COMPOSITIO MATHEMATICA

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*Compositio Mathematica*, tome 30, nº 3 (1975), p. 317-322

<http://www.numdam.org/item?id=CM_1975__30_3_317_0>

COMPOSITIO MATHEMATICA, Vol. 30, Fasc. 3, 1975, pag. 317–322
Noordhoff International Publishing
Printed in the Netherlands

# SOME EXAMPLES AND COUNTER-EXAMPLES IN VALUE DISTRIBUTION THEORY FOR SEVERAL VARIABLES

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This is a compendium of the examples I have run across in thinking about value distribution theory in several variables. Almost all were encountered in ill-fated attempts at proving theorems.

## 1

In [4], it was shown that a holomorphic curve $\mathbb{C} \xrightarrow{f} \mathbb{P}_n$ omitting any $n+2$ distinct hyperplanes must have image lying in a hyperplane. It was variously conjectured that the Ahlfors defect relations might continue to hold for distinct hyperplanes not in general position, and the foregoing seemed evidence for this. The refutation of this, however, is to be found implicitly in the original work of Weyl and Ahlfors. They show (see [1]) that if $f$ can be expressed in homogeneous coordinates in the form $(e^{\lambda_0 z} : e^{\lambda_1 z} : \cdots : e^{\lambda_n z})$, $\lambda_i \in \mathbb{C}$, then $T(f, r)$ differs by a bounded term from $(L/2\pi)r$, where $L$ is the perimeter of the convex hull of the points $\lambda_0, \cdots, \lambda_n$ in $\mathbb{C}$. If $A = (a_0, \cdots, a_n)$ is a hyperplane, then the counting function $N(f, A, r)$ differs by a bounded term from $(L'/2\pi)r$, where $L'$ is the perimeter of the convex hull at the points $\lambda_i$, $i$ such that $a_i \neq 0$. If, for example, $n = 2$ and $\lambda_0 = 0$, $\lambda_1 = 1$, $\lambda_2 = -(1/n)$, then we have $T(f, r) \sim ((1+1/n)/\pi)r$, but $N(f, A, r) \sim (1/n\pi)r$ for any hyperplane $A$ of the form $az_0 + bz_2$, $a, b \neq 0$. Hence any hyperplane in this pencil has defect $(n/n+1)$. There are an infinite number of such hyperplanes, but no three are in general position.

## 2

A theorem of Carlson and Griffiths [3] has as a special case that a holomorphic map $F : \mathbb{C}^n \to \mathbb{P}_n$ omitting a hypersurface $D$ with singulari-

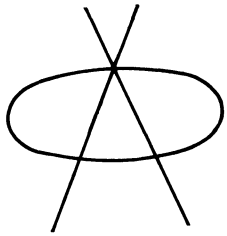
* The author is grateful to Purdue University and the NSF for providing the opportunity to write this paper.

ties no worse than normal crossings and of degree at least $n+2$ must be degenerate in the sense that its Jacobian determinant vanishes identically. That some assumption on the singularities of $D$ is necessary may be seen from this example, which was also found by J. Carlson, F. Sakai, and B. Shiffman. Take $D$ to be the affine curve $\{y = x^d\} \subset \mathbb{C}^2$. The map $F: \mathbb{C}^2 \to \mathbb{C}^2$ given by $F(z, w) = (z, z^d + e^w)$ is non-degenerate and omits the curve. The curve is non-singular in $\mathbb{C}^2$, but possesses a bad singularity when completed to a projective curve.

## 3

A more interesting example of this type has

$$D = \{z_0 = 0\} \cup \{z_1 = 0\} \cup \{(z_0 - z_1)z_2 = (z_0 + z_1)^2\} \subset \mathbb{P}_2.$$



Thus $D$ is a non-singular conic and two non-tangent lines intersecting on the conic. So $D$ has two ordinary double points (normal crossings) and a single ordinary triple point, the nicest singularity not covered by Carlson-Griffiths. We can take our map $F(z, w)$ to be given in homogeneous coordinates by $(e^{\varphi_0}, 1, f_2)$. We omit the conic iff there is a holomorphic exponential $e^{\psi}$ so

$$f_2(e^{\varphi_0} - 1) = (e^{\varphi_0} + 1)^2 + e^{\psi}.$$

Equivalently,

$$f_2 = e^{\varphi_0} + 3 + \frac{e^{\psi} + 4}{e^{\varphi_0} - 1}$$

Taking $\varphi_0 = z$, $\psi = w(e^z - 1) + \log(-4)$, we easily see $f_2$ is entire and the map is non-degenerate, yet omits $D$. So the Carlson-Griffiths result is sharp in this direction.

It is interesting that the example is a map of infinite order. Since $f_2$ is entire iff for all $(z, w)$,

$$\frac{\varphi_0(z, w)}{2\pi i} \in \mathbb{Z} \to \frac{\psi(z, w) - \log(-4)}{2\pi i} \in \mathbb{Z},$$

the existence of a non-degenerate map of finite order omitting $D$ would imply the existence of algebraically independent polynomials $p(z, w)$, $q(z, w)$ so $p(z, w) \in \mathbb{Z} \to q(z, w) \in \mathbb{Z}$. In one variable, given polynomials $p(z)$, $q(z)$ so for all $z$, $p(z) \in \mathbb{Z} \to q(z) \in \mathbb{Z}$, it is easy to show $p$, $q$ are algebraically dependent over $\mathbb{Z}$. So going back to $\mathbb{C}^2$ and restricting to lines through the origin, we can use a category argument to see $p(z, w)$, $q(z, w)$ are algebraically dependent over $\mathbb{Z}$. It follows that no finite order non-degenerate map omitting $D$ exists. We thus have an example of a Picard theorem true for finite order maps, but false for infinite order ones.

## 4

B. Shiffman's defect relations for singular divisors (see [8]) would be shown to be sharp in an interesting particular case if one had a non-degenerate map $\mathbb{C}^2 \to \mathbb{P}_2$ omitting the divisor

$$D = \{z_1^d - z_0(z_1^{d-1} + z_2^{d-1}) = 0\}.$$

Shiffman simplified the map I found, and I give his version here. Take $F(z, w)$ to be given in homogeneous coordinates by

$$f_0 = \frac{1 - e^{w(1+z^{d-1})}}{1 + z^{d-1}}$$

$$f_1 = 1$$

$$f_2 = z.$$

## 5

We will now consider maps which are not equidimensional. For holomorphic curves $\mathbb{C} \xrightarrow{f} \mathbb{P}_n$ omitting a hypersurface $D$ with normal crossings and of degree at least $n+2$, it is natural to conjecture that the image of $f$ must lie in an algebraic hypersurface, a condition we call *algebraic degeneracy*. No counterexample to the conjecture in this form is known. To prove the conjecture, which is unknown even for $n = 2$, it would be helpful to have more explicit information about the hypersurface in which the image is to lie. That this further information, when it comes, will be complicated is indicated by the examples of sections 6–10.

The conjecture is known for maps $f : \mathbb{C} \to \mathbb{P}_2 - D$ in the cases listed here:

(A) $D$ has 4 or more irreducible components, no assumption on singularities (see [5], the case of lines being classical).

(B) $D$ contains three curves in a linear pencil, again no assumption on singularities (this reduces to $\mathbb{P}_1 - \{0, 1, \infty\}$ immediately).

(C) $D$ is a Fermat curve $\{z_0^d + z_1^d + z_2^d = 0\}$, $d \geqq 7$.

(D) $D = \{z_0 = 0\} \cup \{z_1 = 0\} \cup \{z_0^2 + z_1^2 + z_2^2 = 0\}$, provided $f$ is of finite order (see [6]).

(E) $D =$ the dual curve of a curve $D^*$ having no singularities except ordinary multiple points and $\kappa$ cusps, satisfying $\chi(D) + \kappa < 0$. (See [7] for a preliminary version, [2] for the version given here).

The cases known seem to span the spectrum of possible curves. Interestingly, (A), (C), and (D) are shown by purely analytic techniques, while (E) requires some geometry but no analysis. The curves of (E) will always have cusps or worse, indicating that the singularities of $D$ can often work in our favor.

## 6

If $\deg D = 4$, there always exist non-constant holomorphic maps $\mathbb{C} \xrightarrow{f} \mathbb{P}_2 - D$. If $D$ is reducible, $D$ has at least two double points, and the line joining them won't intersect $D$ in any further points by counting degrees. This line $L \cap (\mathbb{P}_2 - D)$ is a $\mathbb{C}^*$. If $D$ is irreducible, but has more than one double point or any triple points, we again get a $\mathbb{C}^*$. If $D$ is irreducible, has at most one singularity and that of multiplicity two, Plucker's formulas imply $D$ has an inflectional tangent (a tangent with order of contact $\geqq 3$ at the point of tangency), which intersects $D$ in at most one other point, again giving a $\mathbb{C}^*$. As $\mathbb{C} \xrightarrow{e^z} \mathbb{C}^*$ is non-constant, we get a non-constant but linearly degenerate map.

This example, simple though it is, illustrates the importance of highly osculating lines of $D$ to our problem. Unfortunately, lines are not enough.

## 7

We lump together here two examples dealing with the Fermat curve $D = \{z_0^d + z_1^d + z_2^d = 0\}$. The first is Peter Kiernan's fundamental example of a non-constant rational map $\mathbb{C} \xrightarrow{f} \mathbb{P}_2 - D$ for arbitrary $d$. The line $z_0 = \mu z_1$, $\mu$ a $d$-th root of $-1$, intersects $D$ in a single point $(\mu, 1, 0)$; in other words this line is a highly inflectional tangent. This line $L \cap (\mathbb{P}_2 - D) = \mathbb{C}$, hence any polynomial function $\mathbb{C} \to \mathbb{C}$ gives the desired map. The second example omits the Fermat curve of degree 4, and lies in the quartic

$$\left\{z_2^4 + \frac{1}{2}\left(z_0 + \frac{z_1}{\sqrt[4]{-1}}\right)^3 \left(z_0 - \frac{z_1}{\sqrt[4]{-1}}\right) = 0\right\}.$$

The map in homogeneous coordinates is

$$\begin{aligned} f_0 &= e^{\varphi} + e^{\psi} \\ f_1 &= \sqrt[4]{-1}(e^{\varphi} - e^{\psi}) \\ f_2 &= \sqrt[4]{-8}e^{3\varphi/4}e^{\psi/4} \end{aligned}$$

For another description, see [5].

## 8

I know of no example where the curve to which we degenerate has degree higher than that which we omit; such an example wouldn't surprise me. We can hope for no better bound on the degree of the degeneracy curve, for taking $D$ to be the non-singular curve

$$\{z_1 z_2^{d-1} = z_0(z_0^{d-1} - z_1^{d-1})\},$$

$D$ can be omitted by the map $(e^{\varphi}, 1, e^{(d/d-1)\cdot\varphi})$, which goes to the curve $\{z_0^d = z_1 z_2^{d-1}\}$, of the same degree as the curve we started with.

## 9

This example will show how the higher degree osculating curves of $D$ come in. Take $D$ to be $\{(z_2^2 + z_1^2 + z_2^2)(z_1^2 + 2z_2^2) + z_0^4 = 0\}$, which is non-singular. If $Q$ is the conic $\{z_0^2 + z_1^2 + z_2^2 = 0\}$, then $Q$ intersect $D$ consists of the points $(0, 1, i)$ and $(0, 1, -i)$. So $Q \cap (\mathbb{P}_2 - D)$ is $\mathbb{P}_1 - 2$ points, hence a $\mathbb{C}^*$. So we get a non-constant map $\mathbb{C} \to \mathbb{P}_2 - D$ lying in $Q$. Note $Q$ has contact 4 with $D$ at both intersections.

## 10

If the conjecture about algebraic degeneracy of holomorphic curves to $\mathbb{P}_2$ omitting a curve $D$ of degree $\geqq 4$ with normal crossings is true, the Picard theorems about maps to Riemann surfaces imply that the curve of degeneracy $P$ is rational (i.e. genus 0) and intersects $D$ in at most two distinct points, where distinctness means on the Riemann surface associated to $P$. One might hope to classify such curves $P$, but the two properties (rationality and number of distinct intersections with $D$) seem hard to inter-relate. The second property, for $D$ irreducible, can be phrased in terms of the Jacobian of $D$. Take $g \in J(D)$ to be the point of the Jacobian corresponding to the linear system of sections of $D$ by lines in $\mathbb{P}_2$. Then there exists $P$ of degree $k$ intersecting $D$ in $k_1 p + k_2 q$ counting multiplicities if and only if $k_1 \varphi(p) + k_2 \varphi(q) = kg$ in $J(D)$, where $\varphi : D \to J(D)$ is the Abel-Jacobi map. We are thus asking if

$$kg \in k_1 \varphi(D) + k_2 \varphi(D).$$

The latter has dimension two, so for genus $(D) \geqq 3$, we expect generically that we can rule out such $P$ without invoking $P$'s rationality (which implies $P$ has lots of multiple points if degree $P$ is high). If genus $(D) \leqq 2$, we do get $P$'s contacting $D$ in only two points for every degree; hopefully these $P$ are non-rational for high degrees.

### BIBLIOGRAPHY

[1] L. Ahlfors: The theory of meromorphic curves. *Acta Soc. Sci. Fenn. (N.S.)* 3 (1941).

[2] J. Carlson: *A Picard theorem for* $\mathbb{P}_2 - D$. Proc. of the A.M.S. Summer Inst. in Diff. Geom., Stanford, 1973.

[3] J. Carlson and P. Griffiths: A defect relation for equidimensional holomorphic mappings between algebraic varieties. *Ann. of Math. 95, No. 3* (May 1972) 557–584.

[4] M. Green: Holomorphic maps into complex projective space omitting hyperplanes, *Trans. A. M. S. 169* (1972) 89–103.

[5] M. Green: Some Picard theorems for holomorphic maps to algebraic varieties. *Am. J. Math. 97* (1975) 43–75.

[6] M. Green: On the functional equation $f^2 = e^{2\varphi^1} + e^{2\varphi^2} + e^{2\varphi^3}$ and a new Picard theorem. *Trans. A.M.S. 195* (1974) 223–230.

[7] M. Green: The complement of the dual of a plane curve and some new hyperbolic manifolds. p. 119–132, *Value-Distribution Theory*. Part A, Marcel Dekker (N. Y. 1974).

[8] B. Shiffman: *Nevanlinna defect relations for singular divisors*. (to appear).

(Oblatum 17–I–1975) Massachusetts Institute of Technology