $\Delta_n \downarrow \phi$. Then $\mu(\Delta) < \infty$ implies $\mu(\Delta_n) \downarrow 0$, so we may assume that $\mu(\Delta_n) \leq \delta$ for all $n \geq n_\delta$. Observe now that

$$(fX_{\Delta_n})^* \leq f^*$$

on $[0, \mu(\Delta_n)] \subset [0, \delta]$, and $(fX_{\Delta_n})^* = 0$ on $(\mu(\Delta_n), \mu(\Delta)]$ for $n \geq n_\delta$. Thus

$$((1-\epsilon)fX_{\Delta_n})^* \leq ((1-\epsilon)f)^* \leq f_1^*$$

holds on $[0, \delta]$ for all $n \geq n_\delta$, and $((1-\epsilon)fX_{\Delta_n})^* = 0$ on $(\delta, \mu(\Delta)]$. Hence

$$\rho((1-\epsilon)fX_{\Delta_n}) \leq \rho(f_1) = 1$$

holds for all $n \geq n_\delta$, so $\lim \rho(fX_{\Delta_n}) \leq (1-\epsilon)^{-1}$ as $n \to \infty$. This holds for all $0 < \epsilon < 1$, so $\lim \rho(fX_{\Delta_n}) \leq 1$ as $n \to \infty$. This shows that L_ρ is a semi-M-space (cf. theorem 5.1).

(ii) Since (Δ, Γ, μ) is atomless we have either $L_\rho^a = L_\rho^b$ or $L_\rho^a = \{0\}$ by lemma 7.3. If $L_\rho^a = L_\rho^b$, then $L_\rho^b = L_\rho$ implies $L_\rho^a = L_\rho$, so L_ρ is a semi-M-space. Assume, therefore, that $L_\rho^a = \{0\}$. Again, let $f_1, f_2 \in L_\rho^+$ satisfy $\rho(f_1) = \rho(f_2) = 1$ and let $f = \sup(f_1, f_2)$. Also, let $\epsilon > 0$ be given. Then there exists a function $f_\epsilon \in L_\rho$ such that

$$0 \leq f_\epsilon \leq f;\ \mu(\mathrm{supp}\,(f_\epsilon)) < \infty;\ \rho(f - f_\epsilon) < \epsilon,$$

since $L_\rho^b = L_\rho$. Setting $A = \text{supp}(f_\epsilon)$ and considering the normed Köthe space $L_\rho(A, \Gamma_A, \mu_A)$, it follows from part (i) that $L_\rho(A, \Gamma_A, \mu_A)$ is a semi-M-space. Hence, if $\{\Delta_n : n = 1, 2, \ldots\} \subset \Gamma$ satisfies $\Delta_n \downarrow \phi$, then

$$\lim \rho(f_\epsilon X_{\Delta_n}) = \lim \rho(f_\epsilon X_{A\cap\Delta_n}) \leq 1 \text{ as } n \to \infty.$$

This implies that

$$\lim \rho(fX_{\Delta_n}) \leq \lim \rho(f_\epsilon X_{\Delta_n}) + \lim \rho((f - f_\epsilon)X_{\Delta_n}) \leq 1 + \epsilon \text{ as } n \to \infty.$$

This inequality holds for all $\epsilon > 0$, so L_ρ is a semi-M-space.

REMARK: If $\mu(\Delta) = \infty$, the condition $L_\rho = L_\rho^b$ in part (ii) of the preceding theorem cannot be changed into $L_\rho^a = \{0\}$. This follows from example 8.3, where a r.i. space M_Λ was considered satisfying $M_\Lambda^a = \{0\}$, but M_Λ is not a semi-M-space. We note that it follows from the last theorem that $M_\Lambda^b \neq M_\Lambda$ must hold in example 8.3.

EXAMPLE 9.2: Let (Δ, Γ, μ) be an atomless measure space such that $\mu(\Delta) < \infty$. Furthermore, let Λ be as defined in section 8 and assume in addition that $\Lambda'(0) < \infty$. Then it follows from lemma 7.3 and lemma 8.1 that the Lorentz space M_Λ, defined in section 8, satisfies $M_\Lambda^a = \{0\}$, and so M_Λ is a semi-M-space by theorem 9.1(i).

We finally note that there exist semi-M-spaces L_ρ satisfying $L_\rho^a = \{0\}$ and $L_\rho^b \neq L_\rho$ (so $\mu(\Delta) = \infty$). For example $L_\infty(\mathbb{R})$ is a space of this kind.

Next, we consider the case $L_\rho^a = L_\rho^b$. It follows from the examples 8.2 and 8.4 that in this case L_ρ is not a semi-M-space in general. According to theorem 4.2 and in view of the remark made at the end of section 5 we now have that L_ρ is a semi-M-space if and only if L_ρ/L_ρ^a is an abstract M-space. Let, therefore, d be the norm in the factor space L_ρ/L_ρ^a, i.e.,

$$d(f) = \inf\{\rho(f - b) : 0 \leq b \leq |f|, b \in L_\rho^b = L_\rho^a\}$$

for all $f \in L_\rho$. Note that d is a Riesz norm in the Riesz space L_ρ/L_ρ^a (cf. [7], th. 62.3).

LEMMA 9.3: *Let ρ be a r.i. function norm, and assume that $L_\rho^a = L_\rho^b$. Furthermore, assume that (a) $fX_A \in L_\rho^a$* for any $f \in L_\rho$ *and any $A \in \Gamma$*

such that $\mu(A)<\infty$, *(b)* $L_\infty \subset L_\rho$ *(so* $X_\Delta \in L_\rho$*). Then* $d(f)=l\rho(X_\Delta)$, *where*

$$l=\lim f^*(x) \text{ as } x\to\infty,$$

for all $f\in L_\rho$.

Proof: First note that if $\mu(\Delta)<\infty$, the statement is obvious. Indeed, according to condition (a) we have $L_\rho^a=L_\rho$ in this case, so $d(f)=0$ for all $f\in L_\rho$. Also, since $f^*(x)=0$ for all $f\in M$ and all $x>\mu(\Delta)$ we have $l=0$ for all $f\in L_\rho$. Hence, we may assume that $\mu(\Delta)=\infty$. Now note that $X_\Delta\in L_\rho$ implies $L_\rho^a\neq L_\rho$, because $X_\Delta\notin L_\rho^a$. Indeed, let $\{\Delta_n: n=1,2,\ldots\}\subset\Gamma$ satisfy $\Delta_n\downarrow\phi$, $\mu(\Delta_n)=\infty$ for all n. Then $X_{\Delta_n}^*=X_\Delta^*$ for all n, so

$$\lim\rho(X_{\Delta_n})=\rho(X_\Delta)>0.$$

This implies $X_\Delta\notin L_\rho^a$.

Let now $f\in L_\rho^+$ and $\epsilon>0$ be given. Furthermore, let $l=\lim f^*(x)$ as $x\to\infty$. Then there exists a number $x_0\geq 0$ such that

$$l\leq f^*(x)<l+\epsilon \text{ for } x>x_0;\, f^*(x)\geq l+\epsilon \text{ for } 0\leq x<x_0.$$

Let $A=\{x\in\Delta: |f(x)|\geq l+\epsilon\}$. Then $\mu(A)<\infty$. Furthermore, let $g_1=fX_A$ and $g_2=f-g_1$. Then $g_1\in L_\rho^a$ by hypothesis, so

$$d(f)\leq\rho(f-g_1)=\rho(g_2).$$

Now note that we have $g_2^*(x)\leq l+\epsilon$ for all $x\geq 0$, so

$$g_2^*\leq((l+\epsilon)X_\Delta)^*$$

on $\mathbb{R}^+$. Thus $d(f)\leq\rho(g_2)\leq(l+\epsilon)\rho(X_\Delta)$. This holds for all $\epsilon>0$, so $d(f)\leq l\rho(X_\Delta)$.

For the inverse inequality we may assume that $l>0$. In this case, let ϵ satisfy $0<\epsilon<l$. Then there exists a set $\Delta_1\in\Gamma$ satisfying $\mu(\Delta_1)=\infty$ and $fX_{\Delta_1}\geq(l-\epsilon)X_{\Delta_1}$ (we note that $f\geq l$ on the whole of Δ does not necessarily hold, as follows from example (c) after definition 7.1). Since d is a Riesz norm, it follows that $d(f)\geq d(fX_{\Delta_1})\geq(l-\epsilon)d(X_{\Delta_1})$. Now note that $\mu(\Delta_1)=\infty$ implies $d(X_{\Delta_1})=d(X_\Delta)$. Since it is also obvious that $d(X_\Delta)=\rho(X_\Delta)$, it follows that $d(f)\geq(l-\epsilon)\rho(X_\Delta)$. This holds for all ϵ such that $0<\epsilon<l$. Hence $d(f)\geq l\rho(X_\Delta)$. This completes the proof.

REMARK: If $L_\rho \subset L_\infty$ (regarded as point sets) and if $L_\rho^a = L_\rho^b$, then condition (a) of the preceding lemma is satisfied. It follows that $d(f) = l\rho(X_\Delta)$ holds in any r.i. space satisfying $L_\rho^a = L_\rho^b$ and $L_\rho = L_\infty$ (as point sets). However, there do exist spaces L_ρ such that $L_\rho^a = L_\rho^b$, the space satisfies the conditions (a) and (b) of lemma 9.3, ρ is r.i. and L_∞ is properly included in L_ρ. This will become clear from example 9.6.

THEOREM 9.4: *Let ρ be a r.i. function norm and assume that $L_\rho^a = L_\rho^b$. Furthermore, assume that $L_\infty \subset L_\rho$ and $fX_A \in L_\rho^a$ for any $f \in L_\rho$ and any $A \in \Gamma$ of finite measure. Then L_ρ/L_ρ^a is an abstract M-space and L_ρ is a semi-M-space.*

PROOF: If $\mu(\Delta) < \infty$, then $L_\rho^a = L_\rho$, so there is nothing to prove. Assume, therefore, that $\mu(\Delta) = \infty$ and let $f_1, f_2 \in L_\rho^+$ be given. We have to show that $d(\sup(f_1, f_2)) = \max(d(f_1), d(f_2))$, so without loss of generality we may assume that $\operatorname{supp}(f_1) \cap \operatorname{supp}(f_2) = \phi$. Let $f = \sup(f_1, f_2) = f_1 + f_2$, and let $l = \lim f^*(x)$, $l_i = \lim f_i^*(x)$ as $x \to \infty$ ($i = 1, 2$). Now assume that $l_1 \geq l_2$. Since $f \geq f_1$ on Δ, it follows that $f^* \geq f_1^*$ on $\mathbb{R}^+$, so $l \geq l_1$. Supposing that $l > m > l_1$ holds for some m, it follows that

$$\mu\{x \in \Delta : f(x) \geq m\} = \infty,$$

i.e. $\mu\{x \in \Delta : f_1(x) + f_2(x) \geq m\} = \infty$. Since f_1 and f_2 are disjoint this shows that

$$\mu\{x \in \Delta : f_1(x) \geq m\} = \infty \text{ or } \mu\{x \in \Delta : f_2(x) \geq m\} = \infty.$$

This implies that $l_1 \geq m$ or $l_2 \geq m$, which contradicts $m > l_1 \geq l_2$. Thus we obtain that $l = l_1 = \max(l_1, l_2)$. It follows then from the preceding lemma that $d(f) = \max(d(f_1), d(f_2))$. It is obvious now that L_ρ is a semi-M-space.

We note that if the condition $L_\infty \subset L_\rho$ (or $X_\Delta \in L_\rho$) is removed, theorem 9.4 does not hold any more. This follows from example 8.4.

COROLLARY 9.5: *If ρ is a r.i. function norm, if (Δ, Γ, μ) is purely atomic with all atoms of equal measure and if $X_\Delta \in L_\rho$, then L_ρ is a semi-M-space.*

PROOF: From lemma 5.2 it follows that $L_\rho^a = L_\rho^b$. Next, note that the characteristic function of an atom always is a member of $L_\rho^b = L_\rho^a$.

Hence, for any $f \in L_\rho$ and any $A \in \Gamma$ of finite measure we have $fX_A \in L_\rho^a$, because $\mu(A) < \infty$ implies that A consists of a finite number of atoms. In view of th. 9.4 it follows therefore that L_ρ is a semi-M-space.

REMARK: If $L_\rho(\Delta)$ satisfies all conditions of corollary 9.5, then $L_\rho = L_\infty$ (as point sets). Indeed, $L_\infty \subset L_\rho$ follows from $X_\Delta \in L_\rho$. Conversely, any $f \in L_\rho$ is essentially bounded. This follows from the fact that the characteristic functions of the atoms all have equal positive norm α, so $|f| \leq \alpha^{-1}\rho(f)$ on Δ.

Example 9.6. *The Lorentz space* L_Λ

Let Λ be as defined in section 8, and let

$$\|f\|_\Lambda = \int_0^\infty \Lambda(\lambda_f(y))dy,$$

for all $f \in M$ (λ_f is defined in def. 7.1(i)). Then $\|\cdot\|_\Lambda$ is a r.i. function norm (cf. [2], [3] or [10]). The normed Köthe space generated by $\|\cdot\|_\Lambda$ is denoted by L_Λ. In ([3] th. 8.2) it was shown that $L_{\Lambda,s}^*$ is an abstract L-space, and hence L_Λ is a semi-M-space. These results can also be derived from th. 9.4. Indeed, in [3], th. 7.1 it was proved that $L_\Lambda^a = L_\Lambda^b$, and in [2], lemma 3.1 it was proved that $fX_A \in L_\Lambda^a$ for all $A \in \Gamma$ of finite measure. Finally, it was shown in [2], th. 3.2 that $L_\Lambda^a \neq L_\Lambda$ if and only if Λ is bounded and $\mu(\Delta) = \infty$. Hence, in case that Λ is unbounded, it follows from $L_\Lambda^a = L_\Lambda$ that L_Λ is a semi-M-space. If Λ is bounded, then $X_\Delta \in L_\Lambda$ (cf. [3], lemma 7.4), so L_Λ is now a semi-M-space in view of theorem 9.4.

REMARK: In [2], th. 2.2, it was proved that the spaces L_Λ and M_Λ (denoted by N_Φ and M_Φ in [2]) are mutually associate in the sense of normed Köthe spaces. It follows from the preceding example and from the examples in section 8 that, although L_Λ is a semi-M-space, this does not necessarily hold for M_Λ. Hence, we may draw the conclusion that L_ρ being a semi-M-space does not necessarily imply that L_ρ' (the first associate space of L_ρ) is also a semi-M-space.

EXAMPLE 9.7: Let Λ be as in section 8 and in addition assume that

$$\lim x/\Lambda(x) < \infty \text{ as } x \to \infty.$$

Then $\|X_\Delta\|_M = \sup\{(\Lambda(x))^{-1}\int_0^x dt: x > 0\} = \sup\{x/\Lambda(x): x > 0\} < \infty$.

Hence $X_\Delta \in M_\Lambda$. It follows that if (Δ, Γ, μ) is purely atomic, all atoms having equal measure then M_Λ is a semi-M-space in view of corollary 9.5.

REFERENCES

[1] T. ANDO: Linear functionals on Orlicz spaces. *Nieuw Archief voor Wiskunde (3) VIII* (1960) 1–16.

[2] E. DE JONGE: A pair of mutually associate Banach function spaces. *Proc. Kon. Ned. Akad. Wet. 77* (1974) 148–156.

[3] E. DE JONGE: The triangle equality for positive singular functionals on some classes of normed Köthe spaces I and II. *Proc. Kon. Ned. Akad. Wet. 78* (1975) 48–69.

[4] E. DE JONGE: Representation of linear functionals on a class of normed Köthe spaces. *J. Functional Analysis 23(2)* (1976) 119–134.

[5] W. A. J. LUXEMBURG: Rearrangement-invariant Banach function spaces. *Queen's papers in pure and applied Mathematics 10* (1967) 83–144.

[6] W. A. J. LUXEMBURG and A. C. ZAANEN: Notes on Banach function spaces VI and VII. *Proc. Kon. Ned. Akad. Wet. 66* (1963) 655–681.

[7] W. A. J. LUXEMBURG and A. C. ZAANEN: Some examples of normed Köthe spaces. *Math. Annalen 162* (1966) 337–350.

[8] W. A. J. LUXEMBURG and A. C. ZAANEN: *Riesz spaces I.* Amsterdam (1971).

[9] M. M. RAO: Linear functionals on Orlicz spaces: General theory. *Pac. Journal of Math. 25 (3)* (1968) 553–585.

[10] M. S. STEIGERWALT and A. J. WHITE: Some function spaces related to L_p spaces. *Proc. London Math. Soc. (3) XXII* (1971) 137–163.

[11] A. C. ZAANEN: *Linear analysis.* Amsterdam (1953).

[12] A. C. ZAANEN: *Integration,* Amsterdam (1967).

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