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ERGODIC MEASURES FOR NON-ABELIAN COMPACT GROUP EXTENSIONS

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0. Introduction

In this paper, we shall be concerned with extending the results of [6] to arbitrary group extensions, and utilizing our previous results to lift dynamical properties of measures. The analytic structure of an ergodic measure sitting over a fixed ergodic measure was examined in detail in [6] for compact abelian groups. However, in most distal extensions, the groups that naturally arise are in general non-abelian. In particular, given a distal extension, one can 'build' it using almost periodic extensions, and each almost periodic extension comes by 'interpolating' between a group extension over the base transformation group (see [5, pp. 255–6] for a precise statement). In the case where the spaces in the original extension are metric, one can show that the group extensions involved also have metric spaces. Although this fact is surely known, we have been unable to locate it in print, and thus will include a proof in an Appendix. With this in mind, the extension of analytic results to general compact group extensions is natural, and we carry this out in Section 2. In fact, we shall carry out our analysis for the case where X and Y are compact T_2, and G is metric. The major result obtained is that every ergodic measure above a fixed one is derived by first an isomorphism and then an ergodic 'Haar lift'. Our isomorphisms are Borel in the case that X is metric or T is countable, and isomorphisms of the completion in the general case. To do this, one needs to construct T-invariant functions which are simultaneously eigenfunctions for irreducible representations of the compact group. Problems arise in obtaining a strictly invariant section in the general case; this yields the weaker type of isomorphism in the general case.

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In Section 3, we consider the following problem: given a free abelian group extension $\pi : (G : X, T) \to (Y, T)$ and an invariant measure ν on Y with some dynamical properties, how can one 'perturb' the action on X to still maintain a G-extension and yet have the Haar lift of ν enjoy the same dynamical property. The simplest type of perturbation comes from a continuous homomorphism $\chi : T \to G$, namely $\rho_\chi(x, t) = \chi(t)(xt)$. We show that for most (in the sense of category) χ, the Haar lift will enjoy the same property with respect to the action ρ_χ. The properties examined are ergodicity and weak-mixing, and analogous results for minimality and topological weak-mixing are indicated in certain cases. We note that Parry and Jones [4] first considered these questions for the most general type of perturbation under the integers and the reals, and also showed that these properties lift for almost all perturbations. However, our actions belong to a set which is first category in their topology (the coboundaries), and hence the results are distinct from one another.

We will generally follow the notation and terminology of [6]. The additional structures required will be presented in the next section.

1. Preliminaries

We begin by establishing some notation. Let (X, T) denote a topological transformation group with X a compact T_2 space and T a locally compact separable group.

We shall briefly review the notation established in [6]. Thus, we will be considering a free G-extension, $(G; X, T)$ of another transformation group (Y, T) where Y and G are compact T_2, and $\pi: (G; X, T) \to (Y, G)$ will be the canonical map. If H is a subgroup of G, we will consider the splitting $\pi = \pi_2 \circ \pi_1$, where $\pi_1: (H; X, T) \to (X/H, T)$ yields a free H-extension and π_2: $(X/H, T) \to (Y, T)$ is the natural homomorphism. Again, $M(X, T)$ will be the collection of T-invariant regular Borel probability measures on X, and $E(X, T)$ the ergodic measures in $M(X, T)$. Also, $\pi^*: M(X, T) \to M(Y, T)$ will be the canonical map given by $\pi^*\nu(B) = \nu(\pi^{-1}B)$. In our case, $\pi^*E(X, T) = E(Y, T)$. Finally, if $\mu \in E(Y, T)$, the Haar lift of μ is given by

$$\tilde{\mu}(f) = \int_X \int_G f(gx)d\lambda(g)d\pi^{-1}\mu,$$

where λ is Haar measure on G, and $\pi^{-1}\mu$ is the induced measure on $\pi^{-1}\mathscr{B}(Y)$.

The main purpose of this paper is to take a fixed ergodic measure $\mu \in E(Y, T)$ and describe analytically the structure of those $\nu \in E(X, T)$ with $\pi^*\nu = \mu$. To this end we have to deal with representations of the group G. As a general reference we refer to [3].

Let $\mathscr{R}(G)$ denote the set of equivalence classes of continuous finite-dimensional irreducible unitary representation of G. With no loss of generality we will assume representations act on a suitable $\mathbb{C}^n$, $\mathbb{C}$ being the complex field, and where necessary we will identify a unitary operator on $\mathbb{C}^n$ with its matrix relative to the standard basis. The dimension of a representation α will be denoted by n_α and the equivalence class of α will be denoted by $[\alpha] \in \mathscr{R}(G)$. We denote by Σ^{n-1} the sphere

$$\Sigma^{n-1} = \{v \in \mathbb{C}^n : \|v\|_n = 1\},$$

$\| \ \|_n$ being the innerproduct norm in $\mathbb{C}^n$.

Let α be an irreducible representation of G with dimension n. A Borel function $f : X \to \Sigma^{n-1}$ is called an *α-function* if

$$f(gx) = \alpha(g) f(x) \qquad (x \in X, g \in G).$$

We first want to show the existence of α-functions. We shall first need Borel sections.

LEMMA (1.1): *Let G be a Lie group. Then π admits local sections, i.e., given $y \in Y$, there exists a neighborhood U of y and a continuous map $s : U \to X$ with $\pi \circ s = id_U$.*

PROOF: [10, page 219].

LEMMA (1.2): *Let G be a Lie group. Then there exists a Borel isomorphism $\varphi : X \to Y \times G$.*

PROOF: By Lemma 1.1, we can write $Y = \bigcup_{i=1}^n U_i$, where U_i is a compact neighborhood, and $p_i : U_i \to X$ is a local section. It easily follows that $\varphi_i : U_i \times G \to \pi^{-1}(U_i)$, $(x, g) \to g p_i(x)$ is a homeomorphism, with inverse

$$\varphi_i^{-1}(z) = (\pi(z), g_{p_i}(z)),$$

where $g_{p_i}(z) \in G$ satisfies $g_{p_i}(z) \cdot p_i \pi(z) = z$. Defining φ by $\varphi(x) = \varphi_i^{-1}(x)$, where i is minimal with respect to $x \in \pi^{-1}(U_i)$, yields the desired isomorphism (φ is actually continuous except on $\bigcup_{i=1}^n \mathrm{Bd.}(U_i)$, a nowhere dense set).

We now set $g_p = pr_2 \circ \varphi: X \to G$. Then g_p is Borel and satisfies $x = g_p(x) \cdot p\pi(x)$, where $p: Y \to X, p(y) = p_i(y)$, where i is minimal, is the induced Borel section, completing the proof.

Let f be an α-function. Then $f(gx) = f(x)$ for $g \in \ker \alpha$, and so f defines on $\bar{\alpha}$-function on the induced extension

$$\pi_2: (G/\ker \alpha, X/\ker \alpha, T) \to (Y, T),$$

where $\bar{\alpha}$ is the induced representation on $G/\ker \alpha$. Conversely, $\bar{\alpha}$-functions yield α functions. Now $G/\ker \alpha \simeq \operatorname{im} \alpha$ is a closed subgroup of a Lie group and hence is a Lie group. So α-functions can be analyzed using Lie group extensions.

LEMMA (1.3): *Let α be an irreducible representation of G with dimension n. Then there are n α-functions $f_1, \ldots, f_n$ such that for each $x \in X$ the values $f_1(x), \ldots, f_n(x)$ form an orthonormal basis of $\mathbb{C}^n$. Every α-function f can then be written in the form*

$$f(x) = \sum_{i=1}^{n} a_i(x) f_i(x)$$

where the functions $a_i(x)$ are G-invariant Borel functions.

PROOF: Using the above comments, we assume WLOG that G is a Lie group. Let $v_1, \ldots, v_n$ be an orthonormal basis on $\mathbb{C}^n$ and let $p: Y \to X$ be a Borel section. Define

$$f_i(x) = \alpha(g_p(x))v_i \qquad i = 1, \ldots, n.$$

It is easily verified that these satisfy the first conditions of the lemma.

Finally, if f is an α-function, define $a_i(x) = (f(x), f_i(x))$, where $(\cdot, \cdot)$ is the inner product. Then clearly a_i is Borel and

$$a_i(gx) = (f(gx), f_i(gx)) = (\alpha(g)f(x), \alpha(g)f_i(x)) = (f(x), f_i(x)) = a_i(x).$$

Since clearly $f(x) = \sum_{i=1}^{n} a_i(x) f_i(x)$, the lemma is proved.

One final result on α-functions

LEMMA (1.4): *Let m be probability measure on X and let $f: X \to \Sigma^{n-1}$ be a Borel function such that $f(gx) = \alpha(g)f(x)$ a.e. m for each g. Then there is a Borel function $f': X \to \Sigma^{n-1}$ such that $f = f'$ a.e. m and f' is an α-function.*

PROOF: Define $h: G \times X \to \mathbb{C}^n$ by $h(g, x) = f(gx) - \alpha(g)f(x)$. Then for every g, $m\{x: h(g, x) \neq 0\} = 0$. Setting $K = \{(g, x): h(g, x) \neq 0\}$, it follows that $\lambda \times m(K) = 0$, and so $\lambda\{g: h(g, x) \neq 0\} = 0$ for $x \in L$, a Borel set with $m(L) = 1$. Hence, if we define

$$\bar{f}(x) = \int_G \alpha(g^{-1}) f(gx) d\lambda(g),$$

then $\bar{f}(gx) = \alpha(g)\bar{f}(x)$ $(g \in G,\ x \in X)$ and clearly $\bar{f}(x) = f(x)$ if $x \in L$. Now $A = \{x | \bar{f}(x) = 0\}$ is Borel and G-saturated and so $L_0 = X - A$ is also G-saturated. Thus, $\bar{f} = f$ on $L_0 \cap L$, $m(L_0 \cap L) = 1$ and $\bar{f}: L_0 \to \Sigma^{n-1}$ is a measurable mapping. In the set $X - L_0$, which has zero m-measure, we replace $\bar{f}$ by any Borel α-function f_0 into Σ^{n-1} restricted to $X - L_0$. Defining $f' = \bar{f}$ on L_0 and $f' = f_0$ on $X - L_0$ completes the proof of the lemma.

2. Ergodic measures for group extensions

In this section we generalize the results of [6] to the case of not necessarily abelian compact group extensions. We begin by giving necessary and sufficient conditions for ergodicity of the Haar lift of an ergodic measure. The same assumptions as Section 1 will be in force.

THEOREM (2.1): *Let $\mu \in E(Y, T)$. Then the Haar lift of μ, $\tilde{\mu}$, belongs to $E(X, T)$ if and only if there are no α-functions, $\alpha \not\equiv 1$, satisfying*

$$f_\alpha \circ t = f_\alpha \qquad \text{a.e. } \tilde{\mu}(t \in T).$$

PROOF: Assume $\tilde{\mu} \in E(X, T)$ and let f_α be an α-function satisfying

$$f_\alpha \circ t = f_\alpha \qquad \text{a.e. } \tilde{\mu}(t \in T).$$

Then there is a constant $a \in \Sigma^{n_\alpha - 1}$ and a Borel set F, $\tilde{\mu}(F) = 1$ such that

$$f_\alpha(x) = a \qquad (x \in F).$$

Now for any x, define $G_x = \{g: gx \in F\}$. Since

$$1 = \tilde{\mu}(F) = \int_X \left(\int_G \chi_F(gx) d\lambda(g) \right) d\pi^{-1}(\mu),$$

then

$$\lambda(G_x) = \int_G \chi_F(gx) d\lambda(g) = 1$$

for $\pi^{-1}\mu$-almost all x. Pick one such $x \in F$, and $g \in G_x$. Then $a = f_\alpha(x) = f_\alpha(gx) = \alpha(g)a$. Since G_x is dense, we have that $\alpha(g)a = a$ for all $g \in G$. Since α is irreducible, it follows that $\alpha = 1$.

Now assume that $f_\alpha \circ t = f_\alpha$ a.e. $\tilde{\mu}$ implies that $\alpha = 1$.

Let $\mathscr{I}$ be the subspace of $L^2(X, \tilde{\mu})$ consisting of T-invariant functions. We identify $L^2(Y, \mu)$ with the subspace of $L^2(X, \tilde{\mu})$ consisting of G-invariant functions. The ergodicity of μ implies that $\mathscr{I} \cap L^2(Y, \mu) = \mathscr{C}$, the subspace of constant functions. Since $\tilde{\mu}$ is G-invariant, the action of G on X induces a unitary representation $\{U_g\}$ of G on $L^2(X, \tilde{\mu})$. Since $\mathscr{I}$ consists of T-invariant functions and G commutes with T, it follows that $\mathscr{I}$ is an invariant subspace for $\{U_g\}$. It follows from general representation theory (see e.g., [9, Theorem 7.8]) that $\mathscr{I}$ decomposes into finite dimensional subspaces on which $\{U_g\}$ is irreducible. Pick any such finite-dimensional subspace and let $f_1, \ldots, f_n$ be an orthonormal basis of it and $\alpha(g)$ the matris of U_g relative to this basis. Define a vector-valued function f to $\mathbb{C}^n$ by $f = \mathrm{col}\,(f_1, \ldots, f_n)$ whenever possible and $\mathbf{0}$ otherwise. Then $f \circ g = \alpha(g)\bar{f}$ in L_2. The set $\{x: f(x) = \mathbf{0}\}$ is both G-invariant and T-invariant, so by the ergodicity of μ has measure 0. Normalizing f in L_2, we obtain $\bar{f}$ to Σ^{n-1} with $\bar{f} \circ g = \alpha(g)\bar{f}$ in L_2. By 1.4, we can assume $\bar{f}: X \to \Sigma^{n-1}$. Since $\bar{f}$ is T-invariant, we have $\alpha = 1$. It follows that $\{U_g\}$ is the identity representation on $\mathscr{I}$. So $\mathscr{I} = \mathscr{I} \cap L^2(Y) = \mathscr{C}$ and $\tilde{\mu}$ is ergodic, completing the proof.

Note that 2.1 does not require that G be metric.

Our next theorem considers the other extreme case when there are the 'maximum' number of T-invariant α-functions. By isomorphism, we shall mean isomorphisms of dynamical systems. This requires the existence of *strictly* T-invariant measurable sections in X over Y, i.e., a Borel set B with $\pi(B)$ Borel, $\mu(\pi B) = 1$, $xt \in B$ whenever $x \in B$, $t \in T$ and $\pi: B \to \pi(B)$ measurable. If either X is metric or T is countable, then π is Borel, and we use the term 'isomorphism mod 0'. In the general case, we will need to consider $\pi: (X, T, \nu_0) \to (Y, T, \mu_0)$, where ν_0, μ_0 are the completions of ν, μ respectively. We then use the expression 'Lebesgue isomorphism mod 0'.

THEOREM (2.2): *For each irreducible representation class* $[\alpha]$ *of dimension* n_α, *let there be* n_α *pointwise orthogonal T-invariant* α*-functions. Let* $\nu \in E(X, T)$ *with* $\pi^*\nu = \mu$. *Then if either* X *is metric or* T *is countable,*

$$\pi: (X, T, \nu) \to (Y, T, \mu)$$

is an isomorphism mod 0. *Otherwise,*

$$\pi: (X, T, \nu_0) \to (Y, T, \mu_0)$$

is a Lebesgue isomorphism mod 0.

PROOF: We will construct a strictly T-invariant measurable section over Y. Let f_α^i, $i = 1, \ldots, n_\alpha$ be the n_α α-functions of the theorem. Since ν is ergodic and $\pi^*\nu = \mu$ there are constant vectors a_α^i, $i = 1, \ldots, n_\alpha$ such that $f_\alpha^i(x) = a_\alpha^i$ a.e. ν. Put

$$X' = \{x \in X : f_\alpha^i(x) = a_\alpha^i, \text{ for all } i, \alpha\}.$$

Since there are only countable many $[\alpha]$, X' is a Borel set and $\nu(X') = 1$. Now let $x \in X'$ and $gx \in X'$, for some $g \in G$. Then

$$a_\alpha^i = f_\alpha^i(gx) = a(g) f_\alpha^i(x) = \alpha(g) a_\alpha^i.$$

Thus, $\alpha(g)$ is the identity map since the vectors a_α^i form a basis of $\mathbb{C}^{n_\alpha}$. This is also true for each α and so $g = e$. Therefore X' intersects each orbit in at most one point.

The remainder of the proof for the case that X is metric follows the final part of the proof of Theorem 3.1.3 of [6], so we sketch the details. We put $X^0 = \{x \in X : xt \in X' \text{ for } \xi\text{-almost all } t \in T\}$, ξ being a left invariant Haar measure on T. Then X^0 is a strictly T-invariant Borel set which intersects each G-orbit in at most one point, πX^0 is a Borel set and $\pi: X^0 \to \pi X^0$ gives the required isomorphism.

For the remaining cases, we write $X^0 = \bigcup_{i=1}^\infty K_i \cup N$, where $\nu(N) = 0$, K_i compact, $K_i \subset K_{i+1}$ and $\pi|K_i$ continuous, by Lusin's Theorem. Then $\pi|K_i$ is a homeomorphism, and setting $K = \bigcup_{i=1}^\infty K_i$, $\pi|K$ is a Borel isomorphism. If T is countable, set $K_0 = \bigcap_{t \in T} Kt$; then K_0 is the desired Borel section over Y. In general, we can assume (see [6, Remark 3.2]) that πX^0 is Borel, and hence $\pi(N)$ is Borel with $\mu\pi(N) = 0$. This will then yield that X^0 is the desired Lebesgue section over Y. The proof is completed.

Finally, we combine Theorem 2.1 and 2.2 to give

THEOREM (2.3): *Let $\nu \in E(X, T)$ with $\pi^*\nu = \mu$. Then there is a closed subgroup $H = H(\nu)$ of G such that if we split the extension by H;*

$$\pi = \pi_2 \circ \pi_1, \pi_1 : (H; X, T) \to (X/H, T), \pi_2 : (X/H, T) \to (Y, T),$$

and put $\nu_1 = \pi_1^* \nu$ *then* π_2 *is a measure* (*either Borel or Lebesgue*) *isomorphism* mod 0 *and* $\tilde{\nu}_1 = \nu$.

Proof: For each irreducible representation class $[\alpha]$ of G let there be precisely l_α pointwise orthogonal T-invariant α-functions $f_\alpha^i, i = 1, \ldots, l_\alpha$ (note that possibly $l_\alpha = 0$). Since ν is ergodic there are constant vectors a_α^i, $i = 1, \ldots, l_\alpha$ such that

$$f_\alpha^i = a_\alpha^i \qquad \text{a.e. } \nu, i = 1, \ldots, l_\alpha.$$

We define a subgroup H by

$$H = G \text{ if for every } \alpha,\ l_\alpha = 0, \text{ and otherwise}$$

$$H = \{g \in G : \alpha(g) a_\alpha^i = a_\alpha^i;\ i = 1, \ldots, l_\alpha;\ [\alpha] \in \mathscr{R}(G)\}$$

and define

$$X' = X \text{ if for each } \alpha,\ l_\alpha = 0, \text{ and otherwise}$$

$$X' = \{x \in X : f_\alpha^i(x) = a_\alpha^i;\ i = 1, \ldots, l_\alpha;\ [\alpha] \in \mathscr{R}(g)\}.$$

Then $\nu(X') = 1$ and given $x \in X'$, $gx \in X'$ if and only if $g \in H$. Now consider the split extension over H

$$X \xrightarrow{\pi_1} X/H \xrightarrow{\pi_2} Y.$$

Since X' is an H-saturated Borel set it follows that $\pi_1(X')$ is a Borel set in X/H with $\nu_1(\pi_1(X')) = 1$. We now note that $\pi_1(X')$ is a Borel section for $\pi_2 : X/H \to Y$. To do this it suffices to show that if $\pi x_1 = \pi x_2$ and $\pi_1 x_1, \pi_1 x_2 \in \pi_1(X')$ then $\pi_1 x_1 = \pi_1 x_2$. This follows, however, directly from the definitions of X' and H. The technique of Theorem 2.2 now allows us to assert that $\pi_2 : (X/H, T, \nu_1) \to (Y, T, \mu)$ is a measure isomorphism mod 0.

We thus have a strictly T-invariant measure section mod 0; $p : Y' \to X/H$ with $\pi_2 \circ p = \mathrm{id}_{Y'}$, $p \circ \pi^t = \pi^t \circ p$ and $\nu(Y') = 1$. We use p to complete the proof by showing that $\tilde{\nu}_1$ is ergodic, and thus $\tilde{\nu}_1 = \nu$.

Let $[\alpha'] \in \mathscr{R}(H)$ with dimension n_α and let $[\alpha] \in \mathscr{R}(G)$ extend $[\alpha']$. Let $f_{\alpha'} : X \to \Sigma^{n_{\alpha'} - 1}$ be a T-invariant α'-function. First we will construct an α-function.

We use $f_{\alpha'}$ to construct an α-function as follows: if $\sigma : X/H \to X$ is any Borel section, then $\sigma \circ p : Y' \to X$ is a section over Y'. If $x \in \pi^{-1} Y'$, then $x = g(x)\sigma p(\pi x)$. Now we can assume that α/H has the form $\left[\begin{smallmatrix}\alpha' & 0\\ 0 & *\end{smallmatrix}\right]$. If 0 is an $(n_\alpha - n_{\alpha'}) \times 1$ zero vector, we define

$$f_\alpha(x) = \alpha(g(x))\begin{bmatrix} f_{\alpha'}(\sigma p(\pi x)) \\ 0 \end{bmatrix}.$$

Since the set $X - \pi^{-1}Y'$ is T and G-invariant and has ν-measure 0, this set is irrelevant and we define f_α to be any α-function on it.

Now we show that f_α is T-invariant. First note that $\sigma(z)t = h(z,t)\sigma(zt)$ for some $h(z,t) \in H$, whenever $z \in X/H$. We then have that

$$g(xt)\sigma((p\pi x)t) = xt = g(x)\sigma(p\pi x)t = g(x)h(p\pi x, t)\sigma((p\pi x)t)$$

which gives that $g(xt) = g(x)h(p\pi x, t)$.

Hence

$$\begin{aligned} f_\alpha(xt) &= \alpha(g(xt))\begin{bmatrix} f_{\alpha'}(\sigma(p(\pi x)t)) \\ 0 \end{bmatrix} \\ &= \alpha(g(x))\begin{bmatrix} \alpha'(h(p\pi x, t)) f_{\alpha'}(\sigma((p\pi x)t)) \\ 0 \end{bmatrix} \\ &= \alpha(g(x))\begin{bmatrix} f_{\alpha'}((\sigma p\pi x)t) \\ 0 \end{bmatrix} \\ &= \alpha(g(x))\begin{bmatrix} f_{\alpha'}(\sigma p\pi x) \\ 0 \end{bmatrix} \qquad \text{a.e. } \nu \\ &= f_\alpha(x) \qquad \text{a.e. } \nu. \end{aligned}$$

Therefore f_α is a T-invariant α-function and so we may write

$$f_\alpha(x) = \sum_{i=1}^{l_\alpha} a_i(x) f_\alpha^i(x).$$

Now the coefficients $a_i(x)$ are both G and T-invariant, and therefore constant a.e. ν. This means that for almost all x $f_\alpha(x)$ belongs to the subspace generated by $a_\alpha^1, \ldots, a_\alpha^{l_\alpha}$. But by the definition of H, $\alpha|_H$ is the identity on this subspace. Therefore α' is the identity representation on H. This shows that $\tilde{\nu}_1$ is ergodic and completes the proof.

It appears from the proof of Theorem 2.3 that the subgroup H depends not only on ν but also on the choice of α-functions. The next result shows that this is not the case.

COROLLARY (2.4): $H(\nu) = \{h: h\nu = \nu\}$, *that is* $H(\nu)$ *is the stabilizer of the ergodic measure* ν.

PROOF: Since ν is a Haar lift through $H(\nu)$ it follows that

$$H(\nu) \subset \{h: h\nu = \nu\} = H_\nu.$$

Let X' be the set defined in the proof of Theorem 2.3. If $g \notin H(\nu)$ then $gX' \cap X' = \emptyset$. But then $\nu(gX') = 0$ and $g\nu(gX') = \nu(g'gX') = \nu(X') = 1$ so $g\nu \neq \nu$ and hence $g \notin H_\nu$. Therefore $H_\nu \subset H(\nu)$ and the proof is complete.

3. Lifting dynamical properties

In this section, we will consider a free G extension $\pi: (G; X, T) \to (Y, T)$, where X and Y are metric, G is abelian and T is locally compact separable abelian. We fix $m \in M(Y, T)$. Consider Hom (T, G), the group of continuous homomorphisms $\chi: T \to G$, provided with the compact open topology. Then Hom (T, G) is a locally compact abelian group. Given $\chi \in$ Hom (T, G), we define a new action of T on X as follows: if ρ is the old action of T on X, we set $\rho_\chi(x, t) = \chi(t)\rho(x, t) = \chi(t)xt$. We shall write xt_χ for $\chi(t)xt$. It is easy to check that $(G; X, T, \rho_\chi)$ is still a bitransformation group and that $(G; X, T, \rho_\chi)$ remains a free G-extension of (Y, T). Hence for every χ, the Haar lift $\tilde{m} \in M(X, T, \rho_\chi)$. Thus, it is meaningful to ask if, given that m is ergodic (weak-mixing), the natural extension $\tilde{m}$ is ergodic (weak-mixing) with respect to ρ_χ. The intent of this section is to show that the first statement holds for a residual set in Hom (T, G), and the second statement has a trivial answer in Hom (T, G).

To prove our results, we recall the following notation: we let $\Gamma(G)$ and $\Gamma(T)$ denote the character groups of G and T respectively, and, if $\gamma \in \Gamma(G)$, f_γ denotes a Borel function of type γ, i.e., $f_\gamma(gx) = \gamma(g)f(x)$. Now consider the following equations:

(1) $$f_\gamma \circ t = f_\gamma[\pi^{-1}m] \qquad (t \in T)$$

(2) $$f_\gamma \circ t = \delta(t)f_\gamma[\pi^{-1}m] \qquad (t \in T), \text{ where } \delta \in \Gamma(T).$$

The basic result used in our analysis is the following

THEOREM:

A. Assume that $m \in \mathscr{E}(Y, T)$. Suppose that whenever f_γ satisfies (1), then $\gamma = 1$. Then $\tilde{m}$ is ergodic.

B. Assume that m is weak-mixing. Suppose that whenever f_γ satisfies (2) for some $\delta \in \Gamma(T)$, then $\gamma = 1$. Then $\tilde{m}$ is weak-mixing.

Statement A. is precisely the statement of [6, Theorem 3.1.2]. With regard to B., this statement appears in [4] for $T = \mathbb{Z}$ or $T = \mathbb{R}$. Although no proof is given, this result even in our situation follows directly from [7, Theorem 3]. For completeness, we sketch a proof. If the condition is satisfied, then $\tilde{m}$ is ergodic by part A. Suppose $f \circ t = \delta(t) f[\pi^{-1}m]$ $(t \in T)$, with f bounded and measurable. Since $|f|$ is invariant, $|f|$ is a constant by ergodicity and so we assume $f: X \to K$. Now if $h: X \to K$ also satisfies $h \circ t = \delta(t)h[\pi^{-1}m](t \in T)$, then h/f is invariant and so $h = cf$, c constant. Fix $g \in G$. Then $(f \circ g) \circ t = \delta(t)(f \circ g)[\pi^{-1}m]$ $(t \in T)$ and so $f \circ g = \gamma(g) f[\pi^{-1}m]$. It is direct to verify that $\gamma \in \Gamma(G)$ and so f is a function of type γ. Hence $\gamma \equiv 1$ and hence f can be regarded as bounded measurable on Y. Thus, f is a constant, completing the proof.

We need one last definition before proving the main result. If U is open in Hom (T, G), then we can assume that U is basic: $U = \{\chi: \chi(C) \subset W\}$, where C is compact in T, and W is a neighborhood of e in G. We say that Hom (T, G) *covers* e if given U as above, $\bigcup \{\chi(T): \chi \in U\} \supset V$, where V is a neighborhood of e in G.

THEOREM (3.1): *Let* $m \in \mathscr{E}(Y, T)$. *Suppose that* G *is connected and* Hom (T, G) *covers e. Then for almost all* χ, $\tilde{m}$ *is ergodic with respect to* (X, T, ρ_χ).

PROOF: Suppose $\tilde{m}$ is not ergodic with respect to (X, T, ρ_χ). Then there exists $\gamma \neq 1$ and an f_γ satisfying (1). Since G is metric, $\Gamma(G)$ is countable. For $\gamma \neq 1$, set $A_\gamma = \{\chi \in \text{Hom}\,(T, G): \chi$ admits a solution to (1) using $\gamma\}$. Now for fixed γ, we define $\hat{\gamma}: \text{Hom}\,(T, G) \to \text{Hom}\,(T, K)$ by $\hat{\gamma}(\chi) = \gamma \circ \chi$. Now if $\chi \in A_\gamma$ and $f_\gamma \circ t_\chi = f_\gamma[\pi^{-1}m]$, then

$$f_\gamma(xt) = f_\gamma(\chi^{-1}(t)\chi(t)xt) = \gamma(\chi^{-1}(t)) f_\gamma(xt_\chi) = \hat{\gamma}(\chi^{-1})(t) f_\gamma(x)[\pi^{-1}m].$$

Hence, $\hat{\gamma}(\chi^{-1})$ is a T-eigenvalue for the original action of T and, since X is metric, a straightforward L_2-argument shows that there are only countably many such eigenvalues. It follows that $\hat{\gamma}(A_\gamma^{-1})$ and thus $\hat{\gamma}(A_\gamma)$ is countable. Setting $\hat{\gamma}(A_\gamma) = \{\hat{\gamma}(\chi_i): i \geqq 1\}$, then

$$A_\gamma \subset \hat{\gamma}^{-1}\hat{\gamma}(A_\gamma) = \bigcup \{\hat{\gamma}^{-1}\hat{\gamma}(\chi_i): i \geqq 1\},$$

and to prove the result, we need only show that $\hat{\gamma}^{-1}\hat{\gamma}(\chi_i)$ does not contain an open set.

Suppose $\hat{\gamma}^{-1}\hat{\gamma}(\chi_i) \supset U_1$, with U_1 open. It follows that if $U = U_1\chi_i^{-1}$, then $\hat{\gamma}|U \equiv 1$. Since $\mathrm{Hom}(T, G)$ covers e, then $\bigcup \{\chi(T): \chi \in U\} \supset V$, where V is open about e. Now choose $g \in V$. Then $g = \chi(t)$ for some $\chi \in U$, and so $\gamma(g) = \gamma(\chi(t)) = \hat{\gamma}(\chi)(t) = 1$. Thus, $\gamma|V \equiv 1$ and so $\gamma(G)$ is finite. Since G is connected, this is a contradiction, and the theorem is proved.

With regard to weak-mixing, it is impossible to perturb a non-weak-mixing flow into a weak-mixing flow within the class of perturbations induced by $\mathrm{Hom}(T, G)$. The next result shows that given $\chi \in \mathrm{Hom}(T, G)$, $(X, T, \rho_\chi; \tilde{m})$ is weak-mixing precisely when $(X, T, \rho; \tilde{m})$ is already weak-mixing.

THEOREM (3.2): *Let (Y, T, m) be weak-mixing. Then the following are equivalent*:

(1) *$(X, T, \rho; \tilde{m})$ is weak-mixing.*

(2) *For some $\chi \in \mathrm{Hom}(T, G)$, $(X, T, \rho_\chi; \tilde{m})$ is weak-mixing.*

(3) *For every $\chi \in \mathrm{Hom}(T, G)$, $(X, T, \rho_\chi; \tilde{m})$ is weak-mixing.*

PROOF: We need only show that (1) $\Rightarrow$ (3). Choose $\chi \in \mathrm{Hom}(T, G)$, and suppose that f_γ satisfies $f_\gamma(xt_\chi) = \delta(t) f_\gamma(x)[\pi^{-1}m]$ for some $\delta \in \Gamma(T)$. Then since

$$f_\gamma(xt_\chi) = f_\gamma(\chi(t)xt) = \hat{\gamma}(\chi)(t) f_\gamma(xt),$$

we have that

$$f_\gamma(xt) = \hat{\gamma}(\chi)^{-1}(t)\delta(t) f_\gamma(xt)[\pi^{-1}m].$$

Since $(X, T, \rho; \tilde{m})$ is weak mixing, this implies that $\gamma = 1$. The conclusion then follows from Theorem B.

COROLLARY 3.3:

A. *Let (Y, T) be uniquely ergodic, and $m \in \mathscr{E}(Y, T)$. Then for almost all χ, (X, T, ρ_χ) is uniquely ergodic, and $\tilde{m} \in \mathscr{E}(X, T)$.*

B. *Let $T = \mathbb{Z}$ or $\mathbb{R}$, and (Y, T, m) be ergodic with m having maximal entropy. Then for almost all χ, $(X, T, \rho_\chi; \tilde{m})$ has maximal entropy.*

PROOF:

A. This follows immediately from Theorem 3.1 and [6, Corollary 2.2.6].

B. Again, this follows immediately by Theorem 3.2 and [6, Proposition 5.4.4].

We now show that the assumption that $\mathrm{Hom}(T, G)$ covers e is always satisfied if T is locally compact separable abelian.

THEOREM (3.4): *If T is locally compact separable abelian, then* $\operatorname{Hom}(T, G)$ *covers e.*

PROOF: First suppose $T = \mathbb{Z}$. Then $\operatorname{Hom}(\mathbb{Z}, G) \simeq G$, and clearly $\operatorname{Hom}(\mathbb{Z}, G)$ covers e. Next, suppose $T = \mathbb{R}$. Since G is connected, then G is a subgroup of some m-dimensional torus K^m, where $1 \leqq m \leqq \infty$. If $G = K^m$, $m < \infty$, then $\operatorname{Hom}(\mathbb{R}, K^m) \simeq \mathbb{R}^m$. This means that the U that we want to consider is transformed to a neighborhood of $0 \in \mathbb{R}^m$. Now it is well known that the one-parameter subgroups cover K^m. Moreover, any one-parameter subgroup has a representation as

$$\chi(x) = (e^{2\pi i r_1 x}, \ldots, e^{2\pi i r_m x}),$$

where $(r_1, \ldots, r_m)$ vary over a fixed neighborhood of $0 \in \mathbb{R}^m$. So

$$\bigcup\{\chi(\mathbb{R}) : \chi \in U\} = K^m$$

and $\operatorname{Hom}(\mathbb{R}, K^m)$ covers e. A similar argument holds for K^∞. Finally, if $G \subset K^\infty$ is a solenoid, then a direct argument involving the representation of G as an inverse limit of tori shows that in the metric case, the one-parameter subgroups cover G, and the same argument holds.

Finally, suppose T is arbitrary. Then $T \simeq \mathbb{R}^n \times \mathbb{Z}^m \times C$, with C compact [2]. Since T is assumed non-compact, either $n \geqq 1$ or $m \geqq 1$. If $n \geqq 1$, we have a homomorphism $\rho: T \to \mathbb{R}$, which gives

$$\hat{\rho}: \operatorname{Hom}(\mathbb{R}, G) \to \operatorname{Hom}(T, G).$$

If U is open about 1_T, then $\hat{\rho}^{-1}(U)$ is open about $1_{\mathbb{R}}$, and by the above,

$$\bigcup\{\chi(\mathbb{R}) : \chi \in \hat{\rho}^{-1}(U)\} \supset V,$$

a neighborhood of e. But $\chi(\mathbb{R}) = \chi(\rho T) = \hat{\rho}(\chi)(T)$, which gives the result. A similar argument holds if $m \geqq 1$, and this completes the proof.

As noted, we now have the results for the classical cases $T = \mathbb{Z}$ and $T = \mathbb{R}$. For $T = \mathbb{Z}$, Theorem 3.1 was already known [7, Theorem 4].

We now note some straightforward extensions of these ideas. Recall that a free G-extension $\pi: (G; X, T) \to (Y, T)$ is *simple* if for every $\gamma \in \Gamma(G)$, there exists a continuous function f_γ of type γ. We now consider the equation

$$f_\gamma \circ t = f_\gamma (t \in T), \; f_\gamma \text{ continuous} \tag{3}$$

$$f_\gamma \circ t = \delta(t) f_\gamma (t \in T), \qquad \text{for some } \delta \in \Gamma(T), \; f_\gamma \text{ continuous.} \tag{4}$$

We then have:

THEOREM:

C. Suppose (Y, T) is minimal and whenever f_γ satisfies (3), $\gamma = 1$. Then (X, T) is minimal.

D. Suppose (Y, T) is minimal and topologically weak-mixing and whenever f_γ satisfies (4), $\gamma = 1$. Then (X, T) is minimal and topologically weak-mixing. If (X, T) is assumed minimal, then weak-mixing is equivalent to this condition.

PROOF:

C. This is a direct consequence of [6, Theorem 6.3].

D. This proof proceeds as in Theorem B, replacing measurable by continuous functions.

We use this to prove the analogue of Theorem 3.1 for minimality.

THEOREM (3.5): *Suppose (Y, T) is minimal and $\pi: (G; X, T) \to (Y, T)$ is a simple G-extension with G connected. Then for almost all χ, (X, T, ρ_χ) is minimal.*

PROOF: This proof follows the lines of Theorem 3.1, and hence will be omitted.

Note that one obvious way of getting a simple extension is to set $X = G \times Y$ and define $g(g_1, y) = (gg_1, y)$. Applying Theorem 3.5 in this setting says that for almost all χ, $\rho_\chi((g, y), t) = (\chi(t)g, yt)$ gives a minimal action on X. Finally, for $T = \mathbb{Z}$, Theorem 3.5 was already known [7, Theorem 2].

We should also note that if (X, T) is minimal, D holds even if G is not abelian (but T is still abelian). Thus, in the case that G is simple, (X, T) is topologically weak-mixing whenever (Y, T) is. Also, a direct copy of Theorem 3.2 shows that again in the topological case, it is impossible to perturb a non-weak-mixing flow into a weak-mixing flow within the class of perturbations induced by $\operatorname{Hom}(T, G)$.

We finally note that if we are willing to give up the continuity of the new actions, we can obtain the same results in the larger group $\operatorname{Hom}(T_d, G)$ of all homomorphisms from T to G (not necessarily continuous). Here we consider T as a discrete group and hence $\operatorname{Hom}(T_d, G)$ is compact in the compact-open topology (being equal to pointwise convergence topology). Carrying through the same proof as Theorem 3.1, we recall that if $\hat{\gamma}^{-1}\hat{\gamma}(\chi_i) \supset U_1$ and $U = U_1\chi_i^{-1}$, then $\hat{\gamma}|U \equiv 1$. In this case, a finite number of left translates of U covers $\operatorname{Hom}(T_d, G)$, and so $\hat{\gamma}(\operatorname{Hom}(T_d, G))$ is a finite subgroup. Now suppose that:

$$(*) \qquad \bigcup \{\chi(T): \chi \in \operatorname{Hom}(T_d, G)\} \supset G.$$

If $k_1, \ldots, k_n$ are the orders of the elements of $\hat{\gamma}(\mathrm{Hom}(T_d, G))$ and $k = \pi_{i=1} k_i$, then $\hat{\gamma}(\chi)^k = 1(\chi \in \mathrm{Hom}(T_d, G))$. If $g \in G$ and $g = \chi(t)$, then $\gamma^k(g) = \hat{\gamma}(\chi)^k(t) = 1$, and thus γ is of finite order. Since G is connected, this is impossible. We use this in showing:

THEOREM (3.6):
I. Let $m \in \mathscr{E}(Y, T)$, and G be connected. Then for almost all $\chi \in \mathrm{Hom}(T_d, G)$, $\tilde{m}$ is ergodic with respect to (X, T, ρ_χ).
II. Let $(Y, T, \tilde{m})$ be uniquely ergodic, and G be connected. Then for almost all $\chi \in \mathrm{Hom}(T_d, G)$, (X, T, ρ_χ) is uniquely ergodic.

PROOF: We need only show that if T_d is the discrete group underlying a locally compact separable abelian topological group, then (*) holds. Following the proof of Theorem 3.4, we need only show this for $T = \mathbb{Z}$ or $T = \mathbb{R}$. The first case is obvious, so we now show that if $g \in G$, $\chi(1) = g$ for some $\chi \in \mathrm{Hom}(\mathbb{R}_d, G)$. By [2, Theorem 25.20], we have that $H = \{\chi(1) | \chi \in \mathrm{Hom}(\mathbb{R}, G)\}$ is a dense subgroup of G. If $g \in G$, choose sequences $(g_n) \in H$ and $(\chi_n) \in \mathrm{Hom}(\mathbb{R}, G)$ with $g_n \to g$ and $\chi_n(1) = g_n$. By compactness, we have that some subsequence (χ_{n_j}) converges in $\mathrm{Hom}(\mathbb{R}_d, G)$, $\chi_{n_j} \to \chi$. Thus $g_{n_j} = \chi_{n_j}(1) \to \chi(1)$ and $\chi(1) = g$, completing the proof.

We end this section by making one observation concerning weak-mixing in the case $T = \mathbb{Z}$, i.e., we have homeomorphisms $\varphi: X \to X$, $\psi: Y \to Y$. In this case, $\mathrm{Hom}(\mathbb{Z}, G) \simeq G$. If $m \geqq 1$, then

$$\pi: (G; X, \varphi^m) \to (Y, \psi^m)$$

and thus, if m is ergodic with respect to ψ^m, then $\tilde{m}$ is ergodic with respect to $g \cdot \psi^m$ for a.e. g, by Theorem 3.6, I. Since a countable intersection of residual sets is residual, it follows that if m is totally ergodic with respect to ψ, then for a.e. g, $\tilde{m}$ is totally ergodic with respect to φ. Since weak-mixing implies total ergodicity, we have shown:

COROLLARY (3.7): *Under the above assumptions, suppose (Y, ψ, m) is weak-mixing. Then for* a.e. *g, $(X, g \cdot \varphi; \tilde{m})$ is totally ergodic.*

In general, we can define total ergodicity for a general flow (Y, T, m) by requiring that (Y, S, m) be ergodic for every syndetic normal subgroup S of T. One can then show that weak-mixing implies total ergodicity. Hence, 3.7 will hold for any countable group T.

Appendix

In this appendix, we show that one application of the general theory developed in Section 2 is to the situation where (X, T) and (Y, T) are metric minimal transformation groups with $\varphi: (X, T) \rightarrow (Y, T)$ a distal extension and (Y, T) uniquely ergodic. To see this, the generalized Furstenberg Structure Theory (see [5] for a precise statement) says that we build X from Y by steps of the form

$$\begin{tikzcd}(X_{\alpha+1}, T) \arrow[d] & (G_\alpha, Y_\alpha, T) \arrow[l] \arrow[d] \\ (X_\alpha, T) = & (Y/G_\alpha, T)\end{tikzcd}$$

where $(X_{\alpha+1}, T)$ is an isometric extension of (X_α, T) in this case (in general, just an almost periodic extension – see [1] for these definitions) interpolating between the group extension (G_α, Y_α, T) of (X_α, T). In general, G_α and Y_α are compact Hausdorff. We show that in the metric case we can choose G_α and Y_α to be compact metric, and thus apply our theory. We again emphasize that some of these results are surely known to dynamicists, but we have been unable to find them in print.

We shall assume familiarity with notation and results of [1], especially Chapter 12.

Let $\mathscr{A} \subset \mathscr{B}$, $A = g(\mathscr{A})$, the group of $\mathscr{A}$, and set $\mathscr{S} = \langle \mathscr{B}\alpha | \alpha \in A \rangle$, the algebra generated by $\{\mathscr{B}\alpha | \alpha \in A\}$. If $B = g(\mathscr{B})$, then

$$g(\mathscr{S}) = \bigcap_{\alpha \in A} \alpha B \alpha^{-1} = S$$

is a normal subgroup of A. If in addition, $\mathscr{B}$ is a distal extension of $\mathscr{A}$, then $(A/S, |\mathscr{S}|, T)$ is a bitransformation group with

$$(|\mathscr{S}|/(A/S), T) \simeq (|\mathscr{A}|, T)$$

[1, Prop. 12.12 and 12.13]. Suppose in this situation, $(H, |\mathscr{F}|, T)$ is another bitransformation group with $\mathscr{B} \subset \mathscr{F}$ and $(|\mathscr{F}|/H, T) \simeq (|\mathscr{A}|, T)$. Then $H \simeq A/F$ where $F = g(\mathscr{F})$, and $\mathscr{F}A \subset \mathscr{F}$. Thus, $\mathscr{B}\alpha \subset \mathscr{F}\alpha \subset \mathscr{F} (\alpha \in A)$, and so $\mathscr{S} \subset \mathscr{F}$. Thus, $\mathscr{S}$ is the 'smallest' bitransformation group over $\mathscr{A} \leqq \mathscr{B}$, and A/S the 'smallest' group. Moreover, in the situation that we are dealing with, namely, an isometric extension of metric transformation groups (or more generally an almost periodic extension), putting the $\tau(\mathscr{S})$-topology on A/S makes $(A/S, \tau(\mathscr{S}))$ into a compact Hausdorff group which acts jointly continuously on $|\mathscr{S}|$ [1, Remark 14.4, Prop. 14.11, 14.26]. We shall show that $(A/S, \tau(\mathscr{S}))$ and $|\mathscr{S}|$ are both metric, giving the result.

PROPOSITION (I): *$(A/S, \tau(\mathscr{S}))$ is compact metric.*

PROOF: We are assuming that the restriction map $r\colon |\mathscr{B}| \xrightarrow{\simeq} |\mathscr{A}|$ is an isometric extension. Now each fibre $r^{-1}r(x)$ $(x \in |\mathscr{A}|)$ is homeomorphic to $(A/B, \tau(\mathscr{B}))$ [1, Prop. 1.2, Notes to Chap. 15] and so $(A/B, \tau(\mathscr{B}))$ is compact metric. Hence, we can assume without loss of generality that this metric is both right and left A-invariant. Letting $I(A/B)$ denote the group of isometries on A/B, it is well known that $I(A/B)$ is metrizable in the topology of uniform convergence. The right invariance of A yields a canonical map $\varphi\colon (A, \tau(\mathscr{B})) \to I(A/B)$, and the left invariance yields that φ is continuous. It is direct to verify that $\ker \varphi = S$ and hence $(A/S, \tau(\mathscr{B})) \simeq \operatorname{im} \varphi$ is compact metric. Since $\tau(\mathscr{B}) \subset \tau(\mathscr{S})$ and $(A/S, \tau(\mathscr{S}))$ is compact, then $(A/S, \tau(\mathscr{S})) = (A/S, \tau(\mathscr{B}))$, completing the proof.

It is easy to verify that the converse: $(A/S, \tau(\mathscr{S}))$ compact metric implies r is isometric, is also true.

PROPOSITION (II): *$|\mathscr{S}|$ is metric.*

PROOF: Recall that $\mathscr{A} \subset \mathscr{A}^{\#} \subset \mathscr{A}^{*}$, where $\mathscr{A}^{\#}$ is the maximal almost-periodic extension of $\mathscr{A}$, and $\mathscr{A}^{*}$ is the maximal distal extension of $\mathscr{A}$. Now $\mathscr{B} \subset \mathscr{S} \subset \mathscr{A}^{*}$ gives $B \supset S$, $\tau(\mathscr{B}) \subset \tau(\mathscr{A}^{*})$. Thus $(A/S, \tau(\mathscr{B}))$ compact metric and $A/S \to A/B$ gives $(A/B, \tau(\mathscr{B}))$ and hence, $(A/B, \tau(\mathscr{A}^{*}))$ compact metric. Pick $\{B\alpha_n | n = 1, 2, \ldots\}$ $\tau(\mathscr{A}^{*})$-dense in A/B and set $\mathscr{S}_1 = \langle \mathscr{B}\alpha_n | n = 1, 2, \ldots \rangle$. Since

$$\mathop{\mathrm{X}}_{n=1}^{\infty} |\mathscr{B}\alpha_n| \supset |\mathscr{S}_1| \quad \text{and} \quad (|\mathscr{B}\alpha_n|, T) \simeq (|\mathscr{B}|, T),$$

$|\mathscr{S}_1|$ is metric. We complete the proof by showing that $\mathscr{S}_1 = \mathscr{S}$.

Choose $\alpha \in A$. Since the canonical map $(A, \tau(\mathscr{A}^{*})) \to (A/B, \tau(\mathscr{A}^{*}))$ is open, if $(B\alpha_{n_j})$ is a sequence in the dense subset with $B\alpha_{n_j} \to B\alpha$, then there exists a sequence $(\beta_j) \in B$ with $\beta_j\alpha_{n_j} \to \alpha$ in $(A, \tau(\mathscr{A}^{*}))$. Next, let $f \in \mathscr{A}^{\#}$. Then $\alpha \to \langle f, \alpha p \rangle$ $(p \in M)$ is continuous on $(A, \tau(\mathscr{A}^{*}))$. This implies that $f\beta_j\alpha_{n_j} \to f\alpha$ pointwise on βT, or $f\beta_j\alpha_{n_j} \to f\alpha$ in $\mathscr{C}(|\mathscr{A}^{\#}|) = \mathscr{A}^{\#}$ with the topology of pointwise convergence. Since

$$(A, \tau(\mathscr{A}^{*})) \times |\mathscr{A}^{\#}| \to |\mathscr{A}^{\#}|$$

is jointly continuous and $\{f\delta | \delta \in A\}$ is compact, $f\beta_j\alpha_{n_j} \to f\alpha$ uniformly on $\mathscr{A}^{\#}$.

Finally, choose $g \in \mathscr{B} \subset \mathscr{A}^{\#}$. Then $g\beta_j\alpha_{n_j} \to g\alpha$ uniformly. Now $g\beta_j = g$ since $\beta_j \in B$ and so $g\alpha_{n_j} \to g\alpha$. Thus, $g\alpha \in \mathscr{S}_1$. Since α is arbitrary,

$\mathscr{S} \subset \mathscr{S}_1$, completing the proof.

In summary, we have shown: $\varphi: (X, T) \precsim (Y, T)$ is isometric iff there exists a group extension (G, Z, T) of (Y, T) with G and Z metric such that $(Z, T) \precsim (X, T)$.

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