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CHEVALLEY-JORDAN DECOMPOSITION FOR A CLASS OF LOCALLY FINITE LIE ALGEBRAS

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In a series of papers [11, 12, 13] we have developed analogues of the classical structure theory of finite-dimensional Lie algebras over a field of characteristic zero for certain classes of infinite-dimensional locally finite Lie algebras. In [11, 12] the class under consideration comprised those algebras generated by a system of finite-dimensional ascendant subalgebras. We discussed radicals and the existence of Levi subalgebras (semisimple complements to the radical), together with certain results on the conjugacy of Levi subalgebras. Extensions of these results to the broader class of Lie algebras generated by a system of finite-dimensional local subideals may be found in Amayo and Stewart [1] chapter 13 pp. 256–273. In [13] we took up the conjugacy question anew for the more restricted class of *ideally finite* Lie algebras, generated by a system of finite-dimensional ideals: for technical reasons the ground field was assumed algebraically closed of characteristic zero. Algebras in this class may be thought of as analogues of *periodic FC-groups* (which are generated by a system of finite normal subgroups, cf. Scott [10] theorem 15.1.12 p. 443), for which there exists a projective limit technique for proving 'local conjugacy' theorems (cf. Kuroš [8] p. 169, Tomkinson [19] pp. 682–686). By using elementary results on algebraic groups we were able to adapt this method to prove conjugacy, under suitable groups of automorphisms, of Levi, Borel, and Cartan subalgebras of ideally finite Lie algebras. The existence of Cartan subalgebras was also proved.

In the present paper we wish to extend to such algebras the technique of 'nilpotent-semisimple splitting', otherwise known as the *Chevalley-Jordan decomposition* (Humphreys [6] p. 17) and to use this to extend the results of Mal'cev [9]. As a byproduct we obtain an alternative proof of the existence of Cartan subalgebras in ideally finite

Lie algebras, which does not require the projective limit methods of [13].

In §2 we develop simple properties of the Fitting and Chevalley-Jordan decompositions (the former relating to 'weight spaces', the latter to 'nilpotent-semisimple splitting) and introduce 'cleft' algebras, generalizing Mal'cev's concept of 'splittable' algebras. In §3 we define a 'torus' and show that in any cleft ideally finite Lie algebra the centralizer of a maximal torus is a Cartan subalgebra. As a corollary we obtain a conjugacy theorem for maximal tori in the spirit of [13]. In §4 we use an embedding process, similar to Mal'cev's, to show that every locally soluble ideally finite Lie algebra has a Cartan subalgebra: it then follows from a result of [13] on Borel subalgebras that the hypothesis of local solubility may be removed. The content of §5 is a technical result weakening the requirements for an algebra to be cleft. It is used in §6 to prove that every ideally finite Lie algebra embeds in a cleft ideally finite Lie algebra, a result underlying everything in Mal'cev [9] in the finite-dimensional case. The proof given here makes no use of Lie group techniques and provides an alternative to Mal'cev's proof in finite dimensions. In §7 we make this construction more precise by introducing the 'cleft envelope' (called the 'splitting' by Mal'cev) of an ideally finite Lie algebra L. It is in some sense a 'minimal' cleft ideally finite algebra $\tilde{L}$ containing L. It always exists, and is unique up to isomorphism. The properties of L and $\tilde{L}$ are closely related: in particular $\tilde{L}^2 = L^2$ and both algebras have the same centre. In §8 this construction is applied to describe the conjugacy classes of maximal locally nilpotent subalgebras.

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1. Notation

Our notation will be consistent with that of [11, 12, 13] and of the book [1], which should be consulted for any unexplained terminology. In particular we shall use $\leq$ and $\triangleleft$ to denote the subalgebra and ideal relations; L^n and $\zeta_n(L)$ will denote the nth terms of the lower and upper central series of the Lie algebra L (with $L^1 = L$ and $\zeta_1(L) =$ the centre of L); and $C_L(X)$ and $I_L(X)$ will denote the centralizer and idealizer of the subset X of L. The Lie algebra of derivations of L is written $\mathrm{Der}(L)$. For any $x \in L$ we write x^* for the adjoint map $y \to [y, x]$ $(y \in L)$: to avoid ambiguity we may also write x^*_L. Commutators are left-normed, so that

$$[x_1, \ldots, x_n] = [\ldots [[x_1, x_2], x_3], \ldots, x_n]$$

$(x_1, \ldots, x_n \in L)$. We write

$$[x, {}_n y] = x(y^*)^n = [x, y, \ldots, y]$$

(n factors y); similarly for subspaces X, Y of L we let

$$[X, {}_n Y] = [X, Y, \ldots, Y]$$

(n factors Y). Triangular brackets denote the subalgebra generated by their contents.

The *Hirsch-Plotkin radical* $\rho(L)$ is the unique maximal locally nilpotent ideal of L (cf. Hartley [5]).

In dealing with linear transformations of a vector space V we shall confuse a field element λ with the corresponding scalar multiplication $\lambda 1_V$, where 1_V is the identity on V. In particular if $f: V \to V$ is linear, we write $(f - \lambda)^n$ instead of $(f - \lambda 1_V)^n$.

As in [13], throughout the paper $\mathfrak{K}$ *denotes an algebraically closed field of characteristic zero.* This convention will be used to shorten statements of theorems.

A Lie algebra is *ideally finite* if it can be generated by a system of finite-dimensional ideals. The class of ideally finite algebras is denoted by $\bar{\mathfrak{F}}$ in [13] but we will avoid this notation here. If L is ideally finite over $\mathfrak{K}$ then $\mathscr{L}(L)$ is the group of *locally inner* automorphisms introduced in §6 of [13].

2. The fitting and Chevalley-Jordan decomposition

Although a suitable choice of hypotheses allows the extension of many of the results of this section to a non-algebraically closed field, we shall state them only in the algebraically closed case since this is simpler and is the only case we need for applications. Most of the proofs are routine extensions of the finite-dimensional case (Freudenthal and de Vries [3] p. 88, Humphreys [6] p. 17, Jacobson [7] pp. 37, 61) and will be referred back to it. Nonetheless we state the results in full to provide a solid foundation for subsequent sections. The traditional terminology with regard to the Chevalley-Jordan decomposition is confusing and conflicts with some of our previous terminology (words like 'semisimple', 'split', 'algebraic' all being used in at least two different senses) and we shall modify it along the lines suggested by Freudenthal and de Vries [3].

Let V be a vector space (usually of infinite dimension) over $\mathfrak{K}$, and $f: V \to V$ a linear map. We say that f is *pure* if V has a basis of f-eigenvectors (or equivalently if V is spanned by f-eigenvectors), and f is *nil* if every $v \in V$ is annihilated by some power of f (perhaps depending on v). If there exists a polynomial $q(t) \in \mathfrak{K}[t]$ for which $q(t) \neq 0$, such that $q(f) = 0$, then f is *algebraic*. There is then a unique monic q of smallest degree such that $q(f) = 0$, the *minimum polynomial* of f. We say that f is *cleft* if we can write

$$f = f_p + f_n$$

where f_p is pure, f_n is nil, and $f_p f_n = f_n f_p$.

LEMMA (2.1): *If $f: V \to V$ is algebraic, then f is cleft, f_p and f_n are unique, and there exist polynomials $q, r \in \mathfrak{K}[t]$ with zero constant term for which $f_p = q(f), f_n = r(f)$. Hence f_p and f_n leave invariant any subspace of V which f leaves invariant, annihilate any subspace of V which f annihilates, and commute with any linear transformation of V with which f commutes.*

PROOF: For all but the uniqueness assertion, mimic Humphreys [6] p. 17 proposition, but use the minimum polynomial instead of the characteristic polynomial. To prove uniqueness argue as in Freudenthal and de Vries [3] p. 89 proposition 18.1.1. The argument in Humphreys [6] for uniqueness cannot be used because it assumes the polynomial property of f_p, f_n: but this will not follow for every choice of f_p and f_n until after uniqueness is proved.

If f is algebraic we call f_p and f_n (now known to be unique) the *pure* and *nil parts*, respectively, of f. The decomposition $f = f_p + f_n$ is called the *Chevalley-Jordan decomposition* of f, following Humphreys [6], or the *cleaving* of f, following Freudenthal and de Vries [3].

If L is a Lie algebra over $\mathfrak{K}$ and $x \in L$ we say that x is *ad-algebraic*, *ad-pure*, or *ad-nil* according as the adjoint map x^* is algebraic, pure, or nil on L. If there exist $x_p, x_n \in L$ such that $x = x_p + x_n$, for which $[x_p, x_n] = 0$ and the decomposition $x^* = x_p^* + x_n^*$ is a cleaving, then we say that x is *ad-cleft* in L. It follows from lemma 2.1 that if x is ad-algebraic and ad-cleft then x_p^* and x_n^* are unique, that is, x_p and x_n are unique modulo the centre of L. If every $x \in L$ is ad-cleft we say that L is *cleft*. If M is an L-module we say that L is *M-cleft* if every $x \in L$ can be written as $x = x_p + x_n$ $(x_p, x_n \in L)$, $[x_p, x_n] = 0$, in such a way that the maps induced on M by x_p and x_n constitute a cleaving of that induced by x. Thus if L is cleft then it is L-cleft, the action of L on itself being the adjoint action.

Next we turn to the Fitting decomposition. Let L be a Lie algebra over $\mathfrak{K}$ with dual space L^*. Let M be any L-module. For any linear form $\lambda \in L^*$ define

$$M_\lambda = \{m \in M\colon \text{for all } x \in L \text{ there exists } n > 0 \text{ such that } m(x^* - \lambda(x))^n = 0\}.$$

We refer to λ as a *weight* of M, and call M_λ its *weight space*. There is of course no reason in general to suppose that any non-zero weight spaces of M exist. However, define M to be *locally finite* if every finite subset is contained in a finite-dimensional L-submodule. (The most important example for us is an ideally finite Lie algebra under the adjoint action of a subalgebra.) We have:

LEMMA (2.2): *Let L be a locally nilpotent Lie algebra over $\mathfrak{K}$ and M a locally finite L-module. Then M is the direct sum of its weight spaces, and these are all L-submodules.*

PROOF: Every finite-dimensional L-submodule X of M is a module for the finite-dimensional nilpotent algebra $L/C_L(X)$ and hence a direct sum of weight spaces under the $L/C_L(X)$-action by Jacobson [7] p. 42 theorem 6. Since the L-action factors through the $L/C_L(X)$-action, it follows that X is the direct sum of weight spaces for L. Since M is the sum of all such X's it follows that M is the sum of its weight spaces. That this sum is direct can be shown either by adapting the usual argument or by looking at the system of finite-dimensional submodules. That the weight spaces are all L-submodules follows from the corresponding statement for finite dimensions (Jacobson [7] p. 42 theorem 6).

If N is a submodule of a locally finite module M it is trivial to verify that for each $\lambda \in L^*$

$$N_\lambda = N \cap M_\lambda \tag{1}$$

$$(M/N)_\lambda = (M_\lambda + N)/N. \tag{2}$$

Further, if L is thought of as an H-module under adjoint action, where H is a locally nilpotent subalgebra, then as in Jacobson [7] p. 64 corollary we obtain

$$(3) \qquad [L_\lambda, L_\mu] \subseteq L_{\lambda+\mu}$$

for all $\lambda, \mu \in L^*$.

The expression of M as a direct sum of weight spaces,

$$M = \bigoplus_{\lambda \in L^*} M_\lambda,$$

is the *Fitting decomposition* of M.

If $m \in M$ spans a 1-dimensional L-submodule then $mx = \lambda(x)m$ for all $x \in L$, where $\lambda \in L^*$. We call m an *L-eigenvector* with *eigenvalue* λ.

LEMMA (2.3): *Let $f: V \to V$ be a linear map such that V is a locally finite $\langle f \rangle$-module and f is pure. Then every weight space V_λ consists entirely of f-eigenvectors with eigenvalue λ.*

PROOF: Let $x \in V_\lambda$. Since f is pure, x is a sum of f-eigenvectors. Each f-eigenvector lies in some weight space, and the sum of the weight-spaces is direct; hence x is an f-eigenvector and λ is its eigenvalue.

COROLLARY (2.4): *Let $f: V \to V$ be a linear map such that V is a locally finite $\langle f \rangle$-module, and let W be a submodule.*

(i) *If f is pure then it induces pure maps on W and on V/W.*

(ii) *If f is nil then it induces nil maps on W and on V/W.*

(iii) *Each cleaving of f on V induces cleavings on W and V/W.*

PROOF: Part (i) follows from lemma 2.3 together with equations (1) and (2) above. Parts (ii) and (iii) are obvious.

3. Toral structure of cleft algebras

A *torus* in a Lie algebra L over $\mathfrak{K}$ is a subalgebra T of L such that every element of T is ad-pure (in its action on L).

LEMMA (3.1): *Every torus of a locally finite Lie algebra over $\mathfrak{K}$ is abelian.*

PROOF: Let L be locally finite, T a torus. We argue as in Humphreys [6] p. 35, and show that $t_T^* = 0$ for all $t \in T$. Since t_T^* is pure by corollary 2.4 (L being a locally finite $\langle t \rangle$-module and T being a submodule) it is sufficient, by lemmas 2.2 and 2.3, to show that t_T^* has no non-zero eigenvalues. Suppose on the contrary that $ut_T^* = \lambda u$ where $0 \neq u \in T$, $0 \neq \lambda \in \mathfrak{K}$. Now

$$t = t_1 + \cdots + t_r$$

where the t_i are linearly independent elements of T such that $[t_i, u] = \lambda_i t_i$ ($\lambda_i \in \mathfrak{K}$), since u_T^* is also pure. Now

$$0 = -\lambda [u, u] = [t, u, u] = \sum \lambda_i^2 t_i$$

so that $\lambda_i = 0$ for all i, and $[t, u] = 0$. But this contradicts $\lambda \neq 0$.

A *maximal torus* in L is a torus not properly contained in another torus. A Zorn's lemma argument shows that maximal tori exist in any Lie algebra. Obviously every maximal torus contains the centre.

We recall some definitions from [13]. A *Cartan subalgebra* (or *locally nilpotent projector* in the language of formation theory) of a Lie algebra L is a subalgebra C such that

(i) C is locally nilpotent,
(ii) If $C \leq H \leq L$, $K \lhd H$, and H/K is locally nilpotent, then $H = K + C$.

A subalgebra Q of L is *quasiabnormal* if for all U, $Q \leq U \leq L$ implies $U = I_L(U)$.

A result of Stonehewer [18] p. 526, or part of lemma 5.6 of Gardiner, Hartley, and Tomkinson [4] p. 203, translates with only verbal alterations to yield:

LEMMA (3.2): *A subalgebra of a Lie algebra is a Cartan subalgebra if and only if it is locally nilpotent and quasiabnormal.*

This leads us to the main theorem of this section:

THEOREM (3.3): *Let L be a cleft ideally finite Lie algebra over $\mathfrak{K}$. If T is a maximal torus of L then $C_L(T)$ is a Cartan subalgebra of L.*

PROOF: By lemma 3.2 it is enough to prove $C = C_L(T)$ locally nilpotent and quasiabnormal.

By the 'annihilation' statement in lemma 2.1, C is L-cleft. Since T

is abelian we have $T \leq C$. Now T contains every $c \in C$ for which c^* is pure on L, for $[T, C] = 0$ and therefore $T + \langle c \rangle$ is a torus for such c. Now for any $c \in C$ we have $c^* = c_p^* + c_n^*$, an ad-cleaving: the above remark shows that c_p^* annihilates C; and c_n^* acts nilpotently. Therefore c^* is nil on C. Engel's theorem, applied to a local system of finite-dimensional subalgebras of C, shows that C is locally nilpotent.

Next suppose that $C \leq U \leq L$ and, for a contradiction, that $U \neq I_L(U)$. Then there exists $x \in L \backslash U$ such that $[U, x] \subseteq U$. Let $V = U + \langle x \rangle$, so that $C \leq U \lhd V \leq L$. Each of U, V is a T-module, and $\dim V/U = 1$. Decomposing U and V into weight spaces for T it follows from (1) that there is a unique weight $\lambda \in T^*$ for which $U_\lambda \neq V_\lambda$, and for this λ we have $\dim V_\lambda / U_\lambda = 1$. If we pick $x' \in V_\lambda \backslash U_\lambda$ then $V = U + \langle x' \rangle$ and x' is a T-eigenvector with eigenvalue λ by lemma 2.3. Now for all $t \in T$,

$$\lambda(t)x' = [x', t] \in [U, x'] \subseteq U$$

so $\lambda(t) = 0$. Hence $x' \in C_L(T) = C$, a contradiction. So C is quasiabnormal.

In [9] Mal'cev proves a conjugacy theorem for maximal tori, which we generalize as:

THEOREM (3.4): *Let L be a cleft ideally finite Lie algebra over $\mathfrak{K}$. Then any two maximal tori of L are conjugate under the group $\mathscr{L}(L)$ of locally inner automorphisms.*

PROOF: Let T and T' be maximal tori of L. Their centralizers C and C' are Cartan subalgebras of L, so by [13] theorem 7.9 there exists $\alpha \in \mathscr{L}(L)$ such that $C^\alpha = C'$. Now the proof of theorem 3.3 shows that T is precisely the set of ad-pure elements of C, and similarly for T'; and since automorphisms of L preserve ad-purity it follows that $T^\alpha = T'$.

4. The existence of Cartan subalgebras

In this section we use an embedding process, suggested by that of Mal'cev, to construct in any ideally finite Lie algebra a Cartan subalgebra. The first step involves a property of linear transformations of finite rank. Let V be a vector space over $\mathfrak{K}$ and let $\boldsymbol{F}(V)$ be the Lie algebra of all linear transformations of V of finite rank (i.e. having finite-dimensional image). It is well known and easy to prove that $\boldsymbol{F}(V)$ is locally finite. It is obvious that each $f \in \boldsymbol{F}(V)$ is algebraic as a linear transformation of V.

LEMMA (4.1): *The Lie algebra $\boldsymbol{F}(V)$ is cleft.*

PROOF: Obviously $\boldsymbol{F}(V)$ is V-cleft: from this we shall deduce that it is cleft. For each $f \in \boldsymbol{F}(V)$ there exists, by lemma 2.1, a unique V-cleaving $f = f_p + f_n$, where f_p and f_n are polynomials in f without constant term. We claim that

$$f^* = f_p^* + f_n^*$$

is an ad-cleaving in $\boldsymbol{F}(V)$. Since $[f_p^*, f_n^*] = [f_p, f_n]^* = 0$ all that is required is that f_p^* be pure on $\boldsymbol{F}(V)$ and f_n^* nil. Now if $g, h \in \boldsymbol{F}(V)$ then an easy induction shows that

$$gh^{*m} = [g, {}_m h] = \sum_{i=0}^{m} (-1)^i \binom{m}{i} h^i g h^{m-i}$$

so that if $h^r = 0$ then $h^{*2r} = 0$. Now an algebraic nil transformation is nilpotent (consider its minimum polynomial) so it follows that f_n^* is nil on $\boldsymbol{F}(V)$.

If $g \in \boldsymbol{F}(V)$ is pure on V we can choose a basis $\{v_i\}_{i \in I}$ of V consisting of g-eigenvectors, so that $v_i g = \lambda_i v_i$ $(\lambda_i \in \mathfrak{K})$. The elementary transformations e_{ij} $(i, j \in I)$ defined by

$$v_k e_{ij} = \delta_{ki} v_j$$

$(k \in I)$, where δ_{ki} is the Kronecker delta, form a basis for $\boldsymbol{F}(V)$. A simple computation shows that

$$e_{ij} g^* = e_{ij} g - g e_{ij} = (\lambda_i - \lambda_j) e_{ij},$$

hence the e_{ij} are g^*-eigenvectors and g^* is pure. Hence f_p^* is pure. The lemma follows.

COROLLARY (4.2): *Any Lie subalgebra of $\boldsymbol{F}(V)$ which contains the pure and nil parts of each of its elements (considered as transformations of V) is cleft.*

LEMMA (4.3): *Let d be a derivation of the Lie algebra L over $\mathfrak{K}$, such* *that d is algebraic on L. Then the pure and nil parts d_p and d_n are derivations of L.*

PROOF: It is easy to see that L is a locally finite $\langle d\rangle$-module, so lemma 2.2 applies. Using (3) we can mimic Humphreys [6] lemma B p. 18 to obtain the result.

Let L be any ideally finite Lie algebra over $\mathfrak{K}$, and let $\{K_i\}_{i\in I}$ be the set of all finite-dimensional ideals of L. Let $\Delta(L)$ be the subalgebra of $\mathrm{Der}(L)$ consisting of those derivations which fix setwise every ideal of L, so that $\Delta(L) \geq \mathrm{Inn}(L)$, the latter being the algebra of inner derivations. For each $i \in I$ define D_i to be the set of all $d \in \Delta(L)$ such that

(i) $Ld \subseteq K_i$,

(ii) $C_L(K_i)d = 0$.

It is clear that inner derivations induced by elements of K_i lie in D_i.

LEMMA (4.4): *With the above notation, each D_i is a finite-dimensional ideal of $\Delta(L)$.*

PROOF: Since K_i has finite dimension, $C_L(K_i)$ has finite codimension, so there exists a finite-dimensional vector space complement W_i to $C_L(K_i)$ in L. Each $d \in D_i$ is (by condition (ii)) uniquely determined by its restriction to W_i. Since $W_i d \subseteq K_i$ by condition (i) we have

$$\dim D_i \leq \dim \mathrm{Hom}(W_i, K_i) < \infty.$$

If $d \in D_i$, $x \in \Delta(L)$, then

(i) $$\begin{aligned} L[d,x] &\subseteq Ldx + Lxd \\ &\subseteq K_i x + Ld \\ &\subseteq K_i + K_i = K_i, \end{aligned}$$

(ii) $$\begin{aligned} C_L(K_i)[d,x] &\subseteq C_L(K_i)dx + C_L(K_i)xd \\ &\subseteq 0x + C_L(K_i)d \\ &= 0. \end{aligned}$$

Hence $D_i \lhd \Delta(L)$.

Hence we can define $\Gamma(L) \lhd \Delta(L)$ by

$$\Gamma(L) = \sum_{i\in I} D_i.$$

If $i, j, k \in I$ and $K_k = K_i + K_j$ then obviously $D_i + D_j \leq D_k$, so in fact

$$\Gamma(L) = \bigcup_{i \in I} D_i.$$

We then have:

THEOREM 4.5: *Let L be ideally finite over $\mathfrak{K}$. Then $\Gamma(L)$ is a cleft ideally finite Lie algebra. The adjoint representation of L induces a homomorphism*

$$\tau : L \to \Gamma(L)$$

whose kernel is $\zeta_1(L)$ and image $\mathrm{Inn}(L)$.

PROOF: By lemma 4.4, $\Gamma(L)$ is ideally finite. If $d \in \Gamma(L)$ then condition (i) shows that d has finite rank, hence $\Gamma(L) \leq \boldsymbol{F}(L)$. Now d is algebraic, so there exists a cleaving $d = d_p + d_n$ by lemma 2.1. By lemma 4.3 each of d_p, d_n is a derivation of L. By the polynomial property and other assertions of lemma 2.1 it is easy to check that d_p and d_n belong to $\Gamma(L)$. Now corollary 4.2 shows that $\Gamma(L)$ is cleft. The remaining assertion is obvious.

Thus there is an embedding $L/\zeta_1(L) \to \Gamma(L)$. For our purposes, where central extensions cause no trouble, this is quite good enough. The question of embedding L, rather than a central quotient, in a cleft ideally finite algebra will be dealt with in §6.

{The definition of the D_i above suggests a method for constructing ideally finite algebras (ensuring an adequate supply of objects to which the theory applies). Namely, take a vector space V, a family $\{V_i\}_{i \in I}$ of finite-dimensional subspaces, and a corresponding family $\{W_i\}_{i \in I}$ of subspaces of finite codimension. For each $i \in I$ let A_i be the Lie algebra of all linear maps $V \to V$ which leave invariant each V_j and $W_j (j \in I)$, map V into V_i, and annihilate W_i. The sum of all the A_i is an ideally finite Lie algebra. If for all $i, j \in I$ there exists $k \in I$ with $V_i + V_j \leq V_k$ and $W_k \leq W_i \cap W_j$ then this algebra is even cleft. Further, every ideally finite algebra is a subalgebra of a central extension of some algebra constructed in this way.}

We return to $\Gamma(L)$ and the map τ. If J is an ideal of L, $j \in J$, and $d \in \Delta(L)$ then

$$[j^*, d] = (jd)^* \in \tau(J)$$

so that $\tau(J)$ is an ideal of $\Gamma(L)$. We wish to use Mal'cev's idea [9] of selecting a 'minimal' cleft subalgebra of $\Gamma(L)$ containing $\tau(L)$. Because of the ambiguity (up to centre) of ad-cleaving the existence of such a subalgebra is not immediately obvious, and we can manage with a less precise statement. (The existence is in fact a corollary of the more general results to be proved in §7.) All we need is that $\tau(L)$ can be embedded in a cleft ideally finite algebra $\hat{L}$ with some of the properties listed by Mal'cev [9] pp. 248–250, namely:

LEMMA (4.6): *If L is an ideally finite Lie algebra over $\mathfrak{K}$ then there exists a cleft subalgebra $\hat{L}$ of $\Gamma(L)$, with $\tau(L) \le \hat{L}$, such that*

$$\hat{L}^2 = (\tau(L))^2.$$

If L is locally soluble so is $\hat{L}$.

PROOF: Let $\Gamma = \Gamma(L)$. We define an increasing sequence of subalgebras

$$\tau(L) = L_0 \le L_1 \le L_2 \le \cdots$$

of Γ, as follows: each L_{i+1} is the subalgebra of Γ generated by all x_p and x_n for $x \in L_i$. The polynomial property of cleavings (lemma 2.1) shows that each L_i is an ideal of Γ. Now define $\hat{L} = \bigcup_{i=0}^{\infty} L_i$. We claim that $L_{i+1}^2 = L_i^2$. For $x \in L_i$ the polynomial property implies that

$$[L_i, x_p] \subseteq \sum_{r=1}^{\infty} [L_{i,r}X] \subseteq L_i^2,$$

and similarly

$$[L_i, x_n] \subseteq L_i^2.$$

Therefore

$$[L_{i+1}, L_i] \le L_i^2.$$

A repetition of this argument shows that

$$L_{i+1}^2 = [L_{i+1}, L_{i+1}] \le [L_{i+1}, L_i] \le L_i^2$$

so that $L_i^2 = L_0^2$ for all i. But $\hat{L}^2 = \bigcup_{i=0}^{\infty} L_i^2 = L_0^2$. It is clear that L is Γ-cleft, hence cleft. The last assertion of the theorem is obvious.

The task of proving existence of Cartan subalgebras is lightened by the following theorem.

THEOREM (4.7): *If L is ideally finite over $\mathfrak{K}$ and C is a subalgebra of L then the following are equivalent*:

(i) *C is a Cartan subalgebra of L.*
(ii) *C is locally nilpotent and quasiabnormal in L.*
(iii) *C is locally nilpotent and self-idealizing.*
(iv) *C is equal to the 0-weight space of its adjoint representation on L.*

PROOF: We have proved (i) $\Leftrightarrow$ (ii) in lemma 3.2. It is obvious that (i) $\Rightarrow$ (iii). We shall show (iii) $\Rightarrow$ (iv) $\Rightarrow$ (ii), from which the result follows.

(iii) $\Rightarrow$ (iv): Let C be locally nilpotent, so that $C \leq L_0$ (the 0-weight space), and suppose $C = I_L(C)$. If $L_0 \neq C$ choose $x \in L_0 \backslash C$. Now x is contained in a finite-dimensional ideal X of L, and $X_0 = L_0 \cap X$ is the 0-weight space of X as C-module, or equivalently as a module for the finite-dimensional nilpotent algebra $C/C_L(C)$. It follows from Jacobson [7] theorem 1 p. 33 that there exists an integer r such that

$$[x, {}_rC] = 0.$$

Choose m maximal subject to

$$[x, {}_mC] \nsubseteq C. \tag{4}$$

Then

$$[[x, {}_mC], C] \subseteq [x, {}_{m+1}C] \subseteq C$$

so that

$$[x, {}_mC] \leq I_L(C) = C,$$

a contradiction. Therefore $C = L_0$ as claimed.

(iv) $\Rightarrow$ (ii): Suppose $C = L_0$. Then Engel's theorem, applied on a local system of finite-dimensional subalgebras, shows that C is locally nilpotent. To prove C quasiabnormal suppose

$$C \leq U < U + \langle x \rangle \leq L$$

where $x \in I_L(U)$. Then U and $U + \langle x \rangle$ are C-modules. Since $C = L_0$ it follows that $x \in U + L_\lambda$ for a non-zero weight λ. Hence the coset $U + x$ is a C-eigenvector in $(U + \langle x \rangle)/U$ with eigenvalue λ. But since

$[x, C] \subseteq [x, U] \subseteq U$ in fact $U + x$ has eigenvalue 0. This contradiction shows that $U = I_L(U)$, hence C is quasiabnormal and (ii) holds.

Had this result, which depends on lemma 2.2, been known at the time [13] was written, it would perhaps have been better to use the more familiar (iii) as a definition of 'Cartan subalgebra' and to prove (i), or what amounts to the same thing, homomorphism invariance of Cartan subalgebras, as above. We would still need (i) to prove conjugacy of Cartan subalgebras, which result we need for the existence proof as will emerge in due course.

A final preliminary result which we need is a generalization of a theorem of Stitzinger [17].

LEMMA (4.8): *Let L be a locally soluble ideally finite Lie algebra over $\mathfrak{K}$, and C a subalgebra. Then the following are equivalent:*

(i) *C is a Cartan subalgebra of L.*

(ii) *C is a maximal locally nilpotent subalgebra of L, and $L = \rho(L) + C$.*

PROOF: That (i) $\Rightarrow$ (ii) is clear from the projector property, since $L/\rho(L)$ is abelian by [11] lemma 3.12 p. 86. To prove (ii) $\Rightarrow$ (i) it is sufficient, by theorem 4.7, to show that C is self-idealizing under hypothesis (ii). If not, then $I = I_L(C) > C$. Suppose if possible that $I \cap \rho(L) = C \cap \rho(L)$. Then

$$\begin{aligned} I &= I \cap (\rho(L) + C) \\ &= (I \cap \rho(L)) + C \\ &= (C \cap \rho(L)) + C \\ &= C, \end{aligned}$$

a contradiction. Therefore $I \cap \rho(L) > C \cap \rho(L)$. Pick $x \in (I \cap \rho(L)) \setminus C$, and consider $M = C + \langle x \rangle$. Now x^* is nil and of finite rank, hence nilpotent. From [15] lemma 3.3.4 p. 319 it follows that

$$C = \bigcup_{r=1}^{\infty} \zeta_r(C).$$

Hence if $x_1, \ldots, x_s \in C$ then $x_1, \ldots, x_s \in \zeta_m(C)$ for some m, and then $\zeta_m(C) + \langle x \rangle$ is nilpotent by [16] lemma 2.1 p. 15. Therefore M is locally nilpotent. Also $L = \rho(L) + M$, which contradicts maximality of C. Hence in fact $C = I_L(C)$ and (ii) $\Rightarrow$ (i).

We may now give an existence proof for Cartan subalgebras, different from corollary 7.5 of [13].

THEOREM (4.9): *Every ideally finite Lie algebra over $\mathfrak{K}$ has a Cartan subalgebra.*

PROOF: Let L be ideally finite over $\mathfrak{K}$, and let B be a Borel subalgebra. The map $\tau : B \to \hat{B}$ embeds $\bar{B} = B/\zeta_1(B)$ in a locally soluble cleft ideally finite algebra $\hat{B}$. Let C be a Cartan subalgebra of $\hat{B}$, which exists by theorem 3.3. Since $\hat{B}/\hat{B}^2$ is abelian, the projector property implies that $\hat{B} = \hat{B}^2 + C$. By lemma 4.6 (ii) $\hat{B}^2 = (\tau(B))^2$, so that

$$\hat{B} = (\tau(B))^2 + C.$$

Intersecting with $\tau(B)$ and using the modular law, it follows that

$$\tau(B) = (\tau(B))^2 + (\tau(B) \cap C).$$

Using τ^{-1} to pull back to $\bar{B}$ we have

$$\bar{B} = \bar{B}^2 + D$$

where D is locally nilpotent. Now $\bar{B}^2 \leqslant \rho(\bar{B})$ by [11] lemma 3.12 p. 86. Since $\zeta_1(B) \leq \rho(B)$ and local nilpotence is preserved by central extensions, it follows that

$$B = \rho(B) + E$$

where E is locally nilpotent. By Zorn's lemma there is a maximal locally nilpotent subalgebra M of B, containing E, and $B = \rho(B) + M$. Lemma 4.8 implies that M is a Cartan subalgebra of B. Since B is a Borel subalgebra of L, we may invoke lemma 8.1 of [13] (whose proof does not depend upon the existence of Cartan subalgebras) to conclude that M is a Cartan subalgebra of L.

5. Semicleaving

The object of this section is to prove a useful technical result. Say that a linear transformation f of a vector space V is *semicleft* if $f = g + h$ where g, h are linear transformations of V, g is pure, and f is nil. This differs from a cleaving only in that g and h need not commute. Call a Lie algebra L *semicleft* if each $x \in L$ can be written $x = p + n$ where $p, n \in L$, p^* is pure, n^* nil. We may now state a generalization of theorem 2 of Mal'cev [9] p. 233. The proof is essentially Mal'cev's but several minor modifications are needed.

THEOREM (5.1): *A locally soluble ideally finite Lie algebra over $\mathfrak{K}$ is cleft if and only if it is semicleft.*

Before giving the proof, we extract a small part which will be used often:

LEMMA (5.2): *Let L be a locally soluble ideally finite Lie algebra over $\mathfrak{K}$. Then $x \in L$ is ad-nil if and only if x belongs to the Hirsch-Plotkin radical $\rho(L)$.*

PROOF: The assertion is well known for L finite-dimensional (cf. Mal'cev [9] p. 233) and the lemma follows by considering a local system of finite-dimensional ideals and using lemma 3.18 of [11].

PROOF OF THEOREM 5.1: Obviously cleft implies semicleft. Let L be semicleft, locally soluble, and ideally finite over $\mathfrak{K}$. Put $Z = \zeta_1(L)$, $R = \rho(L)$. Since L/Z is residually finite by [13] lemma 7.2 it follows that R/Z is residually nilpotent, so that

(5)
$$\bigcap_{m=1}^{\infty} R^m \leq Z.$$

Let $x \in L$, with

(6)
$$x = p + n$$

where $p, n \in L$, p^* is pure, n^* nil. Let X be a finite-dimensional ideal of L containing p and n (and hence also x). Decomposing X into weight spaces for $\langle p \rangle$ we obtain

(7)
$$n = n_0 + n_1 + \cdots + n_s$$

where $n_i \in X$ for all $i = 0, \ldots, s$ and

$$[n_i, p] = \lambda_i n_i$$

for distinct eigenvalues $\lambda_i \in \mathfrak{K}$. We choose notation so that $\lambda_0 = 0$; possibly $n_0 = 0$; but $n_1, \ldots, n_s \neq 0$. If s were zero, we would have $[n, p] = 0$ and hence an ad-cleaving $x = p + n$. Our aim will be to modify the semicleaving (6) until this situation obtains.

Since X is finite-dimensional the decreasing chain

$$X \cap R \geq X \cap R^2 \geq X \cap R^3 \geq \cdots$$

must stop, say at $X \cap R^N$. From (5) it follows that

$$X \cap R^N \leq Z. \tag{8}$$

Lemma 5.2 implies that $n \in R$. Suppose that in (7) we have

$$u = n_1 + \cdots + n_s \in R^k$$

for some integer $k > 0$. The elements $n_1, \ldots, n_s$ also lie in $X \cap R^k$, for this is a $\langle p \rangle$-submodule, and we may use the weight space decomposition in $X \cap R^k$ to write u as a sum of p^*-eigenvectors, and then plead uniqueness. (Alternatively one may argue as in Mal'cev [9] p. 234.) Now

$$\lambda_1^{-1} n_1 + \cdots + \lambda_s^{-1} n_s$$

belongs to R^k, so is ad-nil; we may define the automorphism

$$\alpha = \prod_{i=1}^{s} \exp(\lambda_i^{-1} n_i)$$

of L. Now

$$\begin{aligned}
x^\alpha &= p^\alpha + n_0^\alpha + n_1^\alpha + \cdots + n_s^\alpha \\
&= p - \sum_{i=1}^{s} n_i + \sum_{j=0}^{s} n_j + \sum_{i,j} \lambda_i^{-1}[n_i, n_j] + \cdots \\
&= p + n_0 + n'
\end{aligned}$$

where $n' \in K^{k+1} \cap X$ and $x^\alpha \in X$. Decomposing n' into p^*-eigenvectors in $K^{k+1} \cap X$ we have

$$n' = n_0' + n_1' + \cdots + n_s'$$

where the subscripts correspond to the same eigenvalues $\lambda_0 = 0$, $\lambda_1, \ldots, \lambda_s$ as before. Each $n_i' \in R^{k+1} \cap X$. Now we have

$$x^\alpha = p + (n_0 + n_0') + (n_1' + \cdots + n_s')$$

where p^* is pure, $(n_0 + n_0')^*$ is nil and centralizes p, and $(n_1' + \cdots + n_s')$ belongs to $X \cap K^{k+1}$.

Repeating this process until the superscipt k has been raised to N, and using (8), we are led to a decomposition

$$x^\beta = p + n_0'' + z$$

where n_0'' is ad-nil, $[p, n_0''] = 0$, and $z \in X \cap R^N \leq Z$; while β is an automorphism of L. Then the decomposition

$$x^{\beta} = p + (n_0'' + z)$$

is an ad-cleaving, since $n_0'' + z$ is ad-nil and centralizes p; so that

$$x = p^{\beta^{-1}} + (n_0'' + z)^{\beta^{-1}}$$

is an ad-cleaving of x. Hence L is cleft as required.

It should be noted that the automorphism β, by construction, belongs to the group $\mathscr{E}(L)$ generated by exponentials of strongly ad-nilpotent elements of L, defined in [13] §3; and hence in particular to the group $\mathscr{L}(L)$ of locally inner automorphisms of [13] §6. Hence any semicleaving $x = p + n$ yields a cleaving of the form $p^{\gamma} + n'$, $\gamma \in \mathscr{E}(L)$.

6. The embedding theorem

In this section we show that every ideally finite Lie algebra over $\mathfrak{K}$ embeds in a cleft ideally finite Lie algebra. A partial result in this direction was proved in §4. Our arguments, though based on Mal'cev [9], are somewhat different: in particular we do not have available the apparatus of Lie groups. We do not use Mal'cev's results directly, so our proofs, when specialized to finite dimensions, provide alternatives to his not relying on Lie group methods.

LEMMA (6.1): *Let L be a semisimple ideally finite Lie algebra over $\mathfrak{K}$, and let M be a locally finite L-module. Then every submodule of M is complemented, and M is a direct sum of finite-dimensional irreducible L-submodules.*

PROOF: Each finite-dimensional submodule F of M is a module for the finite-dimensional semisimple algebra $L/C_L(F)$, hence a direct sum of irreducible L-modules (Jacobson [7] p. 79). Thus M is a sum of irreducible L-modules. Easy Zorn's lemma arguments now establish the desired results.

Next we need a generalization of Barnes [2] theorem 2.1 p. 278:

LEMMA (6.2): *Let L be an ideally finite Lie algebra over $\mathfrak{K}$, with*

$H \lhd L$ and C a Cartan subalgebra of H. Then

$$L = H + I_L(C).$$

PROOF: Let $I = I_L(C)$ and consider L/I as C-module. If $I + a \in L/I$ is annihilated by C then $[a, C] \subseteq I \cap H = I_H(C) = C$. Hence $a \in I$, and L/I contains no non-zero element annihilated by C. Now L/I is a locally finite C-module. If the 0-weight space of C on L/I were non-zero then C would act on some finite-dimensional submodule by zero-triangular matrices (Jacobson [7] pp. 34–35) and hence annihilate a non-zero element. Thus the 0-weight space on L/I is zero. On the other hand, C acts trivially on L/H. A contradiction can be avoided only if $H + I = L$, as claimed.

Using this lemma we obtain a substitute for the Lie group argument of Mal'cev [6] p. 247 lines 1–14:

LEMMA (6.3): *Let L be an ideally finite Lie algebra over $\mathfrak{K}$, with S its radical, Λ a Levi subalgebra. Then Λ idealizes some Cartan subalgebra of S. If further S is cleft then Λ also idealizes the maximal torus corresponding to such a Cartan subalgebra.*

PROOF: Let C be any Cartan subalgebra of S. Then lemma 6.2 implies that $L = S + I_L(C)$. If Λ_0 is a Levi subalgebra of $I_L(C)$ it is clear that Λ_0 is also a Levi subalgebra of L, idealizing C. By theorem 4.1 of [13] there is an automorphism α of L, leaving all ideals of L invariant, such that $\Lambda_0^\alpha = \Lambda$. Then C^α is a Cartan subalgebra of S idealized by Λ.

Now suppose S is cleft. Then $C = C_S(T)$ where T is the unique maximal torus of S contained in C and consists of those $c \in C$ for which c^* is pure on S. (This follows from the proof of theorem 3.3.) Since Λ is a direct sum of finite-dimensional simple algebras it is cleft (Humphreys [6] p. 24). The preservation of Chevalley-Jordan decomposition of semisimple algebras by representations (Humphreys [6] p. 29) applied in the usual way to a local system shows that if $x \in \Lambda$ is ad-nil on Λ then x^* is nil on L. Now for each integer n, the automorphism $\exp(nx^*)$ of L leaves C invariant, since Λ idealizes C, and hence leaves T invariant because automorphisms preserve ad-purity. By Hartley [5] lemma 2 p. 262 we have $[T, x] \subseteq T$. But ad-nil elements x generate Λ, since they do in finite dimensions (pick elements corresponding to non-zero roots in a Cartan decomposition), hence $[T, \Lambda] \subseteq T$.

The next lemma asserts the possibility of a 'global semicleaving' in a locally soluble ideally finite algebra.

LEMMA (6.4): *Let L be a cleft locally soluble ideally finite Lie algebra over* $\mathfrak{K}$. *Then*

$$L = \rho(L) + T$$

for any maximal torus T, and

$$\rho(L) \cap T = \zeta_1(L).$$

If T_0 *is any vector space complement to* $\zeta_1(L)$ *in T then*

$$L = \rho(L) \dot{+} T_0$$

and every non-zero element of T_0 *has a non-zero eigenvalue on L.*

PROOF: Let $C = C_L(T)$, a Cartan subalgebra of L. Since $L/\rho(L)$ is abelian the projector property implies that $L = \rho(L) + C$. The ad-nil elements of C lie in $\rho(L)$ by lemma 5.2, the ad-pure elements in T. Hence $L = \rho(L) + T$. The remaining assertions are clear.

The next lemma allows us to concentrate on the locally soluble case.

LEMMA (6.5): *Let L be an ideally finite Lie algebra over* $\mathfrak{K}$. *Then L is cleft if and only if its radical is cleft.*

PROOF: It is not hard to see that if L is cleft then so is $\sigma(L)$. Now let $S = \sigma(L)$ and suppose that S is cleft. Let Λ be a Levi subalgebra of L. By lemma 6.3 there exists a maximal torus T of S idealized by Λ, and by lemma 6.4 $S = R + T$ where $R = \rho(S) = \rho(L)$. By lemma 6.1 there exists a Λ-module complement T_0 to $R \cap T$ in T, and then $S = R \dot{+} T_0$ and T_0 is a torus. Further,

$$[T_0, \Lambda] \subseteq T_0 \cap [S, \Lambda] \subseteq T_0 \cap R = 0$$

from [1] corollary 3.15 p. 87. So Λ centralizes T_0. Now

$$L = R \dot{+} (T_0 \oplus \Lambda)$$

and T_0 is a torus of L since it is a torus of S and centralizes Λ. If $x \in L$ then $x = r + t + l$ where $r \in R$, $t \in T_0$, $l \in \Lambda$. Now Λ is L-cleft by the remarks on preservation of Chevalley-Jordan decomposition in lemma 6.3, so we can write $l = p + n$ where p^* is pure on L, n^* nil (potent), and $[p, n] = 0$. Since t^* and p^* commute, $t + p$ is ad-pure. We claim firstly that $r + n$ is ad-nil. It is clear that if $n^{*k} = 0$ then $L(r + n)^{*k} \subseteq R$. By [16] lemma 2.1 p. 15 $R + \langle n \rangle$ is locally nilpotent, so $r + n$ is ad-nil on R. Since $(r + n)^*$ has finite rank, we have $R(r + n)^{*j} = 0$, hence $L(r + n)^{*k+j} = 0$ and $r + n$ is ad-nil.

Thus we have a semicleaving

$$x = (r + n) + (t + p).$$

We apply theorem 5.1 to the locally soluble algebra

$$J = \langle S, n, p \rangle.$$

Now $J^2 \leq R$. There exists, by the remark after theorem 5.1, an automorphism $\alpha \in \mathscr{E}(J)$ such that

$$x = n' + (t + p)^{\alpha} \tag{9}$$

is an ad-cleaving on J. Now α extends to an automorphism of L by the obvious argument (cf. §3 of [13], or Winter [20] p. 93). Since α is a product of elements exp (j^*) where j is in a non-zero weight space on J, hence in J^2 (look at eigenvectors), hence in R, we have

$$(t + p)^{\alpha} \equiv t + p \pmod{R}$$

and hence

$$n' \equiv n \pmod{R}.$$

The argument that showed $n + r$ ad-nil shows n' ad-nil. Since α is an automorphism $(t + p)^{\alpha}$ is ad-pure. Since (9) is an ad-cleaving of J, $[n', (t + p)^{\alpha}] = 0$. Thus (9) is also an ad-cleaving on L. This completes the proof.

Next we give a criterion for a split extension to be ideally finite.

LEMMA (6.6): *Let $L = H \dot{+} K$ where $H \lhd L$, H and K are ideally finite, H is a locally finite K-module, and K acts on H by maps of finite rank. Then L is ideally finite.*

PROOF: Let I be a K-submodule of H. Then

$$[I, H]K \subseteq [IK, H] + [I, HK] \subseteq [I, H].$$

Hence the ideal

$$I^H = I + [I, H] + [I, {}_2H] + \cdots$$

is also a K-module. Hence every element of H is contained in a finite-dimensional ideal of L. It remains to prove the same for every element $k \in K$. Now $k \in J \lhd K$ where J is finite-dimensional. Since K acts by finite rank maps, $[H, J]$ is finite-dimensional. There is a finite-dimensional K-invariant ideal I of H containing $[H, J]$, and $I + J$ is an ideal of L of finite dimension containing k.

Finally we come to the main result:

THEOREM (6.7): *Every ideally finite Lie algebra over $\mathfrak{K}$ can be embedded in a cleft ideally finite Lie algebra.*

PROOF: Let L be ideally finite, and consider the algebra of finite rank derivations (plus a few other properties) called $\Gamma(L)$ in §4, and the map $\tau : L \to \Gamma(L)$ whose kernel is $Z = \zeta_1(L)$ and image $\operatorname{Inn}(L)$. Let $\Gamma = \Gamma(L)$ and denote images under τ by bars. Let $S = \sigma(L)$, and choose a Levi subalgebra Λ of L. Then

$$\bar{L} = \bar{S} \dot{+} \bar{\Lambda}$$

and $\bar{S} \cong S/Z$. Now Γ is ideally finite and cleft by theorem 4.5. Further $\bar{S} \lhd \Gamma$ so $\bar{S} \leq \sigma(\Gamma)$, and $\sigma(\Gamma)$ is cleft. By lemma 6.3 there is a maximal torus T of $\sigma(\Gamma)$ idealized by $\bar{\Lambda}$. By the definition of Γ, T acts as a Lie algebra of derivations of S, and we may form the split extension

$$Y = S \dot{+}_\Gamma T$$

(the subscript Γ indicating the action of T on S). To make matters clear we shalll adopt the notation (s, t) for an element of Y, where $s \in S$, $t \in T$. We claim that Y is cleft. Now if $s \in S$ then $s^* \in \sigma(\Gamma)$. By lemma 6.4 we can write $s^* = r + t$ where $r \in \rho(\Gamma)$, $t \in T$. Now r is nil in its action on $\bar{S} \cong S/Z$, so acts nilpotently on S. The element $(s, -t)$ of Y acts trivially on Y/S, and acts on S as $s^* - t$ which is nilpotent. Hence $(s, -t)$ is ad-nil on Y. For any $u \in T$ we have

$$(s, u) = (s, -t) + (0, u + t), \tag{10}$$

and $(0, u + t)$ acts purely on S and centralizes T, so acts purely on Y.

Hence (10) is a semicleaving, and by theorem 5.1 Y is cleft. (By lemma 6.6 Y is ideally finite.)

Since $\bar{\Lambda}$ idealizes T it follows that $T + \bar{\Lambda}$ is a subalgebra of Γ. The split extension

$$X = S \dot{+}_{\Gamma} (T + \bar{\Lambda})$$

is ideally finite by lemma 6.6, and obviously its radical is Y. By lemma 6.5 X is cleft. The subalgebra $S \dot{+}_{\Gamma} \bar{\Lambda}$ is easily seen to be isomorphic to $L = S \dot{+} \Lambda$, so that L embeds in X. This completes the proof.

7. The cleft envelope

Let L be ideally finite. A *cleft envelope* of E is a cleft ideally finite Lie algebra $E \geq L$, such that if E' is a subalgebra with $E > E' \geq L$ then E' is not cleft. In [9] Mal'cev shows that finite-dimensional (soluble) algebras over $\mathfrak{K}$ have a cleft envelope (called a 'splitting' by Mal'cev) which is unique up to isomorphism. The existence is obvious in finite dimensions once an embedding theorem is proved: in infinite dimensions more work is required. We shall in fact establish uniqueness first, which allows us to use the existence proof to investigate the structure of a cleft envelope in general.

LEMMA (7.1): *Let $H_1 \leq K_1$, $H_2 \leq K_2$ be ideally finite Lie algebras over $\mathfrak{K}$, with an isomorphism $\phi : H_1 \to H_2$. Suppose that $x \in H_1$ is ad-cleft in K_1, $x = x_p + x_n$, $x_p \notin H_1$; and $y = \phi(x)$ is ad-cleft in K_2, $y = y_p + y_n$, $y_p \notin H_2$. Then there exists an isomorphism*

$$\varphi : \langle H_1, x_p, x_n \rangle \to \langle H_2, y_p, y_n \rangle$$

extending ϕ.

PROOF: We have $[x_n, x_p] = 0$, $x_n = x - x_p$, and x_p^* is a polynomial in x^* with zero constant term. Hence

$$\langle H_1, x_p, x_n \rangle = H_1 + \langle x_p \rangle$$

and $H_1 \lhd H_1 + \langle x_p \rangle$. Similarly

$$\langle H_2, y_p, y_n \rangle = H_2 + \langle y_p \rangle$$

and $H_2 \lhd H_2 + \langle y_p \rangle$. Let $H_{1,\lambda}$ be the λ-weight space of x^* on H_1. Then for $h \in H_{1,\lambda}$ we must have

$$[h, x_p] = hx_p^* = \lambda h$$

because the linear transformation so defined is pure, commutes with x^*, and differs from x^* by a nil transformation by the definition of $H_{1,\lambda}$. Now $\phi(H_{1,\lambda}) = H_{2,\lambda}$ where the latter is the λ-weight space of y^* in H_2. Further, similar reasoning shows that

$$[\phi(h), y_p] = \phi(h)y_p^* = \lambda\phi(h).$$

Since the $H_{1,\lambda}$ span H_1 it is now clear that if we set $\varphi(x_p) = y_p$ and make $\varphi|_{H_1} = \phi$ then φ is an isomorphism of the required kind.

An easy consequence of this, using a Zorn's lemma argument, is:

THEOREM (7.2): *Let H_1 be ideally finite over $\mathfrak{K}$, $\phi: H_1 \to H_2$ be an isomorphism, and K_1 and K_2 be respectively cleft envelopes of H_1 and H_2. Then ϕ extends to an isomorphism $\varphi: K_1 \to K_2$. In particular cleft envelopes are unique up to isomorphism.*

To obtain the existence of a cleft envelope for an ideally finite Lie algebra L over $\mathfrak{K}$ we start with the cleft ideally finite algebra containing L which is constructed in the proof of theorem 6.7. This is of the form

$$S \dot{+} T \dot{+} \Lambda$$

where $S = \sigma(L)$, Λ is a Levi subalgebra of L, T is a torus of $S \dot{+} T$, and $[T, \Lambda] \leq T$. We find a subalgebra of this which is a cleft envelope. By lemma 6.1 we can find a Λ-invariant complement T_0 to $C_T(S)$ in T. Then $S + T_0$ is cleft since it is isomorphic to $(S + T)/C_T(S)$. Since $[\Lambda, T_0]$ is contained in $\rho(S + T)$ by [11] corollary 3.15 p. 87 and is also contained in T_0, we have $[\Lambda, T_0] = 0$. Then $K = S + T_0 + \Lambda$ is a Lie algebra containing $S + \Lambda = L$. Its radical is $S + T_0$ which is cleft, so K is cleft. Since $[S + T_0, K] \leq S$ it follows that any subspace between S and $S + T_0$ is an ideal.

We now resort to a transfinite induction process. Well-order $S + T_0$, and for ordinals α define $S_0 = S$; $S_\lambda = \cup_{\alpha<\lambda} S_\alpha$ if λ is a limit ordinal; and $S_{\alpha+1} = S_\alpha$ if S_α is cleft, $S_{\alpha+1} = S_\alpha + \langle x_p \rangle$ where x is the least element of S_α in the well-ordering such that x^* is not cleft on S_α, $x = x_p + x_n$ being a cleaving in $S + T_0$. On set-theoretic grounds we have $S_{\beta+1} = S_\beta$ for some ordinal β. Obviously S_β is cleft and contains S. We show that S_β is a cleft envelope of S and $S_\beta + \Lambda$ is a cleft envelope of L.

We claim that $\zeta_1(S_{\alpha+1}) = \zeta_1(S_\alpha)$ for all $\alpha < \beta$. For let $z \in \zeta_1(S_{\alpha+1})$. If $z \notin S_\alpha$ then $z = s + \lambda x_p$ $(0 \neq \lambda \in \mathfrak{K})$. On S_α we have $s^* = -\lambda x_p^*$. Consider the decomposition

$$x = (x + \lambda^{-1}s) + (-\lambda^{-1}s)$$

in S_α. Then $-\lambda^{-1}s^* = x_p^*$ is pure, and $(x + \lambda^{-1}s)^* = x_n^*$ is nil. Therefore x is semicleft in S_α, and by theorem 5.1 x is ad-cleft in S_α. This contradicts the choice of x. Therefore $z \in S_\alpha$ so $z \in \zeta_1(S_\alpha)$. Since x_p^* is a polynomial in x^* with zero constant term, it follows that $\zeta_1(S_\alpha)$ centralizes x_p. Therefore $\zeta_1(S_{\alpha+1}) = \zeta_1(S_\alpha)$ as claimed. It follows that $\zeta_1(S_\beta) = \zeta_1(S)$. The same argument about x_p^* being a polynomial in x^* shows that if $u \in C_K(S_\alpha)$ then $u \in C_K(S_{\alpha+1})$: in particular $C_{S_\beta}(S) = \zeta_1(S)$.

Now we show that no subalgebra H of S_β with $S \leq H < S_\beta$ is cleft. Let γ be the least ordinal $\leq \beta$ such that $S_\gamma \nleq H$. Then γ is not a limit ordinal, so $\gamma = \alpha + 1$ for some ordinal α, and we have $S_\alpha \leq H$, $S_{\alpha+1} \nleq H$. Let $x = h_p + h_n$ be an ad-cleaving in H, where $x = x_p + x_n$ is the cleaving used to obtain $S_{\alpha+1}$ from S_α. Now

$$x^* = h_p^* + h_n^* = x_p^* + x_n^*$$

are two cleavings of x^*, thought of as a linear transformation of S_α. Therefore

$$h_p - x_p \in C_{S_\beta}(S_\alpha) = \zeta_1(S)$$

so that $h_p \in S_{\alpha+1}$, and $S_{\alpha+1} \leq H$. This is a contradiction. Therefore S_β is a cleft envelope for S.

Finally suppose that J is a subalgebra of $S_\beta + \Lambda$ with $L = S + \Lambda \leq J \leq S_\beta + \Lambda$. If J is cleft then $\sigma(J)$ is cleft and $S \leq \sigma(J) \leq S_\beta$, so $\sigma(J) = S_\beta$. Hence $J = S_\beta + \Lambda$, and the latter is a cleft envelope for L. This proves:

THEOREM (7.3): *Every ideally finite Lie algebra over $\mathfrak{K}$ possesses a cleft envelope.*

From the uniqueness assertion of theorem 7.2 we may define the *cleft envelope* $\tilde{L}$ of L, and both $\tilde{L}$ and the map $L \to \tilde{L}$ are unique up to isomorphism.

THEOREM (7.4): *Let L be an ideally finite Lie algebra over $\mathfrak{K}$ with* *cleft envelope $\tilde{L}$. Then:*

(i) Every ideal of L is an ideal of $\tilde{L}$.

(ii) $\tilde{L}^2 = L^2$.

(iii) $\zeta_1(\tilde{L}) = \zeta_1(L)$.

(iv) $\tilde{L} = \rho(\tilde{L}) + L$.

(v) $\rho(\tilde{L}) = \rho(L)$ *if and only if* $\tilde{L} = L$.

(vi) $\sigma(\tilde{L})$ *is a cleft envelope for* $\sigma(L)$, and $\tilde{L} = \sigma(\tilde{L}) + \Lambda$ *where* Λ *is any Levi subalgebra of* L.

(vii) *Every automorphism of* L *extends, uniquely modulo* $\zeta_1(L)$, *to an automorphism of* $\tilde{L}$.

PROOF:

(i) The subalgebra $\bigcap_{I \lhd L} I_{\tilde{L}}(I)$ is cleft, by the polynomial property of cleavings, and contains L. Hence by minimality it equals $\tilde{L}$. Alternatively, note that $\Gamma(L)$ leaves invariant every ideal of L, by definition.

(ii) Let L be defined in terms of the S_α above. By transfinite induction we have $S_\alpha^2 = S^2$ and $[S_\alpha, \Lambda] = [S, \Lambda]$. Then $\tilde{L} = S_\beta + \Lambda$ so $\tilde{L}^2 = L^2$.

(iii) This has already been established for S_β.

(iv) We claim that $\rho(\tilde{L}) + L$ is cleft. For if $r \in \rho(\tilde{L})$, $x \in L$, then $x = r' + t$ where $r' \in \rho(\tilde{L})$ and t is ad-pure; and then $r + x = (r + r') + t$ is a semicleaving of x in $\rho(\tilde{L}) + L$. Minimality implies $\tilde{L} = \rho(\tilde{L}) + L$.

(v) This follows from (iv).

(vi) This is true for the cleft envelope constructed above.

(vii) Existence follows from theorem 7.2 and transfinite induction. Uniqueness modulo the centre follows from uniqueness, modulo the centre, of ad-cleavings.

Note that in [9] Mal'cev asserts that automorphisms extend uniquely from L to $\tilde{L}$, but the translator appends an example (credited to Mostow, Chevalley, and Jacobson) to show that this is not the case.

8. Maximal locally nilpotent subalgebras

In this section we classify the $\mathscr{L}(L)$-conjugacy classes of maximal locally nilpotent subalgebras of an ideally finite algebra L, starting with the cleft locally soluble case. We begin by sharpening lemma 6.4.

LEMMA (8.1): *Let L be cleft locally soluble ideally finite over $\mathfrak{K}$, and*

H an L-cleft subalgebra. If U is any torus of L, maximal with respect to $U \leq H$, then

$$H = (H \cap \rho(L)) + U$$

and

$$(H \cap \rho(L)) \cap U = H \cap \zeta_1(L).$$

If U_0 is any vector space complement to $H \cap \zeta_1(L)$ in U then

$$H = (H \cap \rho(L)) \dot{+} U_0$$

and the non-zero elements of U_0 have non-zero eigenvalues on L.

PROOF: For every $h \in H$ we can write

$$h = h_0 + h_1 + \cdots + h_r$$

where

$$[h_i, u] = \lambda_i(u) h_i \tag{11}$$

for distinct eigenvalues $\lambda_i : U \to \mathfrak{K}$, with $\lambda_0 = 0$. If $i > 0$ then (11) implies that $h_i \in H \cap L^2 \leq H \cap \rho(L)$. Since $\lambda_0 = 0$ we have $h_0 \in C_H(U)$. Hence

$$H = (H \cap \rho(L)) + C_H(U).$$

Suppose that

$$H \neq H_1 = (H \cap \rho(L)) + U.$$

Choose $c \in C_H(U) \backslash H_1$, and let $c = c_p + c_n$ be an ad-cleaving in L, such that $c_p, c_n \in H$. Then $c_n \in \rho(L) \cap H$. If $c_p \in H_1$ then $c \in H_1$, a contradiction. Therefore $c_p \notin H_1$. But c_p^* is a polynomial in c^* with zero constant term, so $c_p \in C_H(U)$. But now $U + \langle c_p \rangle$ is a torus of L, contained in H, and larger than U. This is a contradiction, so $H = H_1$ as required. The rest is obvious.

THEOREM (8.2): *Let L be a cleft locally soluble ideally finite Lie algebra over $\mathfrak{K}$, with $L = R \dot{+} T_0$ where $R = \rho(L)$ and T_0 is a torus. Then up to $\mathscr{L}(L)$-conjugacy every maximal locally nilpotent subalgebra M of L has the form*

$$M = C_R(T_1) \dot{+} T_1$$

where $T_1 \leq T_0$ is such that $C_{T_0}(C_R(T_1)) = T_1$.

PROOF: It is not hard to show that any maximal locally nilpotent subalgebra M is L-cleft, taking care to distinguish M-cleaving and L-cleaving, since adding on the pure or nil part of an element of M leads to a locally nilpotent algebra. Hence by lemma 8.1,

$$M = (R \cap M) \dot{+} U_0$$

for a torus U_0 of L. Let $T = T_0 + Z$, where $Z = \zeta_1(L)$. Then T is a maximal torus of L. There exists $\alpha \in \mathscr{L}(L)$ such that $U_0^\alpha \leq T$, and

$$M^\alpha = (R \cap M^\alpha) \dot{+} U_0^\alpha.$$

Now $M \geq Z$ (since $M + Z$ is locally nilpotent) so we can choose U_0 so that $U_0^\alpha \leq T_0$. Put $T_1 = U_0^\alpha$. Then $M^\alpha = (R \cap M^\alpha) \dot{+} T_1$ for $T_1 \leq T_0$. Now T_1 is a torus, and since M^α is locally nilpotent its eigenvalues are all zero on M^α, so $[M^\alpha, T_1] = 0$. Thus the (obviously locally nilpotent) subalgebras

$$C_R(T_1) + T_1,\ C_T(R \cap M^\alpha) + (R \cap M^\alpha),$$

contain M^α. Maximality implies firstly that $M^\alpha = C_R(T_1) + T_1$ and secondly that $C_T(C_R(T_1)) = T_1$, as claimed.

It is clear that algebras of the stated form are maximal locally nilpotent.

Before making a further study of the cleft case, we show that the general locally soluble case reduces to it.

THEOREM (8.3): *Let L be a locally soluble ideally finite Lie algebra over $\mathfrak{K}$ with cleft envelope $\tilde{L}$. Then every maximal locally nilpotent subalgebra of L is contained in a unique maximal locally nilpotent subalgebra of $\tilde{L}$. The $\mathscr{L}(L)$-conjugacy classes of maximal locally nilpotent subalgebras of L are in bijection with a set of $\mathscr{L}(\tilde{L})$-conjugacy classes of maximal locally nilpotent subalgebras of $\tilde{L}$.*

PROOF: First note that, in the terminology of [13] where $\mathscr{L}(L)$ is defined, the strongly ad-nilpotent elements of L lie in L^2. By theorem 7.4(ii) we have $\tilde{L}^2 = L^2$, and it is easy to see that the strongly ad-nilpotent elements of L and $\tilde{L}$ coincide. This allows us to identify the groups $\mathscr{L}(\tilde{L})$ and $\mathscr{L}(L)$.

Next, let $\tilde{L} = R \dot{+} T_0$, as usual, and let H be a maximal locally nilpotent subalgebra of L. There is a unique minimal $\tilde{L}$-cleft subalgebra H_1 of $\tilde{L}$ containing H (by an argument similar to that of theorem 7.2, and noting that $\zeta_1(\tilde{L}) = \zeta_1(L) \leq H$). It is not hard to verify that H_1 is locally nilpotent (by a transfinite induction argument). Hence H_1 is contained in a maximal locally nilpotent subalgebra H_2 of L. By theorem 8.2 we have

$$H_2 = C_R(T_1) \dot{+} T_1$$

where, for some $\alpha \in \mathscr{L}(\tilde{L})$, $T_1^\alpha \leq T_0$.

Let $t \in T_1$. We can find $r \in R$ such that $t + r \in L$, by theorem 7.4(iv). Put

$$r = r_0 + r_1 + \cdots + r_s$$

where

$$[r_i, u] = \lambda_i(u) r_i$$

for all $u \in T_1$, where the $\lambda_i : T_1 \to \mathfrak{K}$ are distinct eigenvalues, and $\lambda_0 = 0$. Now $r_1, \ldots, r_s \in \tilde{L}^2 = L^2$, so $r_0 + t \in L$. Hence $r_0 \in C_R(T_1)$, and $r_0 + t \in H_2 \cap L = H$ by maximality. Therefore $t \in H_1$ by definition of H_1, and $T_1 \leq H_1$.

We claim that there is a unique torus U of L maximal subject to $U \leq H_1$. For if U, U' have this property then $U, U' \leq \zeta_1(H_1)$ since H_1 is locally nilpotent. Therefore $[U, U'] = 0$, from which it follows easily that $U + U'$ is a torus of L. Hence by maximality $U = U'$.

Now it is easy to see that $T_1 + \zeta_1(L)$ is a torus of L, maximal subject to being contained in H_1. The above remark shows that T_1 is unique modulo $\zeta_1(L)$, from which it follows that H_2 is unique.

Since $H_2 \cap L = H$ by maximality, we have an injection from the set of maximal locally nilpotent subalgebras of L to that of $\tilde{L}$. The identification of $\mathscr{L}(L)$ and $\mathscr{L}(\tilde{L})$ allows us to define an induced injection on the corresponding conjugacy classes.

The argument (or even the result) of Mal'cev [9] theorem 5 p. 238 shows that if t is ad-pure, $\gamma \in \mathscr{L}(L)$, and $[t, \tau^\gamma] = 0$, then $t = t^\gamma$. Hence for a fixed maximal torus T, every torus is $\mathscr{L}(L)$-conjugate to a unique subtorus of T. Hence the description given in theorem 8.2 yields inconjugate subalgebras provided that it yields unequal ones. Thus, for a complete description, we have only to classify the subtori T_1 of T_0 with the property $C_{T_0}(C_R(T_1)) = T_1$. It is easy to see that these are precisely the subtori of T_0 of the form $C_{T_0}(N)$, for $N \leq R$. In fact there is a 'Galois correspondence' between the lattice of subalgebras of R and the lattice of subalgebras of T_0, with maps

$$N \mapsto C_{T_0}(N) \qquad (N \leq R)$$

$$T \mapsto C_R(T) \qquad (T \leq T_0)$$

which restrict to a bijection between the set of tori of the form $C_{T_0}(N)$ and the set of N of the form $C_R(T)$.

This opens the way to a more useful description using Mal'cev's concept of 'type'. If L is cleft, locally soluble, and ideally finite over $\mathfrak{K}$ then it has an ascending series of ideals $(L_\alpha)_{\alpha\le\sigma}$ whose factors $L_{\alpha+1}/L_\alpha$ are 1-dimensional. This follows by applying Lie's theorem (Jacobson [7] p. 49) to the $L/C_L(F)$-action on a finite-dimensional ideal F, and using transfinite induction. Further, if we decompose L as usual, $L = \rho(L) \dot{+} T_0$, then there is a basis of T_0-eigenvectors adapted to this series.

Fix such a series. Each $x \in L$ acts on $L_{\alpha+1}/L_\alpha$ by scalar multiplication $\lambda_\alpha(x)$. For $x, y \in L$ we say that $x \ge y$ if for all $\alpha, \beta \le \sigma$, $\lambda_\alpha(x) \ne \lambda_\beta(x)$ implies $\lambda_\alpha(y) \ne \lambda_\beta(y)$. The relation $\sim$ defined by

$$x \sim y \Leftrightarrow x \ge y \text{ and } y \ge x$$

is an equivalence relation: the *type* of $x \in L$ is its equivalence class. The partial semiorder $\ge$ induces a partial order on the set of types, which we also denote by $\ge$.

By considering a basis of T_0-eigenvectors adapted to the series (L_α), it is easy to see that for $x \in T_0$ the centralizer $C_L(x)$ depends only on the type. Hence for $N \le R$ the subtorus $T_1 = C_{T_0}(N)$ is a union of types, and contains along with each type all larger types. Hence we obtain:

THEOREM (8.4): *Let L be cleft locally soluble ideally finite over $\mathfrak{K}$. With the above notation, two subtori T_1 and T_2 of T_0 define $\mathscr{L}(L)$-conjugate maximal locally nilpotent subalgebras if and only if each type in T_2 is $\ge$ some type in T_1, and conversely.*

Finally we consider passage from the locally soluble case to the general case. If we fix a Borel subalgebra B of L it is clear that every maximal locally nilpotent subalgebra M of L has some $\mathscr{L}(L)$-conjugate $M^\alpha \le B$. Now elements of $\mathscr{L}(B)$ extend to give elements of $\mathscr{L}(L)$, so $\mathscr{L}(B)$-conjugacy implies $\mathscr{L}(L)$-conjugacy, and there is a surjection from the set of $\mathscr{L}(B)$-conjugacy classes of maximal locally nilpotent subalgebras of B to the set of $\mathscr{L}(L)$-conjugacy classes of maximal locally nilpotent subalgebras of L. All that remains is the possibility of *fusion*: non-injectivity of this map. We have not been able to solve this problem in general. However, it is clear that fusion does not occur in the two extreme cases: Cartan subalgebras, and Hirsch-Plotkin radicals of Borel subalgebras.

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