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THE LAPLACIAN OPERATOR ON A RIEMANN SURFACE, II

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1. Introduction

This paper is a continuation of [7] and we shall use the notations and results from it continuously. In that paper we obtained the spectral decomposition of a family of Laplace operators Δ_k on the upper half-plane, and from it analogous results on the quotient of the upper half-plane by a Fuchsian group with $\delta(G) < \frac{1}{2}$. Finally, extensions of that theory were described which hold good for finitely generated Fuchsian groups. These extensions are made possible by certain Fourier expansions which generalize the classical Fourier expansion at a cusp. They make it possible to examine the behaviour 'at infinity' of the functions in which we are interested.

The possibility of making such expansions rests on the geometrical description of the fundamental domain, and of the group-theoretical structure of G. This we shall discuss in Section 2. From this the basic eigenfunctions arise naturally. We then have to investigate these and establish various relations between them. In fact, all these relations are consequences of the general theory of [7] applied to elementary groups and the whole procedure can be viewed as a means of passing to general Fuchsian groups through the investigation of certain elementary subgroups.

We shall then investigate the Eisenstein series $E_\zeta(z, s)$ introduced in [7]. Expanding them in the various Fourier expansions we can deduce a convenient form of the functional equation introduced in [7] and thereby effect the analytic continuation of various Poincaré series. The technique has also other applications which are merely alluded to.

Next we consider certain integral relations which the $E_\zeta(z, s)$ satisfy. These give another approach to the proof of the functional equation and the analytic continuation. This approach is also effective if $\delta(G) > \frac{1}{2}$.

This we shall carry out in the final sections of the paper. The application of these results to the spectral decomposition of the Δ_k will not be considered here and as a consequence the analytic continuation cannot be extended to an interval. This lacuna will be removed in the next paper of this series.

In the course of this paper we shall have to consider various conjugates of the Fuchsian group under consideration. For convenience we interpolate here various formulae which will be used repeatedly. So let G be a Fuchsian group and $A \in \mathrm{Con}\,(\boldsymbol{H})$ (the group of conformal maps of $\boldsymbol{H}$ onto itself, isomorphic to PSL $(2, \boldsymbol{R})$). We shall consider the group $A^{-1}GA$.

Let χ be a multiplier system of weight k for G,

$$\chi: G \to U(V)$$

where $U(V)$ is the group of unitary transformations of the (finite dimensional) Hermitian space V. Let f be an automorphic form of weight k, with values in V, and with multiplier χ. So, for $g \in G$,

$$j(g, z)^k f(gz) = \chi(g) f(z).$$

Now let

$$f^A(z) = j(A, z)^k f(Az). \tag{1}$$

Then f^A is an automorphic form for $A^{-1}GA$, of weight k, and with multiplier system χ_A, where

$$\chi_A(A^{-1}gA) = (\sigma^k(g, A)/\sigma^k(A, A^{-1}gA))\chi(g). \tag{2}$$

This formula, along with those of [7] allow us to conjugate freely.

2. Geometric and group-theoretic properties

In this section we recall the description of the fundamental domain which we have used in [5] and have recalled in [7]. It is basic for all that follows. Let G be a finitely generated Fuchsian group acting on the upper half-plane $\boldsymbol{H}$. Let $\boldsymbol{R}^*$ be the boundary of $\boldsymbol{H}$ considered as a circle on the Riemann sphere $\boldsymbol{C}_\infty$. $\boldsymbol{R}^* = \boldsymbol{R} \cup \{\infty\}$. Let $L(G)$ be the limit set of G and $\Omega(G) = \boldsymbol{R}^* - L(G)$. As $L(G)$ is closed in $\boldsymbol{R}^*$, $\Omega(G)$ is open and so can be written as

$$\Omega(G) = \bigcup_{\alpha \in \boldsymbol{A}} \Omega_\alpha$$

where the Ω_α are disjoint open intervals in $\boldsymbol{R}^*$. Then let

$$G_\alpha = \{g \in G : g(\Omega_\alpha) = \Omega_\alpha\}.$$

This is an elementary hyperbolic subgroup of G. The fixed points of G_α are exactly the end-points of Ω_α. There is a finite subset $\{\alpha(1), \alpha(2), \ldots, \alpha(p)\} \subset A$ so that, for $\alpha \in A$, G_α is conjugate to precisely one $G_{\alpha(j)}$ $(1 \leqq j \leqq p)$.

Let $\lambda_\alpha(\theta)$ be the arc of a circle, lying in $\boldsymbol{H}$, joining the end-points of Ω_α and making an internal angle θ with Ω_α. Let $\Lambda_\alpha(\theta)$ be the region in $\boldsymbol{H}$ bounded by Ω_α and $\lambda_\alpha(\theta)$. We shall suppose that $\theta \leqq \pi/2$; then the $\Lambda_\alpha(\theta)$ $(\alpha \in A)$ are mutually disjoint.

Let P be the set of parabolic vertices of G, and for $p \in P$ let G_p be the parabolic subgroup of G fixing p. There is a finite subset $\{p(1), \ldots, p(q)\} \subset P$ so that G_p is conjugate to precisely one $G_{p(j)}$ $(1 \leqq j \leqq q)$. We can construct a horocycle C_p at $p \in P$ so that:

(i) if $p, q \in P$, $p \neq q$, then $C_p \cap C_q = \emptyset$,

(ii) $g(C_p) = C_{g(p)}$ $(g \in G)$, and,

(iii) $C_p \cap \Lambda_\alpha(\theta) = \emptyset$ $(p \in P, \alpha \in A, \theta \leqq \pi/2)$.

If we consider the set

$$\boldsymbol{H} - \Big(\bigcup_{p \in P} C_p \cup \bigcup_{\alpha \in A} \Lambda_\alpha(\theta)\Big)$$

we see that it is invariant under G. We can find a finite-sided fundamental domain

$$D(\theta) = D(\theta, (C_p))$$

for the action of G on this set; $D(\theta)$ is relatively compact in $\boldsymbol{H}$.

It is also known that G has a finite set of generators A_j, B_j $(1 \leqq j \leqq g)$, E_j $(1 \leqq j \leqq r)$, H_j $(1 \leqq j \leqq p)$, π_j $(1 \leqq j \leqq q)$ satisfying the relations

$$[A_1, B_1] \cdot [A_2, B_2] \cdots [A_g, B_g] \cdot E_1 \cdots E_r H_1 \cdots H_p \pi_1 \cdots \pi_q = I$$

$$E_j^{e(j)} = I \qquad (1 \leqq j \leqq r)$$

where $e(j)$ is a positive integer. Any elliptic element of G is conjugate to some power of some E_j $(1 \leqq j \leqq r)$ and this is uniquely determined. H_j is conjugate to a generator of $G_{\alpha(j)}$ and π_j is conjugate to a generator of $G_{p(j)}$. It follows that, if p or q is non-zero,

$$H^2(G, \boldsymbol{Z}) \cong C_{e(1)} \oplus \ldots \oplus C_{e(r)}$$

where C_e is the cyclic group with e elements. From this we see that the structure of multiplier systems for G is quite simple (cf. [6]); if G has no elliptic elements then multipliers of every weight exist. We make this remark in order to show that the introduction of arbitrary weights is not an empty generalization. We shall not need it again in this paper and the actual group-theoretic structure will play no further role. We should note however that $p \neq 0$ if and only if G is of the second kind. This shall be our only concern since the corresponding theory for groups of the first kind is by now well known ([3], [4], [11]). Although it appears to introduce no new conceptual difficulties the presence of parabolic certainly introduces notational difficulties and we shall suppose henceforth that $q = 0$.

The expansion of functions with respect to G_p is well-known; we shall here develop the analogous expansion of functions about G_α. The most convenient method of doing this is to choose $A_j \in \mathrm{Con}(\boldsymbol{H})$ $(1 \leqq j \leqq p)$ so that

$$A_j^{-1}(\Omega_{\alpha(j)}) = \,]0, \infty[.$$

The group $A_j^{-1} G_{\alpha(j)} A_j$ is a hyperbolic group fixing $]0, \infty[$ and so is generated by an element of the form

$$z \mapsto e^{\kappa(j)} z \qquad (\kappa(j) > 0).$$

If we define for $z \in \boldsymbol{H}$,

$$r_j(z) = |A_j^{-1}(z)| \tag{3}$$

and

$$\theta_j(z) = \arg(A_j^{-1}(z)) \in \,]0, \pi[. \tag{4}$$

Observe that $r_j(z)$ is also defined for $z \in \Omega_{\alpha(j)}$, a fact which we shall often use. If $g \in G_{\alpha(j)}$ then there is $n \in \boldsymbol{Z}$ so that

$$r_j(gz) = e^{n\kappa(j)} r_j(z)$$

and

$$\theta_j(gz) = \theta_j(z).$$

Note that

$$\Lambda_{\alpha(j)}(\theta) = \{z\colon \theta_j(z) < \theta\}.$$

Let χ, V, k, $U(V)$ be as in Section 1; let f be an automorphic form of weight k and multiplier χ. Using the notation of Section 1 we set

$$f_j(z) = f^{A_j}(z)$$

which is an automorphic form of weight k, multiplier χ_{A_j}, for the group $A_j^{-1}GA_j$. It is therefore also an automorphic form for $A_j^{-1}G_{\alpha(j)}A_j$ with the restricted multiplier system. Now let

$$X_j = \chi_{A_j}((z \mapsto e^{\kappa(j)}z)). \tag{5}$$

X_j is then a unitary transformation of V, and f_j satisfies

$$f_j(e^{\kappa(j)}z) = X_j f_j(z).$$

We now fix an orthonormal base for V. Then there is $Y_j \in U(V)$ so that

$$Y_j X_j Y_j^{-1} = \operatorname{diag}(e^{i\xi(1,j)}, \ldots, e^{i\xi(n,j)}) = D_j \tag{6}$$

$n = \dim(V)$ and $\operatorname{diag}(x_1, \ldots, x_n)$ is the matrix with entries $x_j\delta_{ij}$. For $x \in \boldsymbol{R}$ we set

$$D_j^x = \operatorname{diag}(e^{ix\xi(1,j)}, \ldots, e^{ix\xi(n,j)}). \tag{7}$$

This is a one-parameter subgroup of $U(V)$ to which D_j belongs. If we set

$$F_j(z) = D_j^{-\log|z|/\kappa(j)} Y_j f_j(z)$$

then

$$F_j(e^{\kappa(j)}z) = F_j(z).$$

Thus, writing $z = re^{i\theta}$, we can expand $F_j(z)$ in a series of the form

$$F_j(z) = \sum_{m=-\infty}^{+\infty} c_m(\theta) e^{2\pi i m \log|z|/\kappa(j)}$$

and

$$c_m(\theta) = \kappa(j)^{-1} \int_{1 < r \leqq e^{\kappa(j)}} F_j(re^{-i}\theta) e^{-2\pi i m \log r/\kappa(j)} r^{-1} dr \tag{8}$$

It is now convenient to introduce some more notation. Let

$$e(x) = e^{2\pi i x}.$$

Let $Z_j = \{(m+\xi(1,j)/2\pi)/\kappa(j), \ldots, (m+\xi(n,j)/2\pi)/\kappa(j)) : m \in \boldsymbol{Z}\}$ and for $\alpha \in Z_j$ we shall write $|\alpha|$ for the corresponding $|m|/\kappa(j)$. We shall write, for any real vector $\alpha = (a_1, \ldots, a_n)$,

$$\mathfrak{E}_\alpha(r) = \operatorname{diag}(e(a_1 \cdot \log r), \ldots, e(a_n \cdot \log r)).$$

Then the Fourier expansion is given by

$$j(A_j, A_j^{-1}z)^k Y_j f(z) = \sum_{\alpha \in Z_j} \mathfrak{E}_\alpha(r_j(z)) c_\alpha(\theta_j(z)) \tag{9}$$

where the $c_\alpha(\theta)$ are certain vectors in V.

For the sake of convenience we shall write $z_j = A_j^{-1}(z)$. We shall also write $\zeta_j = A_j^{-1}(\zeta)$ $(\zeta \in \boldsymbol{R}^*)$ and $\chi_j = \chi_{A_j}$. Finally we shall set

$$Z^* = \{(j, \alpha) : 1 \leqq j \leqq p, \alpha \in Z_j\}$$

and we shall often regard this as a disjoint union of the Z_j.

3. Eisenstein series

The main object of this paper is to study the Eisenstein series introduced in [1] and [7]. This is defined, for $\operatorname{Re}(s) > \delta(G)$, $\zeta \in \Omega(G)$, by

$$E_\zeta(z, s) = \sum_{g \in G} \chi(g)^{-1} j(g, z)^k P(gz, \zeta)^s (gz, \zeta)^k. \tag{10}$$

Our first object is to obtain the Fourier expansion of this as described in Section 2. We let

$$E_\zeta^{(j)}(z, s) = \sum_{g \in G_{\alpha(j)}} \chi(g)^{-1} j(g, z)^k P(gz, \zeta)^s (gz, \zeta)^k \tag{11}$$

which is the corresponding series for the elementary group $G_{\alpha(j)}$. We can, using the results of [7: § 2] split (10) as

$$E_\zeta(z,s) = \sum_{g\in G/G_{\alpha(j)}} \sum_{h\in G_{\alpha(j)}} \chi(h)^{-1}\chi(g)^{-1}\varepsilon(g^{-1}(\infty),\zeta)j(h,z)^k \cdot P(hz, g^{-1}\zeta)^s|(g^{-1})'(\zeta)|^s(hz, g^{-1}\zeta)^k.$$

From this follows

$$E_\zeta(z,s) = \sum_{g\in G/G_{\alpha(j)}} E^{(j)}_{g^{-1}(\zeta)}(z,s)\chi(g)^{-1}\varepsilon(g^{-1}(\infty),\zeta)|(g^{-1})'(\zeta)|^s \tag{12}$$

Thus the problem reduces in the first case to finding the Fourier expansion of $E^{(j)}_\zeta(z,s)$. We retain the notations of Section 2 and set further $H_j = A_j^{-1}G_{\alpha(j)}A_j$. After a short calculation we find

$$j(A_j, z_j)^k E^{(j)}_\zeta(z,s)|A_j'(\zeta_j)|^s = \varepsilon(A_j\infty, A_j\zeta_j)\Big(\sum_{g\in H_j}\chi_j(g)^{-1}P(gz_j,\zeta_j)^s(gz_j,\zeta_j)^k\Big).$$

The second term on the right-hand side is the corresponding Eisenstein series for H_j. Using the results of Section 2 we can express this series as

$$Y_j^{-1}\Big(\sum_{m\in \boldsymbol{Z}} D_j^{-m}P(e^{m\kappa(j)}z_j,\zeta_j)^s(e^{m\kappa(j)}z_j,\zeta_j)^k\Big)Y_j. \tag{13}$$

Proceeding as in Section 2, and writing $z_j = r_je^{i\theta_j}$, we see that this has a Fourier expansion of the form

$$Y_j^{-1}\sum_{\alpha\in Z_j}\mathfrak{E}_\alpha(r_j)e_\alpha(\theta_j, s, \zeta_j)Y_j$$

where

$$\kappa(j)c_\alpha(\theta,s,\zeta) = \int_0^\infty P(re^{i\theta},\zeta)^s(re^{i\theta},\zeta)^k\mathfrak{E}_\alpha(r)^{-1}r^{-1}\,dr.$$

Now let, for $\operatorname{Re}(s) > 0$,

$$Q_k^+(\theta, X, s) = \int_0^\infty P(xe^{i\theta},1)^s(xe^{i\theta},1)^k e(-X\cdot\log x)x^{-1}\,dx \tag{14}$$

$$Q_k^-(\theta, X, s) = \int_0^\infty P(ze^{i\theta},-1)^s(xe^{i\theta},-1)^k e(-X\cdot\log x)x^{-1}\,dx. \tag{15}$$

If $\alpha = (a_1, \ldots, a_n)$ is a given vector we set

$$\mathfrak{Q}_k^\varepsilon(\theta, \alpha, s) = \operatorname{diag}(Q_k^\varepsilon(\theta, a_j, s)).$$

We shall postpone the investigation of the $Q_k^\varepsilon(\theta, X, s)$ for the moment. These functions can be evaluated in terms of hypergeometric functions but it appears in general to be simpler to deduce the properties that we need as special cases of the theory described in [7] as applied to elementary hyperbolic groups.

We see now that (13) has the Fourier expansion

(16) $$\kappa(j)^{-1} Y_j^{-1}\Big(\sum_{\alpha \in Z_j} \mathfrak{Q}_k^{\operatorname{sgn}(\zeta_j)}(\theta_j, \alpha, s)\mathfrak{E}_\alpha(r_j/|\zeta_j|)|\zeta_j|^{-s}\Big) Y_j.$$

We can now obtain the complete expansion of $E_\zeta(z, s)$. First we recall the functional equation

(17) $$E_{g(\zeta)}(z, s)|g'(\zeta)|^s = E_\zeta(z, s)\chi(g)^{-1}\varepsilon(g(\infty), g(\zeta)).$$

From this we may assume that

$$\zeta \in \bigcup_i \Omega_{\alpha(i)}$$

If $\zeta \in \Omega_{\alpha(i)}$ with $i \neq j$ then it follows that for all $g \in G$

$$\operatorname{sgn}(g\zeta_j) = -1$$

whereas if $\zeta \in \Omega_{\alpha(j)}$ then

$$\operatorname{sgn}(g\zeta_j) = +1$$

if and only if $g \in G_{\alpha(j)}$. We shall therefore define $E_\zeta^*(z, s)$ by

(18) $$E_\zeta(z, s) = E_\zeta^{(j)}(z, s)\delta_{ij} + E_\zeta^*(z, s)$$

$E_\zeta^{(j)}(z, s)$, when $\zeta \in \Omega_{\alpha(j)}$, is given by (16) with $\operatorname{sgn}(\zeta_j) = +1$. On the other hand, applying (12) we obtain that

(19) $$E_\zeta^*(z, s) j(A_j, z_j)^k = Y_j^{-1} \sum_{\alpha \in Z_j} \mathfrak{Q}_k^-(\theta_j(z), \alpha, s)\mathfrak{E}_\alpha(r_j(z)) M_\alpha(s, \zeta),$$

where

$$M_\alpha(s,\zeta) = \kappa(j)^{-1} \sum_{g\in G_{\alpha(j)}\backslash G}' (\varepsilon(A_j\infty, g\zeta)|(A_j^{-1})'(g\zeta)|^s)\mathfrak{E}_\alpha(|A_j^{-1}(g\zeta)|^{-1})Y_j \cdot |g'(\zeta)|^s|A_j^{-1}g(\zeta)|^s\chi(g^{-1})^{-1}\varepsilon(g^{-1}\infty,\zeta). \tag{20}$$

Here $\sum'_{g\in G_{\alpha(j)}\backslash G}$ means that the sum is taken over a set of representative cosets, excluding the coset $G_{\alpha(j)}\cdot I$ in the case that $j = i$ (i is, as above, a function of ζ). We shall suppose now that $\zeta\in\Omega_{\alpha(i)}$.

We now intend to write $M_\alpha(s,\zeta)$ as a sum over double cosets $G_{\alpha(j)}\backslash G/G_{\alpha(i)}$. Let $h\in G_{\alpha(i)}$ and $h^* = A_i^{-1}hA_i$. Then

$$\chi((gh)^{-1})^{-1} = \chi(h^{-1}g^{-1})^{-1} = \sigma^k(h^{-1},g^{-1})^{-1}\chi(g^{-1})^{-1}\chi(h^{-1})^{-1}.$$

By (2) $\chi(h^{-1})^{-1} = (\sigma^k(h^{-1},A_i)/\sigma^k(A_i,h^{*-1})\chi_i(h^{-1})^{-1}$. Recall that χ_i is a representation on $A_i^{-1}G_{\alpha(i)}A_i$. Also from the definition of σ^k it follows easily that

$$\sigma^k(A^i,h^{*-1}) = 1.$$

We shall now examine

$$\chi((gh)^{-1})^{-1}\varepsilon((gh)^{-1}(\infty),\zeta) = (\chi(g^{-1})^{-1})(\sigma^k(h^{-1},g^{-1})^{-1}\sigma^k(h^{-1},A_i) \cdot \varepsilon(A_ih^{*-1}A_ig^{-1}(\infty),A_i\zeta_i))\chi_i(h^*).$$

We suppose that $\infty\in L(G)$, and so $\Omega_{\alpha(i)}\subseteq \mathbf{R}$. Then

$$\varepsilon(A_i\zeta_i, h^{-1}g^{-1}(\infty))$$

is constant for $\zeta\in\Omega_{\alpha(i)}$. On the other hand the central term on the right-hand side is, by (17), a multiple of a representation of $G_{\alpha(i)}$. Taking $h = I$ we see that the multiple is $\varepsilon(g^{-1}\infty, A_i\zeta_i)$. However, by equation (6) of [7] it follows that this term is constant except for a possible finite number of exceptions (as a function of h). Now it follows that this term is identically $\varepsilon(g^{-1}\infty, A_i\zeta_i)$. If $g\in G$ write $g_{ji} = A_j^{-1}gA_i$ and then the expression (20) gives

$$M_\alpha(s,\zeta) = \kappa(j)^{-1}\sum_{g\in G_{\alpha(j)}\backslash G/G_{\alpha(i)}}' \sum_{h\in H_i}\mathfrak{E}_\alpha(|g_{ji}(\zeta_i)|)^{-1}|g_{ji}(h\zeta_i)|^s|h'(\zeta_i)|^s \cdot |g_{ji}(h\zeta_i)|^{-s}Y_j\chi(g^{-1})^{-1}\varepsilon(g^{-1}(\infty), A_i\zeta_i)\varepsilon(A_j\infty, gA_i\zeta_i)\chi_i(h).$$

(H_i was defined at the beginning of the section.) Let

$$X^{ji}(g) = Y_j\chi(g^{-1})^{-1}\varepsilon(g^{-1}\infty, A_i\zeta_i)\varepsilon(A_j\infty, gA_i\zeta_i)Y_i^{-1}.$$

This is a unitary matrix and is independent of the choice of $\zeta_i \in]0, \infty[$. Next note that

$$|h'(\zeta_i)|^s|\zeta_i|^s = |h(\zeta_i)|^s.$$

Thus we see that

$$|A_i'(\zeta_i)|^s|\zeta_i|^s M_\alpha(s, \zeta)Y_i^{-1} = \sum_{g\in G_{\alpha(j)}\backslash G/G_{\alpha(i)}} \sum_{h\in H_i} \kappa(j)^{-1}\mathfrak{E}_\alpha(|g_{ji}(h\zeta_i)|)^{-1}$$

$$\cdot |g_{ji}'(h\zeta_i)|^s|h\zeta_i|^s|g_{ij}(h\zeta_i)|^{-s}X^{ji}(g)Y_i\chi_i(h)Y_i^{-1}. \tag{21}$$

It is clear that the right-hand side has a Fourier expansion. Remember that $\zeta_i > 0$, and, that under the conditions of the summation, $g_{ji}(h\zeta_i) < 0$. We obtain for the Fourier expansion of the inner sum an expression of the form

$$\sum_{\beta\in Z_i} N_\beta(g, \alpha, s)\mathfrak{E}_\beta(\zeta_i)^{-1}$$

where

$$\kappa(i)N_\beta(g, \alpha, s) = \int_0^\infty \mathfrak{E}_\alpha(|g_{ji}(r)|)^{-1}(|g_{ji}'(r)/g_{ji}(r)|)^s r^{s-1}X^{ji}(g)\mathfrak{E}_\beta(r)\cdot dr. \tag{22}$$

For a matrix A we shall write A_{uv} for the (u, v)th entry. Let $\alpha = (a_1, \ldots, a_n)$, $\beta = (b_1, \ldots, b_n)$. Then (22) gives

$$N_\beta(g, \alpha, s)_{uv} = \kappa(i)^{-1}\int_0^\infty e(-a_u\cdot \log|g_{ji}(r)| + b_v\cdot \log r)(|g_{ji}'(r)/g_{ji}(r)|)^s r^{s-1}dr.$$

Let now $\gamma \in \mathrm{Con}(\boldsymbol{H})$ be such that $\gamma]0, \infty[\subset]-\infty, 0[$ and let γ be represented by the matrix $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$. Let $X, Y \in \boldsymbol{R}$. Then a straightforward calculation shows that

$$\begin{aligned}\int_0^\infty & e(-X\cdot\log|\gamma(r)| + Y\cdot \log r)(|\gamma'(r)/\gamma(r)|)^s r^{s-1}dr \\ &= |ad|^{-s}\cdot e(-X\cdot\log|\gamma(\infty)| + Y\cdot\log|\gamma^{-1}(0)|) \\ &\quad\cdot B(s+2\pi iX, s+2\pi iY)F(s-2\pi iX, s+2\pi iY; 2s; (ad)^{-1}),\end{aligned} \tag{23}$$

where $F(A, B; C; z)$ is the ordinary hypergeometric function and $B(s, t)$ is the Euler beta-function. If we now write

$$(24)\quad \sigma_{\alpha\beta}(s)_{uv} = \sum_{g\in G_{\alpha(j)}\backslash G/G_{\alpha(i)}}' X^{ij}(g)_{uv}|ad|^{-s}e(-a_u\cdot\log|\gamma(\infty)|-b_v\log|\gamma^{-1}(0)|) \cdot B(s+2\pi i a_u, s+2\pi i b_v)F(s-2\pi i a_u, s+2\pi i b_v; 2s; (ad)^{-1})$$

where $\alpha\in Z_j$, $\beta\in Z_i$ and we regard Z_i, Z_j as subsets of Z^*. Actually this explicit formula for $\sigma_{\alpha\beta}(s)$ will not be used although the function $\sigma_{\alpha\beta}(s)$ (i.e. the matrix with the entries given by (24)) will be central in our discussion. It depends on the multiplier system but we shall not encumber out notations by making this dependence explicit. If we now collect our results together we obtain the following:

THEOREM 1: *Let χ be a multiplier of weight k and let $E_\zeta(z, s)$ be the Eisenstein series defined by* (10). *Then, with the notations introduced above, for i, j* $(1\leqq i, j\leqq p)$ *and* $\zeta\in\Omega_{\alpha(i)}$,

$$j(A_j, z_j)^k Y_j E_\zeta(z, s) Y_i^{-1}|A_i'(\zeta_i)|^s|\zeta_i|^s\kappa(i)\cdot\kappa(j)$$

$$= \kappa(i)\delta_{ij}e(k)\sum_{\alpha\in Z_j}\mathfrak{Q}_k^+(\theta_j(z),\alpha,s)\mathfrak{E}_\alpha(r_j(z))\mathfrak{E}_\alpha(\zeta_i)^{-1} + \sum_{\substack{\alpha\in Z_j\\ \beta\in Z_i}}\mathfrak{Q}_k^-(\theta_j(z),\alpha,s)\cdot\mathfrak{E}_\alpha(r_j(z))\sigma_{\alpha\beta}(s)\mathfrak{E}_\beta(\zeta_i)^{-1}.$$

The series are all absolutely convergent in $\mathrm{Re}(s) > \delta(G)$.

We define the following three functions

$$(25)\qquad D_k(s) = \pi\cdot e^{\pi i k}2^{2-2s}\Gamma(2s-1)/\Gamma(s+k)\Gamma(s-k),$$

$$(26)\qquad q_k^+(s, a) = \Gamma(1-2s)\left(\frac{\Gamma(s-2\pi i a)}{\Gamma(1-s-2\pi i a)} + e(k)\frac{\Gamma(s+2\pi i a)}{\Gamma(1-s+2\pi i a)}\right),$$

and

$$(27)\qquad q_k^-(s, a) = e(k)B(s-2\pi i a, s+2\pi i a).$$

We find that if $\mathrm{Re}(s) > \frac{1}{2}$, as $\theta\to 0$

$$(28)\qquad Q_k^+(\theta, a, s) \sim D_k(s)(\sin\theta)^{1-s}$$

which follows from (2) of [7]. If, instead, $0 < \mathrm{Re}(s) < \frac{1}{2}$,

$$Q_k^+(\theta, a, s) \sim q_k^+(s, a)(\sin\theta)^s \tag{29}$$

as follows from (14). Finally, for $\mathrm{Re}(s) > 0$,

$$Q_k^-(\theta, a, s) \sim q_k^-(s, a)(\sin\theta)^s \tag{30}$$

as follows from (15).

If $\alpha = (a_1, \ldots, a_n)$ we shall write $\mathfrak{q}_k^\varepsilon(s, \alpha)$ for

$$\mathrm{diag}\,(q_k^\varepsilon(s, a_u)).$$

Now let $\eta \in \Omega(G)$. Then

$$\lim_{z \to \eta} E_\zeta(z, s) \cdot \mathrm{Im}\,(z)^{-s} = \sum_{g \in G} \chi(g)^{-1} |g'(\eta)|^s |g\eta - \zeta|^{-2s} \varepsilon(\eta, g^{-1}\infty)\varepsilon(g\eta, \zeta) \tag{31}$$
$$= S(\eta, \zeta; s).$$

$S(\zeta, \eta; s)$ is defined and studied in [7]. The proof of (31) is merely a straightforward calculation. One sees easily that even $E_\zeta(z, s) \cdot \mathrm{Im}\,(z)^{-s}$ is smooth in $\boldsymbol{C}_\infty - L(G)$ except for the set $G\{\zeta\}$. One therefore deduces from theorem 1 the following:

THEOREM 2: *Let* $S(\zeta, \eta; s)$ *be as above. Let* $\mathrm{Re}\,(s) < \frac{1}{2}$, $\zeta \in \Omega_{\alpha(i)}$, $\eta \in \Omega_{\alpha(j)}$. *Then*

$$Y_i S(\zeta, \eta; s) Y_j^{-1} |A_i'(\zeta_i)|^s |\zeta_i|^s |A_j'(\eta_j)|^s |\eta_j|^s \kappa(i)\kappa(j)$$
$$= \kappa(i)\delta_{ij} \sum_{\alpha \in Z_i} \mathfrak{q}_k^+(s, \alpha)\mathfrak{E}_\alpha(\zeta_i)\mathfrak{E}_\alpha(\eta_j)^{-1} + e(-k) \sum_{\substack{\alpha \in Z_i \\ \beta \in Z_j}} \mathfrak{q}_k^-(s, \alpha)\mathfrak{E}_\alpha(\zeta_i)\sigma_{\alpha\beta}(s)\mathfrak{E}_\beta(\eta_j)^{-1}.$$

4. The functional equation

We must now begin to investigate some of the properties of the function $Q_k^\varepsilon(\theta, a, s)$. We see at once that these functions are solutions of a second order differential equation in θ. This fact, and the consequent explicit identification of these functions can be used to obtain the results we need. But it is more in keeping with our general programme to regard these functions as the Fourier coefficients of the Eisenstein series for the hyperbolic groups and then to derive their properties from the general results of [7]. The object of this paragraph is to state the functional equation for the Eisenstein series (as discussed in [7]) in terms of the $\sigma_{\alpha\beta}(s)$. In accordance with our general philosophy we shall do this first for the hyperbolic elementary groups.

THEOREM 3: *The functions* $Q_k^{\varepsilon}(\theta, a, s)$ *satisfy the following*:

$$Q_k^+(\pi-\theta, a, s) = e^{2ik(\theta-\pi)}Q_k^-(\theta, -a, s) \tag{32}$$

$$Q_k^-(\pi-\theta, a, s) = e^{2ik\theta}Q_k^+(\theta, -a, s) \tag{33}$$

$$Q_k^+(\theta, a, s)q_k^+(1-s, a)+Q_k^-(\theta, a, s)q_k^-(1-s, -a)e(-k) = D_k(s)Q_k^+(\theta, a, 1-s) \tag{34}$$

$$Q_k^-(\theta, a, s)q_k^+(1-s, -a)+Q_k^+(\theta, a, s)q_k^-(1-s, a) = D_k(s)Q_k^-(\theta, a, 1-s). \tag{35}$$

PROOF: (32) and (33) follow from (14) and (15) without trouble. By (32) and (33) it follows that (34) and (35) are equivalent. It remains to prove (34). We consider the group G generated by $z \mapsto e^{\kappa}z$, and, for $\xi \in \boldsymbol{R}$, the corresponding Eisenstein series

$$E_{\zeta}(z, s) = \sum_{n\in\boldsymbol{Z}} e(-n\xi)P(e^{n\kappa}z, \zeta)^s(e^{n\kappa}z, \zeta)^k.$$

Let $\Omega_1 = \{x > 0\}$, $\Omega_2 = \{x < 0\}$. If we write $Z_0 = \{(m+\xi/2\pi)/\kappa : m \in \boldsymbol{Z}\}$ and $z = re^{i\theta}$ it then follows from the results of Section 3 that

$$|\zeta|^s E_{\zeta}(z, s) = \kappa^{-1} \sum_{a\in Z_0} Q_k^{\varepsilon}(\theta, a, s)e(a(\log r - \log|\zeta|))$$

where $\varepsilon = +$ if $\zeta \in \Omega_1$ and $\varepsilon = -$ otherwise. Proceeding in the same way as when we deduced theorem 2 from theorem 1 we obtain that if $\zeta, \eta \in \Omega_1$ then

$$|\zeta|^s|\eta|^s S(\zeta, \eta; s) = \kappa^{-1} \sum_{a\in Z_0} q_k^+(s, a)e(a(\log|\zeta| - \log|\eta|)),$$

and, if $\zeta \in \Omega_2$, $\eta \in \Omega_1$, we have, using also (32) that

$$|\zeta|^s|\eta|^s S(\zeta, \eta; s) = \kappa^{-1} \sum_{a\in Z_0} q_k^-(s, -a)e(-k)e(a(\log|\zeta| - \log|\eta|)).$$

From (33) of [7] we see that if $\mathrm{Re}\,(s) > \frac{1}{2}$

$$\int_B E_\zeta(z, s) S(\zeta, \eta; 1-s) d\zeta = D_k(s) E_\eta(z, 1-s).$$

In this case B can be taken to be $[1, e^\kappa[\cup [-e^\kappa, -1[$. On carrying out the integrations if $\eta \in \Omega_1$ we get (34).

As (14) and (15) define $Q_k^\varepsilon(\theta, a, s)$ only for $\mathrm{Re}\,(s) > 0$ it follows that now we have obtained the analytic continuation of these functions. (34) is proved at first only for $\mathrm{Re}\,(s) > \frac{1}{2}$ but it clearly extends to all values of s.

COROLLARY:

$$(36) \qquad q_k^+(s, a) q_k^+(1-s, a) + q_k^-(s, a) q_k^-(1-s, -a) e(-k) = D_k(s) D_k(1-s)$$

$$(37) \qquad q_k^-(s, a) q_k^+(1-s, -a) + q_k^+(s, a) q_k^-(1-s, -a) = 0.$$

PROOF: Using (28), (29), (30) these are the limiting versions of (34), (35). They can also be regarded as consistency relations for (34), (35).

(36), (37) can, of course, be proved directly although not without a little difficulty.

We shall now proceed with the general problem. Suppose that G is a finitely generated Fuchsian group with $\delta(G) < \frac{1}{2}$. Consider the equation (33) of [7]:

$$(38) \qquad \int_B E_\zeta(z, s) S(\zeta, \eta; 1-s) d\zeta = D_k(s) E_\eta(z, 1-s).$$

Theorems 1 and 2 give us the Fourier expansions of both sides of (38). We may take

$$B = \bigcup_{j=1}^{p} A_j([1, e^{\kappa(j)}[).$$

On substituting the various expansions into (38), and simplifying by means of theorem 3 we obtain

$$(39) \quad \sigma_{\alpha\beta}(s) \cdot \mathfrak{q}_k^+(1-s, \beta) + e(-k) \sum_{\gamma \in Z^*} \sigma_{\alpha\beta}(s) \mathfrak{q}_k^-(1-s, \gamma) \sigma_{\gamma\beta}(1-s) \kappa(\gamma)^{-1}$$
$$= \delta_{\alpha\beta} \mathfrak{q}_k^-(1-s, -\beta) \kappa(\alpha) + \mathfrak{q}_k^+(1-s, -\alpha) \sigma_{\alpha\beta}(1-s),$$

which holds for $\alpha, \beta \in Z^*$. For $\alpha \in Z^*$ there is j so that $\alpha \in Z_j$ and we write

$\kappa(\alpha)$ for $\kappa(j)$. (39) is the basic functional equation but so far it is proved only when

$$\tfrac{1}{2} < \operatorname{Re}(s) < 1-\delta(G).$$

By (37) we can replace the multiplier in the last term by

$$-q_k^+(s,\alpha)q_k^-(1-s,\alpha)q_k^-(s,\alpha)^{-1} = q_k^+(1-s,-a).$$

Let

$$\sigma^*_{\alpha\beta}(s) = \delta_{\alpha\beta}q_k^+(s,\alpha)+e(-k)q_k^-(s,\alpha)\sigma_{\alpha\beta}(s)\kappa(\beta)^{-1}.$$

Then (39) becomes

$$\text{(40)}\quad \sum_{\gamma\in Z^*}\sigma^*_{\alpha\gamma}(s)\sigma^*_{\gamma\beta}(1-s) = \delta_{\alpha\beta}(e(-k)\mathfrak{q}_k^-(1-s,\alpha)\mathfrak{q}_k^-(s,\alpha) + \mathfrak{q}_k^+(s,\alpha)\mathfrak{q}_k^+(1-s,\alpha)) = \delta_{\alpha\beta}D_k(s)D_k(1-s)$$

We now let W_0 be the Hilbert space of l^2 sequences on Z^*. This is isomorphic to the space of L^2-functions on

$$\bigcup_i G_{\alpha(i)}\backslash\Omega_{\alpha(i)} = G\backslash\Omega(G).$$

We let $W = W_0\otimes V$; that is the space of L^2 functions on $G\backslash\Omega(G)$ with values in V. Clearly $\sigma_{\alpha\beta}(s)$, $\sigma^*_{\alpha\beta}(s)$ are matrices representing linear maps from W to itself. Representing the corresponding maps by $\Sigma(s)$, $\Sigma^*(s)$ we see that the functional equation becomes

$$\text{(41)}\qquad \Sigma^*(s)\Sigma^*(1-s) = D_k(s)D_k(1-s)\,\mathrm{Id}_W.$$

We must now investigate the nature of $\Sigma^*(s)$ as an operator; we would like to show that it is substantially a Fredholm operator. We write the function $S(\zeta,\eta;s)$, for $\zeta\in\Omega_{\alpha(i)}$, $\eta\in\Omega_{\alpha(j)}$, in

$$S(\zeta,\eta;s) = \delta_{ij}S_j(\zeta,\eta;s)+(S(\zeta,\eta;s)-\delta_{ij}S_j(\zeta,\eta;s)),$$

where $S_j(\zeta,\eta;s)$ is the S-function for the group $G_{\alpha(j)}$. This decomposition corresponds directly to the two terms on the right-hand side of the decomposition given in theorem 2. However from the series definition of

$S(\zeta, \eta; s)$ one sees quite easily that the second term above is smooth function of ζ and η. (In $\{\mathrm{Re}(s) > \delta(G)\}$ one sees that the series of kth derivatives converges absolutely and locally uniformly.)

From the elementary properties of Fourier series there is, for each positive integer K, a constant $C(K)$ so that all the entries of the matrix

$$\mathfrak{q}_k^-(s, \alpha)\sigma_{\alpha\beta}(s)$$

(the Fourier coefficients of the function above by theorem 2) are bounded by

$$C(K) \cdot (1 + |\alpha|^2 + |\beta|^2)^{-K}$$

($|\alpha|$ was defined in section 2). The proof shows even that this holds uniformly in compact subsets in $\{\mathrm{Re}(s) > \delta(G)\}$. By Stirling's formula, as $a \to \infty$

$$q_k^+(s, a) \sim c|a|^{2\,\mathrm{Re}(s) - 1},$$

again uniformly on compact subsets of $\{\mathrm{Re}(s) > 0, s \neq \frac{1}{2}\}$. $q_k^+(s, a)$ has a pole at $\frac{1}{2}$. It follows that only a finite number of the $\mathfrak{q}_k^+(s, \alpha)$ are singular for s lying in a given compact set.

It follows therefore, that if we omit a finite number of rows, the matrix

$$\mathfrak{q}_k^+(s, \alpha)^{-1}\mathfrak{q}_k^-(s, \alpha)\sigma_{\alpha\beta}(s)$$

has no poles in $\{\mathrm{Re}(s) > \delta(G)\}$ and is represented through the Fourier transform by an operator with a smooth kernel.

Let $J_k(s)$ be the operator with the matrix

$$\mathfrak{q}_k^+(s, \alpha)\delta_{\alpha\beta}.$$

We can write

$$\Sigma^*(s) = J_k(s) + \Sigma_0(s)$$

where $\Sigma_0(s)$ has the matrix representation

$$e(-k)q_k^-(s, \alpha)\sigma_{\alpha\beta}(s).$$

Now the functional equation becomes

$$(42) \qquad (I+J_k(s)^{-1}\Sigma_0(s))(I+\Sigma_0(1-s)J_k(1-s)^{-1}) = D_k(s)D_k(1-s) \cdot J_k(s)^{-1}J_k(1-s)^{-1}.$$

Let s be confined to a compact set. Then $J_k(s)^{-1}\Sigma_0(s)$ has only a finite number of poles, and in the matrix form these appear in only a finite number of rows. Outside these it follows from the remarks above that $J_k(s)^{-1}\Sigma_0(s)$ is represented by a compact operator with a smooth kernel and hence the Fredholm determinant exists. Using *det* to denote the Fredholm determinant we set

$$(43) \qquad \Phi_k(s,\chi) = \Phi(s) = \det(I+J_k(s)^{-1}\Sigma_0(s)).$$

which is even analytic (cf. [2]). As the poles of the matrix occur in only a finite number of rows it follows that $\Phi(s)$ extends to a meromorphic function on $\{\mathrm{Re}(s) > \delta(G)\}$.

Next note that, by straightforward estimations,

$$\psi(s) = \det(D_k(s)D_k(1-s)J_k(s)^{-1}J_k(1-s)^{-1})$$

exists. Then, from (42) it follows that

$$(44) \qquad \Phi(s)\Phi(1-s) = \psi(s).$$

This shows first of all that $\Phi(s)$ is not identically zero. From this it follows that, as the inverse of $I+J_k(s)^{-1}\Sigma_0(s)$ exists by Fredholm theory (except, perhaps at a finite set of points) and that it is unique then from (42) this gives a meromorphic continuation of $I+J_k(s)^{-1}\Sigma_0(s)$ to at least the strip $0 < \mathrm{Re}(s) < 1$. In fact this continuation is valid in the whole complex plane – the only fact we need to prove this is that the zeros of the non-trivial matrix elements of $J_k(s)$ form a discrete set, and that for each of these only a finite number of the entries of $J_k(s)$ vanish. This is easily checked. The difficulty outside the strip $0 < \mathrm{Re}(s) < 1$ is that $\Phi(s)$ has many more poles than necessary, poles which arise for essentially trivial reasons.

Where it exists $\Sigma_0(s)$ represents an operator with a smooth kernel, since this follows from the general Fredholm theory in a Banach space (cf. [2]). This can also be deduced without trouble from the explicit formulae for the inverse of an operator. Using this proof one can see that if s_0 is a pole of $\sigma_{\alpha\beta}(s)$ of order N (at most) over all α, β then we have, for a given $K > 0$, a constant $C(K)$ so that, in some neighbourhood of s_0

$$(45) \qquad |\sigma_{\alpha\beta}(s)||(s-s_0)^N < C(K)\cdot(1+|\alpha|^2+|\beta|^2)^{-K}.$$

From this it is now clear that $E_\zeta(z, s)$ and $S(\zeta, \eta; s)$ have analytic continuations to the whole complex plane and that the poles in $0 < \mathrm{Re}(s) < 1$ coincide with the poles of $\Phi(s)$ there.

This completes the investigation here. It is worth noting however that our arguments have been essentially formal. We needed only some identities from the special case of hyperbolic groups and general Fredholm theory. This shows that the results here, although quite striking, do not lie so very deep.

As a result it follows that

$$\sum_{g \in G} P(gz, \eta)^{-s}$$

can be continued to the whole plane. By Landau's theorem it has a singularity at $s = \delta(G)$; the same fact, if $\delta(G) > \frac{1}{2}$ was proved in [5]. This leaves only the case $\delta(G) = \frac{1}{2}$.

5. Maaß-Selberg relations

The results of Section 4 give a quite satisfactory description of the analytic continuation in the case that $\delta(G) < \frac{1}{2}$. For certain reasons, connected with the completeness problem described in [7], we shall need to know more about these functions. Furthermore, we still have to prove the analytic continuation in the case that $\delta(G) \geqq \frac{1}{2}$. Both of these can be solved by means of certain integral relations that the Eisenstein series satisfy. These were first used in this context by Selberg [11] and are often called Maaß-Selberg relations, although this name covers a variety of rather similar relations. We shall need some such here and shall describe them also as Maaß-Selberg relations.

Let $\varphi_j \in]0, \pi/2[$ $(1 \leqq j \leqq p)$. Observe that the set

$$\{z \in \boldsymbol{H}: \text{for all } g \in G, j\ (1 \leqq j \leqq p)\ \theta_j(gz) > \varphi_j\}$$

is invariant under G. Let D^φ be a fundamental domain for the action of G on this set. Let

$$D_j = \{z \in \boldsymbol{H}: \theta_j(z) < \varphi_j, 1 \leqq r_j(z) < e^{\kappa(j)}\} \qquad (1 \leqq j \leqq p).$$

Then

$$D = D^\varphi \cup D_1 \cup D_2 \ldots \cup D_p$$

is a fundamental domain for the action of G on $\boldsymbol{H}$. Let $z \in D$, $\zeta \in \bigcup \Omega_{\alpha(j)}$ and define

$$(46) \qquad \left.\begin{aligned} E_\zeta^\varphi(z, s) &= E_\zeta(z, s) - E_\zeta^{(i)}(z, s) && (z \in D_i,\ \zeta \in \Omega_{\alpha(i)}) \\ &= E_\zeta(z, s) && (\text{otherwise}). \end{aligned}\right\}$$

Here, as before, $E_\zeta^{(i)}(z, s)$ is the Eisenstein series associated to $G_{\alpha(i)}$. For $A \in \operatorname{End}(V)$ we shall write A^* for the Hermitian conjugate to A. Then, for example

$$E_\zeta(z, s)^* = \sum_{g \in G} \chi(g) j(g, z)^{-k} P(gz, \zeta)^{\bar{s}}(gz, \zeta)^{-k}.$$

With these notations, our object is now to evaluate the integrals

$$\int_{D^\varphi} E_\zeta(z, \bar{t})^* E_\zeta(z, s) d\sigma(z)$$

and

$$\int_D E_\zeta^\varphi(z, \bar{t})^* E_\zeta^\varphi(z, s) d\sigma(z).$$

Let us first observe that these integrals converge if $\operatorname{Re}(s)$, $\operatorname{Re}(t) > \delta(G)$, and in the case of the second integral $\operatorname{Re}(s+t) > 1$. Unfortunately the formulae become too complicated if we evaluate them directly and it is best to introduce an auxiliary function. To describe this we note that if $\alpha \in Z^*$ then $\alpha \in Z_j$ for some uniquely defined j; we shall write this j as $\boldsymbol{i}(\alpha)$. Then for $\alpha \in Z^*$ and taking $j = \boldsymbol{i}(\alpha)$, we introduce

$$(47) \qquad E_\alpha^!(z, s) = \int_{\{1 < \zeta_j < e^{\kappa(j)}\}} E_\zeta(z, s) |A_j'(\zeta_j)|^s \zeta_j^{s-1} Y_j^{-1} \mathfrak{E}_\alpha(\zeta_j) d\zeta_j.$$

These are smoothed out versions of the $E_\zeta(z, s)$. From theorem 1 we see that $E_\alpha^!(z, s)$ has the Fourier expansion

$$(48) \qquad Y_i E_\alpha^!(z, s) j(A_i, z_i)^k \kappa(i) = \boldsymbol{e}(k) \delta_{ij} \kappa(i) \mathfrak{Q}_k^+(\theta_i(z), \alpha, s) \mathfrak{E}_\alpha(r_i(z)) + \sum_{\beta \in Z_i} \mathfrak{Q}_k^-(\theta_i(z), \beta, s) \mathfrak{E}_\beta(r_i(z)) \sigma_{\beta\alpha}(s).$$

Here, as above, $j = \boldsymbol{i}(\alpha)$. If $\zeta \in \Omega_{\alpha(j)}$ then we have also

$$E_\zeta(z,s)Y_j^{-1}|A_j'(\zeta_j)|^s|\zeta_j|^s = \sum_{\alpha\in Z_j} \kappa(j)^{-1} E_\alpha^!(z,s)\mathfrak{E}_\alpha(\zeta_j)^{-1}. \tag{49}$$

We can define analogously $E_\alpha^{!\varphi}(z,s)$; this has the same Fourier expansion as in (48) except that the first term on the right-hand side vanishes if $\theta_{i(\alpha)}(z) < \varphi_{i(\alpha)}$.

The technique we use is an application of Green's formula and follows known lines. It is used, for example, in [8]. Thus we shall only sketch the calculations. We start from the fact that the Eisenstein series are eigenfunctions of the Δ_k; from this follows the relation

$$\begin{aligned}(-s(1-s)+t(1-t))&\int_{D^\varphi} E_\beta^!(z,\bar{t})^* E_\alpha^!(z,s)d\sigma(z)\\ &= \int_{D^\varphi} E_\beta^!(z,\bar{t})^*\Delta_k E_\alpha^!(z,s) - (\Delta_k E_\beta^!(z,\bar{t}))^* E_\alpha^!(z,s) d\sigma(z).\end{aligned}$$

The right-hand side can be evaluated by Green's formula. We indicate the form of this theorem that we need. Let R be a relatively compact domain in $\boldsymbol{H}$ with a piece-wise smooth boundary, and let f_1, f_2 be C^2 functions defined on a neighbourhood of R. Then

$$\int_R \Delta_k f_1\cdot \bar{f}_2 - f_1\overline{(\Delta_k f_2)}d\sigma(z) = \int_{\partial R} \frac{\partial f_1}{\partial n}\cdot \bar{f}_2 - f_1\frac{\partial \bar{f}_2}{\partial n}|dz| - 2ki\int_{\partial R} f_1\bar{f}_2 y^{-1}dy$$

($y = \operatorname{Im}(z)$), a formula that is compatible with the action of $\operatorname{Con}(\boldsymbol{H})$, if f_1, f_2 are transformed as forms of weight k. This formula follows by direct calculation but is better regarded as an application of Green's theorem to the universal cover of $\operatorname{Con}(\boldsymbol{H})$. In the formula above n is the outward normal. On the line $\arg(z) = \varphi$, $\partial/\partial n = -\partial/\partial(\arg(z))$, if the region being considered lies in $\arg(z) > \varphi$.

By direct estimates we have, for $z\in D$, that if G has no parabolic elements then

$$E_\alpha^!(z,s) \ll (\sin\theta_{i(\alpha)}(z))^{1-s}$$

and

$$\partial E_\alpha^!(z,s)/\partial\theta_{i(\alpha)} \ll (\sin\theta_{i(\alpha)}(z))^{-s},$$

estimates which are proved by the techniques of [5]. In fact the same results hold true for $E_\zeta^\varphi(z,s)$ and the assertions follow by averaging.

Now let

$$(50)\quad R_k^{\varepsilon\eta}(\theta, s, t, a) = \partial Q_k^{\varepsilon}/\partial\theta(\theta, s, a)\cdot\overline{Q_k^{\eta}(\theta, \bar{t}, a)} - Q_k^{\varepsilon}(\theta, s, a) \cdot \overline{\partial Q_k^{\eta}/\partial\theta(\theta, \bar{t}, a)} + 2kiQ_k^{\varepsilon}(\theta, s, a)\overline{Q_k^{\eta}(\theta, \bar{t}, a)}$$

We shall also let $\mathfrak{R}_k^{\varepsilon\eta}(\theta, s, t, \alpha)$ be the corresponding diagonal matrix if α is a given vector.

From the considerations above and (50) one deduces:

THEOREM 4: *Suppose G is a finitely generated Fuchsian group without parabolic elements. Suppose φ_j, $\varphi'_j \in]0, \pi/2[$ $(1 \leqq j \leqq p)$ with $\varphi'_j \geqq \varphi_j$. If* $\operatorname{Re}(s), \operatorname{Re}(t) > \delta(G)$ *then*

$$(51)\quad (s(1-s)-t(1-t))\int_{D^{\varphi}} E_{\beta}^{!\varphi'}(z, \bar{t})^* E_{\alpha}^{!\varphi'}(z, s)d\sigma(z)$$

$$= \delta_{\alpha\beta}\kappa(\alpha)\mathfrak{R}_k^{++}(\varphi'_{i(\alpha)}, s, t, \alpha) + e(k)\sigma_{\alpha\beta}(\bar{t})^*\mathfrak{R}_k^{+-}(\varphi'_{i(\alpha)}, s, t, \alpha)$$
$$+ e(-k)\mathfrak{R}_k^{-+}(\varphi'_{i(\beta)}, s, t, \beta)\sigma_{\beta\alpha}(s) + \sum_{\gamma\in Z^*}\kappa(\gamma)^{-1}\sigma_{\gamma\beta}(\bar{t})^*\mathfrak{R}_k^{--}(\varphi_{i(\gamma)}, s, t, \gamma)\,\sigma_{\gamma\alpha}(s)$$

If, furthermore, $\operatorname{Re}(s) + \operatorname{Re}(t) > 1$, *then*

$$(52)\quad (s(1-s)-t(1-t))\int_{D} E_{\beta}^{!\varphi}(z, \bar{t})^* E_{\alpha}^{!\varphi}(z, s)d\sigma(s)$$
$$= \delta_{\alpha\beta}\kappa(\alpha)\cdot\mathfrak{R}_k^{++}(\varphi_{i(\alpha)}, s, t, \alpha) + e(k)\sigma_{\alpha\beta}(\bar{t})^*\mathfrak{R}_k^{+-}(\varphi_{i(\alpha)}, s, t, \alpha) + e(-k)\mathfrak{R}_k^{-+}(\varphi_{i(\beta)}, s, t, \beta)\sigma_{\beta\alpha}(s).$$

Observe that in (52) the Fourier coefficients appear only linearly; it is this phenomenon that makes the analytic continuation in $\operatorname{Re}(s) > \frac{1}{2}$ possible. In (51) we have a quadratic form in the Fourier coefficients which turns out to be negative definite; this makes possible the analytic continuation over the whole plane.

As before we require the corresponding results for hyperbolic groups. These are:

THEOREM 5: *For $\theta_1, \theta_2 \in]0, \pi[$ we have*

$$(53)\quad R_k^{\varepsilon\eta}(\theta_1, s, t, a) - R_k^{\varepsilon\eta}(\theta_2, s, t, a) = (s(1-s)-t(1-t))\cdot\int_{\theta_1}^{\theta_2} Q_k^{\varepsilon}(\theta, a, s) \cdot\overline{Q_k^{\eta}(\theta_2, a, \bar{t})}(\sin\theta)^{-2}d\theta.$$

Also

$$R_k^{\varepsilon\eta}(\pi-\theta, s, t, a) = -c_{\varepsilon\eta}R_k^{-\varepsilon,-\eta}(\theta, s, t, -a) \tag{54}$$

where $c_{++} = c_{--} = 1$, $c_{-+} = \overline{c_{+-}} = e(k)$.

As $\theta \to 0$

$$R_k^{--}(\theta, s, t, a) \sim (s-t)q_k^-(s, a)\overline{q_k^-(\bar{t}, a)}(\sin\theta)^{s+t-1}, \tag{55}$$

and, if $\mathrm{Re}(s) > \frac{1}{2}$,

$$R_k^{+-}(\theta, s, t, a) \sim (1-s-t)D_k(s)\overline{q_k^-(\bar{t}, a)}(\sin\theta)^{t-s} \tag{56}$$

whereas, if $\mathrm{Re}(s) < \frac{1}{2}$,

$$R_k^{+-}(\theta, s, t, a) \sim (s-t)q_k^+(s, a)\overline{q_k^-(\bar{t}, a)}(\sin\theta)^{s+t-1}. \tag{57}$$

PROOF: (53) follows from carrying out the same calculations for the hyperbolic group as were carried out in general in theorem 4. It can also be proved using the fact that $Q_k^\varepsilon(\theta, a, s)$ satisfies the differential equation

$$y''(\theta)+2kiy'(\theta)+(-4\pi^2a^2+4\pi ka\cdot\cot\theta)y(\theta) = -s(1-s)(\sin\theta)^{-2}y(\theta).$$

(54) follows from (32), (33) and (50).

It follows at once from (15) that, as $\theta \to 0$,

$$\frac{\partial}{\partial\theta}Q_k^-(\theta, a, s) \sim sq_k^-(s, a)(\sin\theta)^{s-1}.$$

From theorem 3 we deduce that $\partial Q_k^\varepsilon/\partial\theta$, behaves, as $\theta \to 0$, exactly as one deduces from differentiating (28), (29) and (30) formally. Substituting into (50) gives (55), (56) and (57).

COROLLARY:

$$R_k^{++}(\theta, s, s, a) = R_k^{--}(\theta, s, s, a) = 0,$$

$$R_k^{+-}(\theta, s, s, a) = -e(-k)R_k^{-+}(\theta, s, s, a) = e(-4k)(1-2s)D_k(s)q_k^-(s, a),$$

$$\begin{aligned} R_k^{+-}(\theta, s, 1-s, a) &= -e(-k)R_k^{-+}(\theta, s, 1-s, -a)\\ &= (2s-1)q_k^+(s, a)\overline{q_k^-(\overline{1-s}, a)}, \end{aligned}$$

$$R_k^{++}(\theta, s, 1-s, a) = -R_k^{--}(\theta, s, 1-s, a) = -(2s-1)q_k^-(s, a)\overline{q_k^-(1-\bar{t}, a)}.$$

Proof: That all the functions concerned are constant follows from (53). The value of these constants can be evaluated, at least for certain values of s, by (55), (56) and (57). That they hold for all values follows by analytic continuation.

Theorem 6: $(\mathfrak{q}_k^-(s,\alpha)\sigma_{\alpha\beta}(s))^* = e(-3k)\mathfrak{q}_k^-(\bar{s},\beta)\sigma_{\beta\alpha}(s)$.

Proof: In view of theorem 2 we need only show that

$$S(\zeta,\eta;s)^* = e(-k)S(\eta,\zeta;\bar{s}).$$

Remembering that χ is unitary

$$S(\zeta,\eta;s)^* = \sum_{g\in G} \chi(g)|g'(\zeta)|^{\bar{s}}\cdot|g(\zeta)-\eta|^{-2\bar{s}}\overline{\varepsilon(\zeta,g^{-1}\infty)\varepsilon(g\zeta,\eta)}\,\dot{}$$

As χ is a multiplier

$$\chi(g)\cdot\chi(g^{-1})\cdot\sigma^k(g,g^{-1}) = I.$$

However, an easy calculation shows that if $g(\infty)\neq\infty$ then

$$\sigma^k(g,g^{-1}) = e(-k).$$

We choose for simplicity a conjugate of G so that the only element of G fixing ∞ is the identity. If we replace g by g^{-1} and use the identity

$$|(g^{-1})'(\eta)|\cdot|g(\zeta)-\eta|^2 = |g'(\zeta)|\cdot|\zeta-(g^{-1})(\eta)|^2,$$

the desired result follows at once.

Theorem 6 plays a very important rôle since it opens the possibility of using real-variable methods. Theorem 6 can be deduced from the equation (51) of theorem 4 by setting $s = t$ and then using the corollary to theorem 5.

If we now consider the case that $\delta(G) < \frac{1}{2}$, we can set $t = 1-s$ in (51), if $\delta(G) < \mathrm{Re}(s) < 1-\delta(G)$. This gives another relation which, on simplifying by theorems 5 and 6 turns out to be the functional equation investigated in Section 4.

To complete the set of integral relations we need one more. Let $u(z)$ be an automorphic form of weight k and multiplier χ and furthermore that it is an eigenfunction of $-\Delta_k$ with eigenvalue $s(1-s)$ $(0 < \mathrm{Re}(s) < 1)$.

Let j be such that $1 \leqq j \leqq p$ and we shall suppose that u has a Fourier expansion of the form for all j,

$$(58) \qquad Y_j u(z) j(A_j, z_j)^k = \sum_{\alpha \in Z_j} \mathfrak{Q}_k^-(\theta_j(z), \alpha, s)\, \mathfrak{E}_\alpha(r_j(z)) c_\alpha.$$

Proceeding as before

$$(s(1-s)-t(1-t)) \int_{D^\varphi} E^!_\alpha(z, \bar{t})^* u(z) d\sigma(z) \kappa(\alpha)$$
$$= e(-k)\kappa(\alpha)^2 \mathfrak{R}_k^{-+}(\varphi_{i(\alpha)}, s, t, \alpha) c_\alpha + \sum_{\beta \in Z^*} \kappa(\beta) \sigma_{\beta\alpha}(\bar{t})^* \mathfrak{R}_k^{--}(\varphi_{i(\beta)}, s, t, \beta) c_\beta.$$

We take now $t = s$. The left-hand side above vanishes and hence $c_\alpha = 0$. Thus, if $E^!_\alpha(z, s)$ is regular at s, $c_\alpha = 0$. Hence, unless $s = \frac{1}{2}$, if $u \equiv 0$ $E_\zeta(z, s)$ has a pole at s. This argument would even be valid for an analytic continuation of $E^!_\alpha(z, s)$. In the case when $\delta(G) < \frac{1}{2}$, when we have such a continuation we see at once that we must have $\operatorname{Re}(s) < \frac{1}{2}$.

Next take $t = 1-s$. Set

$$c'_\alpha = \kappa(\alpha)^2 \overline{\mathfrak{q}_k^-(1-s, \alpha)} c_\alpha \qquad (\alpha \in Z^*).$$

Using theorems 5 and 6 we obtain

$$0 = -\mathfrak{q}_k^+(s, -\alpha) c'_\alpha + \sum_{\beta \in Z^*} \kappa(\beta)^{-1} e(-k) \mathfrak{q}_k^-(s, \alpha) \sigma_{\alpha\beta}(1-s) c'_\beta.$$

Using (37) we get

$$0 = \mathfrak{q}_k^+(1-s, \alpha) c'_\alpha + \sum_{\beta \in Z^*} \kappa(\beta)^{-1} e(-k) \mathfrak{q}_k^-(1-s, \alpha) \sigma_{\alpha\beta}(1-s) c'_\beta.$$

In other words, if we consider $C = (c'_\alpha)$ as a vector in W then we have

$$\Sigma^*(1-s) C = 0.$$

We had observed that s was a pole of the Eisenstein series and so $\Phi(1-s) = 0$. Thus we expected to be able to find such a vector as C.

If we suppose further that

$$(\sin \theta_j(z))^{-s} j(A_j, z_j)^k u(z)$$

converges uniformly to a continuous function on $\Omega_{\alpha(j)}$ as $\theta_j(z) \to 0$. Then the c'_α are essentially the Fourier coefficients of this function and hence C lies in one of the Banach spaces on which $\Sigma^*(s)$ acts (in section 4 we saw that such spaces are those corresponding to spaces of r-fold differentiable functions.) Thus C lies in a finite dimensional space.

Let us now define a *generalised cusp form* to be a form satisfying all the conditions that we have put on u. The quantity s will be called the parameter of u.

The result here is still subject to the restriction that $\delta(G) < \frac{1}{2}$. In [5] we showed that there is a probability measure μ supported on the limit set of G so that

$$F(z) = \int P(z, \zeta)^{\delta(G)} d\mu(\zeta)$$

is an automorphic function. It is easy to verify that it is a generalised cusp form. Thus, if $\delta(G) < \frac{1}{2}$, the space of such measures is finite dimensional. This was proved in the case that $\delta(G) > \frac{1}{2}$, for in this case F is square-integrable and it is possible to use L^2-theory. The sketch given here is an indication of how this theory can be relieved of the restriction that $\delta(G) > \frac{1}{2}$. We shall not deal with this point here; we do need however, a similar result which is valid without any restriction on $\delta(G)$. This is:

THEOREM 7: *Suppose u is a C^∞-function with an expansion of the form* (58) ($1 \leqq j \leqq p$), *where* $\operatorname{Re}(s) = \frac{1}{2}$, $s \neq \frac{1}{2}$. *Then* $u \equiv 0$.

PROOF: As u is smooth the Fourier series for u and all its derivatives converge uniformly on compact subsets. Then we can calculate ($\langle \cdot , \cdot \rangle$ being the inner product on V)

$$\int_{D^\varphi} (\langle \Delta_k u, u\rangle - \langle u, (\Delta_k u)\rangle) d\sigma.$$

First of all as u is an eigenfunction of Δ_k with a real eigenvalue this vanishes. On the other hand, on applying Green's theorem we obtain that

$$\sum_{\alpha \in Z^*} \langle \mathfrak{R}_k^{--}(\varphi_{i(\alpha)}, s, \bar{s}, \alpha) c_\alpha, c_\alpha \rangle = 0.$$

But as $\bar{s} = 1 - s$ we obtain from the corollary to theorem 5 that

$$\sum_{\alpha \in Z^*} \langle \mathfrak{q}_k^-(s, \alpha) c_\alpha, \mathfrak{q}_k^-(s, \alpha) c_\alpha \rangle = 0.$$

As the left-hand side is positive it follows that $c_\alpha = 0$ for all α; this proves the theorem.

6. Some estimates

We must now study the behaviour of the functions $Q_k^-(\theta, X, s)$ as X becomes large. This is essentially contained in the work of Watson [12] but it proves to be much easier to proceed from first principles.

From (15) after substituting e^z for x we find

$$(59)\quad Q_k^-(\theta, X, s) = e(k)(\sin\theta)^s \int_{-\infty}^{\infty} e(-Xz)(e^{zs}(e^{z+i\theta}+1)^{-s-k} \cdot (e^{z-i\theta}+1)^{-s+k})dz.$$

The integrand is not regular in the whole z-plane and to make it regular we make cuts $L_1 = \{iy\colon y > \pi-\theta\}$ and $L_2 = \{iy\colon y < -\pi+\theta\}$. We shall consider θ only in the region $]0, \pi[$ and X real. We have to distinguish the cases as X is positive or negative.

Suppose first that X is positive. As long as $\mathrm{Re}\,(s) > 0$ we can move the path of integration to the line $\mathrm{Im}\,(z) = -\pi$, but indented around the cut L_2. The contribution from this line, excepting the indentation along L_2, is clearly bounded by

$$|e^{\pi is}|\,|(\sin\theta)^s|e^{-2\pi^2 X}\int_{-\infty}^{+\infty} |e^{sx}(-e^{x+i\theta}+1)^{-s-k}(-e^{x-i\theta}+1)^{-s+k}|dx.$$

Let $s = \sigma+it$. We see that this expression is bounded by

$$e^{\pi|t|}\sin^\sigma\theta e^{-2\pi^2 X}\int_{-\infty}^{+\infty} e^{\sigma x}|e^{2x}-2\cos\theta e^x+1|^{-\sigma}dx$$

since

$$|-e^{x+i\theta}+1| = |-e^{x-i\theta}+1|.$$

Next observe that

$$e^{2x}-2\cos\theta e^x+1 \geqq (e^{2x}+1)(1-\cos\theta)$$

and the bound becomes, after a short calculation,

$$(\tfrac{1}{2})B(\sigma/2, \sigma/2)(\sin\theta/(1-\cos\theta))^{\sigma}e^{\pi|t|}e^{-2\pi^2 X},$$

where $B(\cdot\,,\cdot)$ is the Euler beta-function.

Let now Λ_1 be the path from $-\pi i$ to $-\pi i$, taken on L_2, and going round $-(\pi-\theta)i$ anticlockwise. Then we have shown that

$$Q_k^-(\theta, X, s) = (\sin\theta)^s e(k)\int_{\Lambda_1} e^{-2\pi i X z}(e^{zs}(e^{z+i\theta}+1)^{-s-k}(e^{z-i\theta}+1)^{-s+k})dz + E$$

where

$$|E| \leqq (\tfrac{1}{2})B(\sigma/2, \sigma/2)(\sin\theta/(1-\cos\theta))^{\sigma}e^{\pi|t|}e^{-2\pi^2 X}. \tag{60}$$

We transform the integral by setting $z = -(\pi-\theta)i - iu$ and it becomes

$$t_1 \cdot e^{-2\pi(\pi-\theta)}\int_{\Lambda} e^{-2\pi X u}c(u)du$$

where

$$c(u) = e^{-ius}(e^{-i(u+\pi-2\theta)}+1)^{-s-k}(-e^{-iu}+1)^{-s+k}$$

and

$$t_1 = (\sin\theta)^s e(k)e^{-(\pi-\theta)is}(-i). \tag{61}$$

Λ is the path from θ to θ taken along the positive real axis and circling 0 positively.

$c(u)$ has an expansion of the following form

$$c(u) = \Big(\sum_{j=0}^{N} k_j u^{j-s+k}\Big) + c_N(u)u^{N+1-s+k}$$

where $N-\sigma+k+1 > 0$ and $c_N(u)$ is bounded in a neighbourhood of Λ. Suppose K is a bound for $c_N(u)$ on Λ; if s lies in a compact subset of $\operatorname{Re}(s) > 0$, and if θ lies in a compact subset of $]0, \pi[$ then K can be chosen uniformly. Thus

$$Q_k^-(\theta, X, s) = t_1 e^{-2\pi(\pi-\theta)X}\sum_{j=0}^{N} k_j\int_{\Lambda} e^{-2\pi X u}u^{-s+k+j}du + E_1 + E,$$

where

$$|E_1| \leqq |t_1| e^{-2\pi(\pi-\theta)X} \cdot K \cdot e^{2\pi|t|} \int_0^\theta e^{-2\pi X u} u^{-\sigma+k+N+1} du$$

$$\leqq K|t_1| e^{-2\pi(\pi-\theta)} \Gamma(-\sigma+k+N+2) e^{2\pi|t|} X^{-(k+N+2-\sigma)}.$$

Now let L be the contour from ∞ to ∞ along the positive real axis, encircling 0 positively. Then

$$\left| \int_A e^{-2\pi X u} u^{-\alpha} du - \int_L e^{-2\pi X u} u^{-\alpha} du \right|$$

$$\leqq (1+e^{2\pi \operatorname{Im}(\alpha)}) \int_\theta^\infty e^{-2\pi X u} u^{-\alpha} du$$

$$\leqq (1+e^{2\pi \operatorname{Im}(\alpha)}) X^{-\alpha+1} \int_{\theta X}^\infty e^{-2\pi u} u^{-\alpha} du = O(e^{-(2\pi-\varepsilon)X})$$

for any $\varepsilon > 0$. The implied constant depends on α and θ and can be chosen uniformly on compact subsets. This estimate, like many of the others here can be improved. By the Hankel integral for the gamma function [13: p. 244]

$$\int_L e^{-2\pi X u} u^{-\alpha} du = e^{-\pi i \alpha} 2\pi i \Gamma(\alpha)^{-1} (2\pi X)^{\alpha-1}.$$

Summarising we obtain the following result, that

$$Q_k^-(\theta, X, s) = t^+(s, \theta) e^{-2\pi(\pi-\theta)|X|} |X|^{-s+k+1} + E_2 \tag{62}$$

where

$$|E_2| \leqq c^+(s, \theta) e^{-2\pi(\pi-\theta)|X|} X^{-\sigma+k}. \tag{63}$$

$c^+(s, \theta)$ can be chosen to depend continuously on s, θ.

If X is negative we get instead

$$Q_k^-(\theta, X, s) = t^-(s, \theta) e^{-2\pi(\pi-\theta)|X|} |X|^{-s-k+1} + E_3 \tag{64}$$

where

$$|E_3| \leqq c_2^-(s,\theta)e^{-2\pi(\pi-\theta)|X|}|X|^{-\sigma-k}. \tag{65}$$

Furthermore $t^+(\theta, s)$ (resp. $t^-(s,\theta)$) is non-zero if $s-k$ (resp. $s+k$) is not a negative integer or zero. Explicit expressions for $t^+(s,\theta)$, $t^-(s,\theta)$ can easily be given but for our purposes it is unnecessary.

We can also derive asymptotic formulae for $\partial Q_k^-(\theta, X, s)/\partial\theta$ as above; the result is the formal derivative of (62) or (64) with the corresponding error terms. From this we can deduce estimates for $R_k^{\varepsilon\eta}(\theta, s, t, X)$. These are however not so accurate since the leading terms can cancel. We require a bound for

$$I = R_k^{++}(\theta, s, \bar{s}, X)R_k^{--}(\theta, s, \bar{s}, X) - R_k^{+-}(\theta, s, \bar{s}, X)R_k^{-+}(\theta, s, \bar{s}, X).$$

From (50) after some manipulation we obtain

$$I = |\partial Q_k^+(\theta, X, s)/\partial\theta \cdot Q_k^-(\theta, X, s) - \partial Q_k^-(\theta, X, s)/\partial\theta \cdot Q_k^+(\theta, X, s)|^2.$$

Using theorem 3 to deduce estimates for $Q_k^+(\theta, X, s)$ from those for $Q_k^-(\theta, X, s)$ we deduce that

$$I = O(e^{-4\pi^2|X|}|X|^{-4\sigma+6}) \tag{66}$$

We also require a lower bound for $R_k^{--}(\theta, s, \bar{s}, X)$, at least when $\mathrm{Re}(s) > \frac{1}{2}$. In this case we obtain from theorem 5 that

$$-i(2T)^{-1}(2\sigma-1)^{-1}R_k^{--}(\varphi, s, \bar{s}, X) = \int_0^{\varphi} |Q_k^-(\theta, X, s)|^2(\sin\theta)^{-2}d\theta,$$

with $s = \sigma + iT$. The left-hand side is clearly positive. Hence

$$|(2T)^{-1}(2\sigma-1)^{-1}R_k^{--}(\varphi, s, \bar{s}, X)| \geqq \int_{\varphi/2}^{\varphi} |Q_k^-(\theta, X, s)|^2(\sin\theta)^{-2}d\theta.$$

Applying (62) or (64), we obtain, because of the uniformity of the estimates there, that

$$|R_k^{--}(\varphi, s, \bar{s}, X)| \geqq c(s,\varphi)|X|^{-\beta}e^{-4\pi(\pi-\varphi)|X|}, \tag{67}$$

where β is a number of the form $-2(\sigma \pm k)+1$ and $c(s,\varphi)$ is a constant depending on s, φ.

(66) is valid for s in $\mathrm{Re}(s) > 0$ and uniformly so for θ lying in compact subsets of $]0, \pi[$ and s lying in compact sets. (67) is valid in $\mathrm{Re}(s) > \frac{1}{2}$, but it should be noted that $c(s, \varphi) = 0$ if $\mathrm{Re}(s) = \frac{1}{2}$ or $\mathrm{Im}(s) = 0$. In these exceptional cases $R_k^{--}(\theta, s, \bar{s}, X)$ can be evaluated explicitly by theorem 5 and its corollary.

The following two estimates follow in the same way:

$$R_k^{+-}(\theta, s, \bar{s}, X) = O(e^{-2\pi^2|X|}|X|^{\beta}), \tag{68}$$

$$R_k^{-+}(\theta, s, \bar{s}, X) = O(e^{-2\pi^2|X|}|X|^{\beta}). \tag{69}$$

β is a certain real number. The conclusions about uniformity are the same as before.

7. Analytic continuation, 1st step

Let G be a Fuchsian group with $\delta(G) > \frac{1}{2}$. We shall assume that G has no parabolic elements. Our object is to show that $E_\zeta(z, s)$ can be continued to the whole complex plane as a meromorphic function in s. Our proof follows the lines sketched by Selberg [11]. The first step is therefore to effect the continuation to the region $\{\mathrm{Re}(s) > \frac{1}{2}, \mathrm{Im}(s) \neq 0\}$.

To do this we start from (52) but we require this in a somewhat altered form. Let B_j be a fundamental domain for $G_{\alpha(j)}$ on $\Omega_{\alpha(j)}$ and $B = \bigcup_j B_j$; if $\zeta \in B_j$ we set

$$m_{s,t}(\zeta) = |A_j'(\zeta_j)|^{-1+s+t}|\zeta_j|^{s+t}.$$

Then the relation we need is

$$\begin{aligned}\int_B \int_D \mathrm{Tr}\,(E_\zeta^\varphi(z, \bar{t})^* E_\zeta^\varphi(z, s)) m_{s,t}(\zeta) d\sigma(z) d\zeta \\ = (s(1-s) - t(1-t))^{-1} \sum_{\alpha \in Z^*} \kappa(\alpha)(\kappa(\alpha)\,\mathrm{Tr}\,(\mathfrak{R}_k^{++}(\varphi_{i(\alpha)}, s, t, \alpha)) \\ + e(k)\,\mathrm{Tr}\,(\sigma_{\alpha\alpha}(\bar{t})^* \mathfrak{R}_k^{+-}(\varphi_{i(\alpha)}, s, t, \alpha)) + e(-k)\,\mathrm{Tr}\,(\mathfrak{R}_k^{-+}(\varphi_{i(\alpha)}, s, t, \alpha)\sigma_{\alpha\alpha}(s)))\end{aligned} \tag{70}$$

This follows by expressing the left-hand side in terms of the $E_\alpha^!(z, s)$. The special case $t = \bar{s}$ shows that the resulting sum converges absolutely and hence the right-hand side also. Thus (70) is valid for $\mathrm{Re}(s)$, $\mathrm{Re}(t) > \delta(G)$. *We shall write* $A(s, t)$ *for both sides of* (70).

Now let K be a compact subset of $\{\mathrm{Re}(s) > 0, \mathrm{Im}(s) \neq 0\}$ and suppose

φ fixed. We make the following assumptions about $E_\zeta(z, s)$:
$E(K)$: *$E_\zeta(z, s)$ can be continued to an analytic function in K smooth in z and s. There exists a constant $M(K) > 0$ so that for s lying in some neighbourhood of K*

$$\int_B \int_D \operatorname{Tr}\,(E_\zeta(z, s)^* E_\zeta(z, s) m_{s,\bar{s}}(\zeta) d\sigma(z) d\zeta \leqq M(K) \tag{71}$$

Our method is to show that if $E(K)$ holds on a compact set K then it holds on a strictly larger compact set K. It is clear that $E(K)$ holds for any compact subset of $\operatorname{Re}(s) > \delta(G)$. Let $s_0 \in K$ and let d be the distance of s_0 from the set $\{\operatorname{Re}(s) = \frac{1}{2}\} \cup \{\operatorname{Im}(s) = 0\}$. Let

$$C(s_0, \rho) = \{s : |s - s_0| \leqq \rho\},$$

and suppose that ρ is such that $C(s_0, \rho) \subset K$. We let $\rho' < \sqrt{\rho d}$ and we shall show that $E(K \cup C(s_0, \rho'))$ holds. After a finite number of steps it follows that $E(K)$ holds for any compact subset K of $\{\operatorname{Re}(s) > \frac{1}{2}, \operatorname{Im}(s) \neq 0\}$.

First of all it follows easily that the left-hand side of (70) is an analytic function for $(s, t) \in K \times \bar{K}$ (where $\bar{K} = \{\bar{s} : s \in K\}$). It follows from (71) that, as the Fourier series of theorem 1 converges to a L^2 function that each of the terms is bounded. From this and (62) or (64) we see that for $s \in K$

$$\operatorname{Tr}\,(\sigma_{\alpha\beta}(s)^* \sigma_{\alpha\beta}(s)) \leqq c(\theta) \exp\,(4\pi(\pi - \theta)|\alpha|) \tag{72}$$

where $c(\theta)$ is a positive constant depending (continuously) on θ, and on K. In particular, from (68) and (69), and taking in (72) $\theta > \pi/2$ we see that the right-hand side of (70) converges absolutely, even if the terms are not bracketed together. Hence it too represents a function analytic on $K \times \bar{K}$.

For any matrix $X \in \operatorname{End}(V)$ we write $|X|$ for the maximum of the absolute values of the entries of X. So, (72) shows that

$$|\sigma_{\alpha\alpha}(s)| \leqq \sqrt{c(\theta)} \exp\,(2\pi(\pi - \theta)|\alpha|). \tag{73}$$

Let $s \in C(s_0, \rho)$ where s_0, ρ are as above, and let

$$M_\alpha = \sup |\sigma_{\alpha\alpha}(s')| \qquad (s' \in C(s_0, \rho)). \tag{74}$$

Then, by Cauchy's inequality, $\sigma_{\alpha\alpha}(s)$ has a power series expansion

$$\sum_{n=0}^{\infty} \sigma_n^{(\alpha)}(s-s_0)^n$$

with

$$|\sigma_n^{(\alpha)}| \leqq M_\alpha \rho^{-n}.$$

Now let $d' < d$, $d' > \rho$. Then for $(s, t) \in C(s_0, d') \times C(\bar{s}_0, d')$

$$(s(1-s)-t(1-t))^{-1} \mathfrak{R}_k^{-+}(\varphi_j, s, t, \alpha) \qquad (j = \boldsymbol{i}(\alpha))$$

is analytic; it has an upper bound on this set, say N_α. Then it has a power series expansion

$$\sum_{m,n} r_{m,n}^{(\alpha)}(s-s_0)^n (t-\bar{s}_0)^m,$$

where again by Cauchy's inequality

$$|r_{m,n}^{(\alpha)}| \leqq N_\alpha d'^{-m-n}.$$

Thus

$$(s(1-s)-t(1-t))^{-1} e(-k) \operatorname{Tr}(\mathfrak{R}_k^{-+}(\varphi_j, s, t, \alpha)\sigma_{\alpha\alpha}(s))$$

has a power series expansion of the form

$$\sum a_{m,n,\alpha}^{(3)}(s-s_0)^n (t-\bar{s}_0)^m.$$

where

$$|a_{m,n,\alpha}^{(3)}| \leqq \sum_{k=0}^{n} N_\alpha M_\alpha d'^{-n-k} \rho^{-m+k} \leqq N_\alpha M_\alpha (1-\rho/d') d'^{-n} \rho^{-m}.$$

But by (69), (73), (74) we have that

$$N_\alpha M_\alpha = O(e^{-\theta\pi|\alpha|})$$

for any $\theta \in]0, \pi[$. Hence it follows that

$$(s(1-s)-t(1-t))^{-1} \sum_{\alpha \in Z^*} e(-k) \operatorname{Tr}(\mathfrak{R}_k^{-+}(\varphi_j, s, t, \alpha)\sigma_{\alpha\alpha}(s))$$

has a power series expansion of the form

$$\sum_{m,n} a_{mn}^{(3)}(s-s_0)^n(t-\bar{s}_0)^m$$

where, for some $c_1 > 0$,

$$|a_{mn}^{(3)}| \leqq c_1 d'^{-n}\rho^{-m}.$$

The other two terms on the right-hand side of (70) can be treated in the same way. Thus $A(s,t)$ has a power-series expansion

$$a_{mn}(s-s_0)^n(t-\bar{s}_0)^m$$

where

$$|a_{mn}| \leqq c_2((d')^{-m-n}+(d')^{-m}\rho^{-n}+(d')^{-n}\rho^{-m}).$$

In particular,

$$|a_{mm}| \leqq 3c_2(d'\rho)^{-m}.$$

We consider the Hilbert space $\mathfrak{H}$ consisting of functions on $D\times B$ with values in $\mathrm{End}(V)$, and under the norm

$$\|f\|^2 = \int_B\int_D \mathrm{Tr}\,(f(z,\zeta)^*f(z,\zeta))d\sigma(z)d\zeta.$$

The function

$$E_\zeta^\varphi(z,s)\cdot|A_i'(\zeta_i)|^{-\frac{1}{2}+s}|\zeta_i|^s, \qquad (i\colon \zeta\in B_i)$$

belongs to this space, and the norm is given by (70). This function is analytic is s and has an expansion

$$\sum e_n(z,\zeta)(s-s_0)^n.$$

Applying Cauchy's theorem to the left-hand side of (70) we see that it has the expansion

$$\sum_{m,n}\int_B\int_D \mathrm{Tr}\,(e_m(z,\zeta)^*e_n(z,\zeta))d\sigma(z)d\zeta(s-s_0)^n(t-\bar{s}_0)^m.$$

Thus we have

$$||e_n||^2 = a_{nn} = O((d'\rho)^{-n}).$$

Hence if we choose d' so that $\sqrt{d'\rho} > \rho'$ we see that the series

$$\sum e_n(s-s_0)^n$$

converges strongly in a neighbourhood of $C(s_0, \rho')$ in $\boldsymbol{H}$, and represents an analytic function taking values in $\mathfrak{H}$.

We are now in possession of a function satisfying $E(K \cup C(s_0, \rho'))$ except for the smoothness conditions.

Let $q(z, w)$ be a 'point-pair invariant' of weight k. Suppose that q is smooth and of compact support. Let f be any eigenfunction if $-\Delta_k$, with eigenvalue $s(1-s)$. Then by Selberg's theory [10], there exists an analytic function $p(s)$ (independent of f) so that

$$\int q(z, w)f(w)d\sigma(w) = p(s)f(z). \tag{75}$$

By considering a sequence of such q approximating a δ-function it is clear that we can choose one that does not vanish on $K \cup C(s_0, \rho')$. We shall let L be the hyperbolic diameter of the support of q; i.e. if the hyperbolic distance between z and w $[z, w] > L$ then $q(z, w) = 0$. It is clear that under the restrictions we have so far made we can choose q so that L is arbitrarily small.

Now let

$$Q(z, w) = \sum_{g\in G} \chi(g)j(g, w)^{-k}q(z, gw)$$

and let

$$D_c = \{z \in D\colon [z, w] > L,\ w \in \bigcup_{g\in G} \bigcup_{i} \{z\colon \theta_i(gz) = \varphi_i\}\}.$$

It is easy to see that this is an open subset of D and, since G has no parabolic elements, it follows now that $D - D_c$ is relatively compact in D.

It follows now from (75) that, if $z \in D_c$ then

$$\int_D Q(z, w)E_\zeta^\varphi(w, s)d\sigma(w) = p(s)E_\zeta^\varphi(z, s). \tag{76}$$

Clearly $Q(z, w)$ is bounded, and is even of compact support on D. It follows that as $E_\zeta^\varphi(z, s)$ can be continued to a analytic mapping of $K \cup C(s_0, \rho')$ into $\mathfrak{H}$, that the left-hand side of the above equation represents an analytic function of s and thus we obtain the pointwise continuation of $E_\zeta^\varphi(z, s)$. Also that $E_\zeta^\varphi(z, s)$ is smooth in z and s follows immediately. This holds at least for $z \in D_c$ but by chosing two different φ's it is clear that it extends at once to all of D. By this means we see that all the conditions of $E(K \cup C(s_0, \rho'))$ are satisfied. This then completes the first part of the analytic continuation.

We need to investigate the behaviour with respect to ζ and to show that this is also smooth.

From (72) and Stirling's formula it follows that

$$|\mathfrak{q}_k^-(s, \alpha)\sigma_{\alpha\beta}(s)| < c_3 e^{-2\pi\theta|\alpha|},$$

for any $\theta \in]0, 2\pi[$. This holds uniformly on a compact neighbourhood of s and we have the same inequality holding at $\bar{s}$. Thus by theorem 6 we obtain

$$|\mathfrak{q}_k^-(s, \alpha)\sigma_{\alpha\beta}(s)| < c_4 e^{-2\pi\theta \operatorname{Max}(|\alpha|, |\beta|)}. \tag{77}$$

Using (62) or (64) and Stirling's formula we obtain, for some real $b > 0$,

$$|\mathfrak{Q}_k^-(\varphi, \alpha, s)\sigma_{\alpha\beta}(s)| < c_5 e^{2\pi\varphi|\alpha| - 2\pi\theta \operatorname{Max}(|\alpha|, |\beta|)}(1 + |\alpha|)^b.$$

The sum of the right-hand side over all $\alpha, \beta \in Z^*$ is absolutely convergent if $\theta > \varphi$. Thus the series of theorem 1 is absolutely convergent and the terms are even bounded by an exponentially decreasing factor. Thus $E_\zeta(z, s)$ is smooth in ζ as well as z, s.

8. Analytic continuation, 2nd step

We must now investigate the behaviour of $E_\zeta(z, s)$ as s approaches a point on $\operatorname{Re}(s) = \frac{1}{2}$. The argument here becomes somewhat more involved. First of all we must introduce a certain modified version of $E_\zeta(z, s)$ which is defined as follows: if $\zeta \in \Omega_{\alpha(i)}$

$$\tilde{E}_\zeta(z, s) = E_\zeta(z, s) Y_i^{-1} |A_i'(\zeta^i)|^s |\zeta_i|^s \mathfrak{E}_\alpha(\zeta_i) \kappa(i),$$

where α is a fixed element of Z_i. Then $\tilde{E}_\zeta(z, s)$ is, after the discussion of § 3, invariant under the group $G_{\alpha(i)}$. If we let $\tilde{B}_i = G_{\alpha(i)} \backslash \Omega_{\alpha(i)}$, and $\tilde{B}$ be the disjoint union of the $\tilde{B}_i$ $(1 \leqq i \leqq p)$ then we can regard $\tilde{E}_\zeta(z, s)$ as a function on $\tilde{B}$. Note that $\tilde{B}_i$ is a circle; hence $\tilde{B}$ is a (disconnected) compact manifold.

We observe first that every distribution on $\tilde{B}$ is of finite order (cf. [9; p. 88]). Thus, as we can easily verify by inductive arguments, the series representation for $\tilde{E}_\zeta(z, s)$ derived from (10) converges in the space of test functions in the sense of distribution theory. Furthermore, let D be a distribution on $\tilde{B}$; we shall consider the action of D on $\tilde{E}_\zeta(z, s)$ with z, s held fixed. From theorem 1, if $\zeta \in \Omega_{\alpha(i)}$ we have

$$(78) \quad j(A_j, z_j)^k Y_j \tilde{E}_\zeta(z, s) = \delta_{ij} \tilde{E}_\zeta^{(i)}(z, s) + \sum_{\alpha \in Z_j} \mathfrak{Q}_k^-(\theta_j(z), \alpha, s) \mathfrak{E}_\alpha(r_j(z)) e_\alpha(\zeta, s).$$

Here

$$(79) \qquad \tilde{E}_\zeta^{(j)}(z, s) = e(k) \sum_{\alpha \in Z_j} \mathfrak{Q}_k^+(\theta_j(z), \alpha, s) \mathfrak{E}_\alpha(r_j(z)) \mathfrak{E}_\alpha(\zeta_j)^{-1},$$

and

$$(80) \qquad e_\alpha(\zeta, s) = \kappa(j)^{-1} \sum_{\beta \in Z_\iota} \sigma_{\alpha\beta}(s) \mathfrak{E}_\beta(\zeta_i)^{-1}.$$

By (62), (64), (32) and (77) it follows that each of these series is convergent as a series of test functions. Thus if we let $D\tilde{E}(z, s)$ be the result of letting D act on $\tilde{E}_\zeta(z, s)$ the remarks above give rise to the following three assertions:

(i) if $g \in G$ then $j(g, z)^k D\tilde{E}(gz, s) = \chi(g) D\tilde{E}(z, s)$,
(ii) if $\mathrm{Re}\,(s) > \frac{1}{2}$, $s \notin \left]\frac{1}{2}, \delta(G)\right]$ then

$$j(A_j, z_j)^k Y_j D\tilde{E}(z, s) = D_j \tilde{E}^{(j)}(z, s) + \sum_{\alpha \in Z_j} \mathfrak{Q}_k^-(\theta_j(z), \alpha, s) \mathfrak{E}_\alpha(r_j(z)) \, De_\alpha(\cdot\,, s),$$

where D_j is the restriction of D to $\tilde{B}_j$, and
(iii) $D_j \tilde{E}^{(j)}(z, s)$ is, for s in the range described above, analytic in s and smooth in z, s.

Our object now is to show that, as $\mathrm{Re}\,(s) \to \frac{1}{2}$,

$$\int_{D^\varphi} \mathrm{Tr}\,(D\tilde{E}(z, s)^* D\tilde{E}(z, s)) d\sigma(z)$$

remains bounded, at least if s is in the neighbourhood of a fixed point of $\mathrm{Re}(s) = \frac{1}{2}$, $s \neq \frac{1}{2}$. In order to do this we let $s_0 \in \{\mathrm{Re}(s) = \frac{1}{2}, s \neq \frac{1}{2}\}$ and let (s_n) be a sequence in $\{\mathrm{Re}(s) > \frac{1}{2}, \mathrm{Im}(s) \neq 0\}$ converging to s_0. Let

$$w_m = \left(\int_{D^\varphi} \mathrm{Tr}\,(D\tilde{E}(z, s_m)^* D\tilde{E}(z, s_m))d\sigma(z)\right)^{\frac{1}{2}}$$

and we shall assume that $w_m \to \infty$ as $m \to \infty$. From this we shall deduce a contradiction.

We introduce the Hilbert space $H(\varphi)$ of functions on D^φ, taking values in $\mathrm{End}\,(V)$, and with the inner product

$$\|f\|^2 = \int_{D^\varphi} \mathrm{Tr}\,(f(z)^* f(z))d\sigma(z).$$

In this Hilbert space the sequence

$$f_m(z) = w_m^{-1} D\tilde{E}(z, s_m)$$

is of norm 1. Thus a subsequence converges weakly to a limit f, say, in $H(\varphi)$. Suppose, for convenience of notation, that this sequence is all of (s_m).

Suppose that $\varphi = (\varphi_1, \ldots, \varphi_p)$ with $\varphi_j < \pi/2$ $(1 \leqq j \leqq p)$. We shall assume φ to be sufficiently small for the following argument. We choose also φ', φ'' so that $\varphi' = (\varphi_1', \ldots, \varphi_p')$, $\varphi'' = (\varphi_1'', \ldots, \varphi_p'')$ and

$$0 < \varphi_j < \varphi_j' < \varphi_j'' < \pi/2.$$

We choose also a 'point-pair invariant' $q(z, w)$ of weight k which is smooth and of compact support. We let L be the diameter of the support of q; this will be chosen small. Let Q, $p(s)$ be as in § 7; we assume also, as there, that $p(s_0) \neq 0$. We shall also assume that L is so small, and the φ, φ', φ'' are so chosen that:

if $z \in D^\varphi$, $w \in D - D^{\varphi'}$, $g \in G$ then $[z, g(w)] > L$, and

if $z \in D^{\varphi'}$, $w \in D - D^{\varphi''}$, $g \in G$ then $[z, g(w)] > L$.

Then, if $z \in D^{\varphi'}$ we have

$$\int_{D^\varphi} Q(z, w) f_m(w) d\sigma(w) = p(s_m) f_m(z). \tag{81}$$

Clearly $Q(z, w)$ is bounded on $D^\varphi \times D^\varphi$. Hence as (f_m) converges weakly to f the left-hand side converges to

$$\int_{D^{\varphi}} Q(z, w) f(w) d\sigma(w).$$

It follows also from (81) that if $z \in D^{\varphi'}$ then $f_m(z)$ is uniformly bounded. Applying (81) with φ, φ' replaced by φ', φ'' we see that for $z \in D^{\varphi''}$ $(f_m(z))$ converges uniformly to

$$\int Q(z, w) f(w) d\sigma(w).$$

Thus, in the limit, for $z \in D^{\varphi''}$,

$$\int_{D^{\varphi'}} Q(z, w) f(w) d\sigma(w) = p(s_0) f(z).$$

Thus, as q, and hence Q, is smooth it follows that f is also smooth on $D^{\varphi''}$.

On the other hand, from (78)

$$(82) \qquad j(A_j, z_j)^k Y_j f_m(z) = w_m^{-1} D_j E^{(j)}(z, s_m) + \sum_{\alpha \in Z_j} \mathfrak{Q}_k^-(\theta_j(z), \alpha, s_m) \mathfrak{E}_\alpha(r_j(z)) \cdot w_m^{-1} De_\alpha(\cdot\, , s_m).$$

The left-hand side and the first term on the right-hand side are clearly bounded for $\theta_j(z) = \pi/2$; thus, for some $c_1 > 0$

$$|\mathfrak{Q}_k^-(\pi/2, \alpha, s_m) w_m^{-1} De_\alpha(\cdot\, , s_m)| < c_1.$$

Using (62), (64) we see that there are $c_2 > 0$, $b > 0$ so that

$$|w_m^{-1} De_\alpha(\cdot\, , s_m)| \leqq c_2 e^{\pi^2|\alpha|}(1+|\alpha|)^b.$$

But then using (62), (64), (82) it follows that in the region

$$\{z \in D \colon 0 < \gamma_1 < \theta_j(z) < \gamma_2 < \pi/2\} \qquad (j \text{ fixed})$$

(82) converges, and the weak limit also converges to a smooth function which then extends f to this region. Otherwise expressed, f_m converges to f uniformly on compact subsets of D.

The Fourier expansion of f can be deduced from (82); because $w_m \to \infty$ we have

$$j(A_j z_j)^k Y_j f(z) = \sum_{\alpha\in Z_j} \mathfrak{Q}_k^-(\theta_j(z), \alpha, s_0)\mathfrak{E}_\alpha(r_j(z))e_\alpha^D.$$

As D^φ is relatively compact it follows that in $H(\varphi)$

$$\|f_m\| \to \|f\|.$$

But $\|f_m\| = 1$ and so $\|f\| = 1$. In particular $f \not\equiv 0$. But this contradicts theorem 7. Hence the assumption that $w_m \to \infty$ leads to a contradiction. So $D\tilde{E}(\cdot\,, s)$ remains bounded as $s \to s_0$. We can now repeat the argument given above and we deduce that if we are given a sequence (s_m'), $(s_m' \to s_0)$ there is a subsequence (s_m) so that $D\tilde{E}(z, s_m)$ converges uniformly on compact subsets in z to a limit. From (78) we deduce that if this limit is denoted by $f^D(z)$ then

$$(83)\quad j(A_j, z_j)Y_j f(z) = D_j\tilde{E}^{(j)}(z, s_0) + \sum_{\alpha\in Z_j} \mathfrak{Q}_k^-(\theta_j(z), \alpha, s_0)E_\alpha(r_j(z))e_\alpha^D$$

where

$$e_\alpha^D = \lim De_\alpha(\cdot\,, s_m).$$

From theorem 7 we see at once that there is only one such limit. Thus $f^D(z)$ is uniquely determined and as $s \to s_0$

$$D\tilde{E}(z, s) \to f^D(z)$$

$$De_\alpha(\cdot\,, s) \to e_\alpha^D.$$

This holds for all distributions D and so by the Banach-Steinhaus theorem for distributions ([9; pp. 69, 70]) there are functions $\tilde{E}_\zeta(z, s_0)$, $e_\alpha(\zeta, s_0)$ so that

$$f^D(z) = D\tilde{E}(z, s_0)$$

and

$$e_\alpha^D = De_\alpha(\cdot\,, s_0).$$

Furthermore, as $s \to s_0$,

$$\tilde{E}_\zeta(z, s) \to \tilde{E}_\zeta(z, s_0)$$

uniformly in z lying in compact subsets of D and ζ.

We can now apply the same argument again but allow the (s_m) to lie in $\{\mathrm{Re}\,(s) \geqq \frac{1}{2}, \mathrm{Im}\,(s) \neq 0\}$. From this we deduce that $\tilde{E}_\zeta(z, s)$ is continuous in s on $\{\mathrm{Re}\,(s) = \frac{1}{2}\}$, except, perhaps, at $s = \frac{1}{2}$.

From this we see, as in Section 7, that (77) holds uniformly on a neighbourhood of s_0 lying in $\{\mathrm{Re}\,(s) \geqq \frac{1}{2}, \mathrm{Im}\,(s) \neq 0\}$. In particular we see that the operator introduced in Section 4,

$$I + J_k(s)^{-1}\Sigma_0(s)$$

is Fredholm; also its Fredholm determinant $\Phi(s)$ is analytic in $\{\mathrm{Re}(s) > \frac{1}{2}, \mathrm{Im}\,(s) \neq 0\}$ and continuous in $\{\mathrm{Re}\,(s) \geqq \frac{1}{2}, \mathrm{Im}\,(s) \neq 0\}$. We deduce also that (51) holds if $s, t \in \{\mathrm{Re}\,(s') \geqq \frac{1}{2}, \mathrm{Im}\,(s') \neq 0\}$. As we have observed before, on taking $\mathrm{Re}\,(s) = \frac{1}{2}$, $t = 1-s = \bar{s}$ this implies that

$$(84) \quad (I + J_k(s)^{-1}\Sigma_0(s))(I + \Sigma_0(1-s)J_k(1-s)^{-1}) \\ = D_k(s)D_k(1-s)J_k(s)^{-1}J_k(1-s)^{-1}.$$

If $\mathrm{Re}\,(s) \geqq \frac{1}{2}$, $\Phi(s) \neq 0$ then $(I + \Sigma_0(s)J_k(s)^{-1})^{-1}$ exists and is analytic in s in $\{\mathrm{Re}\,(s) > \frac{1}{2}, \mathrm{Im}\,(s) \neq 0\}$, and is continuous on $\{\mathrm{Re}\,(s) \geqq \frac{1}{2}, \mathrm{Im}\,(s) \neq 0\}$. Thus

$$D_k(s)D_k(1-s)J_k(s)^{-1}J_k(1-s)^{-1}(I + \Sigma_0(1-s)J_k(1-s)^{-1})^{-1}$$

gives the analytic continuation of $I + J_k(s)^{-1}\Sigma_0(s)$ to the whole complex plane.

Note that, if $\mathrm{Re}\,(s) = \frac{1}{2}$ then, by (84),

$$|\Phi(s)|^2 = \det\,(D_k(s)J_k(s)^{-1}(D_k(s)J_k(s)^{-1})^*)$$

and is therefore non-zero, which shows that Φ is non-trivial. Furthermore we see that if $0 < \mathrm{Re}\,(s) < \frac{1}{2}$, $s \notin [1-\delta(G), \frac{1}{2}[$ the function

$$\Phi(1-s)(I + J_k(s)^{-1}\Sigma_0(s))$$

is regular.

We can now, as in Section 4, use the general Fredholm theory to deduce the properties of $I + J_k(s)^{-1}\Sigma_0(s)$ when $\mathrm{Re}\,(s) \leqq \frac{1}{2}$. In particular (77) holds uniformly on compact subsets of C away from the poles and the line $[1-\delta(G), \delta(G)]$. Thus, outside this line one sees that $E_\zeta(z, s)$ is meromorphic (this requires a little care at the poles). Now we can summarise the results of Sections 4, 7, 8 into the following theorem, which is the main result of this paper.

THEOREM 8: *Suppose G is a finitely generated Fuchsian group of the second kind without parabolic elements. The function $E_\zeta(z, s)$ can be analytically continued, as a meromorphic function, to the region*

$$\boldsymbol{C} - [1-\delta(G), \delta(G)] \qquad (\boldsymbol{C} \text{ if } \delta(G) < \tfrac{1}{2}).$$

It satisfies a functional equation which can be written as

$$(J_k(s) + \Sigma_0(s))(J_k(1-s) + \Sigma_0(1-s)) = D_k(s) D_k(1-s) I.$$

Let $\Phi(s)$ be the Fredholm determinant of $I + J_k(s)^{-1}\Sigma_0(s)$. Then if $0 < \operatorname{Re}(s) < 1$, $\Phi(s)E_\zeta(z, s)$ is regular. $E_\zeta(z, s)$ is smooth in ζ. If s lies in a compact set in which $E_\zeta(z, s)$ is regular then, for any $\theta < \pi$ there exists a constant $c(\theta) > 0$ so that uniformly

$$|\mathfrak{q}_k^-(s, \alpha)\sigma_{\alpha\beta}(s)| \leqq c(\theta) e^{-2\pi\theta|\alpha|}.$$

9. Concluding remarks

Theorem 8 does not represent the final truth about $E_\zeta(z, s)$. In particular the behaviour in the neighbourhood of the line segment $[1-\delta(G), \delta(G)]$ has not yet been investigated. This is, however, closely associated with the spectral decomposition of the Laplace operator. Accordingly, we shall have to investigate the behaviour of the generalised eigenfunctions $E_\zeta(z, s)$, especially when $|s-\frac{1}{2}|$ is either large or small. This involves another detailled arguments which we shall not consider here. It will be the subject of the next paper in this series.

The results which we have already obtained have several applications in the theory of Fuchsian groups; essentially because they sharpen the existing knowledge about certain Poincaré series. The most important case is $k = 0$, $\chi = I$.

The restriction that G be without parabolic elements is a technical simplification. The formalism is clear from the foregoing argument but since the Eisenstein series at parabolic vertices interact essentially with those considered in this paper the whole matter becomes womewhat more complex.

The restriction that G be finitely generated seems to be more essential. It is quite possible that the series can be continued to the region $\{\operatorname{Re}(s) > \frac{1}{2},$

Im $(s) \neq 0\}$, but there seems to be no possibility of a functional equation or of an analytic continuation substantially beyond the region mentioned above.

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