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ON MODULI OF ALGEBRAIC VARIETIES III. FINE MODULI SPACES

Herbert Popp

Dedicated to the memory of Professor O. F. G. Schilling

This paper deals mainly with fine moduli spaces. We consider over the complex numbers the following types of algebraic varieties with level n-structure, where n is a sufficiently large integer,

(a) canonically polarized algebraic varieties with very ample canonical sheaf and fixed Hilbert polynomial,

(b) polarized $K-3$ surfaces with fixed Hilbert polynomial,

(c) stable curves of genus $g \geqq 2$ [1],

and show that there exist algebraic spaces of finite type over $\boldsymbol{C}$ which are over the complex numbers fine moduli spaces for these objects.

More precisely, for large n there exist algebraic spaces $M^{(n)}$ of finite type over $\boldsymbol{C}$ and proper, flat families $\Gamma^{(n)} \to M^{(n)}$ of polarized varieties of the above mentioned types which are universal with respect to proper, flat and polarized families of such varieties with level n-structure. (Compare also [12] p. 97.)

In chapter I a representablility theorem for functors in the category of noetherian algebraic $\boldsymbol{C}$-spaces is proved. This theorem is essential to obtain the families $\Gamma^{(n)} \to M^{(n)}$. It is related to theorem 4.7 in [1; 11] and also to [12], proposition 7.5, and states that under certain conditions fine moduli spaces are obtained as geometric quotients from a covering of a Hilbert-scheme via the action of PGL(N).

Chapter II contains the existence of the coarse moduli spaces for canonically polarized algebraic varieties and for polarized varieties with irregularity 0 (excluding certain ruled varieties) and furthermore the construction of the families $\Gamma^{(n)} \to M^{(n)}$ for varieties of type (a) and (b).

The case of stable curves is treated separately in chapter III. The universal family $\bar{\Gamma}^{(n)} \to \bar{M}^{(n)}$ of stable curves of genus g with level n-structure is constructed by compactifying the universal family $\Gamma^{(n)} \to M^{(n)}$

[1] For abelian varieties see remark (2.4), for Enriques surfaces the remark at the end of chapter II.

for smooth curves of genus g with level n-structure. (See remark (2.4) for the existence of $\Gamma^{(n)} \to M^{(n)}$ for smooth curves.)

The philosophy is as follows. The papers [18] and [24] show that a fibre space $X \to S$ of curves of genus g is determined essentially by the smooth part $X_0 \to S_0$ and the monodromy action of $\pi_1(S_0)$ on the integral homology of $X_0 \to S_0$. So, wanting a good universal family for stable curves one should add in a global way the monodromy.

One expects that if to the universal family $\Gamma_H \to H$ of 3-canonical stable curves of genus g the monodromy is added, the new family has better properties. Using the monodromy action on the level n-structure of the smooth part $\Gamma_{H_0} \to H_0$ of $\Gamma_H \to H$ we construct a finite normal (ramified) covering $\rho : H^{(n)} \to H$ of H where $\rho : \rho^{-1}(H_0) \to H_0$ is etale and $\rho^{-1}(H_0) = H_0^{(n)}$ is the scheme which parametrizes the smooth 3-canonical curves with level n-structure. So $H^{(n)}$ is nothing but the normalisation of H in the function field of $H_0^{(n)}$. But the important fact is a geometric interpretation of the points of $H^{(n)}$ which results from our monodromy considerations. Roughly speaking, a point $Q \in H^{(n)}$ is determined by $\varphi(Q) \in H$ and a symplectic basis of $H^1(\Gamma_{\rho(Q)}, Z/n)$, where $\Gamma_{\rho(Q)}$ is the fibre in $\Gamma_H \to H$ over $\rho(Q)$.

This fact allows to show that the natural operation of $PGL(N)$ on $H^{(n)}$ is fixpoint free if $n \geqq 3$. Having this available we look to the family $\Gamma_H^{(n)} = \Gamma_H \times_H H^{(n)} \to H^{(n)}$. The group $PGL(N)$ operates on $\Gamma_H^{(n)} \to \mathrm{H}^{(n)}$ properly and fixpoint free if $n \geqq 3$, yielding that the quotient family $\bar{\Gamma}^{(n)} \to \bar{M}^{(n)}$ which exists in the category of algebraic spaces is a family of stable curves with good universal properties. The algebraic spaces $\bar{\Gamma}^{(n)}$ respectively $\bar{M}^{(n)}$ are proper over $\boldsymbol{C}$ and compactifications of $\Gamma^{(n)}$ respectively $M^{(n)}$.

It seems to be necessary to indicate the significance of universal families and in particular the universal family $\bar{\Gamma}^{(n)} \to \bar{M}^{(n)}$ of curves for classification theory of algebraic varieties. The m-canonical mappings (compare Itaka's theorem [20], 5.10) relate algebraic varieties X of Kodaira dimension $= \dim(X) - 1$ to families of elliptic curves. Surfaces with Kodaira dimension 1, the elliptic surfaces of general type, are thus related to pencils of elliptic curves. The Albanese mapping $\alpha : X \to \mathrm{Alb}(X)$ of an algebraic variety X determines a canonical fibration of X. This fibration is a fibration into curves if $\dim(\alpha(X)) = \dim X - 1$. If X is of general type, the general fibre of α is by [20], theorem 5.11, also of general type and hence, if it is a curve, of genus $\geqq 2$. Both facts lead to the problem of classifications of algebraic varieties which carry a family of curves and more general of families of curves. The idea of classification is as follows.

Let $\lambda : X \to S$ first be a 1-parameter family of curves of genus $g \geqq 2$

where X and S are smooth schemes of dimension 2 respectively 1 and where λ is a proper and flat morphism which is smooth when restricted to $\lambda^{-1}(S-\{P_1,\ldots,P_n\})$; the P_i are closed points in S, sufficiently many. Furthermore, the surface X shall be minimal with respect to λ. The stable reduction theorem leads in this case to a galois covering $f: S' \to S$ with galois group G which is unramified outside the points P_i and to a family $\lambda': X' \to S'$ of stable curves such that

(1) If $U = S-\{P_1,\ldots,P_n\}$ and $U' = f^{-1}(U)$ then the families $\lambda: X\times_U U' \to U'$ and $\lambda': X'\times_{S'} U' \to U'$ are isomorphic.

(2) G operates on X' and λ' is a G-map.

(3) The quotient $X'^G \to S'^G = S$ is a fibre space over S which, restricted to U, is isomorphic to $X\times_S U \to U$. From $X'^G \to S$ the family $X \to S$ is obtained by resolving the singularities of X'^G in a minimal way.

Having these facts available the problem becomes to classify stable curves over a proper (1-dimensional) base S and secondly to describe the resolution of the singularities on X'^G by invariants.

To solve the first problem a 'universal' family of stable curves of genus g over a *compact base* is needed. The best to expect would be to have a good family of stable curves over the moduli space for stable curves available. Such a family however does not exist, it even does not exist for smooth curves. But for smooth curves with level n-structure, $n \geqq 3$, universal families $\Gamma^{(n)} \to M^{(n)}$ do exist. So, one might ask if it is possible to introduce for stable curves the notion of a level n-structure which is consistent with the smooth case and to obtain universal families for stable curves with level n-structure. Chapter III contains the construction of these families.

Concerning the second problem, describing the minimal resolution of the singularities of $X'^G \to S$ by invariants, we refer to the thesis of Viehweg [21]. There, for 1-parameter families, invariants are introduced using the Hilbert scheme of 3-canonical stable curves of genus g, which allow to describe the closed fibres of a family $\Gamma \to \operatorname{Spec}(R)$ over a discrete valuation ring R with residue characteristic $> 2g+1$. It is not difficult to rewrite these results using the family $\bar{\Gamma}^{(n)} \to \bar{M}^{(n)}$.

For smooth, projective algebraic varieties X which carry a fibration $\lambda: X \to S$ into curves of genus $g > 1$ with a proper base S of dimension > 1, at least a *birational classification* is possible using the universal family $\bar{\Gamma}^{(n)} \to \bar{M}^{(n)}$ as follows:

We may assume that S is projective and smooth. Consider the smooth part $\lambda: X_0 \to S_0$ of $\lambda: X \to S$ and let $S_0^{(n)}$ be the galois covering of S_0, parametrizing the level n-structures of the fibres of $\lambda: X_0 \to S_0$. Denote by S' an irreducible component of the normalisation of S in the ring of rational functions of $S_0^{(n)}$, and by $\varphi : S^{(n)} \to S$ the covering map. Let G be the galois group of this covering. By changing S birationally we may assume that there exist morphism $\rho : S \to M$ and $\rho^{(n)} : S^{(n)} \to M^{(n)}$ such that $\rho^{(n)}$ extends ρ. Let $\lambda^{(n)} : X^{(n)} = \Gamma^{(n)} \times_{M^{(n)}} S^{(n)} \to S^{(n)}$ be the pullback of $\Gamma^{(n)} \hookrightarrow M^{(n)}$. Then G operates on $X^{(n)}$ such that $\lambda^{(n)}$ is a G-map and the quotient $X^{(n)G}$ which exists as a scheme is birationally isomorphic in X.

Classification theory is a birational theory. Its general principals are described in [19], [20] and [24]. According to these principals, fine moduli spaces and a compactification of these spaces together with a good interpretation of the boundary points are needed for algebraic varieties with Kodaira dimension 0 and for algebraic varieties of general type. We have indicated above how one can obtain these fine moduli spaces and how one should use them to classify higher dimensional varieties. Compare in this respect also the paper [18]. Recently, Y. Namikawa [23] has constructed for this purpose a compactification of the family of polarized abelian varieties with level n-structure.

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Chapter I. A representability theorem for functors

We work entirely over the complex numbers $\boldsymbol{C}$. All schemes and algebraic spaces are $\boldsymbol{C}$-spaces.

The notion of a (smooth) polarized algebraic variety $(V, \mathfrak{X})$ is as introduced in [14], 2.1, hence we deal in this chapter with inhomogeneous polarized varieties. (See [14], p. 25.) $h_V(x)$ denotes the Hilbert polynomial of $(V, \mathfrak{X})$ ([14], p. 23) and $\mathfrak{M}_{h(x)}$ the set of isomorphy classes of polarized algebraic varieties defined over $\boldsymbol{C}$ with $h(x)$ as Hilbert polynomial. We assume in the following that $\mathfrak{M}_{h(x)}$ satisfies one of the two conditions a) or b)[1], i.e. we restrict $\mathfrak{M}_{h(x)}$ if necessary:

(a) All elements $(V, \mathfrak{X})$ of $\mathfrak{M}_{h(x)}$ are canonically polarized, i.e. the polar set $\mathfrak{X}$ contains a very ample (positive) multiple of a canonical divisor of V.

(b) For every $(V, \mathfrak{X}) \in \mathfrak{M}_{h(x)}$ the irregularity of V is zero.

We assume furthermore that the polarization of the varieties $(V, \mathfrak{X})$ is sufficiently ample. (See [14], p. 26, for this assumption.)

[1] The case of abelian varieties is excluded. See remark (2.4).

For a noetherian $\boldsymbol{C}$-scheme S we consider smooth, proper, (inhomogeneous) polarized families $(V/S, \mathfrak{X}/S)$over S, [14], p. 24, with the property that every geometric fibre of V/S together with the induced polarization belongs to $\mathfrak{M}_{h(x)}$.

Let $\mathscr{M}(S)$ be the set of isomorphy classes of such families. The collection of sets $\mathscr{M}(S)$ form a contravariant functor from the category of noetherian $\boldsymbol{C}$-schemes to the category of sets. The sheafification of the functor $S \to \mathscr{M}(S)$ with respect to the etale topology of schemes and its extension to the category of noetherian algebraic $\boldsymbol{C}$-spaces (see [8], p. 103) is also denoted by $\mathscr{M}$ and is considered in the following. $\mathscr{M}$ is then a contravariant functor from the category of noetherian algebraic $\boldsymbol{C}$-spaces to the category of sets, which is a sheaf with respect to the etale topology and which is called the *functor of polarized varieties of* $\mathfrak{M}_{h(x)}$.

Recall that for a noetherian algebraic $\boldsymbol{C}$-space S an element $\Gamma \to S$ of $\mathscr{M}(S)$ is (up to isomorphism) determined by a commutative diagram

$$\begin{tikzcd} \Gamma_R \arrow[r, shift left, "\mathscr{M}(\pi_1)"] \arrow[r, shift right, "\mathscr{M}(\pi_2)"'] \arrow[d] & \Gamma_{S'} \arrow[r] \arrow[d] & \Gamma \arrow[d] \\ R \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & S' \arrow[r] & S. \end{tikzcd}$$

In this diagram $R \rightrightarrows S' \to S$ and $\Gamma_R \rightrightarrows \Gamma_{S'} \to \Gamma$ are representable etale coverings of S and Γ, respectively, $(R, S', \Gamma_R$ and $\Gamma_{S'}$ are schemes); $\Gamma_{S'} \to S'$ and $\Gamma_R \to R$ are polarized families belonging to $\mathscr{M}(S')$ and $\mathscr{M}(R)$, respectively, such that $\Gamma_R \to R$ is the pullback (as polarized family) of $\Gamma_{S'} \to S'$ with respect to π_1 and π_2.

REMARK: The functor $\mathscr{M}$ introduced above is in general different from the deformation functor considered in [14]; they are however related. It is easily checked that the deformation functor from [14] is an open subfunctor of $\mathscr{M}$ and that $\mathscr{M}$ is the direct sum of finitely many deformation functors for appropriately chosen elements in $\mathfrak{M}_{h(x)}$.

For a polarized algebraic variety $(V, \mathfrak{X}) \in \mathfrak{M}_{h(x)}$ we consider projective embeddings $\Phi : V \to \mathbb{P}^N$ where Φ is a map determined by the multiple of the canonical sheaf ω_V which is in $\mathfrak{X}$, if $(V, \mathfrak{X})$ is canonically polarized, i.e. we are in case (a), and by any sheaf of $\mathfrak{X}$ in case V has irregularity 0.

The assumption that the polarization $(V, \mathfrak{X})$ is sufficiently ample yields that the varieties $(V, \mathfrak{X})$ are all mapped by Φ into the same projective space and that the image variety has $h(x)$ as Hilbert polynomial. Let $H^{h(x)}_{\mathbb{P}^N}$ be the Hilbert scheme which parametrizes the proper flat families of closed subschemes of $\mathbb{P}^N$ with $h(x)$ as Hilbert polynomial and $\Gamma \to H^{h(x)}_{\mathbb{P}^N}$

the corresponding universal family.

By the arguments of [14], propositions 2.11 and 2.12, there exists a locally closed subscheme H of $H^{h(x)}_{\mathbb{P}^N}$ such that the following statements are satisfied.

(1) Let $\Gamma_H \to H$ be the pullback of the universal family $\Gamma \to H^{h(x)}_{\mathbb{P}^N}$ to H; then $\Gamma_H \to H$ with the polarization induced from the hyperplane sections of $\mathbb{P}^N$ belongs to $\mathscr{M}(H)$.

(2) Every family $(V/S, \mathfrak{X}/S)$ of $\mathscr{M}(S)$ is locally, with respect to the etale topology, a pullback of $\Gamma_H \to H$.

(3) The group $PGL(N)$ operates on H and Γ_H in a natural way such that the following holds: Let S be a noetherian $\boldsymbol{C}$-scheme and Γ_1/S, Γ_2/S two pullback families of Γ_H/H via S-valued points $f_i : S \to H$, $i = 1, 2$. Then Γ_1/S and Γ_2/S are isomorphic as polarized families if and only if the S-valued points f_i are equivalent with respect to the action of $PGL(N)$.

(4) If the fibres of $\Gamma_H \to H$ are unruled varieties (this is the case if the varieties are canonically polarized; for the irregularity zero case compare Chap. II) the operation of $PGL(N)$ on H is proper and with finite stabilizers. Hence, the geometric quotient $\bar{H}$ of H by $PGL(N)$ exists as an algebraic space and is a coarse moduli space for the functor $\mathscr{M}$ in the sense of [14], Definition 2.8. (This is obtained by [15], theorem 1.4, together with the considerations in [14], chapter II. Notice that the restriction to H_{red}, which was necessary in [14], is no longer needed.)

The statements 1–3 imply:

PROPOSITION (1.1): *The etale sheafification $\mathscr{H}$ of the quotient functor $H/PGL(N)$ with respect to the etale topology and its extension to the category of noetherian algebraic $\mathbb{C}$-spaces coincides with the functor $\mathscr{M}$.*

The following theorem will give in many cases the representability of $\mathscr{M}$.

THEOREM (1.2): *Let X be a quasi-projective algebraic variety*[1] *over $\boldsymbol{C}$ and G an algebraic group which acts on X properly and freely. Then the geometric quotient $\bar{X}$ which exists as an algebraic space, represents the etale sheafification $\bar{\mathscr{X}}$ of the quotient functor*[2] *X/G.*

PROOF: There exists a natural map of functors $\lambda : \bar{\mathscr{X}}(S) \to \bar{X}(S)$, which is defined as follows. Consider the diagram from [14], p. 12

[1] The theorem still holds if X is a separated algebraic space of finite type over $\mathbb{C}$. Compare [15], p. 55.

[2] The S-valued points of the quotient functor X/G are $X(S)/G(S)$.

$$(*)\qquad \begin{tikzcd} G\times\bar{R} \arrow[r, "\sim"] \arrow[dr, "\mathrm{proj}_1"'] & R \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\varphi''"] & X' = \coprod (G\times U_i) \arrow[r] \arrow[d, "\varphi'"] & X \arrow[d] \\ & \bar{R} \arrow[r, shift left] \arrow[r, shift right] & U = \coprod U_i \arrow[r] & \bar{X} \end{tikzcd}$$

where the horizontal arrows are etale and $\bar{R} \rightrightarrows \coprod U_i$ determines the geometric quotient $\bar{X}$ as an algebraic space.[1] The isomorphism $G\times\bar{R} \xrightarrow{\sim} R$ is obtained as described in [14], p. 13, if X is reduced.

For the non-reduced case one has to use in addition arguments as in [15], p. 67.

Let $\bar{s}\in\bar{\mathscr{X}}(S)$ and $R_{S'} \rightrightarrows S' \to S$ a representable etale covering of S together with an S'-valued point $s' : S' \to X$ which determines $\bar{s}$. We may assume that s' factors through X'. Then $\varphi\circ s' : S' \to \coprod U_i$ determines the S-valued point $\lambda(s)$ of $\bar{X}$ via the diagram

$$\begin{tikzcd} R_{S'} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & S' \arrow[r] \arrow[d, "s'"] & S \\ R \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & X' \arrow[d, "\varphi'"] & \\ \bar{R} \arrow[r, shift left] \arrow[r, shift right] & \coprod U_i \arrow[r] & \bar{X}. \end{tikzcd}$$

The diagram (*) implies that λ is surjective. For the injectivity of λ the following fact is needed.

Claim: *The graph map $\Psi : G\times X \to X\times X$ is an isomorphism of $G\times X$ and $X\times_{\bar{X}} X$.*

Proof: Recall that $R\times_{\bar{R}} R \rightrightarrows X'\times_U X' \to X\times_{\bar{X}} X$ is a representable etale covering of $X\times_{\bar{X}}X$. The graph maps $\Psi_1 : G\times X' \to X'\times X'$ and $\Psi_2 : G\times R \to R\times R$ determine by [12], proposition 0.9, isomorphisms of $G\times X'$ and $X'\times_U X'$ respectively $G\times R$ and $R\times_{\bar{R}} R$. The diagram

$$\begin{tikzcd} R\times_{\bar{R}} R \arrow[r, shift left] \arrow[r, shift right] & X'\times_U X' \arrow[r] & X\times_{\bar{X}} X \\ G\times R \arrow[r, shift left] \arrow[r, shift right] \arrow[u, "\Psi_2"] & G\times X' \arrow[r] \arrow[u, "\Psi_1"] & G\times X \arrow[u, "\Psi"] \end{tikzcd}$$

implies therefore the claim.

Let now $\bar{s}_1, \bar{s}_2 \in \bar{\mathscr{X}}(S)$ such that $\lambda(\bar{s}_1) = \lambda(\bar{s}_2)$. Let $R_{S'} \rightrightarrows S' \to S$ be a properly choosen etale covering of S and $s_i' : S' \to X$ be S'-valued points which define $\bar{s}_i$, $i = 1, 2$. Then $\lambda(\bar{s}_1) = \lambda(\bar{s}_2)$ implies that the morphisms

[1] The reducedness of X is because of [15], chapter 1 not needed.

$\varphi \circ s'_i : S' \to \bar{X}$, $i = 1, 2$, induce a map $(s'_1, s'_2) : S' \to X \times_{\bar{X}} X$. As $X \times_{\bar{X}} X = \Psi(X \times G) \simeq X \times G$ holds, the S'-valued points s_i are in the same orbit with respect to G, hence $\bar{s}_1 = \bar{s}_2$ and the injectivity of λ follows. Q.E.D.

To apply theorem (1.2) to the functor $\mathscr{M}$, one of the essential facts is the fixpoint free action of $PGL(N)$ on H. In practice this is, due to the existence of non trivial automorphisms of the considered algebraic varieties, almost never the case. One has to modify the situation by considering varieties with level n-structure.

For an arbitrary smooth, proper family V/S over a $\boldsymbol{C}$-scheme S of finite type we constructed in [14], p. 32, an etale covering $P^{(n)}(V/S)$ of S. (The assumption made there that V/S is pullback of Γ_H/H is not needed.) A level *n-structure* of V/S is a section of $P^{(n)}(V/S)$ over S.

The covering $H^{(n)} = P^{(n)}(\Gamma_H/H)$ of H is of particular interest. We have shown in [14] that $PGL(N)$ operates on $H^{(n)}$ in a natural way such that the map $\rho : H^{(n)} \to H$ is a $PGL(N)$ map[1]. This fact can be used to define the notion of a level n-structure for an arbitrary noetherian $\mathbb{C}$-scheme S and a family V/S which is pullback of Γ_H/H.

Definition (1.3): Let $f : S \to H$ be a map and V/S the pullback family of Γ_H/H via f. Let $S^{(n)} = H^{(n)} \times_H S$ be the pullback of $H^{(n)}$ also with respect to f. Then a section α of $S^{(n)}$ over S is called a *level n-structure* of V/S.

It is clear that the pair of maps (f, α) determines a unique map $F : S \to H^{(n)}$ such that the diagram

$$\begin{tikzcd} S^{(n)} \arrow[r] & H^{(n)} \arrow[d] \\ S \arrow[u, "\alpha"] \arrow[ur, "F"] \arrow[r, "f"'] & H \end{tikzcd}$$

is commutative and, conversely, that F determines (f, α). See [14], p. 33.) We write $F = (f, \alpha)$ in the following. Continuing, if $\mathscr{H}^{(n)}(S)$ is the set of polarized families which are pullbacks of Γ_H/H with level n-structure, this set corresponds functorially to the set of S valued points of $H^{(n)}$ and the functors $\mathscr{H}^{(n)}(S)$ and $\operatorname{Hom}(S, H^{(n)})$ are equal. The group $PGL(N)(S)$ operates therefore on $\mathscr{H}^{(n)}(S)$ via this correspondence as follows.

[1] The important fact to be used is that $P^{(n)}(V/S)$ is functorial in the following sense. Let V_1/S and V_2/S be families of $\mathscr{M}(S)$ and $\varphi : V_1/S \to V_2/S$ an isomorphism of families. Then φ induces in a natural way an isomorphism $P^{(n)}(V_2/S) \to P^{(n)}(V_1/S)$.

Let $(V/S, \mathfrak{X}/S)^{(n)} \in \mathscr{H}^{(n)}(S)$ and $F : S \to H^{(n)}$ be the corresponding S-valued point. Let $F = (f, \alpha)$ with $f : S \to H$ and α a level n-structure. If φ is a S-valued point of $PGL(N)$, the image $(V^{\varphi}/S, \mathfrak{X}^{\varphi}/S)^{(n)}$ of $(V/S, \mathfrak{X}/S)^{(n)}$ under φ is determined by the S-valued point $F^{\varphi} = (f^{\varphi}, \alpha^{\varphi^{-1}})$ of $H^{(n)}$. As V/S and V^{φ}/S are the pullbacks of Γ_H/H with respect to f and f^{φ} we obtain that the family V/S is mapped isomorphically to V^{φ}/S by the projective transformation φ of $\mathbb{P}^N/S$. In particular V/S and V^{φ}/S are isomorphic as polarized algebraic varieties. $\alpha^{\varphi^{-1}}$ is a level n-structure of V^{φ}/S which is called the image of α.

The interpretation of $\alpha^{\varphi^{-1}}$ is much more natural if S is a scheme of finite type over $\boldsymbol{C}$. Then we can use the schemes $P^{(n)}(V/S)$ and $P^{(n)}(V^{\varphi}/S)$, and the isomorphism $\varphi : V/S \to V^{\varphi}/S$ induces a natural isomorphism $\varphi : P^{(n)}(V^{\varphi}/S) \to P^{(n)}(V/S)$. (See [14], p. 34.) Via the inverse φ^{-1} of this isomorphism the level n-structure α of V/S is mapped to the level n-structure $\alpha^{\varphi^{-1}}$ of V^{φ}/S.

With the usual notion of isomorphism the following holds. For a $\boldsymbol{C}$-scheme S of finite type two families of $\mathscr{H}^{(n)}(S)$ are isomorphic if and only if they are equivalent with respect to the action of $PGL(N)$.

This leads to consider the quotient functor

$$\mathscr{H}^{(n)}(S)/PGL(N)(S) = H^{(n)}/PGL(N)(S).$$

We sheafify this functor with respect to the etale topology. The sheaf thus obtained extends to the category of noetherian algebraic $\boldsymbol{C}$-spaces, yielding to a functor $\mathscr{M}^{(n)}$ of this category (with values in sets), called the *functor of polarized varieties of* $\mathfrak{M}_{h(x)}$ *with level n-structure.* The representability of $\mathscr{M}^{(n)}$ is established by the following Theorem.

THEOREM (1.4): *If* $PGL(N)$ *operates properly and fixpoint free*[1] *on* $H^{(n)}$, *the geometric quotient* $\bar{H}^{(n)}$ *of* $H^{(n)}$ *by* $PGL(N)$, *which exists by* [15], *Theorem* 1.4, *as an algebraic space, represents the functor* $\mathscr{M}^{(n)}$.

PROOF: Proposition (1.1) and Theorem (1.2) yield that the geometric quotient $\bar{H}^{(n)}$ represents $\mathscr{M}^{(n)}$. The universal family for $\mathscr{M}^{(n)}$ is obtained as follows. Look at the pullback family $\Gamma^{(n)} \to H^{(n)}$ of $\Gamma_H \to H$ to $H^{(n)}$. The operations of $PGL(N)$ on $H^{(n)}$ and Γ_H induce an operation of $PGL(N)$ on $\Gamma^{(n)} = \Gamma_H \times_H H^{(n)}$ via the components such that $\Gamma^{(n)} \to H^{(n)}$ is a $PGL(N)$ map. The proper action of $PGL(N)$ on $H^{(n)}$ implies (use [6], II, 5.4.3) that the operation of $PGL(N)$ on $\Gamma^{(n)}$ is also proper. The action is moreover fixpoint free.

[1] For $K-3$ surfaces and varieties with a very ample canonical sheaf, this is the case for sufficiently large n. Compare Chapter II.

This facts yield by [15], Theorem 1.4, that the geometric quotient $\bar{\Gamma}^{(n)}$ of $\Gamma^{(n)}$ with respect to the action of $PGL(N)$ exists in the category of algebraic $\boldsymbol{C}$-spaces. If U_i, $i = 1, \ldots, s$, are locally closed subschemes of $H^{(n)}$ (see [15], p. 64) such that $\coprod U_i \to \bar{H}^{(n)}$ is a representable etale covering, the pullback families $\Gamma_{U_i} = \Gamma^{(n)} \times_{H^{(n)}} U_i \to U_i$ will determine $\bar{\Gamma}^{(n)}$ as an algebraic space. More precisely, there exists an etale map $\coprod_i \Gamma_{U_i} \to \bar{\Gamma}^{(n)}$ which is a representable etale covering of $\bar{\Gamma}^{(n)}$. (Use that $\Gamma^{(n)} \to H^{(n)}$ is smooth and therefore Γ_{U_i} transversal to the orbits of points of Γ_{U_i} in connection with [15], Theorem 1.4.) The map $\Gamma^{(n)} \to H^{(n)}$ induces a map $\bar{\Gamma}^{(n)} \to \bar{H}^{(n)}$ yielding that $\bar{\Gamma}^{(n)} \to \bar{H}^{(n)}$ is a family which belongs to $\mathscr{M}^{(n)}(\bar{H}^{(n)})$. The family $\bar{\Gamma}^{(n)} \to \bar{H}^{(n)}$ is the universal family for $\mathscr{M}^{(n)}$ denoted in the introduction by $\Gamma^{(n)} \to M^{(n)}$.

REMARK (1.5): In the construction of the universal family $\bar{\Gamma}^{(n)} \to \bar{H}^{(n)}$ described above the smoothness of $\Gamma^{(n)} \to H^{(n)}$ is not essential. One needs only that for the families $\Gamma_{U_i} \to U_i$ the following is satisfied.

For any point $Y \in \Gamma_{U_i}$ the scheme Γ_{U_i} is transversal to the orbit O_Y of Y. (Orbit with respect to the action of $PGL(N)$ on $\Gamma^{(n)}$.) This fact is also available for the universal family $\Gamma^{(n)} \to H^{(n)}$ of 3-canonical stable curves of genus $g \geqq 2$ with level n-structure (see chapter III for the definition of this family) if the U_i are choosen appropriately, and is obtained as follows.

Let $Q \in H^{(n)}$ and let $P = \varphi(Q) \in H$ where $\varphi : H^{(n)} \to H$ is the natural covering map (H is the scheme parametrizing 3-canonical stable curves of genus g, see chapter III). Let U be a transversal section to the orbit O_P of P on H at the point P. Then the considerations in Chap. III (use that the covering $H^{(n)} \xrightarrow{\varphi} H$ is ramified along a divisor with strong normal crossings) yield that $\varphi^{-1}(U) = U^{(n)}$ is at Q transversal to the orbit O_Q of Q on $H^{(n)}$. We obtain by [15], Chapter I, that $H^{(n)}$ is locally at Q isomorphic to $U^{(n)} \times PGL(N)$. Consider the pullback family $\Gamma_U = \Gamma_H \times_H U \to U$. As this family is, by the results of [2], locally at P isomorphic to the Kuranishi family of the fibre Γ_P of $\Gamma_H \to H$ we obtain that $\Gamma^{(n)} \to H^{(n)}$ is locally at Q the pullback of $\Gamma_U \to U$ with respect to the morphism

$$(H^{(n)} \underset{\text{locally}}{\simeq}) U^{(n)} \times PGL(N) \xrightarrow{pr_1} U^{(n)} \xrightarrow{\varphi} U_p.$$

From this it is seen that the family $\Gamma^{(n)} \to H^{(n)}$ is locally at Q isomorphic to $\Gamma_{U^{(n)}} \times PGL(N)$ where $\Gamma_{U^{(n)}} = \Gamma_U \times_U U^{(n)}$. Hence if we choose the subscheme $U_i \subset H^{(n)}$ in such a way that $\varphi(U_i) \subset H$ is transversal to the orbit O_P for every point $P \in \varphi(U_i)$ the families $\Gamma_{U_i} \to U$ will have the desired properties.

Chapter II. Applications to moduli of algebraic varieties

a. Coarse moduli spaces for algebraic varieties with an ample canonical sheaf. Fine moduli spaces for algebraic varieties with a very ample canonical sheaf

Let $\mathfrak{M}$ be the set of isomorphy classes of smooth, projective, algebraic varieties V defined over $\boldsymbol{C}$ such that a positive multiple of the canonical sheaf ω_V is very ample.

By [9], Proposition 3.1, $\chi(V, \omega_V^{\otimes t})$ is a polynomial $h_V(t)$ in t called the *Hilbert polynomial* of the (canonically polarized) variety V.

We subdivide the set $\mathfrak{M}$ according to the Hilbert polynomials. Let $\mathfrak{M}_{h(t)} = \{V \in \mathfrak{M}; h_V(t) = h(t)\}$, then $\mathfrak{M} = \bigcup \mathfrak{M}_{h(t)}$.

For a noetherian $\boldsymbol{C}$-scheme S let $\mathscr{M}_{h(t)}(S)$ be the set of isomorphy classes of smooth, proper families V/S with the property that the geometric fibres of V/S are in $\mathfrak{M}_{h(t)}$.

The sets $\mathscr{M}_{h(t)}(S)$ form in a obvious way a contravariant functor which we sheafify with respect to the etale topology and extend to the category of noetherian algebraic $\boldsymbol{C}$-spaces. The functor thus obtained is also denoted by $\mathscr{M}_{h(t)}$ or $\mathscr{M}_h$ and is called the *functor of canonically polarized varieties* with $h(t)$ as Hilbert polynomial.[1] The sheafified functor $\mathscr{M}_h$ is dealt with in the following.

THEOREM (2.1): *There exists an algebraic space $M_{h(t)}$ of finite type over $\boldsymbol{C}$ which is a coarse moduli space for $\mathscr{M}_{h(t)}$ in the sense of* [14], 2.8.

PROOF: By [10] there exists an integer $c > 0$, depending only on $h(t)$, such that for every $V \in \mathfrak{M}_{h(t)}$ the sheaf $\omega_V^{\otimes m}$ is very ample and even sufficiently ample if $m \geqq c$.

We pick an integer $m \geqq c$ and we consider for a noetherian $\mathbb{C}$-scheme S isomorphy classes of inhomogeneous polarized families $(V/S, \mathfrak{X}/S)$ with $V/S \in \mathscr{M}_h(S)$ such that $\omega_{V/S}^{\otimes m} \in \mathfrak{X}/S$, where $\omega_{V/S}$ is the canonical sheaf of V/S. Families $(V/S, \mathfrak{X}/S)$ of this type are called *m-canonically polarized families* with $h(t)$ as Hilbert polynomial.[2]

Let ${}_m\mathscr{M}_h(S)$ be the functor of isomorphy classes of m-canonical polarized families $(V/S, \mathfrak{X}/S)$ such that $V/S \in \mathscr{M}_h$. The sheafification of ${}_m\mathscr{M}_h$ with respect to the etale topology and the extension of the resulting sheaf to the category of noetherian $\boldsymbol{C}$-spaces is considered in the following

[1] Notice, the families $V/S \in \mathscr{M}_h(S)$ carry a natural polarization determined by the canonical sheaf of V/S. There is therefore methodically no difference between this section and the next sections of this chapter.

[2] The Hilbert polynomial of the fibres of $(V/S, \mathfrak{X}/S)$ in the sense of Chapter I is $h(m \cdot t)$.

and denoted also by ${}_m\mathscr{M}_h$. By construction, for every family $V/S \in \mathscr{M}_h(S)$ the m-canonical sheaf $\omega_{V/S}^{\otimes m}$ of V/S is restricted to the fibres sufficiently ample. This implies, see [14], p. 27, that V/S is locally (with respect to the etale topology) a pullback of a family in ${}_m\mathscr{M}_h(S)$, yielding that the functors $\mathscr{M}_h$ and ${}_m\mathscr{M}_h$ are equal. Hence we can work with ${}_m\mathscr{M}_h$ in the following. But for ${}_m\mathscr{M}_h$ the considerations from page 5 apply. We consider m-canonical embeddings of the varieties of $\mathfrak{M}_{h(t)}$, i.e., embeddings of varieties $V \in \mathfrak{M}_h$ determined by a birational map which is associated to the sheaf $\omega_V^{\otimes m}$.

Let $\Gamma_H \to H$ be the universal family parametrizing these m-canonical embedded algebraic varieties (see page 6). Then the geometric quotient $\bar{H}$ of H by $PGL(N)$ exists as an algebraic space and is a coarse moduli space for $\mathscr{M}_h$. Put $\bar{H} = M_{h(t)}$. Q.E.D.

Let $\mathfrak{M}^*_{h(t)} \subseteq \mathfrak{M}_{h(t)}$ be the set of isomorphy classes of smooth, canonically polarized, algebraic varieties over $\boldsymbol{C}$, such that the canonical sheaf ω_V of $V \in \mathfrak{M}^*_{h(t)}$ is very ample. We assume that $\mathfrak{M}^*_{h(t)}$ is not empty. Let m be an integer $\geqq c$ where c is as above and $\Gamma_H \to H$ the universal family parametrizing the m-canonical embedded algebraic varieties of $\mathfrak{M}_{h(t)}$.

Let n be a *sufficiently large* integer, $H^{(n)} = P^{(n)}(\Gamma_H/H)$ the etale covering of H considered in chapter I and $\Gamma^{(n)} \to H^{(n)}$ the pullback family. Then $PGL(N)$ operates on $H^{(n)}$ properly and with finite stabilizers. Furthermore the stabilizer of a point $P \in H^{(n)}$ is the identity if the fibre Γ_P in $\Gamma^{(n)} \to H^{(n)}$ belongs to $\mathfrak{M}^*_h$. The last fact follows because the automorphisms of a variety $V \in \mathfrak{M}^*_h$ operate faithfully on the integral cohomology of V and n is taken sufficiently large. (See [14], p. 37.)

Claim: *The set of points $P \in H^{(n)}$ where the stabilizer is the identity form an open subscheme $H_0^{(n)}$ of $H^{(n)}$ which contains in particular all points $P \in H^{(n)}$ where the corresponding fibre in $\Gamma^{(n)} \to H^{(n)}$ is in $\mathfrak{M}^*_h$.*

Proof: Consider the graph map

$$PGL(N) \times H^{(n)} \xrightarrow{\Psi} H^{(n)} \times H^{(n)}$$

which is proper.

Let $\Delta_{H^{(n)}}$ be the diagonal of $H^{(n)} \times H^{(n)}$ and $\psi^{-1}(\Delta_{H^{(n)}}) = Y$ the fibre of Ψ over $\Delta_{H^{(n)}}$. Look at

$$\Psi : Y \to \Delta_{H^{(n)}} \cong H^{(n)},$$

and notice that if P, P' are points of $H^{(n)}$ such that P' is a specialization of P over C with $I_{P'} = \{Id\}$, then also $I_P = \{Id\}$ (I_P = stabilizer of P with respect to $PGL(N)$), or equivalently that if $\Psi^{-1}(P')$ consists of only one point then also $\Psi^{-1}(P)$.

Let $H_1, \ldots, H_r$ be all of those irreducible components of $\Delta_{H^{(n)}}$ such that the generic fibres of the maps $Y_i = \Psi^{-1}(H_i) \to H_i$ contain only one point. By $\Psi^{-1}(H_i) = Y_i$ the fibre of $\Psi : Y \to \Delta_{H^{(n)}}$ over H_i is denoted. Let $Y_i^{(1)}, \ldots, Y_i^{(s_i)}$ be the irreducible components of Y_i; then only one of them, say $Y_i^{(1)}$, is mapped surjectively to H_i by the map Ψ and the open subscheme

$$H_0 = \bigcup_{i=1}^{r} H_i - \bigcup_{\substack{v=2, \quad ,s_i \\ i=1, \quad ,r}} (Y_i^{(v)})$$

satisfies the claim.

Consider the pullback $\Gamma_0 \to H_0$ of $\Gamma_H \to H$ to H_0. Using $\Gamma_0 \to H_0$ and the etale covering $H_0^{(n)} = P^{(n)}(\Gamma_0/H_0)$ of H_0 we define as in chapter I an operation of $PGL(N)$ on $H_0^{(n)}$, consider the sheafification $\mathscr{M}_0^{(n)}$ of the quotient functor $H_0^{(n)}/PGL(N)$ with respect to the etale topology and its extension to noetherian algebraic C-spaces. By construction, the natural operation of $PGL(N)$ on $H_0^{(n)}$ is proper and fixpoint free as n is sufficiently large. We may apply Theorem (1.4) and obtain

THEOREM (2.2): *The functor $\mathscr{M}_0^{(n)}$ is represented by the quotient family $\bar{\Gamma}_0^{(n)} \to \bar{H}_0^{(n)}$ of $\Gamma_0^{(n)} \to H_0^{(n)}$ with respect to the action of $PGL(N)$.*

REMARK (2.3): By construction, the fibres of the families Γ/S of $\mathscr{M}_0^{(n)}(S)$ have the property that its automorphism groups operate faithfully on the integral cohomology. In particular varieties from $\mathfrak{M}_h^*$ appear in $\mathscr{M}_0^{(n)}$. It might be that $\mathscr{M}_0^{(n)}$ is equal to $\mathscr{M}_h^{(n)}$. To decide this is an interesting question which is open for dim $\geqq 2$.

REMARK (2.4): Over the complex numbers for smooth curves of genus $g \geqq 2$ with level n-structure the above considerations lead to universal families $\Gamma^{(n)} \to M^{(n)}$, provided $n \geqq 3$. Also in the case of polarized abelian varieties of dimension g with level n-structure the above method leads to universal families $\Gamma^{(n)} \to M^{(n)}$ of algebraic spaces. However some modifications are necessary for abelian varieties and one needs the important fact, to be found in Mumford [12], p. 120 ff., that on a polarized abelian variety $(V, \mathfrak{X})$ one can pick in $\mathfrak{X}$ an ample sheaf in a functorial way. But more is known for curves and abelian varieties. By the work of Mumford it is known that the families $\Gamma^{(n)} \to M^{(n)}$ exist as schemes and as such even over $\boldsymbol{Z}$.

b. Coarse moduli spaces for non ruled polarized algebraic varieties with irregularity 0. *Fine moduli spaces for* $K-3$ *surfaces*

The methods used for the construction of the coarse moduli spaces for algebraic varieties with ample canonical sheaf together with results from [9] and [11] yield to the existence of coarse moduli spaces for homogeneously polarized algebraic varieties with Hilbert polynomial h and irregularity 0. Certain ruled varieties have to be excluded. (Use [9], Theorem 6.)

For a polarized $K-3$ surface X we have shown in [14] that their automorphisms operate faithfully on $H^2(X, \boldsymbol{Z})$. The considerations of Chapter I imply therefore the existence of fine moduli spaces for homogeneously polarized $K-3$ surfaces with Hilbert polynomial h and with level n-structure, provided n is sufficiently large.

REMARK ON ENRIQUES SURFACES. It is not difficult to obtain by the above method and the results on Enriques surfaces in [17] an algebraic $\boldsymbol{C}$-space which is over the complex numbers a coarse moduli space for Enriques surfaces. However, the existence of a universal family for Enriques surfaces with level n-structure is an open and interesting problem.

Chapter III. Fine moduli spaces for stable curves with level *n*-structure

We shall work again entirely over the complex numbers $\boldsymbol{C}$. Consider 3-canonically embedded stable curves of genus g over $\boldsymbol{C}$. All these curves live in the projective space $\mathbb{P}^N$, where $N = 5g-6$, and have $h(x) = (6x-1)(g-1)$ as Hilbert polynomial.

Let $H^{h(x)}_{\mathbb{P}^N}$ be the Hilbert scheme which parametrizes the closed subschemes of $\mathbb{P}^N$ with $h(x)$ as Hilbert polynomial and let H be the locally closed subscheme of $H^{h(x)}_{\mathbb{P}^N}$ parametrizing the 3-canonical embedded stable curves (see [2] and [15]).

Denote by $\Gamma_H \to H$ the universal family of 3-canonical embedded stable curves of genus g. For a noetherian $\boldsymbol{C}$-scheme S let $\mathscr{h}(S)$ be the set of families $V/S \hookrightarrow \mathbb{P}^N \times S$ of 3-canonical stable curves which are pullbacks of $\Gamma_H \to H$. On the set $\mathscr{h}(S)$ the group $PGL(N)(S)$ operates in a functorial way. The sheafification $\mathscr{M}_g$ of the quotient functor $\mathscr{h}(S)/PGL(N)(S)$ (sheafify with respect to the etale topology) and its extension to the category of noetherian $\boldsymbol{C}$-spaces is called the *functor of stable curves* *of genus g*. (Note that $\mathcal{M}_g$ does not depend on the 3-canonical stable curves. We could have used instead m-canonical stable curves for any $m \geqq 3$.) As the operation of $PGL(N)$ on H is proper and with finite stabilizers, see [15], the geometric quotient $\bar{H}$ of H by $PGL(N)$ exists and is a coarse moduli space for $\mathcal{M}_g$.

Let H_0 be the maximal open subscheme of H such that the pullback family $\Gamma_H \times_H H_0 = \Gamma_{H_0} \to H_0$ is smooth. By the results in [2] the scheme H is smooth and irreducible and $D = H - H_0$ is a divisor on H with strong normal crossings as singularities.

For smooth families of curves of genus g the notion of a level n-structure was introduced in [14], p. 33, using an arbitrary basis of the first integral cohomology groups of the curves. The more convenient level n-structure for curves is obtained if one uses the construction of [14] and instead of arbitrary basis symplectic basis of the first integral cohomology groups. For smooth families $V \to S$ of curves of genus g with a scheme S of finite type over $\boldsymbol{C}$ as base, one constructs then as in [14] for every integer $n \geqq 1$ a finite etale covering, denoted again by

$$\rho : P^{(n)}(V/S) = S^{(n)} \to S$$

of S such that the points of the fibre $\rho^{-1}(P)$ over $P \in S$ are in $1-1$ correspondence to the symplectic basis of $H^1(V_P, \boldsymbol{Z}/n)$, where V_P is the fibre of $V \to S$ over P. A *level n-structure* of V/S is again a section of $S^{(n)}$ over S with respect to the morphism $S^{(n)} \to S$.

We deal in the following with level n-structures using a symplectic basis. The properties shown in [14] and Chapter I for the coverings $P^{(n)}(V/S)$ are also valid for the new coverings. In particular, the arguments in Chapter I yield, that the functor $\mathscr{h}_0^{(n)}(S)$ of smooth families of 3-canonical curves of genus g with level n-structure is represented by

$$H_0^{(n)} = P^{(n)}(\Gamma_{H_0}/H_0).$$

To obtain the notion of a level n-structure for proper and flat families of stable curves of genus g we first look at 3-canonical stable curves and in particular to the universal family $\Gamma_H \to H$ of 3-canonical stable curves. Let $H_0^{(n)}$ be the $\boldsymbol{C}$-scheme from above and let $\boldsymbol{C}(H_0^{(n)})$be the field of rational functions on $H_0^{(n)}$[1]. Denote by $H^{(n)}$ the normalization of H in $\boldsymbol{C}(H_0^{(n)})$ and $H^{(n)} \to H$ the natural map. Then $H^{(n)}$ is a reduced $\boldsymbol{C}$-scheme and $\rho : H^{(n)} \to H$ is a covering which is unramified outside the divisor $D = H - H_0$. Let V/S be a family of 3-canonical embedded stable curves

[1] $H_0^{(n)}$ is irreducible by Teichmüller theory [1], 7–07.

of genus g and $f: S \to H$ the morphism induced by $V \to S$. Let $S^{(n)} = S \times_H H^{(n)}$ be the pullback with respect to f and $S^{(n)} \to S$ the covering map induced by ρ. We define

DEFINITION (3.1): A level n-structure of a family of 3-canonical embedded stable curves $V \to S$ is a section of $S^{(n)}$ over S with respect to the map $\rho : S^{(n)} \to S$.

For a noetherian $\boldsymbol{C}$-scheme S we denote by $\mathscr{h}^{(n)}(S)$ the set of 3-canonical embedded stable curves with level n-structure. The collection of sets $\mathscr{h}^{(n)}(S)$ is a functor represented by $H^{(n)}$. As in [14] one shows that $PGL(N)$ operates on $H_0^{(n)}$ such that the map $H_0^{(n)} \to H_0$ is a $PGL(N)$ map. Using the fact that H is normal (it is even smooth) and [17], Lemma 6.1, we conclude that this operation of $PGL(N)$ on $H_0^{(n)}$ extends to an operation of $PGL(N)$ on $H^{(n)}$ such that the map $\rho : H^{(n)} \to H$ is also a $PGL(N)$ map. In particular we obtain that the group $PGL(N)$ operates on $\mathscr{h}^{(n)}$. We consider the sheafification of the quotient functor $\mathscr{h}^{(n)}(S)/PGL(N)(S)$ with respect to the etale topology of schemes, and extend this sheaf to the category of noetherian $\boldsymbol{C}$-spaces. The obtained functor is denoted by $\mathscr{M}_g^{(n)}$ and is called the *functor of stable curves of genus g with level n-structure*. To justify the name of the functor $\mathscr{M}_g^{(n)}$ we must show that $\mathscr{M}_g^{(n)}$ is independent of the 3-canonical stable curves used in its definition, i.e. that we obtain the same functor if we use m-canonical stable curves instead of 3-canonical stable curves, $m \geqq 3$.

Let $m \geqq 3$ and $\Gamma_m \to H_m$ be the universal family of m-canonical stable curves of genus g. Let $H_m^{(n)} \to H_m$ be the covering of H_m which is constructed according to the above procedure and which parametrizes the m-canonical embedded stable curves with level n-structure.

CLAIM: *$H_m^{(n)}$ is locally the pullback of $H_3^{(n)}$.*

PROOF: Let $P \in H_m$ and $\hat{O}_P$ the completion of the local ring O_{P, H_m}. Let $T = \operatorname{Spec}(\hat{O}_P)$ and identify the closed point of T with P. Using the formal moduli space $\mathscr{C} \to \mathscr{T}$ for the fibre Γ_P of the family $\Gamma_m \to H_m$ over P it is shown in [2], p. 82 ff., that $T = \mathscr{T} \times \operatorname{Spec}(\hat{O}_{e, PGL(N(m))})$ where $\hat{O}_{e, PGL(N(m))}$ is the local ring of the identity element e of the group $PGL(N(m))$.

The pullback $\Gamma_m^* \to T$ of $\Gamma_m \to H_m$ to T allows a 3-canonical embedding. Fixing one, we obtain a map

$$f: T \to H_3.$$

Let $f(P) = Q$ and $T' = \operatorname{Spec}(\hat{O}_{Q, H_3})$. Then

$$T' = \mathscr{T} \times \operatorname{Spec}(\hat{O}_{e, PGL(N(3))})$$

and there are unique morphisms

$$T \xrightarrow{\alpha} \mathscr{T}, \qquad T' \xrightarrow{\alpha'} \mathscr{T}$$

such that $\Gamma_m^* = \mathscr{C} \times_{\mathscr{T}} T$ and $\Gamma_3^* = \mathscr{C} \times_{\mathscr{T}} T'$ where the isomorphisms between the families are such that they induce the identity on the closed fibre. The map f factors through T' and one may choose f in such a way that the diagram

$$\begin{tikzcd}
\mathscr{T} \times \operatorname{Spec}(\hat{O}_{e, PGL(N(m))}) = T \arrow[d, "f"'] \arrow[dr, "\alpha"] & \\
\mathscr{T} \times \operatorname{Spec}(\hat{O}_{e, PGL(N(3))}) = T' \arrow[r, "\alpha'"'] & \mathscr{T}
\end{tikzcd}$$

is commutative and that f maps $\mathscr{T} \times \{e\}$ to $\mathscr{T} \times \{e\}$ and is on $\mathscr{T}$ the identity. Knowing this we look at the diagram

$$\begin{tikzcd}
 & T \underset{H_3}{\times} H_3^{(n)} \arrow[r] \arrow[d] & H_3^{(n)} \arrow[d] \\
 & & T' \arrow[d] \\
\mathscr{T} \times \operatorname{Spec}(\hat{O}_{e, PGL(N(m))}) = & T \arrow[ur, "f"] \arrow[r] & H_3 .
\end{tikzcd}$$

The map ρ restricted to the inverse image $\rho^{-1}(O_Q)$ of the orbit O_Q of Q by $PGL(N(3))$ is etale. Also the map $f : T \to T'$ maps

$$\{P\} \times \operatorname{Spec}(\hat{O}_{e, PGL(N(m))}) \quad \text{to} \quad \{Q\} \times \operatorname{Spec}(\hat{O}_{e, PGL(N(3))}).$$

This yields

$$T \times_{H_3} H_3^{(n)} = ((\mathscr{T} \times \{e\}) \times_{H_3} H_3^{(n)}) \times (\{P\} \times \operatorname{Spec}(\hat{O}_{e, PGL(N(m))}) \times_{H_3} H_3^{(n)})$$

and that $T \times_{H_3} H_3^{(n)}$ is normal since $(\mathscr{T} \times \{e\}) \times_{H_3} H_3^{(n)}$ and

$$(\{P\} \times \operatorname{Spec}(\hat{O}_{e, PGL(N(m))}) \times_{H_3} H_3^{(n)}$$

are normal. The first two facts follow if one uses that the ramification locus of the covering $\rho : H_3^{(n)} \to H_3$ has strong normal crossings as singularities and Abhyankar's Lemma [7], p. 279.

The third fact results from $H_3^{(n)} \to H_3$ being etale over the orbit of Q. This yields that $H_m^{(n)}$ is locally a pullback of $H_3^{(n)} \to H_3$. To make this more precise we recall that Γ_m/H_m is locally a pullback of $\Gamma_3 \to H_3$, i.e. for every point $P \in H_m$ there exists an affine open neighborhood U_P such that $\Gamma_m \times U_P \to U_P$ is pullback of $\Gamma_3 \to H_3$.

But then $H_m^{(n)} \times_{H_m} U_P \to H_3^{(n)} \times_{H_3} U_{P'}$ since $H_3^{(n)} \times_{H_3} U_P$ is normal by the above considerations and since the schemes $H_m^{(n)} \times_{H_m} U_P$ and $H_3^{(n)} \times_{H_3} U_P$ are clearly birationally isomorphic. (Use the smooth part $\tilde{\Gamma}_m \to \tilde{U}_P$ of $\Gamma_m \to U_P$ and the etale covering $P^{(n)}(\tilde{\Gamma}_m/\tilde{U}_P)$ of $\tilde{U}_P$.)

Our considerations yield that every 3-canonical family of stable curves with level n-structure is locally a pullback of m-canonical stable curves with level n-structure. As we can interchange m and 3 in the above argument, we obtain that $\mathscr{M}_g^{(n)}$ is independent of the 3-canonical stable curves used in its definition.

In the following we deal again with 3-canonical stable curves. Our aim is to show that $\mathscr{M}_g^{(n)}$ is representable, if $n \geqq 3$.

PROPOSITION (3.2): *Let $H^{(n)}$ be the scheme from above which parametrizes the 3-canonical stable curves of genus g with level n-structure. If $n \geqq 3$ the natural action of $PGL(N)$ on $H^{(n)}$ is fixpoint free.*

For the proof of the proposition a geometric interpretation of the stalks in the map $\rho : H^{(n)} \to H$ ist needed. For this purpose we give another description of the topological space consisting of the $\boldsymbol{C}$-valued points of $H^{(n)}$ with the complex topology. We define a set $\mathscr{H}^{(n)}(P)$ by

(1) If $P \in H_0$, i.e., Γ_P is smooth, $\mathscr{H}^{(n)}(P) =$ set of symplectic isomorphisms $\alpha : H^1(\Gamma_{P'} \boldsymbol{Z}/n) \to (\boldsymbol{Z}/n)^{2g}$,

(2) if $P \in H - H_0 = D$, i.e., Γ_P is singular, $\mathscr{H}^{(n)}(P) =$ set of symplectic homomorphism $\alpha : H^1(\Gamma_P, \boldsymbol{Z}/n) \to (\boldsymbol{Z}/n)^{2g}$ which are injective but not necessarily surjective.

Let $\mathscr{H}^{(n)} = \bigcup_{P \in H} \mathscr{H}^{(n)}(P)$ be the disjoint union and $f : \mathscr{H}^{(n)} \to H$ the natural map of sets.

LEMMA (3.3): *There exists a topology on $\mathscr{H}^{(n)}$ such that the map $f : \mathscr{H}^{(n)} \to H$ is an analytic covering of H in the sense of Grauert-Remmert* [3], *Def.* 3.

PROOF OF LEMMA (3.3): Let $P \in H - H_0$, i.e., Γ_P is singular. d shall be the number of nodes of Γ_P. Then the number of irreducible components of the divisor D at P is d. As D has normal crossings at P, we can pick a small neighborhood $U = \{(t_1, \ldots, t_N); |t_i| < \epsilon\}$ of P on H such that $U - D$ is isomorphic to $(E')^d \times E^{N-d}$, where $E = \{t \in \boldsymbol{C}; |t| < \varepsilon\}$ and $E' = \{t \in \boldsymbol{C}; 0 < |t| < \varepsilon\}$. The fundamental group $\pi_1((E')^d \times E^{N-d})$ of $(E')^d \times E^{N-d}$ is the free abelian group $\mu_1 \boldsymbol{Z} \oplus \mu_2 \boldsymbol{Z} \oplus \ldots \oplus \mu_d \boldsymbol{Z}$, where

μ_i is the homotopy class of a circle rounding the divisor $t_i = 0$ through P counterclockwise. (See [13].) We call a neighborhood of P on H of the above type a distinguished neighborhood of P. Clearly the distinguished neighborhoods generate the complex topology on H. Let $Q \in U - D$ and Γ_Q be the fibre of the family $\Gamma_H \to H$ over Q. Then $\pi_1(U-D)$ operates on $H^1(\Gamma_Q, \mathbf{Z})$ by the usual monodromie action (see [13]), and also on the set $\mathscr{H}^{(n)}(Q)$ defined above. We define $\tilde{\mathscr{H}}^{(n)}(P) = \mathscr{H}^{(n)}(Q)/\pi_1(U-D) =$ set of residue classes of $\mathscr{H}^{(n)}(Q)$ by the action of $\pi_1(U-D)$. (Clearly $\tilde{\mathscr{H}}^{(n)}(Q) = \mathscr{H}^{(n)}(Q)$ holds for $Q \in H_0$.) $\tilde{\mathscr{H}}^{(n)}(P)$ is as a set independent of the choice of the point Q as long as we restrict to distinguished neighborhoods of P.

Denote by $\alpha_{Q,P} : \mathscr{H}^{(n)}(Q) \to \tilde{\mathscr{H}}^{(n)}(P)$ the natural quotient map. It is then obvious by the definition of $\tilde{\mathscr{H}}^{(n)}(P)$ that for every $P' \in D \cap U$ there exists a map $\alpha_{P',P} : \tilde{\mathscr{H}}^{(n)}(P') \to \tilde{\mathscr{H}}^{(n)}(P)$.

Claim I: $\tilde{\mathscr{H}}^{(n)}(P) = \mathscr{H}^{(n)}(P), \forall P \in H.$

Proof: Consider the natural surjective homomorphismus

$$\rho : H_1(\Gamma_Q, \mathbf{Z}/n) \to H_1(\Gamma_P, \mathbf{Z}/n).$$

Let $\alpha_1, \ldots, \alpha_{g'}, \beta_1, \ldots, \beta_{g'}, \mu_1, \ldots, \mu_t, \delta_1, \ldots, \delta_t$ be a symplectic base of $H_1(\Gamma_Q, \mathbf{Z}/n)$ as in [13], such that $\varphi(\alpha_i)$, $\varphi(\beta_i)$, $\varphi(\gamma_j)$ form a symplectic basis of $H_1(\Gamma_P, \mathbf{Z}/n)$ and $\delta_1, \ldots, \delta_t$ a basis for the vanishing cycles.

If γ is a distinguished generator of $\pi_1(U-D)$ the monodromy operation $h = h_\gamma : H_1(\Gamma_Q, \mathbf{Z}/n) \to H_1(\Gamma_Q, \mathbf{Z}/n)$ is the identity or is characterized by the following properties (compare [4] and [13])

(1) $h(\alpha_i) = \alpha_i$, $h(\beta_i) = \beta_i$, $i = 1, \ldots, g'$.
(2) There exists j_0 such that $h(\gamma_{j_0}) = \gamma_{j_0} \pm \delta_{j_0}$, $h(\gamma_j) = \gamma_j$, $\forall j \neq j_0$.
(3) $h(\delta_i) = \delta_i$, $\forall i$.

Passing to the dual spaces $H^1(\Gamma_Q, \mathbf{Z}/n)$ and $H^1(\Gamma_P, \mathbf{Z}/n)$ we obtain an injection $H^1(\Gamma_P, \mathbf{Z}/n) \hookrightarrow H^1(\Gamma_Q, \mathbf{Z}/n)$. If α_i^*, β_i^*, γ_j^*, δ_j^* is the dual basis to α_i, β_i, γ_j, δ_j, the space $H^1(\Gamma_P, \mathbf{Z}/n)$ is spanned by α_i^*, β_i^*, γ_i^* and the monodromy operation h^* induced by $\gamma \in \pi_1(U-D)$ is the identity or is determined by

(1) $h^*(\alpha_i^*) = \alpha_i^*$, $h^*(\beta_i) = \beta_i^*$.
(2) $\forall_j : h^*(\gamma_j^*) = \gamma_j^*$.
(3) For $j_0 : h^*(\delta_{j_0}^*) = \delta_{j_0}^* \pm \gamma_{j_0}^*$, $\forall_j, j \neq j_0 : h^*(\delta_j) = \delta_j$.

These facts imply (using elementary results on symplectic geometry) that any two symplectic basis of $H^1(\Gamma_Q, \boldsymbol{Z}/n)$ which extend the same symplectic basis of $H^1(\Gamma_P, \boldsymbol{Z}/n)$ are conjugate by the monodromy operation of $\pi_1(U-D)$ and imply the claim.

Now we define for every point $P' \in \mathscr{H}^{(n)}$ a system of sets which will be a base for the neighborhood filter of P' of the desired topology on $\mathscr{H}^{(n)}$. For a distinguished neighborhood U of $P = f(P')$ we consider

$$W_{P',U} = \{Q' \in \mathscr{H}^{(n)}; f(Q') = Q \in U \text{ and } \alpha_{Q,P}(Q') = P'\}$$

and then take $W_{P',U}$ to be an open neighborhood of P'. As P' and U vary the sets $W_{P',U}$ define a topology on $\mathscr{H}^{(n)}$ such that $f : \mathscr{H}^{(n)} \to H$ becomes an analytic covering of H. (Note that f is obviously proper. Moreover, for a distinguished open set U of P the following holds: if W is a connected component of $f^{-1}(U)$ then $W - f^{-1}(D)$ is also connected. These two facts imply immediately that $\mathscr{H}^{(n)} \to H$ satisfies [3], Definition 3. Lemma (3.3) is proved.

By [3], § 3, $\mathscr{H}^{(n)}$ carries a unique algebraic structure such that $\mathscr{H}^{(n)}$ with this structure is a normal algebraic $\boldsymbol{C}$-scheme. Furthermore the map $f : \mathscr{H}^{(n)} \to H$ is a finite map in the sense of algebraic geometry, i.e., $\mathscr{H}^{(n)} \to H$ is a ramified normal covering of H, [7].

CLAIM II: *The covering $\mathscr{H}^n \to H$ is isomorphic to the covering $H^{(n)} \to H$.*

PROOF: Clearly, the covering $\mathscr{H}_0^{(n)} = f^{-1}(H_0^{(n)}) \to H_0$ is isomorphic to $H_0^{(n)} \to H_0$. This implies that $\mathscr{H}^{(n)}$ and $H^{(n)}$ have the same field K of rational functions. But then $\mathscr{H}^{(n)}$ and $H^{(n)}$ are both isomorphic to the normalisation of H in K. Q.E.D.

We identify in the following the $\boldsymbol{C}$-schemes $\mathscr{H}^{(n)}$ and $H^{(n)}$. For the proof of proposition (3.2) it remains to show that the action of $PGL(N)$ on $H^{(n)}$ is fixpoint free for $n \geqq 3$. Let $\sigma \in PGL(N)$ be a $\boldsymbol{C}$-valued point and $P' \in \mathscr{H}^{(n)} = H^{(n)}$ with $\sigma(P') = P'$. Let $P = f(P') \in H$ be the image point. Then σ induces an automorphism of the fibre Γ_P over P of the universal curve $\Gamma_H \to H$. This automorphism leaves the symplectic basis of $H^1(\Gamma_P, \boldsymbol{Z}/n)$ fixed which is determined by P'. The following Lemma (3.4) implies then that σ is the identity automorphism on Γ_P. But Γ_P is a 3-canonical curve in $\mathbb{P}^N$ and as such not contained in any hyperplane of $\mathbb{P}^N$, hence σ has to be the identity of $PGL(N)$.

LEMMA (3.4): *Let Γ/C be a stable curve of genus $g \geqq 2$ and σ an automorphism of Γ. Let σ^* be the automorphism of $H^1(\Gamma, \boldsymbol{Z}/n)$, $n \geqq 3$, induced by σ. If σ^* is the identity, also σ is the identity.*

PROOF: $H^1(\Gamma, \boldsymbol{Z}/n)$ is canonical isomorphic to the dual of the $\boldsymbol{Z}/n$-module of n-partition points $\mathrm{Pic}^0(\Gamma/\boldsymbol{C})^{(n)}$ of the connected component $\mathrm{Pic}^0(\Gamma/\boldsymbol{C})$ of $\mathrm{Pic}(\Gamma/\boldsymbol{C})$. It suffices therefore to show that the induced action of an automorphism $\sigma, \sigma \neq \mathrm{Id}$, on these n-partition points is not the identity. Now, [2], 1.13, yields that the automorphism σ^* of $\mathrm{Pic}^0(\Gamma/\boldsymbol{C})$ induced by σ is not the identity. As $\mathrm{Pic}^0(\Gamma/\boldsymbol{C})$ is a group scheme without unipotent radical, [2], 2.3, and, because σ^* is of finite order, we obtain by [5], IX, 4.7.1, the Lemma.

Having Proposition (3.2) available, the situation for stable curves with level n-structure is similar to the situation for smooth varieties with very ample canonical bundle and level n-structure which was considered in Chapter II. We obtain along the same lines: Let $\mathscr{M}_g^{(n)}$ be the functor of stable curves of genus g with level n-structure, $n \geqq 3$, and $\Gamma^{(n)} \to H^{(n)}$ be the universal family of 3-canonical curves with level n-structure. Denote by $\bar{\Gamma}^{(n)} \to \bar{H}^{(n)}$ the quotient family of $\Gamma^{(n)} \to H^{(n)}$ with respect to the action of $PGL(N)$. (Notice, $\bar{\Gamma}^{(n)}$ and $\bar{H}^{(n)}$ are by [15] proper algebraic $\mathbb{C}$-spaces.) Then the following Theorem holds.

THEOREM (3.5): *The functor $\mathscr{M}_g^{(n)}$ is represented by the family $\bar{\Gamma}^{(n)} \to \bar{H}^{(n)}$.*

PROOF: Follows from Theorem (1.4) together with Remark (1.5) and [15], p. 70.

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