COMPOSITIO MATHEMATICA

D. R. LEWIS
P. WOJTASZCZYK

Symmetric bases in Minkowski spaces

Compositio Mathematica, tome 32, n° 3 (1976), p. 293-300
<http://www.numdam.org/item?id=CM_1976__32_3_293_0>

COMPOSITIO MATHEMATICA, Vol. 32, Fasc. 3, 1976, pag. 293–300
Noordhoff International Publishing
Printed in the Netherlands

SYMMETRIC BASES IN MINKOWSKI SPACES

D. R. Lewis* and P. Wojtaszczyk

An analog of a well-known property of symmetric bases in infinite dimensional Banach spaces is established for finite dimensional spaces. It is shown that if a finite dimensional space E has a basis with the property that each permutation of indices naturally induces an isomorphism of norm at most λ, then E has a (possibly different) 9λ-unconditional basis. Restated in terms of symmetry parameters this answers a question posed by Gordon [2], which is implicit in the paper of Gurarii, Kadec and Macaev [4]. Some examples are given to show the non-isometric nature of the result.

Let $B = (b_i)_{i\varepsilon I}$ be a basic sequence (finite or countably infinite) in a normed space E. For π a permutation of I with $\pi(i) \neq i$ only finitely often, g_π is the isomorphism of E defined by $g_\pi(b_i) = b_{\pi(i)}$, $i\varepsilon I$; and for $(\varepsilon_i)_{i\varepsilon I}$ a sequence of scalars with $|\varepsilon_i| = 1$ for all $i\varepsilon I$ and $\varepsilon_i \neq 1$ only finitely often, g_ε is the operator defined by $g_\varepsilon(b_i) = \varepsilon_i b_i$. Three symmetry parameters of B are defined as follows:

the *unconditional basis constant* of B is $x(B) = \sup_\varepsilon \|g_\varepsilon\|$;

the *diagonal symmetry constant* of B is $\delta(B) = \sup_\pi \|g_\pi\|$;

and the *total symmetry constant* of B is $t(B) = \sup_{\varepsilon, \pi} \|g_\pi g_\varepsilon\|$.

Clearly $x(B) \leqq t(B)$ and $\delta(B) \leqq t(B)$ for every basis B, and it is known that $x(B) \leqq 2\beta\delta(B)^2$, where β is the basis constant of B (cf. [7]). But also observe that no inequalities of the form $x(B) \leqq f(\delta(B))$ or $\delta(B) \leqq f(x(B))$ are valid for all bases, where f indicates a real function independent of the particular basis. A simple sequence of examples showing that the first relation cannot hold may be given as follows.

* Partially supported by NSF GP-42666

AMS (MOS) subject classification (1970)
Primary 46B15
Secondary 46B05

Key words and phrases, Minkowski space, unconditional constant, symmetry constant, diagonal symmetry constant.

For n odd let B be the unit vector basis of $\mathbb{R}^n$ considered with the norm $\|x\| = \max |\langle x, \varepsilon\rangle|$, where the maximum is taken over all *n-tuples* $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ of signs with $\sum_{i \leqq n}\varepsilon_i = 1$. Then $\delta(B) = 1$ and $x(B) = n$.

For p any one of the parameters x, δ or t define the corresponding symmetry parameter of E by $p(E) = \inf_B p(B)$, with the infimum taken over all bases for E.

The Banach Mazur distance between isomorphic spaces E and F is defined as $d(E, F) = \inf \|u\| \|u^{-1}\|$, the infimum being taken over all isomorphisms u between E and F. It is immediate that each of the three symmetry parameters of E defined above is continuous in the sense that $p(E) \leqq d(E, F)p(F)$ holds for all E and F.

Although the diagonal and total symmetry constants of a particular basis in a finite dimensional space may behave quite differently, the diagonal and total symmetry constants of the space itself are equivalent. More precisely,

THEOREM 1: *The relations $\delta(E) \leqq t(E) \leqq 9\delta(E)$ hold in every finite dimensional space E.*

The first inequality is obvious. To prove the second it is convenient to first consider the case in which E has a basis $B = (b_i)_{i \leqq n}$ with $\delta(B) = 1$. The coefficient functionals of the basis are denoted by $(b_i')_{i \leqq n}$, and m is the greatest integer satisfying $2m \leqq n$. The group of all permutations of $\{1, 2, \ldots, k\}$ is written S_k. The proof of the special case requires two lemmata, the first of which is given without proof.

LEMMA 1: *Let $w_1, w_2, \ldots, w_m$ and $v_1, v_2, \ldots, v_m$ be two finite sequences of non-negative reals with $v_i \geqq v_{i+1}$ for $1 \leqq i < m$. For $1 \leqq k \leqq m$ set $u_{2k} = u_{2k-1} = v_k$, and $u_n = 0$ if n is odd. Then*

$$\max_{\pi \in S_n} \sum_{k \leqq m} [u_{\pi(2k)} + u_{\pi(2k-1)}] w_k = 2 \max_{\tau \in S_m} \sum_{k \leqq m} v_{\tau(k)} w_k.$$

LEMMA 2: *Let $|\ |$ be any norm on E for which $\delta(B \subset (E, |\ |)) = 1$. Then*
(a) *$q = n^{-1}(\sum_{i \leqq n} b_i') \otimes (\sum_{k \leqq n} b_k)$ is a norm one projection,*
(b) *$p = 2^{-1}\sum_{k \leqq m}(b_{2k}' - b_{2k-1}') \otimes (b_{2k} - b_{2k-1})$ is a norm one projection,*
(c) *$1_E - q = (n-1)m^{-1}(n!)^{-1}\sum_{\pi \in S_n} g_\pi^{-1} p g_\pi$, and*
(d) *$t((b_{2k} - b_{2k-1})_{k \leqq m} \subset (E, |\ |)) = 1$.*

PROOF OF LEMMA 2: Part (a) follows from the equality $q = (n!)^{-1}\sum_\pi g_\pi$, and (b) is true since $p = 2^{-1}(1 - g_\tau)$, where $\tau \in S_n$ interchanges $2k$ with

$2k-1$ for each $k = 1, 2, \ldots, m$.

To verify (c) write $w = (n!)^{-1}\sum_\pi g_\pi^{-1} p g_\pi$. Since w commutes with each g_π, $w = s1_E + tq$ for some scalars s and t. Write H for the kernel of q. Then

$$s1_H = (n!)^{-1} \sum_\pi (g_\pi|H)^{-1}(p|H)(g_\pi|H)$$

and $p|H$ is a projection onto an m dimensional subspace of H, so

$$s(n-1) = \operatorname{trace}(s1_H) = \operatorname{trace}(p|H) = m.$$

Also

$$sn+t = \operatorname{trace}(w) = \operatorname{trace}(p) = m,$$

so that $s = -t = m(n-1)^{-1}$.

Finally, given signs $\delta_1, \delta_2, \ldots, \delta_m$ and $\tau \in S_m$ let $\pi \in S_n$ be the permutation which maps $\{2k, 2k-1\}$ onto $\{2\tau(k), 2\tau(k)-1\}$, $1 \leqq k \leqq m$, and which satisfies $\pi(2k) = 2\tau(k)$ if $\delta_k = 1$, $\pi(2k) = 2\tau(k)-1$ if $\delta_k = -1$. For each $k = 1, 2, \ldots, m$, $\delta_k(b_{2\tau(k)} - b_{2\tau(k)-1}) = g_\pi(b_{2k} - b_{2k-1})$, which proves (d).

Proof of Theorem 1: Assume $\delta(B) = 1$ and let (( )) be the norm on E defined by

$$((x)) = \max_{\pi,\varepsilon} \|pg_\varepsilon g_\pi(x)\|,$$

where $\|\ \|$ is the given norm on E and the maximum is over all $\pi \in S_n$ and n-tuples of signs ε. Let F denote E under (( )). Notice that each operator $g_\varepsilon g_\pi$ is an isometry of F, and hence $t(F) = \delta(B \subset F) = 1$, so the assumptions of Lemma 2 are satisfied by the basis B in both norms, (( )) and $\|\ \|$.

The first claim is that $((x)) = \|x\|$ for all x in $[b_{2k} - b_{2k-1}]_{k \leqq m}$, the span of the vectors $b_{2k} - b_{2k-1}$. The inequality $((x)) \geqq \|x\|$ is immediate, and for the other direction it is enough, by Lemma 2(d), to consider vectors of the form $x = \sum_{k \leqq m} a_k(b_{2k} - b_{2k-1})$ with $|a_k| \geqq |a_{k+1}|$ for $1 \leqq k < m$. For ε an n-tuple of signs and $\pi \in S_n$, choose $x' \in E'$ so that $\|x'\| = 1$ and

$$\begin{aligned}\|pg_\varepsilon g_\pi^{-1}(x)\| &= |\langle pg_\varepsilon g_\pi^{-1}(x), x'\rangle| \\ &\leqq 2^{-1} \sum_{k \leqq m} [|\langle x, b'_{\pi(2k)}\rangle| + |\langle x, b'_{\pi(2k-1)}\rangle|]|\langle b_{2k} - b_{2k-1}, x'\rangle|\end{aligned}$$

Applying Lemma 1 with $v_k = |a_k|$ and $w_k = |\langle b_{2k} - b_{2k-1}, x' \rangle|$ shows that

$$\|pg_\varepsilon g_\pi^{-1}(x)\| \leqq \max_{\tau\in S_m} \sum_{k\leqq m} |a_{\tau(k)}^{-1}||\langle b_{2k} - b_{2k-1}, x'\rangle|$$

$$= \max_{\tau, |\delta|=1} |\langle \sum_{k\leqq m} \delta_k a_k (b_{2\tau(k)} - b_{2\tau(k)-1}), x' \rangle| \leqq \|x\|,$$

the last by part (d) of Lemma 2.

We next assert that the inequalities

$$\|(1-q)(x)\| \leqq 2((x)) \quad \text{and} \quad (((1-q)(x))) \leqq 2\|x\|$$

hold for all $x \in E$. For the first, applying Lemma 2 with both $\| \; \|$ and $((\;))$ yields

$$\begin{aligned}
\|(1-q)(x)\| &\leqq (n-1)m^{-1}(n!)^{-1} \sum_\pi \|g_\pi^{-1} pg_\pi(x)\| \\
&\leqq 2(n!)^{-1} \sum_\pi \|pg_\pi(x)\| \\
&= 2(n!)^{-1} \sum_\pi ((pg_\pi(x))) \\
&\leqq 2(n!)^{-1} \sum_\pi ((g_\pi(x))) \\
&= 2((x)),
\end{aligned}$$

and the other inequality follows by interchanging the rôles of $\| \; \|$ and $((\;))$.

Now let λ be the constant satisfying

$$\lambda \|\sum_{i\leqq n} b_i\| = ((\sum_{i\leqq n} b_i))$$

and define $u: F \to E$ by $u = 1_E + (\lambda - 1)q$. Since $((q(x))) = \|q(x)\|\lambda$ for all $x \in E$, $\|u(x)\| \leqq \|(1-q)(x)\| + ((q(x))) \leqq 3((x))$ by the preceeding paragraph and Lemma 2, and hence $\|u\| \leqq 3$. But $u^{-1} = 1_E + (\lambda^{-1} - 1)q$ so the same proof gives $((u^{-1}(x))) \leqq 3\|x\|$ and thus $d(E, F) \leqq 9$. Then

$$t(E) \leqq t(F)d(E, F) \leqq 9.$$

More generally for $B \subset E$ any basis let H be E with the norm $|x| = \max_\pi \|g_\pi(x)\|$. Since $\delta(B \subset H) = 1$ and $d(E, H) \leqq \delta(B)$ the special case shows that $t(E) \leqq d(E, H)t(H) \leqq 9\delta(B)$.Taking the infimum over all possible bases for E completes the proof of the theorem.

REMARK 1: In [4] Gurarii, Kadec and Macaev define the symmetry parameter α of a finite dimensional space E by $\alpha(E) = \inf_B x(B)\delta(B)$. The theorem implies that $\delta(E) \leqq \alpha(E) \leqq 9\delta(E)^2$, answering a question raised by Gordon [2] and by Lewis [5].

REMARK 2: Let E be a Banach space with a diagonally symmetric basis $B = (b_i)_{i\geqq 1}$. Then for any $\varepsilon > 0$ and any finite dimensional $F \subset E$ there is finite dimensional W with $F \subset W \subset E$ and $t(W) \leqq (9+\varepsilon)\delta(B)$. This follows from a routine pertubation argument and the fact that $t([b_1, b_2, \ldots, b_n]) \leqq 9\delta(B)$ for all n. Thus, although the unconditional basis constant of B depends on the basis constant of B, the local unconditional structure of E depends only on $\delta(B)$.

Following [1] define the *asymmetry constant* of a finite dimensional space E by

$$s(E) = \inf_G \sup_{g\in G} \|g\|,$$

with the infimum taken over all compact groups G of isomorphisms of E which have the property that only scalar multiples of the identity commute with the elements of G.

It is clear that $s(E) \leqq t(E)$, so the following theorem strongly indicates the non-isometric nature of the relationship between $\delta(E)$ and $t(E)$.

THEOREM 2: *There is a sequence $(E_n)_{n\geqq 5}$ of Minkowski spaces with* $\dim E_n = n$, $\delta(E_n) = 1$ *and* $\liminf_n s(E_n) \geqq (2^{-1}+2^{-\frac{1}{2}})^{\frac{1}{2}}$.

PROOF: Let $e_i \in l_\infty^{n+1}$ and $e'_i \in l_1^{n+1}$ be the unit vectors, $1 \leqq \lambda \leqq n$ and $E_n \subset l_\infty^{n+1}$ be the kernel of $\sum_{i\leqq n} e'_i + \lambda e'_{n+1}$ (a sequence of values for λ will be specified later). The basis $b_i = e_i - \lambda^{-1}e_{n+1}$, $1 \leqq i \leqq n$, has diagonal symmetry constant one so $\delta(E_n) = 1$. To estimate $s(E_n)$ from below we use the inequality [1]

$$s(E_n)^2 \geqq n^{-1}\gamma_\infty(E_n)\pi_1(E_n),$$

with $\gamma_\infty(E_n)$ and $\pi_1(E_n)$ denoting, respectively, the projection constant of E_n and the 1-absolutely summing norm [6] of the identity on E_n. Write G for the group of isometries of l_∞^{n+1} of form $g(e_i) = e_{\pi(i)}$ for some $\pi \in S_{n+1}$ with $\pi(n+1) = n+1$.

If w is a projection of l_∞^{n+1} onto E_n with $\|w\| = \gamma_\infty(E_n)$, then $u = |G|^{-1}\sum_{g\in G} g^{-1}wg$ is also a projection onto E_n with norm $\gamma_\infty(E_n)$. Since u commutes with each element of G

$$u = 1-(\sum_{i\leqq n} e_i'+\lambda e_{n+1}') \otimes (t\sum_{i\leqq n} e_i+se_{n+1})$$

for some scalars s and t with $tn+\lambda s = \operatorname{trace}(1-u) = 1$. Thus

$$\begin{aligned}\gamma_\infty(E_n) &= \|u'\| = \max\{|1-t|+(n+\lambda-1)|t|, n|s|+|1-s\lambda|\}\\ &\geqq \inf_t \max\{|1-t|+(n+\lambda-1)|t|, n\lambda^{-1}|1-nt|+n|t|\}\\ &= 2n(n-1)(n^2-2\lambda+\lambda^2)^{-1}.\end{aligned}$$

To estimate $\pi_1(E_n)$ from below, there is by Pietsch's Theorem [6] a measure μ on $\Omega = \{e_i'|E_n: 1 \leqq i \leqq n+1\}$ such that $\|\mu\| = \pi_1(E_n)$ and $\|x\| \leqq \mu(|\langle x,\cdot\rangle|)$ for all $x\in E_n$. Let v be a measure on Ω given by $v(f) = |G|^{-1}\sum_{g\in G}\mu(fog)$, so that $\|v\| = \pi_1(E_n)$, $\|x\| \leqq v(|\langle x,\cdot\rangle|)$ for $x\in E_n$ and $v(f) = v(fog)$ for all $f\in C(\Omega)$ and $g\in G$. The last implies that $s = v(\{b_i'\})$ is independent of i, $1\leqq i\leqq n$. Setting $t = v(\{b_{n+1}'\})$ gives scalars s and t satisfying $\pi_1(E_n) = sn+t$ and

$$\|x\| \leqq s\sum_{i\leqq n}|\langle x, e_i'\rangle|+t|\langle x, e_{n+i}\rangle|, x\in E_n.$$

Substituting $e_1-\lambda^{-1}e_{n+1}$ and e_1-e_2 in the last inequality shows that $s+t\lambda^{-1}\geqq 1$ and $2s\geqq 1$, so that

$$\pi_1(E_n) = sn+t \geqq 2^{-1}(n+\lambda).$$

Now vary λ with n by taking $\lambda_n = [2n(n-1)]^{\frac{1}{2}}-n$ for $n\geqq 5$. Combining inequalities yields the desired lower estimate.

REMARK 3: As is observed above every Minkowski space satisfies $x(E)\leqq t(E)$ and $s(E)\leqq t(E)$. Some other possible relations between the three parameters x, t and s are known to be false. The space $A_n = l_1^n\oplus l_2^n$ has unconditional basis constant one but $s(A_n)$ and $t(A_n)$ behave asymptotically like $n^{\frac{1}{4}}$ [1], and the tensor product $B_n = l_2^n\hat{\otimes} l_2^n$ has asymmetry constant one but $x(B_n)$ and $t(B_n)$ both act asymptotically like $n^{\frac{1}{2}}$ [3]. Such examples suggest the following problem. Is there a real function f of two variables such that $t(E)\leqq f(x(E), s(E))$ for all finite dimensional E?

The answer to this problem is negative as is shown by the following example due to J. Lindenstrauss.

EXAMPLE: Let

$$E_n = (\sum_{k=1}^{n} E_k^n)_{l_4}$$

where each E_k^n is isometric to an n^2 dimensional Hilbert space. Then $s(E_n) = x(E_n) = 1$ but $t(E_n) \to_{n\to\infty} \infty$. The first two statements are clear so we will prove only the last one. Let us start with the observation that E_n is isometric to a subspace of $L_4[0, 1]$.

LEMMA 1: *Let $B = (b_i)_{i \leqq n}$ be a normalized basic sequence in $L_4[0, 1]$ with $t(B) = \alpha$. Then* span $\{b_i\}_{i \leqq n}$ *is α-isomorphic to a subspace of $l_2^n \oplus_4 l_4^n$.*

PROOF: The expression

$$\int_0^1 |\sum_{i=1}^{n} \lambda_i b_i|^4$$

is α-equivalent to

$$\frac{1}{2^n n!} \sum_{\varepsilon} \sum_{\sigma} \int_0^1 |\sum_{i=1}^{n} \varepsilon_i \lambda_i b_{\sigma(i)}|^4$$

where $\varepsilon = (\varepsilon_i)_{i=1}^n$ ranges over all 2^n choices of signs and σ over all $n!$ permutations of $\{1, 2, \ldots, n\}$. This latter sum is of the form

$$a \sum_{i=1}^{n} \lambda_i^4 + c(\sum_{i=1}^{n} \lambda_i^2)^2$$

for suitable positive a and c.

LEMMA 2: *Let* $E \subset l_4^n$ $\dim E \geqq n^{\frac{2}{3}}$ *then* $d(E, l_2^{\dim E}) \to_{n\to\infty} \infty$.

This Lemma follows immediately from Corollary 3.1 of [8]. Using those two Lemmas we will estimate $t(E_n)$. Suppose $t(E_n) \leqq C$ for $n = 1, 2, 3, \ldots$ then E_n embeds uniformly into $l_2^{n^3} + l_4^{n^3}$. Denote

$$\beta_k = \|P \circ \varphi | E_k^n\|$$

where φ is an isomorphic embedding from E_n into $l_2^{n^3} \oplus l_4^{n^3}$ and P is a projection from $l_2^{n^3} \oplus l_4^{n^3}$ onto $l_2^{n^3}$ annihilating $l_4^{n^3}$. Let

$$z_k \in E_k^n, \qquad \|z_k\| = 1, \qquad \|P \circ \varphi(z_k)\| = \beta_k.$$

Then for some ε_k, $|\varepsilon_k| = 1$

$$n^{\frac{1}{4}} = \|\sum_{k=1}^{n} \varepsilon_k z_k\| \geqq \frac{1}{C}\|P \circ \varphi(\sum_{k=1}^{n} \varepsilon_k z_k)\| \geqq \frac{1}{C}(\sum_{k=1}^{n} |\beta_k|^2)^{\frac{1}{2}}.$$

But this implies that for n big enough at least one β_k must be very small. Then an easy perturbation argument implies that $l_4^{n^3}$ contains uniformly $l_2^{n^2}$ which contradicts Lemma 2.

REFERENCES

[1] D. J. H. GARLING, Y. GORDON: Relations between some constants associated with finite dimensional Banach spaces. *Israel J. Math. 9* (1971) 346–361.

[2] Y. GORDON: Asymmetry and projection constants of Banach spaces. *Israel J. Math. 14* (1972) 50–62.

[3] Y.GORDON, D. R. LEWIS: Absolutely summing operators and local unconditional structures, *Acta Math. 133* (1974) 27–48.

[4] V. I. GURARII, M. I. KADEC, V. I. MACAEV: On Banach-Mazur distance between certain Minkowski spaces. *Bull. Acad. Pol. Sci. Ser. Math. Astron. Phy. 13* (1965) 719–722.

[5] D. R. LEWIS: A relation between diagonal and unconditional basis constant. Math. Ann. 218 (1975) 193–198.

[6] A. PIETSCH: Absolut p-summierende Abbildungen in normierten Räumen. *Studia Math. 28* (1967) 333–353.

[7] I. SINGER: *Bases in Banach spaces.* Berlin-New York, Springer Verlag 1970.

[8] A. PELCZYNSKI, H. P. ROSENTHAL: Localization techniques in L_p spaces. *Studia Math. 52* (1975) 263–289.

(Oblatum 1–VII–1975)

The Ohio State University, Columbus, Ohio 43210
University of Florida, Gainesville, Florida 32611
and
The Ohio State University, Columbus, Ohio 43210
Institut of Mathematics of
Polish Academy of Sciences, Warsaw