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A CONSTRUCTION FOR ALGEBRAS SATISFYING THE MAXIMAL CONDITION FOR SUBALGEBRAS

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Introduction

A not necessarily associative algebra A is said to satisfy the maximal condition for subalgebras if it has no infinite strictly ascending chains of subalgebras or, equivalently, if every subalgebra of A is finitely generated. Finite-dimensional algebras of course have this property, and the general problem is whether a given class of algebras has infinite-dimensional members with the property.

The aim of this paper is to construct a general infinite-dimensional algebra which with certain restrictions on the multiplication constants of its basis elements satisfies the maximal condition for subalgebras. We thus reduce the above-mentioned problem to the (not necessarily easier) one of determining whether subject to these restrictions the multiplication constants can be chosen in such a way that our algebra will be a member of the class in question.

The method enables us to prove (the motivation for our paper) that over fields of characteristic zero there exist infinitely many infinite-dimensional Lie algebras satisfying the maximal condition for subalgebras and thus giving an affirmative answer to Problem 1 of Amayo and Stewart [1], when the field has characteristic zero. Another consequence is the seemingly hitherto unknown property that the polynomial algebra in one variable over any field satisfies the maximal condition for subalgebras. This is false for two variables. We also show that not only does the infinite cyclic group satisfy the maximal condition for subgroups but also its group algebra satisfies the maximal condition for subalgebras.

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Notation:

Unless otherwise specified $\boldsymbol{k}$ will denote an arbitrary field. We let Z denote the set of all integers. If A is an algebra and $S_1, S_2, \ldots$ subsets of A we let $\langle S_1, S_2, \ldots \rangle$ denote the subalgebra generated by these subsets. For n-fold products we introduce for convenience the non-standard notation: $[a, {}_0b] = a$, $[a, b] = ab$, $[a, {}_{n+1}b] = [[a, {}_nb], b]$ for all $n \geqq 0$; here a, b are elements of the algebra and ab denotes the given product.

1. Construction and preliminary results

Let $\lambda : Z \times Z \to \boldsymbol{k}$ be a map and let $A = A(\lambda)$ be the infinite-dimensional algebra over $\boldsymbol{k}$ with basis $\{a_i : i \in Z\}$ and bilinear product defined by

$$a_i a_j = \lambda(i, j) a_{i+j} \tag{1}$$

$i, j \in Z$. The result we wish to prove is

THEOREM A: *Let $A(\lambda)$ be the algebra defined above and suppose that the following condition holds*:

$$\textit{if } i \neq j \textit{ then } \lambda(i, j) \neq 0. \tag{2}$$

Then $A(\lambda) = \langle a_{-2}, a_{-1}, a_2 \rangle$ and $A(\lambda)$ satisfies the maximal condition for subalgebras.

The proof of theorem A is given in section 3. For now we deduce several consequences of (1) and (2), which we suppose to hold throughout.

Evidently every non-zero element x of A can be uniquely expressed in the form $x = \alpha_r a_r + \alpha_{r+1} a_{r+1} + \ldots + \alpha_s a_s$, where $r \leqq s$, $\alpha_r \alpha_s \neq 0$ and the α_i's are scalars from $\boldsymbol{k}$. In this case we say that x has lowest weight r and highest weight s. We write $l.wt(x) = r$ and $h.wt(x) = s$. We thus have a map

$$\mu : A(\lambda) \backslash 0 \to Z \times Z$$

defined by $\mu(x) = (\mu_1(x), \mu_2(x)) = (l.\,wt(x), h.\,wt(x))$. For a subset S of A with $S \backslash 0$ not empty we let $\mu(S)$ denote the image of $S \backslash 0$ under this mapping, and $\mu_1(S)$, $\mu_2(S)$ the sets of first and second components respectively.

We note that the elements with the same lowest and highest weights are precisely the non-zero scalar multiples of the a_i's.

T1. *Let a, b be non-zero elements of A.*
(i) *If* $\mu(a) \neq \mu(b)$ *then* $ab \neq 0$.
(ii) *If* $\mu_i(a) \neq \mu_i(b)$ *then* $\mu_i(ab) = \mu_i(a)+\mu_i(b)$ *for* $i = 1, 2$.

T1 is of course an immediate consequence of (1) and (2).

T2. *Let M be a subset of Z and let*

$$A(\lambda, M) = \langle a_i : i \in M \rangle.$$

Then $A(\lambda, M)$ is a finitely generated subalgebra of A.

This is trivial if M is a finite subset. So let M be infinite and let M_1, M_2 denote the subsets of positive and negative integers of $M \backslash \{0\}$ respectively. If M_1 has at least two elements let $n < m$ be its first two elements. Then any other i in M_1 is such that $i > m$ and so may be expressed in the form

$$i = m+k(i)n+r(i), \quad \text{where} \quad k(i) \geqq 0 \quad \text{and} \quad 0 \leqq r(i) \leqq n-1.$$

Let $J = \{s : 0 \leqq s \leqq n-1 \text{ and } s = r(i) \text{ for some } i \text{ in } M_1\}$ and for each s in the finite set J let

$$N_s = \{i : i \in M_1 \quad \text{and} \quad r(i) = s\}.$$

For each such s we denote by $j(s)$ the least member of N_s. We claim that

$$(*) \qquad A(\lambda, M_1) = \langle a_n, a_m, \{a_{j(s)} : s \in J\} \rangle.$$

For if M_1 contains an element i other than n or m then i is in some N_s. Thus $i \geqq j(s)$ and so $k(i) \geqq k(j(s))$. Using T1 repeatedly we see that the element $[a_{j(s), k(i)-k(j(s))} a_n]$ lies in the subalgebra on the right-hand side of (*) and has lowest and highest weight both equal to

$$j(s)+(k(i)-k(j(s)))n = m+k(j(s))n+s+(k(i)-k(j(s))n,$$

and this is i. Thus a_i is in the right hand side for each i and (*) follows. Similarly we can show that $A(\lambda, M_2)$ is also finitely generated (or this can be deduced from our next result since this shows that $A(\lambda, M_2) \cong A(\lambda', -M_2)$ for some λ' satisfying (2)). Thus $A(\lambda, M)$ is also finitely generated and T2 is proved.

For each $n \in Z$ let

$$Z_{\leqq n} = \{i : i \in Z \text{ and } i \leqq n\} \text{ and } Z_{\geqq n} = \{i : i \in Z \text{ and } i \geqq n\}.$$

T3. *Let $\lambda^* : Z \times Z \to \boldsymbol{k}$ be the map defined by*

$$\lambda^*(i, j) = \lambda(-i, -j). \tag{3}$$

Then λ^* satisfies (2) if and only if λ satisfies (2). The linear map $A(\lambda) \to A(\lambda^*)$ defined by $a_i \to a_{-i}$ is an algebra isomorphism of $A(\lambda)$ onto $A(\lambda^*)$. Under this isomorphism, $A(\lambda, M) \cong A(\lambda^*, -M)$ for any subset M of Z.

The statement about λ^* is trivial. The given map of $A(\lambda)$ onto $A(\lambda^*)$ is obviously a linear isomorphism. For convenience let b_i's denote the basis elements of $A(\lambda^*)$ with product as defined by (1). Then our linear map is $\theta : a_i \to b_{-i}$. Then using (3),

$$b_{-i}b_{-j} = \lambda^*(-i, -j)b_{-i-j} = \lambda(i, j)b_{-i-j} = (\lambda(i, j)a_{i+j})\theta = (a_i a_j)\theta.$$

Thus by bilinearity of the product we have that θ preserves products and so is an algebra isomorphism. The remaining statement is obvious.

2. Proof of the result for $A(\lambda, Z_{\geqq 0})$

For $n \geqq 0$ set $A(\lambda, n) = A(\lambda, Z_{\geqq n})$. Then $A(\lambda, n)$ is simply the vector space with basis $\{a_i : i \geqq n\}$ and product as defined by (1). Since $A(\lambda, n)$ has co-dimension $n+1$ in $A(\lambda, 0)$ it suffices to show that every subalgebra of $A(\lambda, n)$ is finitely generated. Indeed we need only assume that only the restriction of λ to $Z_{\geqq n} \times Z_{\geqq n}$ satisfies (2). With this assumption the statements T1–T3 hold, with the necessary modifications.

We now assume that $n \geqq 1$ and we let B be a subalgebra of $A(\lambda, n)$, where the restriction of λ to $Z_{\geqq n} \times Z_{\geqq n}$ satisfies (2) and product is defined by (1). We wish to show that B is finitely generated.

We may without loss of generality assume that $0 \neq B \neq A(\lambda, m)$ for any $m \geqq n$, since by T2 these subalgebras are finitely generated. Let

$$I = \mu_1(B) = \{i : \mu_1(b) = l.\,wt(b) = i \text{ for some } b \in B\backslash 0\},$$

$$M = \{i : \mu_1(b) = \mu_2(b) = i \text{ for some } b \text{ in } B\backslash 0\}$$

and

$$C = A(\lambda, M).$$

Then I is a set of positive integers, M is a subset of I and consists of the integers i for which a_i is in B, and C is a finitely generated subalgebra of B, by T2. Clearly if $I = M$ then $C = B$ and we are finished. So assume that $C \neq B$, whence $I \neq M$. This means that there is at least one i and an element b of B with $i = l.\,wt(b) \neq h.\,wt(b)$. For a fixed but arbitrary element i of $I \backslash M$ let

$$I(i) = \{j : \mu_1(b) = i \text{ and } \mu_2(b) = j \text{ for some } b \text{ in } B\backslash 0\},$$

and

$$f(i) = \inf(I(i))$$

be the least integer in $I(i)$. Then $i+1 \leqq f(i)$, since otherwise $f(i) = i$ and so i is in M, a contradiction.

S1. (i) *Let j be any element of $I\backslash M$. If b is an element of B with $\mu_2(b) = f(j)$ then $\mu_1(b) < f(j)$. In particular $f(j) \notin M$.*
(ii) *If i, j are elements of $I\backslash M$ then $f(i) = f(j)$ if and only if $i = j$.*

For part (i) suppose to the contrary that for some such j a b exists with $\mu_1(b) = \mu_2(b) = f(j)$. Then $f(j)$ is in M and $a_{f(j)}$ is in B. Now by definition B contains an element x with lowest weight j and heighest weight $f(j)$. If α is the coefficient of $a_{f(j)}$ in x then B contains $y = x - \alpha a_{f(j)}$. We have $y \neq 0$ since $j \neq f(j)$. Further $k = h.\,wt(y) < f(j)$ and k is in $I(j)$,since $l.\,wt(y) = j$. But this contradicts the definition of $f(j)$ as the least member of $I(j)$. Thus no such b exists.

For part (ii) the implication in one direction is trivial. Now suppose that $f(i) = f(j)$ but $i \neq j$. Without loss of generality we may suppose that $i < j$. Now B contains elements

$$x = \alpha_i a_i + \alpha_{i+1} a_{i+1} + \ldots + \alpha_{f(i)} a_{f(i)},$$

$$y = \beta_j a_j + \beta_{j+1} a_{j+1} + \ldots + \beta_{f(j)} a_{f(j)},$$

with $\alpha_i \alpha_{f(i)} \beta_j \beta_{f(j)} \neq 0$. Thus, as $i < j$ and $f(i) = f(j)$, B contains the element $w = x - \beta_{f(j)}^{-1} \alpha_{f(i)} y$, which has lowest weight i and highest weight k for some $k < f(i)$. But then k is in $I(i)$ and this contradicts the minimality of $f(i)$. Thus if $f(i) = f(j)$ then $i = j$. This completes the proof of S1.

For each i in $I \backslash M$ it is clear that B contains elements of the form $a_i + \alpha_{i+1} a_{i+1} + \ldots + \alpha_{f(i)} a_{f(i)}$, with $\alpha_{f(i)} \neq 0$. So we may define

$$D(i) = \{x : x \in B \text{ and } x = a_i + \alpha_{i+1} a_{i+1} + \ldots + \alpha_{f(i)} a_{f(i)} \text{ and } \alpha_{f(i)} \neq 0\}.$$

Then $D(i)$ is a non-empty subset of B for each i in $I \backslash M$.

S2. *For each i in $I \backslash M$ let d_i be a fixed but arbitrary element of $D(i)$. Let $E = E(C, \{d_i : i \in I \backslash M\}) = C + \sum_{i \in I \backslash M} \mathbf{k} d_i$ be the vector subspace of B spanned by the elements of C and the chosen d_i's (just one d_i for each distinct i in $I \backslash M$). Then $E = B$.*

To prove S2 we take x in $B \backslash 0$ and use induction on $l(x) = \mu_2(x) - \mu_1(x)$ to show that x is in E. If $l(x) = 0$ then $x \in C \subseteq E$. Let $l(x) > 0$ and suppose inductively that $y \in E$ whenever $y \in B \backslash 0$ and $l(y) < l(x)$. Let

$$x = \beta_r a_r + \beta_{r+1} a_{r+1} + \ldots + \beta_s a_s,$$

where $\beta_r \beta_s \neq 0$. If $r \in M$ then $a_r \in C$ and so $y = x - \beta_r a_r \in B \backslash 0$ and $l(y) = l(x) - 1$, whence $y \in E$ and $x = y + \beta_r a_r \in E$. If $r \notin M$ then $r \in I \backslash M$ and $s \in I(s)$ and so $s \geqq f(r)$. Then $y = x - \beta_r d_r = 0$ or $y \in B \backslash 0$ and has highest weight not exceeding s and lowest weight greater than r and so $l(y) < l(x)$. In either case $y \in E$ and so $x = y + \beta_r d_r \in E$. This completes our induction and proves that $E = B$.

It is clear from S2 that our contention that B is finitely generated will be proved if we can find a finitely generated subalgebra F of B with the properties:

$$C \subseteq F \quad \text{and} \quad F \cap D(i) \neq \emptyset \qquad \text{for each } i \in I \backslash M. \tag{4}$$

Then such an F will equal B. The rest of the proof is aimed at obtaining such an F, and for this we need to define a few more sets. It follows from S2 that if $I \backslash M$ is a finite set then we may take F as one of the E's above. So we assume that $I \backslash M$ is an infinite set. Further let us fix one choice $\{d_i : i \in I \backslash M\}$ of the d_i's and let $E = B$ be the corresponding E as defined in S2.

Let $n < m$ be the first two integers in $I \backslash M$ and let $J = I \backslash M \cup \{n, m\})$. Then any i in J has the form

$$i = m + k(i)n + r(i), \qquad k(i) \geqq 0, \qquad 0 \leqq r(i) \leqq n - 1. \tag{5}$$

So we may define the finite set

$$R = \{s : 0 \leqq s \leqq n-1 \text{ and } s = r(i) \text{ for some } i \text{ in } J\}.$$

For each s in R let

$$R(s) = \{i : i \in J \text{ and } r(i) = s\}$$

and

$$j(s) = \inf(R(s))$$

be the least integer in $R(s)$. It follows from S1 that the set

$$R^*(s) = \{i : i \in R(s),\ f(i) \leqq f(j(s))\}$$

is a finite set. If $i \in R(s) \backslash R^*(s)$ then $f(i) > f(j(s))$ and so has the form

$$f(i) = f(j(s)) + p(i) f(n) + q(i), \tag{6}$$

where $p(i), q(i)$ are integers with $p(i) \geqq 0$, $0 \leqq q(i) \leqq f(n)-1$. Let $Q(s) = \{t : 0 \leqq t \leqq f(n)-1 \text{ and } t = q(i) \text{ for some } i \in R(s) \backslash R^*(s)\}$, and for each t in the finite set $Q(s)$ let

$$R(s, t) = \{f(i) : i \in R(s) \backslash R^*(s) \text{ and } q(i) = t\}$$

and

$$j(s, t) = f(i(s, t)) = \inf(R(s, t)).$$

Then $j(s, t)$ is the least integer in $R(s, t)$ and $i(s, t)$ is that element of $R(s) \backslash R^*(s)$ for which $j(s, t) = f(i(s, t))$. Define

$$I^* = \{n, m\} \cup \Big(\bigcup_{s \in R} R^*(s)\Big) \cup \{i(s, t) : s \in R,\ t \in Q(s)\}.$$

Then I^* is a finite set and so

$$G = \langle \{d_i : i \in I^*\} \rangle$$

is a finitely generated subalgebra of B. Finally let F be the vector space sum of the two finitely generated subalgebras C and G:

$$F = C + G.$$

We will show that the vector space F satisfies the conditions exhibited in (4) and so must equal B, whence B is finitely generated. Before doing this we need

S3. *Let H be a subspace of B such that $C \subseteq H$. Let $i \in J$ and suppose that $H \cap D(j) \neq \emptyset$ for all $j \in I \backslash M$ with $j < i$. If H contains an element of highest weight $f(i)$ then $H \cap D(i) \neq \emptyset$.*

Among the elements x of H with highest heighest weight $f(i)$ pick x_0 of minimal length $l(x_0) = \mu_2(x) - \mu_1(x)$. Then of course $\mu_2(x_0) = f(i)$ and $\mu_1(x) = j$ for some j in I. If $j = i$ then multiplication of x_0 by a suitable scalar yields an element of $H \cap D(i)$ and we are done. Let α denote the coefficient of a_j in x_0. Suppose if possible that $j \neq i$. Now $j \neq f(i)$ since this would contradict S1. Thus $l(x_0) > 0$. If $j \in M$ then $a_j \in C \subseteq H$ and so H contains $y = x - \alpha a_j$, an element of highest weight $f(i)$ and length at most $l(x_0) - 1$. This contradicts the choice of x_0. Thus $j \in I \backslash M$, and hence $f(i) \in I(j)$ and $f(j) \leqq f(i)$. Since $j \neq i$ we have $f(j) < f(i)$, by S1. We contend that $j < i$. For otherwise $i < j < f(j) < f(i)$, and then subtraction of a suitable scalar multiple of x_0 from d_i would yield an element w of B with $\mu_1(w) = i$, $\mu_2(w) = r < f(i)$. But then $r \in I(i)$ and this would contradict the definition of $f(i)$ as the least integer in $I(i)$. Thus $j < i$ and so $H \cap D(j) \neq \emptyset$, by hypothesis. Let

$$e_j = a_j + \beta_{j+1} a_{j+1} + \dots + \beta_{f(j)} a_{f(j)} \in H \cap D(j).$$

Since $f(j) < f(i)$, the element $x_1 = x_0 - \alpha e_j$ has highest weight $f(i)$. It has lowest weight greater than j and so $l(x_1) = f(i) - \mu_1(x_1) < f(i) - j = l(x_0)$. Since also $x_1 \in H$ we get a contradiction to the choice of x_0. Therefore $j = \mu_1(x_0) = i$ and S3 is proved.

We are now ready to prove that

$$(*) \qquad \text{if } i \in I \backslash M \text{ then } F \cap D(i) \neq \emptyset.$$

For this we use induction on i. If $i = n$ or m then by definition G and so F contains d_i and so (*) holds in these cases. Suppose that $i > m$ and suppose inductively that (*) holds for all j in $I \backslash M$ with $j < i$. Then $i \in J$ and hence by (5) we have that $i \in R(s)$ for some s in R. If $i \in R^*(s)$ then by definition we have $d_i \in G \subseteq F$ and so (*) holds for i. Suppose that $i \in R(s) \backslash R^*(s)$. Then $f(i) > f(j(s))$ and so by (6) we have $f(i) \in R(s, t)$ for some t in $Q(s)$. Thus $f(i) \geqq j(s, t) = f(i(s, t))$ and so $p(i) \geqq p(i(s, t))$, since

$q(i) = q(i(s, t)) = t$. Let $r = p(i) - p(i(s, t))$. Then G and so F contains the element

$$x = [d_{i(s,t)}, {}_r d_n].$$

Now $j(s, t) \neq f(n)$, since $i(s, t) > m > n$, by S1. Therefore

$$j(s, t) + kf(n) \neq f(n) \qquad \text{for any } k \geqq 0.$$

Thus applying T1 repeatedly we see that x has highest weight $\mu_2(x) = j(s, t) + rf(n)$. Using (6), the definition of r, and the fact that i and $i(s, t)$ are in $R(s, t)$ we have

$$\begin{aligned} \mu_2(x) &= f(i(s, t)) + (p(i) - p(i(s, t)))f(n) \\ &= f(j(s)) + p(i)f(n) + t = f(i). \end{aligned}$$

Thus F has an element of highest weight $f(i)$ and so satisfies the hypothesis of S3, whence $F \cap D(i) \neq \emptyset$ and (*) holds for i. This completes our induction. Thus (*) holds and so $F = B$ and B is finitely generated.

It is clear from the proof above that in place of F we could just as well have taken $F^* = C + G^*$, where G^* is the subspace of G defined by

$$G^* = \sum_{n=0}^{\infty} \Big(\sum_{i \in I^*} \boldsymbol{k}([d_i, {}_r d_n]\Big).$$

A look at the proof of T2 shows that a similar expression holds for $A(\lambda, M)$ in general and in particular for C.

We can now state the result we have proved:

THEOREM B: *Let $\lambda : Z_{\geqq 0} \times Z_{\geqq 0} \to \boldsymbol{k}$ be a map and let $A(\lambda, Z_{\geqq 0})$ be the algebra defined over $\boldsymbol{k}$ with basis $\{a_i : i \in Z_{\geqq 0}\}$ and product defined by $a_i a_j = \lambda(i, j) a_{i+j}$ for all $i, j \geqq 0$. If there exists an i_0 such that $\lambda(i, j) \neq 0$ whenever $i \neq j$ and $i, j \geqq i_0$ then $A(\lambda, Z_{\geqq 0})$ satisfies the maximal condition for subalgebras. In this case every subalgebra B of $A(\lambda, Z_{\geqq 0})$ contains a finite number $b_1, b_2, \ldots, b_k$ of elements such that $B = B_0 + B^*$, where B_0 is a subspace of dimension not exceeding i_0 and B^* is the subspace*

$$B^* = \sum_{r=0}^{\infty} \Big(\sum_{i=1}^{k} (\boldsymbol{k}([b_i, {}_r b_1]) + \boldsymbol{k}([b_i, {}_r b_2])\Big).$$

Furthermore B^* is an ideal of B and is the set of elements of B of the form $\alpha_j a_j + \alpha_{j+1} a_{j+1} + \ldots$, where $j \geqq i_0$.

If we wish to be more explicit then b_1 and certain of the b_j's give us our C and certain of the b_j's and b_2 give G^*.

3. Proof of theorem A

The proof here follows the same lines as that in section 2, except for a couple of steps. Let $A_1 = A(\lambda, Z_{\leqq 0})$ and $A_2 = A(\lambda, Z_{\geqq 0})$. Then by theorem B and T3 we know that A_1 and A_2 satisfy the maximal condition for subalgebras. Let B be a subalgebra of $A(\lambda)$ and let $B_1 = B \cap A_1$, $B_2 = B \cap A_2$ and $C = \langle B_1, B_2 \rangle$. Then C is a finitely generated subalgebra of B. Suppose that $C \neq B$. Then B contains elements with lowest weight negative and highest weight positive. If $C = 0$ set

$$\mu(C) = \mu_1(C) = \mu_2(C) = \emptyset.$$

Let

$$I = \{(i,j) : i, j > 0 \text{ and } (-i, j) \in \mu(B),\ -i \notin \mu_1(B_1), j \notin \mu_2(B_2)\}.$$
$$I^1 = \{i : i > 0 \text{ and } (i,j) \in I \text{ for some } j > 0\}$$

and

$$I^2 = \{j : j > 0 \text{ and } (i,j) \in I \text{ for some } i > 0\}.$$

We let $n < m$ be the first two elements of I^1. Then

(7) $\quad i > m, i \in I^1 \Rightarrow i = m + k(i)n + r(i),\ k(i) \geqq 0, 0 \leqq r(i) \leqq n-1.$

Let

$$R^1 = \{s : 0 \leqq s \leqq n-1 \text{ and } s = r(i) \text{ for some } i \in I^1\}.$$

Then this is a finite set. For each i in I^1 let

$$I^2(i) = \{j : j \in I^2 \text{ and } (i,j) \in I\}$$

and

$$f(i) = \inf(I^2(i)).$$

R1. (i) *Let $j \in I^1$. If $b \in B$ and $\mu_2(b) = f(j)$ then*

$$\mu_1(b) \leqq -j < 0 < f(j).$$

Furthermore $a_{f(j)} \notin B$.

(ii) *If* $i, j \in I^1$ *then* $f(i) = f(j)$ *if and only if* $i = j$.

That $f(j)$ is positive follows from the definition of I and $I^2(j)$. We know that B contains an element x of lowest weight $-j$ and highest weight $f(j)$. If a b exists with $\mu_1(b) > -j$ and $\mu_2(b) = f(j)$ then subtraction of a suitable scalar multiple of b from x yields an element x_1 with $\mu_1(x_1) = -j$ and $\mu_2(x_1) < f(j)$. If $\mu_2(x_1) \in \mu_2(B_2)$ then the subtraction of a suitable multiple of an element of B_2 yields x_2 with $\mu_1(x_2) = -j$ and

$$0 < \mu_2(x_2) < \mu_2(x_1) < f(j); \quad (\text{if } \mu_2(x_1) \text{ or } \mu_2(x_2)$$

were non-positive we would have $-j \in \mu_1(B_1)$ and thus contradict the definition of I and I^1). Continuing in this way we arrive after a finite number of steps at an element x_r with $\mu_1(x_r) = -j$ and $\mu_2(x_r) = k < f(j)$ and $k \notin \mu_2(B_2)$. But this implies that

$$(j, k) \in I, k \in I^2(j) \quad \text{and} \quad k < f(j) = \inf(I^2(j)),$$

a contradiction. Thus $\mu_1(b) \leqq -j < 0$. In particular $a_{f(j)}$ is not in B, since $\mu_1(a_{f(j)}) = f(j) > 0$. This proves part (i) of R1. Part (ii) follows in the same way as for S1 or follows from part (i) above since $f(i) = f(j)$ would then imply $-i \leqq -j \leqq -i$.

If the set I is not empty then $I^2(i)$ is not empty for each i in I^1. Thus

$$D(i) = \{x : x \in B \text{ and } x = a_{-i} + \alpha_{-i+1} a_{-i+1} + \dots + \alpha_{f(i)} a_{f(i)}, \quad \text{with all } \alpha_j \text{ in } \boldsymbol{k} \text{ and } \alpha_{f(i)} \neq 0\}$$

is a non-empty subset of B for each i in I^1.

R2. *For each* i *in* I^1 *let* d_i *be a fixed but arbitrary element of* $D(i)$. *Let*

$$C^* = B_1 + B_2$$

be the subspace of C *spanned by the elements of* B_1 *and* B_2 *and let*

$$E = E(C^*, \{d_i : i \in I^1\}) = C^* + \sum_{i \in I^1} \boldsymbol{k} d_i$$

be the subspace of B *spanned by the elements of* C^* *and the* d_i's. *Then* $E = B$.

The proof is the same as before for S2. We induct on the length of $x \in B\backslash 0$ to show that $x \in E$. If $l(x) = 0$ then $x \in B_1$ or B_2 and we are done. Let $l(x) > 0$ and assume that E contains all y in B with $l(y) < l(x)$. If x is in B_1 or B_2 we are done. If not then $\mu_1(x) = -i$ and $\mu_2(x) = j$ with $i, j > 0$. If $-i \in \mu_1(B_1)$ or $j \in \mu_2(B_2)$ then substaction of a suitable element of B_1 or B_2 from x yields an element $y \neq 0$ with y in E and $l(y) < l(x)$.Thus y and so x lies in E. If $-i \notin \mu_1(B_1)$ and $j \notin \mu_2(B_2)$ then $(i, j) \in I$, $i \in I^1$, $j \in I^2(i)$ and so $j \geqq f(i)$. Subtraction of a suitable scalar multiple of d_i from x then gives an element w which is either 0 or else has $\mu_1(w) > \mu_1(x)$ and $\mu_2(w) \leqq \mu_2(w)$. Thus $w = 0$ or $l(w) < l(x)$ and $w \in B$, so $w \in E$ and hence $x \in E$. Therefore E contains all elements of B and so equals B.

So to show that B is finitely generated we need only find a finitely generated subalgebra F of B and a subspace F^* of F with

$$C^* \subseteq F^* \quad \text{and} \quad F^* \cap D(i) \neq \emptyset \qquad \text{for all } i \text{ in } I^1. \tag{8}$$

We now set out to find such an F. Here the proof diverges a bit from that in section 2.

If I^1 or I^2 is a finite set then $\{f(i) : i \in I^1\}$ is a finite set and so in view of R1 we have that I^1 is a finite set and so we may take $F^* = E$ as defined in R2 and $F = C + F^* = \langle C, \{d_i : i \in I^1\}\rangle$. For then $F^* = B$ and so $F = B$. So let us assume that both I^1 and I^2 are infinite sets and let us fix one choice of the d_i's and E.

R3. (i) *Let $j \in I^1$. If $b \in B\backslash 0$ and $\mu_1(b) = -j$ then $\mu_2(b) \geqq f(j)$.*
(ii) *Let H be a subspace of B with $C^* \subseteq H$. Suppose that $i \in I^1$ and $H \cap D(j) \neq \emptyset$ for all j in I^1 with $f(j) < f(i)$. If H contains an element of highest weight $f(i)$ then $H \cap D(i) \neq \emptyset$.*

For part (i): Suppose to the contrary that there is a b with $\mu_1(b) = -j$ and $\mu_2(b) < f(j)$. Among the elements b of B with this property choose one b_0 with $k = \mu_2(b_0) < f(j)$ as small as possible. If $k \in \mu_2(B_2)$ then subtraction of a suitable element of B_2 from b_0 would yield an element b_1 with $\mu_1(b_1) = -j$ and $\mu_2(b_1) < k$, thus contradicting the choice of b_0. If on the other hand k is not positive then b_0 is in B_1, so $-j \in \mu_1(B_1)$, a contradiction to the definition of I. So we have k positive, $(j, k) \in I$, $k \in I^2(j)$, $k < f(j)$, and this contradicts the definition of $f(j)$. So we must have $\mu_2(b) \geqq f(j)$, and (i) is proved.

For part (ii): Let x be an element of H with $\mu_2(x) = f(i)$. By R1 we know that $\mu_1(x) \leqq -i$. Choose such an x with $\mu_1(x)$ as large as possible. Then $\mu_1(x) \notin \mu_1(B_1)$ for then the subtraction of a suitable element of B_1 and so H would yield a y in H with $\mu_1(y) > \mu_1(x)$ and $\mu_2(y) = f(i)$, a contradiction to the choice of x. Thus if $\mu_1(x) = -k$ then $(k, f(i)) \in I$, $f(i) \in I^2(k)$ and so $f(k) \leqq f(i)$. If $k \neq i$ then by R1 we have $f(k) < f(i)$ and so by hypothesis there exists an element

$$e_k = a_{-k} + \beta_{-k+1} a_{-k+1} + \dots + \beta_{f(k)} a_{f(k)}$$

in $H \cap D(k)$. If we subtract a suitable multiple of e_k from x we obtain w in H with $\mu_2(w) = f(i)$ and $\mu_1(w) > -k = \mu_1(x)$, thus contradicting the choice of x. Thus $k = i$ and a suitable multiple of x is in $H \cap D(i)$.

The stage is now set for the proof of theorem A. We define as before (recall that $n < m$ are the first two integers in I^1)

$$J^1 = I^1 \backslash \{n, m\}.$$

Using (7) we define for each s in R^1,

$$R^1(s) = \{i : i \in J^1 \text{ and } r(i) = s\}$$

and set

$$j(s) = \inf(R^1(s)).$$

Then

$$R^{1*}(s) = \{i : i \in R^1(s) \text{ and } f(i) \leqq f(j(s))\}$$

is, by R1, a finite set. For each i in $R^1(s) \backslash R^{1*}(s)$ we have

$$f(i) = f(j(s)) + p(i) f(n) + q(i), \tag{9}$$

where $p(i) \geqq 0$ and $0 \leqq q(i) \leqq f(n) - 1$. So we define

$$Q(s) = \{t : 0 \leqq t \leqq f(n) - 1 \text{ and } t = q(i) \text{ for some } i \in R^1(s) \backslash R^{1*}(s)\}$$

and set

$$R^1(s, t) = \{f(i) : i \in R^1(s) \backslash R^{1*}(s) \text{ and } q(i) = t\}.$$

We let

$$j(s, t) = f(i(s, t)) = \inf(R^1(s, t)).$$

Define

$$I^{1*} = \{n, m\} \cup (\bigcup_{s \in R^1} R^{1*}(s)) \cup \{i(s,t) : s \in R^1, t \in Q(s)\}.$$

Then I^{1*} is a finite subset of I^1 and hence

$$G = \langle \{d_i : i \in I^{1*}\} \rangle$$

is a finitely generated subalgebra of B. Let

$$G^* = \sum_{r=0}^{\infty} (\sum_{i \in I^{1*}} (k([d_i, {}_r d_n])),$$

$$F^* = C^* + G^*, \qquad F = C + G.$$

We claim that F^* satisfies (8). Suppose to the contrary that for some i in I^1 we have $F^* \cap D(i) = \emptyset$. Then as d_n, d_m are in G^* we must have $i \in J^1$. Now choose i such that $f(i)$ is minimal with respect to $F^* \cap D(i) = \emptyset$. We have $i \in R^1(s)$ for some s in R^1 and so $f(i) > f(j(s))$, since $d_k \in G^*$ if $f(k) \leqq f(j(s))$. Thus $f(i) \in R^1(s, t)$ for some t in $Q(s)$. We also have by the choice of i that $F^* \cap D(j) \neq \emptyset$ if $f(j) < f(i)$. Now $f(i) > j(s,t) = f(i(s,t))$. Thus $p(i) > p(i(s,t))$ and so $r = p(i) - p(i(s,t)) > 0$. As $i(s,t) \neq n$ we have by R1, $j(s,t) \neq f(n)$. Thus using T1 repeatedly we see that G^* contains the element $w = [d_{i(s,t)}, {}_r d_n]$ and this has highest weight $j(s,t) + rf(n) = f(i)$, on using (9). But then by T3 we have $F^* \cap D(i) \neq \emptyset$, a contradiction. Thus F^* satisfies (8) and so

$$B = F^* = F \subseteq \langle C, G \rangle \subseteq B$$

is a finitely generated subalgebra. This proves theorem A.

Using theorem B and the definition of G^* and F^* above we may state a more descriptive form of theorem A as:

THEOREM A*: *Let $A(\lambda)$ be the infinite-dimensional algebra defined by* (1) *and suppose that $\lambda(i,j) \neq 0$ whenever $i \neq j$. Then $A(\lambda) = \langle a_{-2}, a_{-1}, a_2 \rangle$ satisfies the maximal condition for subalgebras. In this case every subalgebra B of A contains elements $b_1, b_2, b_3, b_4, b_5, \ldots, b_n$ such that*

$$B = \sum_{r=0}^{\infty} (\sum_{i=1}^{n} \sum_{j=1}^{5} k([b_i, {}_r b_j])).$$

If there is an integer m with $\lambda(m,m) = 0$ then $A(\lambda)$ has no non-trivial one-sided ideals.

PROOF: That a_{-2}, a_{-1}, a_2 or a_{-2}, a_1, a_2 generate A is trivial. Evidently if a one-sided ideal B contains an a_n then it contains a_0 and so by (1) it contains all a_i. Suppose that B contains no a_i and let

$$w = \alpha_r a_r + \alpha_{r+1} a_{r+1} + \dots + \alpha_s a_s$$

be an element of minimal length in B (assuming that $B \neq 0$), where $\alpha_r \alpha_s \neq 0$. By applying a_{m-r} on the left or right if necessary we may assume that $r = m$. But then application of a_m to w yields a non-zero element of B of length less than $l(w)$, a contradiction.

4. Applications

Let L be the algebra over $\boldsymbol{k}$ with basis $a_{-1}, a_0, a_1, a_2, \dots$, and multiplication given by

$$a_i a_j = \lambda(i, j) a_{i+j},$$

where

$$\lambda(-1, -1) = 0, \qquad \lambda(-1, 0) = -\lambda(0, -1) = 1$$

and

$$\lambda(i, j) = \binom{i+j}{i+1} - \binom{i+j}{j+1}, \qquad \text{for } \{i, j\} \neq \{-1, 0\},$$

and $\binom{m}{n}$ is the usual binomial coefficient with the understanding that $\binom{m}{n} = 0$ whenever $n > m, m < 0$ or $n < 0$. Thus $\lambda(-1, j) = 1 = -\lambda(j, -1)$ for all $j > -1$ and, for $i, j > -1$,

$$\lambda(i, j) = (i+j+1)!(j-i)/(i+1)!(j+1)!$$

It can also be checked (or see [2]) that

$$\lambda(i, j)\lambda(i+j, k) + \lambda(j, k)\lambda(j+k, i) + \lambda(k, i)\lambda(k+i, j) = 0 \tag{10}$$

for all i, j, k. Thus L is a simple Lie algebra.

If the field has characteristic $p > 0$ then L is a locally finite-dimensional Lie algebra and so cannot satisfy the maximal condition for subalgebras. This is so because if $n \geqq 1$ and $1 \leqq i \leqq p^n - 2$ then $\lambda(i, p^n - i - 1) = 0$ (see [2]). Thus condition (2) of theorem A is necessary.

If the field has characteristic zero then clearly $\lambda(i,j) = 0$ if and only if $i = j$. Thus by theorem A or B we have that L satisfies the maximal condition for subalgebras. In this case let, for each $n \geqq -1$, L_n be the subspace of L spanned by the basis elements $a_n, a_{n+1}, a_{n+2}, \ldots$. Then each L_n is a subalgebra of L and L_n is an ideal of L_0 for all $n \geqq 0$. If $L_n^2 = [L_n, L_n]$ is the derived algebra of L_n then L_n/L_n^2 has dimension precisely $n+1$. Thus $L_n \cong L_m$ if and only if $n = m$. So we have

THEOREM C: *Over any field of characteristic zero there exists a countable infinity of pairwise non-isomorphic infinite-dimensional Lie algebras satisfying the maximal condition for subalgebras.*

Over a field $\boldsymbol{k}$ of characteristic zero the Lie algebra W (see [1] pp. 209 for its other properties) with basis $\{w_i : i \in Z\}$ and multiplication $w_i w_j = (j-i)w_{i+j}$ is a simple Lie algebra satisfying the maximal condition for subalgebras (by theorem A).

At the present time we have been unable to find λ with $\lambda(i,j) = -\lambda(i,j)$, $\lambda(i,i) = 0$, and satisfying (2) and (10) over a field of characteristic $p > 0$. It might be possible to make further progress by using a more complicated definition of multiplication, but a proof that every subalgebra is finitely generated would become correspondingly more difficult.

If we take $A(\lambda)$ with $\lambda(i,i) = 0$, $\lambda(i,j) = 1$ or -1 according as $i < j$ or $i > j$ then we obtain an anti-symmetric algebra and in much the same way as before we have

THEOREM D: *Over any field there exist infinitely many pairwise non-isomorphic infinite-dimensional anti-symmetric algebras satisfying the maximal condition for subalgebras.*

THEOREM E: *The polynomial algebras* $\boldsymbol{k}[t]$, $\boldsymbol{k}[t, t^{-1}]$ *the maximal condition for subalgebras.*

PROOF: Using theorem B or A, take $a_i = t^i$ and $\lambda(i,j) = 1$ for all i, j.

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