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A DECOMPOSITION THEOREM FOR COMODULES

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Injective comodules over coalgebras can be decomposed as a direct sum of indecomposable injective comodules, in a fashion similar to the dual decomposition of projective modules over algebras, [1]. This paper gives an elementary proof of this theorem, avoiding the use of idempotents.

1. Preliminaries and definitions

Let k be a field of unspecified characteristic. A *coalgebra* (C, Δ, e) is a k-space C together with a comultiplication or diagonal map $\Delta: C \to C \otimes C$, and a counit (or augmentation) $e: C \to k$ such that the following properties are satisfied.

CA 1. $(\Delta \otimes I)\Delta = (I \otimes \Delta)\Delta$ Coassociativity

CA 2. $(e \otimes I)\Delta = (I \otimes e)\Delta = I$

A *comodule* (W, T) for a coalgebra C is a k-space W together with a map $T: W \to W \otimes C$ such that the following properties are satisfied.

CM 1. $(T \otimes I)T = (I \otimes \Delta)T$

CM 2. $(I \otimes e)T = I$

A *subcomodule* (*subcoalgebra*) is a subspace which has a comodule (coalgebra) structure under the restricted structure maps. If S is a subset of a comodule (coalgebra) the subcomodule (subcoalgebra) *generated* by S, denoted by $\langle\langle S \rangle\rangle$ is defined to be the smallest subcomodule (subcoalgebra) containing S. If S is a finite set

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or spans a finite dimensional subspace, $\langle\langle S\rangle\rangle$ is in fact a finite dimensional subcomodule (subcoalgebra).

If W is a comodule and V is a subcomodule, then W/V has a comodule structure. If (W, T) and (W', T') are comodules and f: $W \to W'$ is a k-map, then f is a comodule map if $(f \otimes I)T = T'f$. The usual isomorphism theorems hold.

A comodule (coalgebra) will be called *simple* if it contains no proper non-zero subcomodules (subcoalgebras). Every comodule contains a simple comodule, and every coalgebra contains a simple subcoalgebra. If W is a comodule for C, define the *socle* of W, $s(W)$ to be the sum of all simple subcomodules of W. Define the *coradical* R of the coalgebra C to be the sum of all simple subcoalgebras of C. If C is considered as the C-comodule (C, Δ), then $s(C) = R$. If V is a subcomodule of W such that $T(V) \leq V \otimes R$, then $V \leq s(W)$. $s(W)$ has the property that it decomposes as a direct sum of simple subcomodules. R decomposes as a direct sum of simple subcoalgebras.

The notion of the socle can be extended. Define $s_n(W)$ inductively by setting $s_0(W) = 0$, and $s_n(W)/s_{n-1}(W) = s(W/s_{n-1}(W))$. Since every non-zero subcomodule contains a simple subcomodule, the chain $s_0(W) \leq s_1(W) \leq s_2(W) \leq \dots$ is strictly ascending unless $s_k(W)$ is the whole of W for some k. Since every element w of W is contained in the finite dimensional subcomodule $\langle\langle w\rangle\rangle$, $W = \bigcup_{n=1}^{\infty} s_n(W)$.

The socle can be described in another way. For subspaces $X \leq W$, and $Y \leq C$, define the *wedge* of X and Y, $X \wedge Y$ to be the kernel of the map

$$W \xrightarrow{T} W \otimes C \longrightarrow W/X \otimes C/Y$$

Thus $X \wedge Y = T^{-1}(W \otimes Y + X \otimes C)$. It can be shown that $0 \wedge R = s(W)$. If we define $\wedge_w^0 R = 0$ and $\wedge_w^n R = (\wedge_w^{n-1} R) \wedge R$, then it follows that $\wedge_w^n R = s_n(W)$.[1]

A comodule (I, T) is injective if for every comodule (W, T') and every subcomodule $U \leq W$, every comodule map f: $U \to I$ extends uniquely to a map f: $W \to I$. C itself is an injective C-comodule. Direct summands of injective comodules are injective.

2. The theorem

THEOREM: *Let (W, T) be an injective comodule. Let $s(W) = \Sigma_{\mu \in M} X_\mu$ be a direct decomposition of the socle of W as a sum of*

[1]For elementary properties of comodules and coalgebras, see Sweedler, [2].

simple subcomodules. This decomposition of $s(W)$ can be extended to a direct decomposition of W as a sum of indecomposable injective subcomodules, $W = \Sigma_{\mu \in M} J_\mu$ such that $s(J_\mu) = X_\mu$.

The theorem is proved by constructing inductively a decomposition of $s_n(W)$ which extends the decomposition of $s_{n-1}(W)$.

For every μ in M, let $J_\mu^1 = X_\mu$. Suppose we have J_μ^{n-1} defined for some $n \geq 2$ such that

(i) $s(J_\mu^{n-1}) = X_\mu$

(ii) $\sum_{\mu \in M} J_\mu^{n-1} = s_{n-1}(W)$

(iii) The sum $\sum_{\mu \in M} J_\mu^{n-1}$ is direct.

We wish to define J_μ^n. Set $Z_\mu = \Sigma_{\lambda \in M \setminus \mu} X_\lambda$. Define

$$\mathcal{B}_\mu = \{S \leq J_\mu^{n-1} \wedge R \colon S \geq J_\mu^{n-1}, S \cap Z_\mu = 0\}$$

$\mathcal{B}_\mu$ is nonempty, since J_μ^{n-1} is in $\mathcal{B}_\mu$, and by Zorn's lemma $\mathcal{B}_\mu$ has maximal elements. Choose J_μ^n to be a maximal element of $\mathcal{B}_\mu$. It remains to show that the set $\{J_\mu^n\}_{\mu \in M}$ satisfies the three conditions of the inductive hypothesis.

(i) $s(J_\mu^n) \geq X_\mu$, since $J_\mu^n \geq J_\mu^{n-1}$. If $s(J_\mu^n) \not\geq X_\mu$, it follows that $J_\mu^n \cap Z_\mu \neq 0$, a contradiction. So $s(J_\mu^n) = X_\mu$.

(ii) It is enough to show that the sum $\Sigma_{\lambda \in \Lambda} J_\lambda^n$ is direct for all finite subsets $\Lambda \leq M$. This can be done by induction on $|\Lambda|$. Assume now that for any subset Λ of M with $|\Lambda| < r$, the sum $\Sigma_{\lambda \in \Lambda} J_\lambda^n$ is direct. If $\Gamma \leq M$, $|\Gamma| = r$, and the sum $\Sigma_{\lambda \in \Gamma} J_\lambda^n$ is not direct then there is some λ in Γ and some simple comodule $U \leq J_\lambda^n$ such that $U = X_\lambda \leq s(\Sigma_{\mu \in \Gamma \setminus \lambda} J_\mu^n) = \Sigma_{\mu \in \Gamma \setminus \lambda} s(J_\mu^n) \leq \Sigma_{\mu \in \Gamma \setminus \lambda} X_\mu \leq Z_\lambda$, which contradicts the directness of the decomposition of the socle, and completes the inductive step. (The second equality follows from the directness of the sum $\Sigma_{\mu \in R \setminus \lambda} J_\mu^n$, by the inductive hypothesis.)

(iii) This condition is shown in three steps.

Step 1. $J_\mu^{n-1} \wedge R = J_\mu^n \oplus Z_\mu$

Step 2. $\sum_{\mu \in M} J_\mu^n = \sum_{\mu \in M} (J_\mu^{n-1} \wedge R)$

Step 3. $\sum_{\mu \in M} (J_\mu^{n-1} \wedge R) = \left(\sum_{\mu \in M} J_\mu^{n-1}\right) \wedge R = s_{n-1}(W) \wedge R = s_n(W)$.

Step 1. Clearly $J_\mu^n + Z_\mu \leq J_\mu^{n-1} \wedge R$. To see the converse, it is sufficient to show that if $U \geq J_\mu^{n-1}$ is a subcomodule of W such that U/J_μ^{n-1} is simple, then $U \leq J_\mu^n + Z_\mu$. Suppose that $U \nleq J_\mu^n + Z_\mu$. Then $U + J_\mu^n \gneq J_\mu^n$. Moreover, $U + J_\mu^n \leq J_\mu^{n-1} \wedge R$ so by the maximality of J_μ^n in $\mathcal{B}_\mu$ it must be that $(U + J_\mu^n) \cap Z_\mu \neq 0$. We may pick $z \neq 0$ in Z_μ such that $z = u + j$ with u in U and j in J_μ^n. Now u is not in J_μ^n (otherwise z would be in $J_\mu^n \cap Z_\mu$ contrary to the conditions in $\mathcal{B}_\mu$) and hence not in J_μ^{n-1}. Therefore $u + J_\mu^{n-1}$ must generate U/J_μ^{n-1}. Thus

$$U = \langle\langle u \rangle\rangle + J_\mu^{n-1} \leq \langle\langle j \rangle\rangle + \langle\langle z \rangle\rangle + J_\mu^{n-1} \leq J_\mu^n + Z_\mu$$

which is a contradiction. Thus it must be that $U \leq J_\mu^n + Z_\mu$, and therefore $J_\mu^n + Z_\mu = J_\mu^{n-1} \wedge R$. Since J_μ^n is in $\mathcal{B}_\mu$, $J_\mu^n \cap Z_\mu = 0$ and the sum is direct.

Step 2. This is a direct consequence of step 1 and the definition of J_μ^n.

Step 3. The last equality is a property of the wedge, the second uses the inductive hypothesis, that $\Sigma_{\mu \in M} J_\mu^{n-1} = s_{n-1}(W)$. Since $J_\mu^{n-1} \leq \Sigma_{\lambda \in M} J_\lambda^{n-1}$, we have that $J_\mu^{n-1} \wedge R \leq (\Sigma_{\lambda \in M} J_\lambda^{n-1}) \wedge R$ for all μ in M, and $\Sigma_{\mu \in M}(J_\mu^{n-1} \wedge R) \leq (\Sigma_{\mu \in M} J_\mu^{n-1}) \wedge R$.

Now let $U \leq (\Sigma_{\mu \in M} J_\mu^{n-1}) \wedge R$. We may assume that U is finite dimensional. Then

$$U + \sum_{\mu \in M} J_\mu^{n-1} \Big/ \sum_{\mu \in M} J_\mu^{n-1} \cong U/U \cap \left(\sum_{\mu \in M} J_\mu^{n-1} \right) \cong U + \sum_{\mu \in M'} J_\mu^{n-1} \Big/ \sum_{\mu \in M'} J_\mu^{n-1}$$

Where M' is a finite subset of M such that $U \cap (\Sigma_{\mu \in M} J_\mu^{n-1}) \leq \Sigma_{\mu \in M'} J_\mu^{n-1}$. Since $U \leq (\Sigma_{\mu \in M} J_\mu^{n-1}) \wedge R$, $U + \Sigma_{\mu \in M'} J_\mu^{n-1} / \Sigma_{\mu \in M'} J_\mu^{n-1}$ is completely reducible. Let

$$U + \sum_{\mu \in M'} J_\mu^{n-1} \Big/ \sum_{\mu \in M'} J_\mu^{n-1} \cong \sum_{i=1}^{k} \left(U_i \Big/ \sum_{\mu \in M'} J_\mu^{n-1} \right)$$

be a direct decomposition as simple comodules. It is sufficient to show each U_i is contained in $\Sigma_{\mu \in M}(J_\mu^{n-1} \wedge R)$.

Take $U = U_i$, and set $Q = \Sigma_{\mu \in M'} J_\mu^{n-1}$, and $Q_\mu = \Sigma_{\lambda \in M', \lambda \neq \mu} J_\lambda^{n-1}$, for all μ in M'. We have projections (which are comodule maps)

$$p_\mu \colon U \to U/Q_\mu \text{ for all } \mu \text{ in } M'.$$

These can be used to get a comodule homomorphism

$$p\colon U \to \sum_{\mu \in M'} U/Q_\mu \text{ (external direct sum).}$$

If a is in $\ker(p)$, then $p_\mu(a) = 0$ for all μ in M'. That is, a is in Q_μ for all μ in M'. But the sum $\Sigma_{\mu \in M'} J_\mu^{n-1}$ is direct, and so $\bigcap_{\mu \in M'} Q_\mu = 0$, whence $a = 0$ and p is injective.

Let $U' = \operatorname{im}(p)$ in $\Sigma_{\mu \in M'} U/Q_\mu$. p is an isomorphism of U onto U'. Let r_0: $U' \to W$ be the inverse to p on U'. Since W is injective we can extend r_0 to a map

$$r\colon \sum_{\mu \in M'} U/Q_\mu \to W$$

$\operatorname{Im}(r) \geq U$ and $\operatorname{im}(r) \leq \Sigma_{\mu \in M'} r(U/Q_\mu)$.

It remains to show that $r(U/Q_\mu)$ is contained in $J_\mu^{n-1} \wedge R$. We have a series

$$U/Q_\mu \geq Q/Q_\mu \geq 0$$

The bottom factor is isomorphic to J_μ^{n-1} and the top factor $(U/Q_\mu)/(Q/Q_\mu)$ is simple. Moreover,

$$r(Q/Q_\mu) = r_0(p(J_\mu^{n-1})) = J_\mu^{n-1}$$

(Notice that $p_\lambda(J_\mu^{n-1}) = 0$ if $\lambda \neq \mu$, and thus $p(J_\mu^{n-1}) \leq Q/Q_\mu \leq U/Q_\mu$.) We have an induced homomorphism

$$\bar{r}\colon U/Q_\mu/Q/Q_\mu \to r(U/Q_\mu)/r(Q/Q_\mu) = r(U/Q_\mu)/J_\mu^{n-1}$$

Thus $r(U/Q_\mu)/J_\mu^{n-1}$ is a homomorphic image of a simple comodule and must therefore be simple or 0. If $r(U/Q_\mu)/J_\mu^{n-1}$ is simple, then $r(U/Q_\mu) \leq J_\mu^{n-1} \wedge R$, by a property of the wedge. If $r(U/Q_\mu)/J_\mu^{n-1} = 0$, then $r(U/Q_\mu) \leq J_\mu^{n-1} \leq J_\mu^{n-1} \wedge R$.

Thus $r(U/Q_\mu) \leq J_\mu^{n-1} \wedge R$ for all μ in M' and $U \leq \Sigma_{\mu \in M'} r(U/Q_\mu) \leq \Sigma_{\mu \in M} (J_\mu^{n-1} \wedge R)$, which completes step 3.

Let $J_\mu = \bigcup_{n=1}^{\infty} J_\mu^n$. The sum $\Sigma_{\mu \in M} J_\mu$ is direct, since the sum $\Sigma_{\mu \in M} J_\mu^n$ is direct for all n, and it is the whole of W since $\Sigma_{\mu \in M} J_\mu^n = s_n(W)$ and $\bigcup_{n=1}^{\infty} s_n(W) = W$. $s(J_\mu) = (\Sigma_{\lambda \in M} J_\lambda^1) \cap J_\mu = J_\mu^1$, by directness of the sum $\Sigma_{\lambda \in n} J_\lambda$. The J_μ are indecomposable since each J_μ contains a unique simple subcomodule. Each J_μ is injective since direct summands of injective comodules are injective.

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