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DEFORMATION OF DETERMINANTAL SCHEMES

Dan Laksov

1. Introduction

We shall in the following work deal with the problem of constructing (global, flat) deformations of determinantal subschemes of affine spaces. Our main contribution is the construction of deformations, such that the generic member of each flat family has a stratification consisting of determinantal schemes, each member of the stratification being the singular locus of the preceding. With the exception of the determinantal schemes of codimension one, a stratification of the above type is generally the best structure that can be obtained. Indeed, T. Svanes has proved (in [17]) that with the above exception, the natural stratification of a generic determinantal variety, given by determinantal subvarieties corresponding to matrices of successively lower rank, is rigid. Hence all members of a family deforming a generic determinantal variety, have the same stratification. The determinantal subschemes of codimension one in an affine space are simpler. Indeed, by a particular case of Bertini's theorem, they can be deformed into smooth schemes.

The most spectacular consequence of our construction and a globalization (due to M. Schaps [16]) of a well known result by L. Burch, is the result that every Cohen-Macaulay scheme of pure codimension two in an affine space of dimension less than six can be deformed into a smooth scheme. This result was first treated by M. Schaps in her doctoral dissertation [16]. Another construction of a splitting of points in the plane into distinct simple points can be found in J. Briancon and A. Galligo's article [3], however, the existence of such deformations already follows from the work of J. Fogarty [8].

Consider a morphism

$$f : X \to M$$

of smooth varieties and a subvariety D of M. Denote by Y the scheme theoretic inverse image $f^{-1}(D)$ of D by f. One way of obtaining a family of subschemes of X parametrized by a variety G and having Y as a member, is by constructing a morphism

$$F : G \times X \to M$$

such that for some rational point e of G the restriction of F to $(e \times X) \cong X$ coincides with the morphism f. Then the resulting morphism

$$q_D : F^{-1}(D) \to G,$$

induced by the projection of $G \times X$ onto the first factor determines such a family. Indeed, by the associativity formula the fiber

$$q_D^{-1}(g) = g \times_G [(G \times X) \underset{M}{\times} D]$$

of q_D at a point g of G is isomorphic to the subscheme

$$f_g^{-1}(D) = (g \times X) \underset{M}{\times} D$$

of $g \times X$, where f_g denotes the restriction of the morphism F to the scheme $(g \times X)$. In particular the fiber $q_D^{-1}(e)$ is isomorphic to the scheme $Y = f_e^{-1}(D)$.

In sections 2 and 3 below we shall deal with two different aspects of the family q_D. First we consider the singular locus of the generic fiber of q_D, in other words we study the transversality of the morphisms f_g with respect to the regular locus of the scheme D. By a method used in S. L. Kleimans article [12] we prove a generalization and sharpening of a transversality theorem presented there ([12], Theorem 10). One difference between our version and the one in Kleimans article, which is essential for the application of the result to determinantal varieties in positive characteristic, is that the scheme G need not be a group scheme acting on M. Another difference is that we not only prove that in general the morphism f_g is transversal to the complement $(D - D')$ of the singular locus D' of D, but that in general $f_g^{-1}(D)$ is nonempty whenever $\dim(D) + \dim(X) - \dim(M)$ is non-negative. Then we impose conditions on F and D assuring that the morphism q_D is flat in a neighbourhood of the fiber $q_D^{-1}(e) = Y$, the crucial requirement being that D is Cohen-Macaulay. Together with the transversality theorem we obtain a useful criterion for q_D to induce a deformation of Y with 'nice generic fibers'.

Given a morphism $f : \mathbb{A}^p \to \mathbb{A}^q$ of affine spaces. We construct in section four a morphism $F : G \times \mathbb{A}^p \to \mathbb{A}^q$, where G is a smooth variety with a distinguished point e, such that the morphism f coincides with the restriction of F to the scheme $(e \times \mathbb{A}^p) \cong \mathbb{A}^p$ and such that F satisfies the conditions imposed on the corresponding morphism in the criterion described above. There are several possible choices of the scheme G and the corresponding morphism F satisfying the requirements of the criterion, the particular one presented here is also used in M. Schaps' work mentioned above.

In Section 5 we apply the results of the previous sections to obtain deformations of determinantal subschemes of affine spaces. We like to point out that in this context the crucial property of determinantal schemes is that they are Cohen-Macaulay schemes. In Section 5 we also treat the exceptional case of determinantal schemes of codimension one.

2. A transversality theorem

Fix an algebraically closed field k.

THEOREM 1: *Let G, X and M be regular, irreducible algebraic schemes and*

$$F : G \times X \to M$$

a morphism. Given rational points x and m of X and M, we denote by $G(x, m)$ the fiber of the morphism $F|(G \times x)$ at the point m.

Then the morphism F induces a natural morphism

$$T(x, m) : G(x, m) \to \mathrm{Hom}\,(T_x(X), T_m(M))$$

from the scheme $G(x, m)$ to the scheme

$$\mathrm{Hom}\,(T_x(X), T_m(M)) = \mathbb{V}(T_x(X) \otimes T_m(M)^{\check{}})$$

of homomorphisms from the tangent space $T_x(X)$ of X at x to the tangent space $T_m(M)$ of M at m.

Assume that for each pair of rational points x and m of X and M the following conditions are satisfied:

(*) *The morphism $F|(G \times x) : G \to M$ is faithfully flat.*

(**) *The morphism $T(x, m)$ is flat.*

Then for each unramified morphism $h : D \to M$ from an irreducible scheme D to M, there exists an open dense subset U of G, such that for each rational point g of U the scheme $(g \times X) \times_M D$ is of pure dimension

$\dim(X)+\dim(D)-\dim(M)$ *(empty if* $\dim(X)+\dim(D)-\dim(M)$ *is negative), here* $g\times X$ *is considered as an* M *scheme via the morphism* $F|(g\times X)$. *Moreover, let* D' *denote the singular locus of the scheme* D. *Then* $(g\times X)\times_M D'$ *is the singular locus of the scheme* $(g\times X)\times_M D$.

In particular the morphism $f_g = F|(g\times X)$ is transversal to the morphism $h|\Delta$ for each rational point g of U, here Δ denotes the complement of D' in D.

REMARK: The condition (*) implies that F is faithfully flat. In fact, since the morphism $F|(G\times x)$ is faithfully flat for each rational point x of X it follows that $G(x, m)$ is of pure dimension equal to $\dim(G)-\dim(M)$ for all rational points m of M (A-K, V, (2.10) or EGA IV_2, (6.1.4)). However, the scheme $G(x, m)$ is clearly isomorphic to the fiber of the projection $(G\times X)\times_M m \to X$ at x. Since the product $G\times X$ is irreducible (EGA, IV_2, (4.5.8)) it follows that the fiber $(G\times X)\times_M m$ of F at m is of pure dimension $\dim(G)+\dim(X)-\dim(M)$ (A-K, V, (2.10) or EGA, IV_2 (6.1.4)). Consequently, since M and $G\times X$ are regular schemes (EGA, IV_2) (6.8.5), the morphism F is flat (A-K, V, (3.5) or EGA IV_2, (6.1.5)).

PROOF: We first prove that for each rational point g of an open dense subset U_1 of G, the scheme $(g\times X)\times_M \Delta$ is empty or regular of pure dimension $\dim(X)+\dim(D)-\dim(M)$.

The morphism F determines a map $T(G\times X)\to F^*T(M)$ of tangent sheaves and consequently, a map $\alpha : p_2^* T(X)\to F^*T(M)$, of locally free sheaves, where p_2 denotes the projection of the product $G\times X$ onto the second factor. Pulling back the map α by the morphism $(\mathrm{id}_{G\times X}\times(h|\Delta))$, we obtain a map

$$T : p_X^* T(X) \to p_\Delta^*(h|\Delta)^* T(M)$$

of locally free sheaves on $(G\times X)\times_M \Delta$, here p_X and p_Δ denotes the projections onto X and Δ.

Note that, given rational points x and d of X and Δ, the morphism $T(x, h(d))$ is obtained in the following way: Restrict T to a map $l^*T_x(X)\to l^*T_m(M)$ of locally free sheaves on $G(x, h(d))$ via the inclusion $(G\times x)\times_M d\to(G\times X)\times_M D$, here l denotes the structure morphism of G and $m = h(d)$. We deduce a map $l^*T_x(X)\otimes l^*T_m(M)^{\check{}}\to 0_{G(x,h(d))}$ into the structure sheaf of $G(x, h(d))$ and consequently a section

$$s : G(x, h(d)) \to \mathbb{V}(l^*T_x(X) \otimes l^*T_m(M)^{\check{}}).$$

The morphism $T(x, h(d))$ is the composition of s with the projection

$$\mathbb{V}(l^* T_x(X) \otimes l^* T_m(M)^\vee) \to \mathbb{V}(T_x(X) \otimes T_m(M)^\vee).$$

Since the morphism h is unramified, the corresponding map $T(\Delta) \to (h|\Delta)^* T(M)$ of tangent sheaves on Δ is locally split (A-K, VI, (3.6) or EGA IV_4, (17.3.6)). Pulling back this injection by the projection p_Δ we obtain a locally split map $i: p_\Delta^* T(\Delta) \to p_\Delta^*(h|\Delta)^* T(M)$ of locally free sheaves on the scheme $(G \times X) \times_M \Delta$. Denote by S the support of the map obtained by composing T with the quotient map

$$p_\Delta(h|\Delta)^* T(M) \to p_\Delta^*(h|\Delta)^* T(M)/ip_\Delta^* T(\Delta).$$

We shall determine the dimension of the scheme S. Fix rational points x and d in X and Δ. Then a rational point (g, x, d) of the scheme $G(x, h(d)) \cong (G \times x) \times_M d$ is in S if and only if the composite map

$$T_x(X) \xrightarrow{T(x, h(d))(g)} T_m(M) \to T_m(M)/iT_d(\Delta)$$

is not injective. However, it is well known that the (determinantal) subscheme of Hom $(T_x(X), T_m(M)/iT_d(\Delta))$ of homomorphisms of rank less than min (dim (X), dim $(M) -$ dim (D)) is irreducible and of codimension dim $(X) +$ dim $(D) -$ dim $(M) + 1)$ (eg. [13], (4.13) p. 425). Since the morphism $T(x, h(d))$ is flat by assumption, it easily follows that $S \cap G(x, h(d))$ is either empty or of pure codimension equal to dim $(X) +$ dim $(D) -$ dim $(M) + 1$ in $G(x, h(d))$ (A-K, V, (2.10) or EGA IV_2, (6.1.4)).

We noted above that $G(x, h(d))$ is of pure dimension dim $(G) -$ dim (M). Consequently $S \cap G(x, h(d))$ is empty or of pure dimension

$$\dim(G) - \dim(X) - \dim(D) - 1.$$

However, $S \cap G(x, h(d))$ is the fiber of the morphism $S \to X \times D$ induced by the projection $(G \times X) \times_M D \to X \times D$. Hence each component of S is of dimension less than dim (G) (A-K, V, (2.10) or EGA, IV_2, (6.1.4)).

Consider the morphism

$$q_D : (G \times X) \times_M D \to G$$

induced by projection of $G \times X$ onto the first factor. Since S is of dimension less than dim (G) there is an open dense subset U_1 of G contained in the complement of the image of S by q_D.

Let g be a rational point of U_1 and let x' and d' be a pair of rational points of X and D such that the relation $F(g, x') = h(d')$ holds. Then the vector space $T_{h(d)}(M)$ is spanned by the images of the subspaces $T_x(X)$ and $T_d(\Delta)$ by the maps $T(x', h(d'))$ and i. It follows that the fiber $(g \times X) \times_M D$ of q_D at g is regular of dimension $\dim(X) + \dim(D) - \dim(M)$ at the point (x', d'). In fact, this is a local matter and since h is unramified we may assume, after passing to the completions, that D is a closed subscheme of M (A-K, VI, (3.7) or EGA, IV_4, (17.4.4)) the result then follows from EGA, IV_4, (17.13.2).

We now prove that there exists an open dense subset U' of G such that for each rational point g of U' the scheme $(g \times X) \times_M D$ is nonempty and of pure dimension $\dim(X) + \dim(D) - \dim(M)$.

By the remark following the theorem the morphism F is faithfully flat. Hence the morphism q_D obtained by base extension by h is faithfully flat. It follows that $(G \times X) \times_M D$ is of pure dimension

$$\dim(D) + \dim(X) + \dim(G) - \dim(M)$$

(A-K, V, (2.10) or EGA, IV_2, (6.1.4)). Let U_2 be an open dense subset of G over which q_D is flat (A-K, V, (5.2) or EGA, IV_2, (6.9.1), then at each point of U_2, the fiber of q_D is empty or of pure dimension equal to $\dim(D) + \dim(X) - \dim(M)$ (A-K, V, (2.10) or EGA, IV_2, (6.1.4)).

Assume that $\dim(X) + \dim(D) - \dim(M)$ is not negative. Then it follows from the first part of the proof that there exists a rational point (g, x, d) of $G(x, h(d))$ not contained in S and that the local ring of the fiber $q_D^{-1}(g) = (g \times X) \times_M D$ is regular and of dimension

$$\dim(X) + \dim(D) - \dim(M)$$

at the point (g, x, d). Hence the fiber of q_D at g has a component Y of this dimension. Denote by H an irreducible component of the scheme $(G \times X) \times_M D$ whose intersection with the fiber $q_D^{-1}(g)$ has Y as a component and by G' the image of H by q_D. Then every component of each fiber of the morphism $(q_D|H)$ has dimension at least $\dim(H) - \dim(G')$ (A-K, V, (2.10) or EGA, IV_2, (6.1.4)). In particular

$$\begin{aligned} \dim(Y) = \dim(X) + \dim(D) - \dim(M) &\geqq \dim(H) - \dim(G') \\ &\geqq \dim(X) + \dim(D) + \dim(G) - \dim(M) - \dim(G'), \end{aligned}$$

that is $\dim(G') \geqq \dim(G)$. Consequently the closure of G' is the scheme G and since G' is constructible it easily follows that there exists an open dense subset U_3 of G contained in G'. For each point g in U_3 the fiber of

q_D at g is nonempty.

We have proved that the subset $U = U_1 \cap U_2 \cap U_3$ satisfies all the assertions of the theorem except the assertion that $(g \times X) \times_M D'$ is exactly the singular locus of $(g \times X) \times_M D$. This last assertion follows (after passing to the completions) from the following general result.

LEMMA: *Let $f : X \to M$ be a morphism of regular irreducible algebraic schemes. Moreover, let D be an irreducible closed subscheme of M.*

Assume that the scheme $X \times_M D$ is of pure dimension

$$\dim (X) + \dim (D) - \dim (M).$$

Then every point of $X \times_M D$, which maps to a singular point of D by the projection onto the second factor, is itself singular.

PROOF: Consider the cartesian diagram

$$\begin{tikzcd} X \arrow[r, "f"] & M \\ X \times_M D \arrow[u, "j"] \arrow[r, "g"] & D \arrow[u, "i"'] \end{tikzcd}$$

Denote by I the ideal in the structure sheaf $\mathcal{O}_M$ of M defining the closed subscheme D of M and by J the ideal in $\mathcal{O}_X$ defining the closed subscheme $X \times_M D$ of X. Then we have a commutative diagram of coherent sheaves on $X \times_M D$,

$$\begin{tikzcd} g^*I/I^2 \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & g^*i^*\Omega^1_M \arrow[r] \arrow[d] & g^*\Omega^1_D \arrow[r] \arrow[d] & 0 \\ J/J^2 \arrow[r, "\beta"] & j^*\Omega^1_X \arrow[r] & \Omega^1_{X \times_M D} \arrow[r] & 0, \end{tikzcd}$$

(EGA, IV_4, (17.13.1.2)).

The horizontal lines are exact (A-K, VI, (1.8) or EGA, IV_4, (16.4.21)) and since the above diagram is cartesian, it is easily verified that the left vertical map γ is surjective (EGA, IV_4, (16.2.2, (iii))).

Let d be a singular point of D. Then $\dim (\Omega^1_D(d))$ is greater that $\dim (D)$ (A-K, VII, (6.4) or EGA, IV_4, (17.15.5)). Hence at d the image of $\alpha(d)$ is of dimension less than $\dim (M) - \dim (D)$. Let X be a point of X satisfying the relation $f(x) = d$. Since the map γ is surjective, it follows that the image of β is of dimension less than $\dim (M) - \dim (D)$. Consequently $\dim \Omega^1_{X \times_M D}(x)$ is greater than $\dim (X) + \dim (D) - \dim (M)$, and since $X \times_M D$ is of pure dimension $\dim(X)+\dim(D)-\dim(M)$ by assumption, it follows that x is a singular point of $X \times_M D$ (A-K, VII, (6.4) or EGA, IV_4, (17, 15.5)).

COROLLARY (S. L. Kleiman [12], Theorem 10): *Let G be an integral algebraic group scheme and M a regular algebraic scheme with a transitive G-action. Moreover let $f: X \to M$ and $h: \Delta \to M$ be unramified morphisms of regular, irreducible algebraic schemes.*

Assume that for each rational point m of M the induced homomorphism

$$G(m) \to G1(T_m(M)),$$

from the stability group of G at m to the general linear group of the tangent space of M at m, is surjective.

Then there exists an open subset U of G such that for each rational point g of U, the fibered product $(g \times X) \times_M \Delta$ is regular of dimension $\dim(X)+\dim(\Delta)-\dim(M)$ (*empty if* $\dim(X)+\dim(\Delta)-\dim(M)$ *is less than zero*).

PROOF: Consider the surjective morphism

$$F: G \times X \to M$$

obtained by composing the morphism $(\mathrm{id}_G \times f): G \times X \to G \times M$ with the action $G \times M \to M$ of G on M. To prove the corollary it is sufficient to verify that the morphism F satisfies the conditions (*) and (**) of the transversality theorem.

Let x and m be rational points of X and M and denote by F_x the restriction of the morphism F to the scheme $(G \times x) \cong G$. Since the scheme M is irreducible there is an open dense subset M' of M such that the morphism F_x is flat over M'. (A-K, V, (5.2) or EGA, IV_2, (6.9.1)). Consequently the morphism $F_x | F_x^{-1}(gM'): F_x^{-1}(gM') \to gM'$ is flat for all rational points g of the scheme G. Since the group G acts transitively on M the sets gM', as g runs through the rational points of G, form an open covering of M. Hence the morphism F_x is flat.

The morphism

$$T(x, m): G(x, m) \to \mathrm{Hom}\,(T_x(X), T_m(M))$$

is the composite of the following three morphisms:

(1) The isomorphism $G(x, m) \to G(m)$ given by $g \to gg'$ where g' is a rational point of G satisfying the relation $g'm = f(x)$.

(2) The surjection $G(m) \to Gl(T_m(M))$.
(3) The morphism $Gl(T_m(M)) \to \mathrm{Hom}\,(T_x(X), T_m(M))$ given by $\alpha \to \alpha \circ t$, where t is the map $T_x(X) \to T_m(M)$ of tangent spaces determined by the morphism f. Since f is an unramified morphism and X and M are regular, the map t is injective, (A-K, VI, (3.6) or EGA, IV_2, (17.3.6)).

It easily follows that the third morphism maps $Gl(T_m(M))$ onto the open subset of $\mathrm{Hom}\,(T_x(X), T_m(M))$ consisting of all injective homomorphisms.

By the argument we used above to prove that the morphism $F|G \times x)$ is flat it easily follows that the second and third morphism above are flat. Hence the morphism $T(x, m)$ is flat.

3. Deformations of Cohen-Macaulay schemes

LEMMA: *Let G, X and M be irreducible, regular algebraic schemes and*

$$F : G \times X \to M$$

a faithfully flat morphism. Moreover, let D be an irreducible Cohen-Macaulay subscheme of M. Denote by G' the subset of G consisting of the points g of G such that the inverse image $f_g^{-1}(D)$ of D by the morphism $f_g = F|(g \times X)$ is of pure codimension codim (D, M).

Then, for each rational point g of G', the morphism

$$q_D : F^{-1}(D) \to G$$

induced by projection of $G \times X$ onto the first factor is flat along the fiber $q_D^{-1}(g)$.

PROOF: The scheme $G \times X$ is regular and irreducible since G and X are regular and irreducible and the ground field is algebraically closed (EGA, IV_2, (4.5.8) and (6.8.5)). Since the morphism F is faithfully flat it follows that the scheme $F^{-1}(D)$ is of pure codimension codim (D, M) in $G \times X$. (A-K, V, (2.10) or EGA, IV_2, (6.1.4)). Hence, since D is a Cohen-Macaulay scheme, we conclude that $F^{-1}(D)$ is a Cohen-Macaulay scheme [10] Lemma 9, p. 160.

The fiber $g \times_G (G \times X) \times_M D$ of q_D at g is isomorphic to the scheme $f_g^{-1}(D) \cong (g \times X) \times_M D$, by the associativity formula. Hence for all points g of the set G' the fiber $q_D^{-1}(g)$ is of dimension $\dim (X) + \dim (D) - \dim (M)$ by our dimension assumption. Since G is regular and the scheme $F^{-1}(D)$ is Cohen-Macaulay of pure dimension

$$\dim(G)+\dim(X)+\dim(D)-\dim(M)$$

it follows that the morphism q_D is flat along each fiber $q_D^{-1}(g)$ for all points g in G'. (A-K, V, (3.5) or EGA, IV_2, (6.1.5)).

THEOREM 2: *Let G, X and M be irreducible, regular algebraic schemes and*

$$F : G \times X \rightarrow M$$

a morphism. Moreover, let e be a distinguished point of G and let

$$\Phi = D_0 \subseteq D_1 \subseteq \cdots \subseteq D_c = D$$

be a sequence of irreducible subschemes of M.

Denote by V the open subset of the scheme $F^{-1}(D) = (G \times X) \times_M D$ *where the morphism*

$$q_D : F^{-1}(D) \rightarrow G,$$

induced by the projection of $G \times X$ *onto the first factor, is flat* (A-K, V, (5.5) *or* EGA, IV_3, (11.1.1)). *Moreover, for each rational point g of the scheme G, we denote by* f_g *the restriction of the morphism F to the scheme* $(g \times X) \cong X$.

Note that by the associativity formula, the fiber

$$q_D^{-1}(g) = g \times_G (G \times X) \times_M D$$

is isomorphic to the inverse image $f_g^{-1}(D) = (g \times X) \times_M D$ *of D by* f_g.

Assume that the following conditions hold:

(i) *The morphism F satisfies the conditions* (*) *and* (**) *of the transversality theorem.*

(ii) $f_e^{-1}(D)$ *is a subscheme of X of pure codimension* codim (D, M).

(iii) *D is a Cohen-Macaulay scheme.*

(iv) D_{i-1} *is the singular locus of the scheme* D_i *for* $i = 1, \cdots, e$.

Then the fiber $q_D^{-1}(e)$ *is contained in V. Moreover, there exists an open dense subset U of G such that for each rational point g of U the following assertions hold:*

(a) *The fiber* $q_D^{-1}(g) \cong f_g^{-1}(D)$ *is contained in V.*

(b) *Each scheme* $f_g^{-1}(D_i)$ *in the sequence*

$$\Phi = f_g^{-1}(D_0) \subseteq f_g^{-1}(D_1) \subseteq \cdots \subseteq f_g^{-1}(D_c) = f_g^{-1}(D)$$

is of pure codimension codim (D_i, M) *in* X (*empty if* codim (D_i, M) *is greater than* dim (M)).

(c) $f_g^{-1}(D_{i-1})$ *is the singular locus of the scheme* $f_g^{-1}(D_i)$ *for* $i = 1, \ldots, c$.

PROOF: By the remark following the transversality theorem we have that F is faithfully flat. Consequently, by the assumption (ii) and (iii), it follows from the lemma above that the fiber $q_D^{-1}(e)$ is contained in V.

Since F satisfies the conditions (*) and (**) of the transversality theorem and since D_{i-1} is the singular locus of the scheme D_i it follows from the transversality theorem that there exists an open dense subset U_i of G such that for each rational point g of U_i the scheme $f_g^{-1}(D_i)$ is of pure codimension codim (D_i, M) in X and that $f_g^{-1}(D_{i-1})$ is the singular locus of the scheme D_i. In particular the fiber $q_D^{-1}(g) \cong f_g^{-1}(D)$ is of pure dimension $\dim(X) + \dim(D) - \dim(M)$. Hence we conclude, as in the first part of the proof, that the fiber $q_D^{-1}(g)$ is contained in V. Consequently, the assertions (a), (b) and (c) hold for each rational point in the open dense subset $U = U_1 \cap \ldots \cap U_c$ of G.

The situation is simpler when the ground field k is of characteristic zero. Then we have the following transversality result (see [12], Theorem (2)).

PROPOSITION: *Assume that the ground field k is of characteristic zero. Let G be an integral algebraic group scheme and M an algebraic scheme with a transitive G-action. Moreover, let* $f: X \to M$ *and* $g: \Delta \to M$ *be morphisms of irreducible, regular algebraic schemes.*

Then there exists an open dense subset U of G such that for each rational point g of U the fibered product $(g \times X) \times_M \Delta$ *is either empty, or regular of pure dimension* $\dim(X) + \dim(\Delta) - \dim(M)$.

THEOREM 3: *Assume that the ground field k is of characteristic zero. Let G be an integral algebraic group scheme and M an algebraic scheme with a transitive G-action. Moreover, let* $f: X \to M$ be a morphism of irreducible, regular algebraic schemes and let

$$\Phi = D_0 \subseteq D_1 \subseteq \ldots \subseteq D_c = D$$

be a sequence of irreducible subschemes of M.

Assume that the following conditions hold:

(i) $f^{-1}(D)$ *is a subscheme of* X *of pure codimension equal to* codim (D, M).

(ii) D *is a Cohen-Macaulay scheme.*

(iii) *D_{i-1} is the singular locus of the scheme D_i, for $i = 1, \ldots, c$.*
Then there exists a faithfully flat morphism

$$q : V \to W$$

from an algebraic scheme V to an open dense subset W of G and an open dense subset U of W such that the following assertions hold:

(a) *W contains the identity e of G and $q^{-1}(e)$ is isomorphic to $f^{-1}(D)$.*

(b) *For each rational point g in U the fiber $q^{-1}(g)$ is isomorphic to the scheme $f_g^{-1}(D)$, where f_g is the morphism $X \to M$ given by $x \to gx$.*

(c) *For each rational point g in U each scheme $f_g^{-1}(D_i)$ in the sequence*

$$\Phi = f_g^{-1}(D_0) \subseteq f_g^{-1}(D_1) \subseteq \ldots \subseteq f_g^{-1}(D_c) = f_g^{-1}(D)$$

is empty or of pure codimension equal to codim (D_i, M) *in X. Moreover $f_g^{-1}(D_{i-1})$ is the singular locus of $f_g^{-1}(D_i)$ for $i = 1, \ldots, c$.*

PROOF: Consider the surjective morphism

$$F : G \times X \to M$$

obtained by composing the morphism $(\mathrm{id}_G \times f) : G \times X \to G \times M$ with the action $G \times M \to M$ of the group G on M. Like in the proof of the corollary to the transversality theorem we show that the morphism $F|(G \times x) : G \to M$ is flat for all rational points x of X. Hence F is faithfully flat by the note following the transversality theorem.

Let V be the open subset of the scheme $F^{-1}(D) = (G \times X) \times_M D$ where the morphism

$$q_D : F^{-1}(D) \to G$$

obtained by projection of $G \times X$ onto the first factor, is flat (A-K, V, (5.5) or EGA, IV_3, (11.1.1)). Moreover, let W be the image of V by q_D. Then W is open since $q_D|V$ is flat (A-K, V, (5.1) or EGA, IV_2, (2.4.6)). Then Theorem 3 follows from the above proposition and the lemma of this section by the same procedure as Theorem 2 followed from the transversality theorem and the above lemma.

4. Construction of deformations of affine schemes

PROPOSITION: *Let $f : \mathbb{A}^p \to \mathbb{A}^q$ be a morphism of affine spaces of* *dimensions p and q. Denote by G the affine space of* $(p+1)\times q$*-matrices, and by e the rational point of G corresponding to the matrix with all entries equal to zero.*

Then there exists a morphism

$$F : G \times \mathbb{A}^p \to \mathbb{A}^q$$

satisfying the conditions (*) *and* (**) *of the transversality theorem and such that the morphism f is the restriction of F to the scheme* $(e \times \mathbb{A}^p) \cong \mathbb{A}^p$.

Construction of F (M. Schaps)

Put $\mathbb{A}^p = \operatorname{Spec} k[X_1, \ldots, X_p]$ and $\mathbb{A}^q = \operatorname{Spec} k[Y_1, \ldots, Y_q]$. Denote by $f_j(X)$ the image of Y_j by the homomorphism of polynomial rings $k[Y_1, \ldots, Y_q] \to k[X_1, \ldots, X_p]$ corresponding to the morphism f. Moreover, put $G = \operatorname{Spec} k[U_{1,1}, U_{1,2}, \ldots, U_{p,q}, V_1, \ldots, V_q]$. Define a homomorphism of polynomial rings

$$\Phi : k[Y_1, \cdots, Y_q] \to k[X_1, \cdots, X_p, U_{1,1}, U_{1,2}, \cdots, U_{p,q}, V_1, \cdots, V_q]$$

by

$$\Phi(Y_j) = \Big(\sum_{i=1}^{p} U_{i,j} X_i + V_j + f_j(X)\Big).$$

We let F be the morphism of affine spaces corresponding to Φ.

Proof of the proposition: Let $x = (x_1, \cdots, x_p)$ be a rational point of $\mathbb{A}^p$. Then the fiber $G(x, m)$ of the morphism $F|(G \times x) : G \to \mathbb{A}^q$ at a rational point $m = (y_1, \cdots, y_q)$ of $\mathbb{A}^q$ is the closed subscheme of G defined by the ideal I generated by the elements $(\sum_{i=1}^{p} U_{i,j} x_i + V_j + f_j(x) - y_j)$ for $j = 1, \cdots, q$. Clearly the morphism

$$\mathbb{A}^{p \cdot q} = \operatorname{Spec} k[U'_{1,1}, U'_{1,2}, \cdots, U'_{p,q}] \to G(x, m),$$

defined by sending $U_{i,j}$ to the indeterminate $U'_{i,j}$ and V_j to

$$-\Big(\sum_{i=1}^{p} U'_{i,j} x_i + f_j(x) - y_j\Big),$$

is an isomorphism, having the morphism defined by sending $U'_{i,j}$ to $U_{i,j}$ as an inverse. Since the schemes G and $\mathbb{A}^q$ are regular it follows that the morphism $F|(G \times x)$ is flat (A-K, VII, (4.7) or EGA, IV_1, (17.3.3)) and even smooth (A-K, VII, (4.8) or EGA, IV_1, (17.3.3)).

To determine the morphism $T(x, m)$ we choose the basis $\partial/\partial x_j$ of the tangent space $T_x(\mathbb{A}^p)$ and the basis $\partial/\partial Y_j$ of $T_m(\mathbb{A}^q)$. We introduce coordinates into the scheme

$$\operatorname{Hom}(T_x(\mathbb{A}^p), T_m(\mathbb{A}^q)) = \mathbb{V}(T_x(\mathbb{A}^p) \otimes T_m(\mathbb{A}^q)^{\vee})$$

with respect to this choice of basis and denote by $W_{i,j}$ the corresponding coordinate functions. The morphism $T(x, m)$ is then given by the ring homomorphism

$$k[W_{1,1}, W_{1,2}, \cdots, W_{p,q}] \to k[U_{1,1}, U_{1,2}, \cdots, U_{p,q}, V_1, \cdots, V_q]/I$$

sending the indeterminate $W_{i,j}$ to $(U_{i,j} + (\partial f/\partial x_i)(x))$. Taking the composite of this homomorphism with the ring homomorphism defining the above isomorphism $\mathbb{A}^{p \cdot q} \to G(x, m)$, we easily conclude that $T(x, m)$ is an isomorphism.

5. Applications to determinantal schemes

Denote by $M(a, b)$ the affine $(a \cdot b)$-dimensional space of all $(a \times b)$-matrices. Moreover, denote by $D_c(a, b)$ the generic determinantal scheme of all $(a \times b)$-matrices whose minors of order c all vanish, here a, b and c are positive integers and $c \leqq \min(a, b)$. Put

$$M(a, b) = \operatorname{Spec} k[X_{1,1}, \cdots, X_{a,b}],$$

then $D_c(a, b)$ is the closed subscheme of $M(a, b)$ defined by the ideal generated by the determinants of all $(c \times c)$-submatrices of the $(a \times b)$-matrix $[X_{i,j}]$.

We shall need the following facts about generic determinantal schemes:

(a) $D_c(a, b)$ is an irreducible algebraic variety of codimension $(a-c+1)(b-c+1)$ in $M(a, b)$ (see eg. [13], (4.13), p. 425.)
(b) $D_{c-1}(a, b)$ is the singular locus of the scheme $D_c(a, b)$. (Although this result seems to be part of the folklore on determinantal schemes it is difficult to find good references. One may obtain a proof by combining Theorem (4.10) (p. 424) with Corollary (6.3) p. 428 of [13]).
(c) $D_c(a, b)$ is a Cohen-Macaulay scheme. ([6] Theorem 1, p. 1023. Similar algebraic proofs of a more general result can be found in

[11], Corollary (3.13), p. 53, [14] Theorem 12, p. 7, and [15]. A beautiful geometric proof can be found in [9] Theorem 2, p. 7).

Let $X = \operatorname{Spec} A$ be an affine scheme and $M = [a_{i,j}]$ an $(a \times b)$-matrix with entries from A. We denote by $D_c(M)$ the scheme where the minors of M of order c vanish. That is $D_c(M)$ is the closed subscheme of X defined by the ideal generated by the determinants of all $(c \times c)$-submatrices of M. We say that the scheme $D_c(M)$ is determinantal if it is of pure codimension $(a-c+1)(b-c+1)$ in X (empty if $(a-c+1)(b-c+1)$ is strictly greater than $\dim(X)$).

Clearly, a subscheme Y of X is determinantal if and only if there exists a morphism

$$f : X \to M(a, b)$$

such that Y is the scheme theoretic inverse image of a generic determinantal subscheme $D_c(a, b)$ of $M(a, b)$ and Y is of pure codimension $(a-c+1)(b-c+1)$ in X.

THEOREM 4: *Let M be an $(a \times b)$-matrix with entries in a polynomial ring $k[X_1, \cdots, X_p]$. Assume that the subscheme $D_c(M)$ of*

$$\mathbb{A}^p = \operatorname{Spec} k[X_1, \cdots, X_p],$$

where the minors of M of order c vanish, is determinantal (that is a subscheme of $\mathbb{A}^p$ of pure codimension $(a-c+1)(b-c+1)$).

Then there exists a faithfully flat morphism

$$q : V \to W$$

from an algebraic scheme V to a regular, irreducible algebraic scheme W of dimension $(p+1)ab$ and an open dense subset U of W such that the scheme $D_c(M)$ is isomorphic to the fiber of q at some rational point of W and such that for each rational point g of U the following assertions hold:

(a) *There exists an $(a \times b)$-matrix $M(g)$ with entries in the polynomial ring $k[X_1, \cdots, X_p]$ such that the fiber $q^{-1}(g)$ is isomorphic to the scheme $D_c(M(g))$.*

(b) *Each scheme $D_i(M(g))$ in the sequence*

$$\Phi = D_0(M(g)) \subseteq D_1(M(g)) \subseteq \cdots \subseteq D_c(M(g))$$

is a determinantal subscheme of the affine space

$$\mathbb{A}^p = \operatorname{Spec} k[X_1, \cdots, X_p].$$

(c) *$D_{i-1}(M(g))$ is the singular locus of the scheme $D_i(M(g))$ for $i = 1, \cdots, c$.*

PROOF: By assumption $D_c(M)$ is a determinantal subscheme of $\mathbb{A}^p$. Hence there exists a morphism

$$f : \mathbb{A}^p \to M(a, b)$$

such that the scheme $D_c(M)$ is the inverse image of the generic determinantal scheme $D_c(a, b)$. Put $q = a \cdot b$. Denote by G the $(p+1) \cdot q$-dimensional affine space of all $(p+1) \times q$-matrices and by e the rational point corresponding to the matrix with all entries equal to zero. Then it follows from the proposition of section 4 that there exists a morphism

$$F : G \times \mathbb{A}^p \to \mathbb{A}^q$$

satisfying the conditions (*) and (**) of the transversality theorem and such that the morphism f is the restriction of F to the scheme $(\mathbb{A}^p \times e) \approxeq \mathbb{A}^p$. Because of the facts (a), (b) and (c) above, about generic determinantal schemes, and the assumption that $D_c(M)$ is a subscheme of $\mathbb{A}^p$ of pure codimension $(a-c+1)(b-c+1)$ we can apply Theorem 2 to the sequence

$$\Phi = D_0(a, b) \subseteq D_1(a, b) \subseteq \cdots \subseteq D_c(a, b)$$

of subschemes of $M(a, b)$. Let W be the image of V by the morphism q_D of Theorem 2. Then q_D induces a morphism

$$q : V \to W$$

and W is an open subset of G since the morphism $(q_D|V)$ is flat. (A-K, V, (5.1) or EGA, IV_2, (2.4.6)). Then the assertions of Theorem 4 follow immediately from the corresponding assertions of Theorem 2.

Let Y be an algebraic scheme. A faithfully flat morphism $q : V \to W$ of algebraic schemes such that Y is isomorphic to the fiber of q at some rational point of W we call a (global) deformation of Y. We state the next result in this terminology.

COROLLARY: *Let Y be a Cohen-Macaulay subscheme of pure codimension two in a p-dimensional affine space* $\mathbb{A}^p = \operatorname{Spec} k[X_1, \cdots, X_p]$.

Then there exists a deformation

$$q : V \to W$$

of Y, where W is a regular scheme of dimension $n(n-1)$ for some integer n, and an open dense subset U of G such that for each rational point g of U the following assertions hold:

(a) *There exists an $(n-1) \times n$-matrix $M(g)$ with entries in the polynomial ring $k[X_1, \cdots, X_p]$ such that the fiber $q^{-1}(g)$ is isomorphic to the scheme $D_{n-1}(M(g))$.*

(b) *Each scheme $D_i(M(g))$ in the sequence*

$$\Phi = D_0(M(g)) \subseteq D_1(M(g)) \subseteq \cdots \subseteq D_{n-1}(M(g))$$

is a determinantal subscheme of $\mathbb{A}^p$ (that is, a subscheme of $\mathbb{A}^p$ of pure codimension $(n-i+1)(n-i)$).

(c) *$D_{i-1}(M(g))$ is the singular locus of the scheme $D_i(M(g))$ for $i = 1, \cdots, n$.*

In particular the scheme $D_{n-2}(M(g))$ is of codimension six in $\mathbb{A}^p$, hence empty if p is less than six. Consequently, a Cohen-Macaulay subscheme of pure codimension two in an affine space of dimension less than six has non-singular deformations.

Proof: The corollary follows immediately from the theorem and from the following well known result.

Lemma: (M. Schaps' globalization of a theorem of L. Burch [5] Theorem 5, p. 944). *Let Y be a Cohen-Macaulay subscheme of pure codimension two in a p-dimensional affine space $\mathbb{A}^p = \operatorname{Spec} k[X_1, \cdots, X_p]$.*

Then for some integer n there exists an $(n-1) \times n$-matrix M with entries in the polynomial ring $k[X_1, \cdots, X_p]$ such that Y is isomorphic to the scheme $D_{n-1}(M)$ where the minors of M of order $(n-1)$ vanish.

Proof: Denote by I the ideal in the polynomial ring $P = k[X_1, \cdots, X_p]$ defining the subscheme Y of $\mathbb{A}^p = \operatorname{Spec} k[X_1, \cdots, X_p]$. Choose a partial resolution $P^r \xrightarrow{\alpha} P \to P/I \to 0$ of the quotient ring P/I, where P^r is the direct sum of P with itself r-times. Let Q be a maximal ideal in P containing I. Since P_Q is a regular local ring of dimension p we have that depth (P_Q/IP_Q) + proj. dim $(P_Q/IP_Q) = p$ (A-K, III, (5.19) or EGA, IV_1 (17.3.4)). Moreover, we have that depth $(P_Q/IP_Q) = \dim (P_Q/IP_Q) = (p-2)$ since Y is Cohen-Macaulay of pure codimension two. Hence the kernel of the homomorphism $\alpha_Q : P_Q^r \to P_Q$ is locally free (A-K, III, (5.2)).

We conclude that the kernel K of the homomorphism α is a projective P-module. By a well known result from algebraic K-theory every projective module over a polynomial ring P can be imbedded in a free module with a free complement (in fact this follows easily from the result that the Grothendieck group $K_0(P)$ of P is canonically isomorphic to the integers [2], chapter XII, Theorem (3.1)). Hence we may write $K \oplus P^m = P^{n-1}$ for some integers m and n. Denote by β the homomorphism $P^m \oplus P^r \to P$ which is zero on P^m and coincides with α on P^r. Then clearly the kernel of β is the free module P^{n-1}. Consequently we have an exact sequence

$$0 \to P^{n-1} \xrightarrow{\gamma} P^{m+r} \to P \to P/I \to 0.$$

Localizing at the prime ideal (0) of P we see that $m+r = n$.

Let $[a_{i,j}]$ be the $(n-1) \times n$-matrix representing the homomorphism γ with respect to a choice of basis $e_1, \cdots, e_{n-1}$ and $f_1, \cdots, f_n$ of P^{n-1} and P^n. Denote by $\delta : P^n \to P$ the homomorphism sending f_i to the determinant of the $(n-1) \times (n-1)$-matrix obtained from the matrix $[a_{i,j}]$ by omitting the i'th column and denote by J the image of δ. We easily check that the composite $\delta \cdot \gamma$ is the zero homomorphism (indeed, this corresponds to the fact that the determinant of the $n \times n$-matrix, obtained by adding to the matrix $[a_{i,j}]$ one of its rows, is zero). Hence there exists a homomorphism $I \to J$ such that the composite homomorphism $P^n \to I \to J$ coincides with δ. To prove the lemma it suffices to show that the ideals I and J are the same. This can be accomplished by following the original proof by Burch ([5] proof of Theorem 5, p. 944), however, we prefer to present a proof which follows closer to a method used by D. A. Buchsbaum to study ideals of projective dimension one ([4] section 3, p. 268).

By assumption the scheme Y is of pure codimension two in the affine space Spec P. Hence the length $\text{depth}_I(P)$ of the longest P-regular sequence contained in I is equal to two. It follows that $\text{Ext}^i_p(P/I, P)$ is zero for $i = 0, 1$. (A-K, III, (3.8)). Hence, by the long exact sequence of Ext's derived from the short exact sequence $0 \to I \xrightarrow{\varepsilon} P \to P/I \to 0$, we conclude that the homomorphism $\text{Hom}(\varepsilon, \text{id}_p) : \text{Hom}(P, P) \to \text{Hom}(I, P)$ is an isomorphism. In particular, there exists a homomorphism $g : P \to P$ making the diagram

$$\begin{tikzcd}
0 \arrow[r] & P^{n-1} \arrow[r] \arrow[d, equal] & P^n \arrow[r] \arrow[d, equal] & I \arrow[r, "\varepsilon"] \arrow[d] & P \arrow[d, "g"] \\
0 \arrow[r] & P^{n-1} \arrow[r] & P^n \arrow[r, "\delta"] & J \arrow[r] & P
\end{tikzcd}$$

commutative. Since every homomorphism $P \to P$ is of the form $f \to f \cdot g$ for some polynomial g of P we conclude that J is contained in a principal ideal hP of P. Then, either h is an invertible element of P and hence I coincides with J, which is the desired result, or J is contained in a proper principal ideal of P.

We shall exclude the last possibility. Choose a prime ideal Q of P not containing the ideal I. Then the inclusion $\varepsilon_Q : I_Q \to P_Q$ is an isomorphism and consequently the sequence $0 \to P_Q^{n-1} \xrightarrow{\varepsilon_Q} P_Q^n \to P_Q \to 0$ splits. It easily follows that the homomorphism $\delta_Q : P_Q^n \to P_Q$ is surjective (indeed, we can choose a basis $f_1', \cdots, f_n'$ of P_Q^n, such that the elements $f_1', \cdots, f_{n-1}'$ is a basis of P_Q^{n-1}. Then the element f_n' maps to the unity of P_Q by the homomorphism δ). Consequently the ideal Q is not contained in the support of the P-module P/J. We have thus proved that the support of P/J is contained in the set of prime ideals containing I. Hence the height of the ideal J of P is at least equal to the height of the ideal I which is two by our assumptions. Consequently J cannot be contained in a proper principal ideal. This finishes the proof of the lemma.

To treat the exceptional case of determinantal schemes of codimension one, we shall need the following version of Bertini's theorem.

Proposition: *Let h be an element of the polynomial ring $k[X_1, \cdots, X_p]$. Then there exists a deformation*

$$q : V \to W$$

of the scheme $Y = \operatorname{Spec}(k[X_1, \cdots, X_p]/h)$, where W is a regular scheme of dimension $(p+1)$, and an open dense subset U of W such that for each rational point g of U the fiber of the morphism q at g is a regular scheme.

Proof: Denote by φ the homomorphism $k[Y] \to k[X_1, \cdots, X_p]$ of polynomial rings which sends the indeterminate Y to the polynomial h, and let $f : \mathbb{A}^p \to \mathbb{A}^1$ be the morphism of affine spaces corresponding to the homomorphism φ. Clearly, the scheme Y is the scheme theoretic inverse image of the point (0) in $\mathbb{A}^1$ by the morphism f.

Denote by G the affine $(p+1)$-dimensional space of $(1 \times p)$-matrices and by e the point corresponding to the matrix $(0, \cdots, 0)$. Then, by the proposition of section four, there exists a morphism $F : G \times \mathbb{A}^p \to \mathbb{A}^1$ satisfying the conditions (*) and (**) of the transversality theorem and such that the morphism f is the restriction of F to the scheme $(\mathbb{A}^p \times e) \cong \mathbb{A}^p$. The assertion of the proposition then follows immediately from the corresponding assertions of Theorem 2 (applied to the subscheme (0) of $\mathbb{A}^1$).

Corollary: *Let M be an $a \times a$-matrix with entries in the polynomial* ring $k[X_1, \cdots, X_p]$. Then the subscheme $D_a(M)$ of the affine space $\mathbb{A}^p = \operatorname{Spec} k[X_1, \cdots, X_p]$, *where the determinant of M vanishes, can be deformed into a regular scheme.*

REFERENCES

[1] A-K. A. Altman and S. L. Kleiman: *Introduction to Grothendieck duality theory.* Lecture notes in mathematics no. 146, Springer Verlag, 1970.

[2] H. Bass: *Algebraic K-theory.* Benjamin, New York (1968).

[3] J. Briancon and A. Galligo: *Déformations distinguées d'un point de* $\mathbb{C}^2$*ou* $\mathbb{R}^2$. Astérisque no. 7–8 (1973) 129–138.

[4] D. A. Buchsbaum: *Complexes associated with the minors of a matrix.* Symposia Mathematica vol. IV, Bologna (1970) 255–283.

[5] L. Burch: *On ideals of finite homological dimension in local rings.* Proc. Cambridge Phil. Soc. vol. 64 (1968) 941–952.

[6] J. A. Eagon and M. Hochster: Cohen-Macaulay rings, invariant theory, and the generic perfection of determinantal loci. *Am. Journ. Math. vol. 93* (1971) 1020–1058.

[7] EGA. A. Grothendieck (with J. Dieudonné), *Eléments de géométrie algébrique.* Chap. IV, 1^{ere}, 2^{eme}, 3^{eme}, 4^{eme} parties, Publ. Math. de I.H.E.S., no. 20, 24, 28, 32 (1964, 1965, 1966, 1967).

[8] J. Fogarty: Algebraic families on an algebraic surface. *Am. Journ. Math., vol. 90* (1968) 511–521.

[9] G. Kempf: *Schubert methods with an application to algebraic curves.* Mathematisch Centrum, Amsterdam (1971).

[10] G. Kempf and D. Laksov: The determinantal formula of Schubert calculus. *Acta Matematica, vol. 132* (1974) p. 153–162.

[11] M. Hochster: Grassmannians and their Schubert subvarieties are arithmetically Cohen-Macaulay. *Journ. Algebra 25* (1973) 40–57.

[12] S. L. Kleiman: The transversality of a general translate. *Compositio Math., vol. 28* (1974) 287–297.

[13] S. L. Kleiman and J. Landolfi: Geometry and deformation of special Schubert varieties. *Compositio Math., vol. 23* (1971) 407–434.

[14] D. Laksov: The arithmetic Cohen-Macaulay character of Schubert schemes. *Acta Mathematica vol. 129* (1972) 1–9.

[15] C. Musili: Postulation formula for Schubert varieties. *Journ. Indian Math. Soc. 36* (1972) 143–171.

[16] M. Schaps: *Non singular deformations of space curves using determinantal schemes.* Doctoral dissertation, Harvard University, Cambridge, Mass. (1972).

[17] T. Svanes: *Coherent cohomology on flag manifolds and rigidity.* Doctoral dissertation, M.I.T., Cambridge, Mass. (1972).

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