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A. J. VAN DER POORTEN

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A NOTE ON THE ZEROS OF EXPONENTIAL POLYNOMIALS

A. J. van der Poorten

1. Introduction

In a recent paper *The Zeros of Exponential Polynomials* [1], C. J. Moreno gave precise information as to the location of the strips containing the zeros of exponential sums of the shape

$$\sum_{j=1}^{m} p_j e^{\alpha_j z} \qquad p_j \in \mathbb{C},\ \alpha_j \in \mathbb{R}. \tag{1}$$

In particular he showed that the real parts of the zeros of exponential sums of the shape (1) are dense in the intervals of the real line which lie entirely inside a strip of zeros. Moreno conjectured that the appropriate generalisation of this density result would hold for exponential sums with complex frequencies $\alpha_1, \ldots, \alpha_m$. He remarks that it seems difficult to obtain generalisations of his result for exponential polynomials

$$F(z) = \sum_{j=1}^{m} p_j(z) e^{\alpha_j z} \qquad p_j(z) \in \mathbb{C}[z],\ \alpha_j \in \mathbb{C}. \tag{2}$$

It is the purpose of this brief note to describe how the ideas employed by Moreno in [1] can be applied to obtain the generalisation of the results of [1] for exponential polynomials $F(z)$ of the shape (2). For brevity we avoid a detailed repetition of the results of [1] and also refer the reader to [1] for references to the relevant literature.

2. A simplification of the general case

It will be sufficient to suppose that we may write

$$p_j(z) = p_j z^{\mu_j}(1+\varepsilon_j(z)), \qquad p_j \neq 0, \qquad (j = 1, 2, \ldots, m) \tag{3}$$

where $\varepsilon_j(z) \to 0$ as $|z| \to \infty$.

Our remarks therefore apply to a somewhat wider class of functions (2) than just the class with polynomial coefficients.

We now recall the well-known result[1] that zeros of (2) with large absolute value lie in strips in the complex plane where at least two of the terms $p_k(z)e^{\alpha_k z}$, $p_l(z)e^{\alpha_l z}$, are of similar size, dominating the size of the remaining terms. Indeed such strips lie perpendicular to the convex hull determined in the complex plane by the points $\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_m$, the complex conjugates of the frequencies.

For convenience write for $j = 1, 2, \ldots, m$

$$\alpha_j = a_j + ib_j \qquad a_j, b_j \in \mathbb{R}$$

We make our remark precise as follows:

All but finitely many of the zeros of (2) *lie in logarithmic strips of the shape*

$$I_{\theta, c} = \left\{ z = x + t \exp i\left(\theta + \frac{c \log t}{t}\right) : x_0 < x < x_1, t_0 < t < \infty \right\},$$

where, θ, $0 \leqq \theta < 2\pi$ *and* c *are such that:*

(a) *there exist* k, l $(k \neq l)$ *in* $\{1, 2, \ldots, m\}$ *such that*

$$(a_k - a_l) \cos \theta - (b_k - b_l) \sin \theta = 0, \tag{4}$$

and

$$a_k \cos \theta - b_k \sin \theta \geqq a_j \cos \theta - b_j \sin \theta \qquad \text{for all } j = 1, 2, \ldots, m.$$

Write

$$s_\theta = \{j \in \{1, 2, \ldots, m\} : (a_k - a_j) \cos \theta - (b_k - b_j) \sin \theta = 0\}.$$

(b) $c((a_k - a_l) \sin \theta + (b_k - b_l) \cos \theta) - (\mu_k - \mu_l) = 0$, *and*

$$c(a_k \sin \theta + b_k \cos \theta) - \mu_k \leqq c(a_j \sin \theta + b_j \cos \theta) - \mu_l, \quad \text{for all } j \in S_\theta. \tag{5}$$

We write

$$S_{\theta, c} = \{l \in S_\theta : c((a_k - a_l) \sin \theta + (b_k - b_l) \cos \theta) - (\mu_k - \mu_l) = 0\}.$$

[1] See, say, the survey article R. E. Langer on the zeros of exponential sums and integrals; *Bull. Amer. Math. Soc.*, *37* (1931), 213–239. A more recent source is D. G. Dickson, Asymptotic distribution of zeros of exponential sums; *Publ. Math. Debrecen*, *11* (1964), 295–300.

With k as above, we now write

$$F_k(z) = z^{-\mu_k} e^{-\alpha_k z} F(z) = \sum_{l \in S_{\theta,c}} p_l z^{\mu_l - \mu_k} e^{(\alpha_l - \alpha_k)z}(1+\varepsilon_l(z)) + \delta_k(z),$$

and observe that by virtue of the inequalities (4) and (5) if

$$z = x + t \exp i\left(\theta + \frac{c \log t}{t}\right)$$

then $|\delta_k(z)| \to 0$ as $t \to \infty$. Similarly, for $l \in S_{\theta,c}$ and $z \in I_{\theta,c}$ as above, we see by virtue of the equations (4) and (5) and some simple manipulation that

$$|\exp((\alpha_l - \alpha_k)z + (\mu_l - \mu_k)\log z)| = |\exp(\alpha_l - \alpha_k)x|(1 + v_l(z))$$

where $|v_l(z)| \to 0$ as $t \to \infty$.Recalling that $|\varepsilon_l(z)| \to 0$ as $|z| \to \infty$ and hence as $t \to \infty$, we can summarise the situation as follows:

Write

$$F_k\left(x + t \exp i\left(\theta + \frac{c \log t}{t}\right)\right) = f_k(x;t).$$

Then for every $\delta > 0$ there is a $t_0 = t_0(\delta)$ such that for $t > t_0$, $x_0 < x < x_1$,

$$|f_k(x;t) - \sum_{l \in S_{\theta,c}} p'_l \exp(x(\alpha_l - \alpha_k) + it\beta_l)| < \delta, \tag{6}$$

where the p'_l are given by $p_l \exp i(\mu_l - \mu_k)\theta$ so that $|p'_l| = |p_l|$, and a simple calculation shows that the β_l are given by

$$\beta_l = (a_l - a_k)\sin\theta + (b_l - b_k)\cos\theta + \gamma_l(t) = \pm|\alpha_l - \alpha_k| + \gamma_l(t), \tag{7}$$

where $t\gamma_l(t) \to 0$ as $t \to \infty$.

3. Main result

It is now convenient to state the main result of this note.

THEOREM: *Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be distinct complex numbers and*

$$F(z) = \sum_{j=1}^{m} p_j(z)e^{\alpha_k z}, \qquad p_j(z) \in \mathbb{C}[z],\ \alpha_j \in \mathbb{C}$$

an exponential polynomial; suppose that the $p_j(z)$ are of exact degree μ_j respectively with leading coefficient p_j. Let θ, c and the index k be defined by the conditions (a) and (b) of section 2. Let the numbers $|\alpha_l - \alpha_k|$, $l \neq k$, $l \in S_{\theta,c}$ be irrationals linearly independent over $\mathbb{Q}$. Then a necessary and sufficient condition for $F(z)$ to have infinitely many zeros near any curve

$$C_{\theta,c,x} = \left\{ z = x + t \exp i\left(\theta + \frac{c \log t}{t}\right) : t_0 < t < \infty \right\}$$

is that

$$|p_h e^{x\alpha_h}| \leqq \sum_{\substack{l \in S_{\theta,c} \\ l \neq h}} |p_l e^{x\alpha_l}|, \qquad \text{all } h \in S_{\theta,c} \tag{8}$$

REMARKS:

(a) The linear independence condition on the α_j is stronger than is required for the truth of the theorem; but it avoids degenerate cases.

(b) By 'a zero near any curve $C_{\theta,c,x}$' we mean that if $x \in \mathbb{R}$ satisfies (8) then given any $\varepsilon > 0$ there exists a x' satisfying $x - \varepsilon < x' < x + \varepsilon$ and a $t' > t_0$ such that

$$F\left(x' + t' \exp i\left(\theta + \frac{c \log t'}{t'}\right)\right) = 0.$$

PROOF: We can apply our argument to the function $F_k(z) = z^{-\mu_k}e^{-\alpha_k z}F(z)$ which has the same zeros as $F(z)$ in the region under consideration. Our proof consists of indicating that the proof of [1], pages 73–74, can be applied *mutatis mutandis* to this case. For the sufficiency argument we notice that the lemma of [1], pages 73–74, requires only a reformulation to apply to the function $F_k(z)$ and to the curves $C_{\theta,c,x}$ within the logarithmic strips $I_{\theta,c}$ mentioned in section 2. Moreover in section 2 we 'simplified' $F_k(z)$ to show in (6) that $F_k(z)$ behaves on the curves $C_{\theta,c,x}$ like the exponential sum

$$\sum_{l \in S_{\theta,c}} p'_l \exp\left(x(\alpha_l - \alpha_k) + it\beta_l\right). \tag{9}$$

We may therefore apply the Kronecker-Weyl theorem so as to construct a sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ so that the sum (9) $\to 0$ as $n \to \infty$ when we replace t by t_n in (9); it follows that

$$F_k\left(x+t_n \exp i\left(\theta+\frac{c \log t_n}{t_n}\right)\right) \to 0$$

as $n \to \infty$ as required. The linear independence condition guarantees the required linear independence of the β_l as given by (7).

The necessity argument of [1], page 73, applies similarly since by (6) $F_k(z)$ can vanish infinitely many times near the curve $C_{\theta,c,x}$ only if the sum (9) is arbitrarily small for some $t > t_0$ for all t_0 sufficiently large. Finally, the sufficiency argument requires that $F_k(z)$ be bounded below on segments of the curves $C_{\theta,c,x}$ and again we can apply the argument of [1] page 76 to the sum (9). Alternatively the bound is available directly for $F(z)$ from the results of Tijdeman [2].

4. Remarks

The theorem of course includes the assertion of section 2 since, in all but degenerate cases, (8) will be satisfied by x in some intervals $x_0 < x < x_1$. We have also proved the conjecture of [1], page 71, to the effect that when the coefficients $p_j(z)$ are constants p_j then near every line parallel to the sides of a strip of zeros of $F(z)$ lie infinitely many zeros of $F(z)$. Indeed the statement of the theorem is the natural generalisation of this conjecture to the case of coefficients $p_j(z)$ satisfying (3).

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Mathematisch Instituut
Rijksuniversiteit Leiden
Wassenaarseweg 80
Leiden, Nederland.