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CONJUGACY THEOREMS FOR A CLASS OF LOCALLY FINITE LIE ALGEBRAS

Ian Stewart

The object of this paper is to extend the classical conjugacy theorems for Levi, Borel, and Cartan subalgebras of finite-dimensional Lie algebras over an algebraically closed field $\mathfrak{K}$ of characteristic zero, to a class of infinite-dimensional locally finite Lie algebras. The subject-matter lies in an area where interactions between the theories of groups and of Lie algebras have been, and may be, used to advantage. The classical concept of a Cartan subalgebra was extended by Carter [9] to finite soluble groups, and broadly generalized in the 'formation theory' of Gaschütz [13]. Stonehewer [33, 34, 35] extended much of this theory to certain classes of infinite groups; Wehrfritz [39] to linear groups; and Tomkinson [36] to periodic FC-groups (in which all conjugacy classes are finite). Gardiner, Hartley, and Tomkinson [12] found a simultaneous generalization of the work of Stonehewer and of Wehrfritz; and recently Klimowicz [18, 19, 20] has developed a general axiomatic setting for all of these theories. On the other hand, the theory has returned to its origins, in that Barnes and Gastineau-Hills [3], Barnes and Newell [4], and Stitzinger [32] have developed a version of formation theory for finite-dimensional Lie algebras.

This work, especially that of Tomkinson [36], is very suggestive as regards certain infinite-dimensional Lie algebras. For a group is a periodic FC-group if and only if it is generated by finite normal subgroups. Let us define an analogous class (called $\overline{\mathfrak{F}}$ in [1] p. 258) of Lie algebras, consisting of those algebras which can be generated by a system of finite-dimensional ideals. We shall call algebras in $\overline{\mathfrak{F}}$ *ideally finite* (since this is a property similar to, but stronger than, being locally finite). For some time it has been suspected that generalizations of the classical conjugacy theorems to this class might be possible, and some tentative steps in this direction were taken in [26, 27, 28, 1], where much less is proved but in a more general setting. We shall confirm this suspicion here, provided that the ground field is algebraically closed and of characteristic zero. This restriction, which is natural in the context of the conjugacy theorems but less so for the existence theorems, arises on purely technical grounds as follows.

In the study of FC-groups a prominent role is played by the theorem that a projective limit of non-empty finite sets is non-empty (equivalent to, but more convenient than, the theorem of Kuroš [21] p. 167 on 'projection sets'). For example the local conjugacy of Sylow p-subgroups of a periodic FC-group is proved by finding a system of conjugating elements in each finite normal subgroup, and passing to a projective limit to obtain an automorphism of the whole group with the correct property. For ideally finite Lie algebras there is an obstacle, in that the sets to which a projective limit argument must be applied are no longer finite. In many instances, however, they have a natural structure as algebraic varieties, suggesting a topological approach by way of the Zariski topology. The usual non-emptiness theorem for projective limits of topological spaces applies only to compact Hausdorff spaces (Bourbaki [8] p. 89) and cannot be used since the Zariski topology, though compact, is not Hausdorff. Instead we use a theorem of Serre [22] p. 15, based on a criterion of Bourbaki [7] p. 138. Serre's theorem is stated for homogeneous spaces arising from abelian algebraic groups, and requires an algebraically closed field. The proof extends to a slightly more general situation which is sufficient for our purposes. It turns out to be advantageous to work with a topology weaker than the Zariski topology, but still non-Hausdorff.

The restrictions on the field also occur in connection with a second technical problem, of finding the 'correct' group of automorphisms. For Levi and Borel subalgebras this is easy. But for Cartan subalgebras we require a more delicate approach to cope with extension and lifting arguments. This problem is dealt with in sections 3 and 6.

Because the results on projective limits are crucial for everything that follows, we give a relatively self-contained discussion of them in section 2 below. In section 3 we recall some necessary facts about automorphisms of finite-dimensional Lie algebras. In section 4 we discuss Levi subalgebras (a term preferable to the usual 'Levi factor'), that is, semisimple complements to the locally soluble radical. The existence of Levi subalgebras follows from [26, 27] which establish it in the wider class $\acute{N}\mathfrak{F}$ of Lie algebras generated by finite-dimensional ascendant subalgebras. We prove that Levi subalgebras are conjugate under a group of automorphisms analogous to the 'locally inner automorphisms' which occur in the group-theoretic results. In section 5 we define Borel subalgebras, the existence of which is trivially true, and prove them conjugate. Section 6 develops lifting and extension properties of a particular type of automorphism, preparatory to the results of section 7 on Cartan subalgebras. These are defined in that section as 'locally nilpotent projectors' in the sense of formation theory; and we prove existence and conjugacy. I am indebted to the referee for the existence proof given here. Between the initial submission of this paper (which lacked an existence proof) and the referee's report (supplying one) I found another much longer proof using a version of nilpotent-semisimple splitting and properties of maximal tori. This approach is of some interest in its own right, and is being written up as [29]. Finally in section 8 we give a conjugacy theorem for Borel-Cartan pairs (B, C) where B is a Borel subalgebra and C is a Cartan subalgebra of B; show that C is in fact a Cartan subalgebra of the whole algebra; and sharpen the conjugacy theorem for Levi subalgebras so that the same group of automorphisms occurs in all cases.

Levi, Borel, and Cartan subalgebras have been treated separately, rather than some kind of formation-theoretic generalization, for several reasons. The first is that we do not wish to be restricted to the locally soluble case. A second is that over an algebraically closed field of characteristic zero the only non-trivial saturated formations (in the sense of Barnes and Gastineau-Hills [3]) are the classes of nilpotent or soluble algebras; so the special cases of Borel and Cartan subalgebras exhaust the interesting possibilities. A third is that such generalization, at the present stage, would introduce more extraneous notions than the results would warrant. The situation might be improved if a version of formation theory could be developed for insoluble finite-dimensional Lie algebras (with the usual restrictions on the field). There are slight hints that this may be possible, for instance Levi subalgebras are precisely the projectors for the class of semisimple algebras, but nothing definite seems to be known.

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1. Notational conventions

Notation for infinite-dimensional Lie algebras, now fairly standard, will be as in [25] p. 291 or in [1]. For convenience the more common notation will be stated below. In addition the notation and results of [26, 27] will be used freely.

Throughout this paper, $\mathfrak{K}$ *will denote an algebraically closed field of characteristic zero and* $\mathfrak{k}$ *a field of characteristic zero.* This convention will be used to shorten the statements of theorems, and should be borne in mind when reading them.

A Lie algebra L is *ideally finite* if it can be generated by a system of finite-dimensional ideals. We let $\overline{\mathfrak{F}}$ denote the class of ideally finite Lie algebras. Clearly every ideally finite Lie algebra is *locally finite*, that is, every finite subset is contained in a finite-dimensional subalgebra. The class $\overline{\mathfrak{F}}$ is a subclass of $\acute{\mathrm{N}}\mathfrak{F}$, defined in [26], so the results of [26, 27] apply to $\overline{\mathfrak{F}}$-algebras over $\mathfrak{k}$. It is easy to find examples of ideally finite Lie algebras: for instance any subquotient of a direct sum of finite-dimensional algebras.

Let L be a Lie algebra. The notation $H \leqq L$ (respectively $H \lhd L$) will mean that H is a subalgebra (respectively ideal) of L. If $x \in L$ then x^* denotes the adjoint map $L \to L$, given by $yx^* = [y, x]$ for $y \in L$. Other notations in the literature are ad(x) or $\dot{x}$. To avoid ambiguity we write x_L^*. If α is an automorphism of L we write x^α for the image of $x \in L$ under α. This distinguishes automorphisms from other maps, which will be written on the right (or sometimes left) in the usual way. If $X \subseteq L$ we define the *centralizer*

$$C_L(X) = \{x \in L : [x, X] = 0\}$$

and the *idealizer*

$$I_L(X) = \{x \in L : [x, X] \subseteq X\}.$$

If $X = I_L(X)$ we say that X is *self-idealizing*. We use $\zeta_n(L)$, L^n, and $L^{(n)}$ respectively to denote the nth terms of the upper central, lower central, and derived series of L; so that $\zeta_1(L)$ is the centre of L and $L^2 = [L, L] = L^{(1)}$. We write $\rho(L)$ for the *Hirsch-Plotkin radical* of L, namely the unique maximal locally nilpotent ideal of L (see Hartley [15] p. 265) and $\nu(L)$ for the *Fitting radical*, which is the sum of the nilpotent ideals of L. If L is locally finite there exists a unique maximal locally soluble ideal (see [26] p. 83) which we call $\sigma(L)$. If L is finite-dimensional then $\rho(L) = \nu(L)$ is the nil radical and $\sigma(L)$ is the soluble radical. We say that L is *semisimple* if $\sigma(L) = 0$.

By an *algebraic group* we shall always mean an affine algebraic group over $\mathfrak{K}$ (or $\mathfrak{k}$) of finite dimension. A *morphism* of algebraic groups is a morphism of algebraic varieties which is also a group homomorphism. If G is an algebraic group acting morphically on a variety V, and $W \subseteq V$, then we define the *stabilizer* (or *normalizer*) of W in G to be

$$N_G(W) = \{\alpha \in G : W^\alpha = W\}.$$

If G is a group of automorphisms of L, we say that two subsets X and Y of L are *G-conjugate* if there exists $\alpha \in G$ such that $X^\alpha = Y$.

2. Projective limits

Much of the material in this section is standard, but we shall give enough details to make it clear which definitions we use, and to make the paper relatively self-contained.

Let I be a directed set with partial ordering $\leqq$. A *projective limit system* over I (of topological spaces) is a set of topological spaces $\{X_i\}_{i \in I}$ together with continuous maps $\{f_{ij} : X_i \to X_j : i, j \in I,\ i \geqq j\}$ such that if $i \geqq j \geqq k$ are elements of I then

$$f_{ij} f_{jk} = f_{ik},$$

and f_{ii} is the identity map on X_i for each $i \in I$. We write

$$\boldsymbol{X} = \{X_i, f_{ij}\}$$

to denote such a system. The *projective limit*

$$X = \text{proj lim } X_i = \text{proj lim } \boldsymbol{X}$$

is the subset of $\prod_{i \in I} X_i$ consisting of those elements (x_i) such that $f_{ij}(x_i) = x_j$ for all $i, j \in I$ with $i \geqq j$. Restriction of the coordinate projections to X gives canonical maps $f_i : X \to X_i$. We may equip X with the weakest topology making all f_i continuous, which is the same as the subspace topology on X induced by the Tychonoff topology on $\prod_{i \in I} X_i$ (see Bourbaki [7] p. 52 § 4 no. 4).

To settle terminology, let us call a topological space *compact* if every open cover has a finite subcover; or equivalently if every collection of closed sets, of which finite intersections are non-empty, has non-empty intersection. (Bourbaki's term is *quasi-compact*.) A T_1-*space* is one in which singletons are closed. A map $f : X \to Y$, where X and Y are topological spaces, is *closed* if $f(F)$ is closed in Y for every closed set $F \subseteq X$.

The following result (apart from (d)) can be read off from Bourbaki [7] p. 138, appendix. For completeness we give a proof. It is fundamental for the whole paper.

THEOREM (2.1): *Let $\mathbf{X} = \{X_i, f_{ij}\}$ be a projective limit system of topological spaces X_i, such that*

(i) *Each X_i is non-empty, compact, and T_1;*
(ii) *The maps f_{ij} are closed.*

Then

(a) $X = \text{proj lim } \mathbf{X}$ *is non-empty;*
(b) *The image of the canonical projection $f_i : X \to X_i$ is*

$$f_i(X) = \bigcap_{j \geqq i} f_{ji}(X_j);$$

(c) *If bars denote closures then for $A \subseteq X$,*

$$\bar{A} = \text{proj lim } \overline{f_i(A)}$$

and if A is closed then

$$A = \text{proj lim } f_i(A) = \text{proj lim } \overline{f_i(A)};$$

(d) *X is compact in the aforementioned topology.*

PROOF: We use Zorn's lemma. Let $\mathscr{S}$ be the set of subsystems $\{A_i\}$ of X such that A_i is a non-empty closed subset of X_i and $f_{ij}(A_i) \subseteq A_j$ for all $i, j \in I$ with $i \geqq j$. Define a partial ordering $\leqslant$ on $\mathscr{S}$ such that $\{A_i\} \leqslant \{B_i\}$ if $A_i \subseteq B_i$ for all $i \in I$. By compactness it follows that $\mathscr{S}$ satisfies the hypotheses of Zorn's lemma with respect to the inverse ordering $\geqslant$, so there exists a minimal (with respect to $\leqslant$) member $\{A_i\}$ of $\mathscr{S}$. By (ii) it follows that

$$B_i = \bigcap_{j \geqq i} f_{ji}(A_j)$$

is closed in A_i, from which it is easy to see that $\{B_i\} \in \mathscr{S}$. By minimality $B_i = A_i$ for all i, or equivalently the maps f_{ij} are surjective when restricted to the A_i. For a fixed $i \in I$ choose $x_i \in A_i$. If we let

$$\begin{aligned} C_j &= f_{ji}{}^{-1}(x_i) \cap A_j \qquad &(j \geqq i) \\ C_j &= A_j &(j \ngeqq i) \end{aligned}$$

then the T_1 condition implies that $\{C_i\} \in \mathscr{S}$. By minimality $A_i = C_i = \{x_i\}$. Then $(x_i) \in \prod X_i$ belongs to proj lim X_i, which proves (a).

For (b) take $x_i \in \bigcap_{j \geqq i} f_{ji}(X_j)$. Replace the X_j by $f_{ji}^{-1}(x_i)$ if $j \geqq i$ and apply part (a) to the resulting system.

To prove (c) let

$$A' = \bigcap_{i \in I} f_i^{-1}(\overline{f_i(A)}).$$

This is closed in X and contains A, hence contains $\bar{A}$. We show that each point of A' is a closure point of A. Since the Tychonoff topology on $\prod X_i$ is defined so that its non-empty open sets are unions of sets of the form $\prod G_i$, with G_i open in X_i and all but a finite number of $G_i = X_i$, it suffices to prove that each neighbourhood of $x \in A'$ which is of the form $f_i^{-1}(G_i)$ for G_i open in X_i must intersect A. For then every neighbourhood of x must intersect A. Now $f_i(x) \in G_i$, and so $G_i \cap \overline{f_i(A)} \neq \phi$, so $A \cap f_i^{-1}(G_i) \neq \emptyset$. Hence

$$\bar{A} = \bigcap_{i \in I} f_i^{-1}(\overline{f_i(A)}) = \text{proj lim}\, \overline{f_i(A)}.$$

If further A is closed then

$$A \subseteq \text{proj lim}\, f_i(A) \subseteq \text{proj lim}\, \overline{f_i(A)} = \bar{A} = A.$$

Finally we prove part (d). Let $\{F^\lambda\}_{\lambda \in \Lambda}$ be a set of closed subsets of X with all finite intersections non-empty. Then

$$F^\lambda = \text{proj lim}\, f_i(F^\lambda) = \text{proj lim}\, \overline{f_i(F^\lambda)}$$

for each $\lambda \in \Lambda$. For fixed i and variable λ the sets $\overline{f_i(F^\lambda)}$ have all finite intersections non-empty, so by compactness of X_i

$$B_i = \bigcap_{\lambda \in \Lambda} \overline{f_i(F^\lambda)} \neq \emptyset.$$

Then $\{B_i\}_{i \in I}$ forms a projective limit system whose maps are the restrictions of the f_{ij}. Because hypotheses (i) and (ii) carry over to $\{B_i\}$, we have proj lim $B_i \neq \emptyset$. If we take $x \in$ proj lim B_i then $x \in F^\lambda$ for all $\lambda \in \Lambda$, hence $x \in \bigcap_{\lambda \in \Lambda} F^\lambda$ which is therefore non-empty. This proves that X is compact.

In the sequel we shall require only part (a) of this theorem, but it is likely that the other parts will be required in future developments. Note that (d) does not contradict a statement of Serre [22] p. 16 since we are assuming the f_{ij} to be closed.

We wish to apply the above theorem to certain varieties associated with affine algebraic groups, and it is convenient to define a topology weaker than the Zariski topology. Let G be an affine algebraic group over $\mathfrak{K}$. Let $\mathscr{Z}$ denote the Zariski topology on G. Define a topology $\mathscr{W}$ as follows: a closed subbase conists of all cosets xH of $\mathscr{Z}$-closed subgroups H of G. Thus a closed set in $\mathscr{W}$ is a finite union

$$x_1 H_1 \cup \cdots \cup x_n H_n$$

where $x_1, \cdots, x_n \in G$ and $H_1, \cdots, H_n$ are algebraic (or $\mathscr{Z}$-closed) subgroups of G.

Clearly $\mathscr{W} \subseteq \mathscr{Z}$ so that G is compact in $\mathscr{W}$ (indeed noetherian, cf. Borel [6]). Since the identity subgroup is $\mathscr{Z}$-closed, $\mathscr{W}$ is T_1. (Similar remarks apply to $\mathscr{W}'$, defined analogously but using cosets Hx instead of xH, or to $\mathscr{W}''$ generated by $\mathscr{W}$ and $\mathscr{W}'$.) The advantage of $\mathscr{W}$ over $\mathscr{Z}$ stems from:

LEMMA (2.2): *Let G and K be affine algebraic groups over $\mathfrak{K}$, and $\alpha : G \to K$ an affine algebraic group morphism. If G and K are equipped with the $\mathscr{W}$-topology, then α is both continuous and closed.*

PROOF: Certainly α is $\mathscr{Z}$-continuous. If H is a $\mathscr{Z}$-closed subgroup of K then $\alpha^{-1}(H)$ is a $\mathscr{Z}$-closed subgroup of G. Let $x \in K$, and pick $z \in \alpha^{-1}(x)$ if this is non-empty. It is easy to check that

$$\alpha^{-1}(xH) = z\alpha^{-1}(H)$$

(independently of the choice of z) and the latter is $\mathscr{W}$-closed in G. Hence α is $\mathscr{W}$-continuous.

Now let L be a $\mathscr{Z}$-closed subgroup of G, and $g \in G$. Then $\alpha(gL) = \alpha(g)\alpha(L)$. Since $\mathfrak{K}$ is algebraically closed it follows (Borel [6] p. 88 corollary 1.4(a)) that $\alpha(L)$ is $\mathscr{Z}$-closed in H, hence $\alpha(gL)$ is $\mathscr{W}$-closed in H. The same goes for finite unions of cosets, so α is a closed map.

Algebraic closure of $\mathfrak{K}$ is essential here. If we consider the multiplicative group G of non-zero reals as an algebraic group over the reals, then the map $\alpha : G \to G$ with $\alpha(x) = x^2$ ($x \in G$) is a morphism, but $\alpha(G)$ is the group of positive reals which is not algebraic, so α is not closed. If instead of $\mathscr{W}$

we use $\mathscr{Z}$, then the analogue of lemma 2.2 does not hold, again because α need not be $\mathscr{Z}$-closed.

Next consider the case of a homogeneous space G/H where H is a $\mathscr{Z}$-closed subgroup of G. If $\alpha : G \to K$ is an algebraic group morphism and $\alpha(H)$ is contained in a $\mathscr{Z}$-closed subgroup L of K there is induced a map

$$\tilde{\alpha} : G/H \to K/L.$$

If we equip G/H and K/L with their quotient topologies relative to $\mathscr{W}$ (which we still call $\mathscr{W}$-topologies) then it remains the case that $\tilde{\alpha}$ is continuous and closed. We define a *coset variety* over $\mathfrak{K}$ to be any $\mathscr{W}$-closed subset of a homogeneous space G/H where G is an affine algebraic group over $\mathfrak{K}$ and H is a $\mathscr{Z}$-closed subgroup. A map $\tilde{\alpha} : G/H \to K/L$, or its restriction to a coset variety contained in G/H, will be called *affine* if it is induced as above from an algebraic group morphism $\alpha : G \to K$ such that $\alpha(H) \subseteq L$. It follows that affine maps between coset varieties are continuous and closed in the $\mathscr{W}$-topology.

We may apply theorem 2.1 to this situation, to obtain the following variant of a theorem of Serre [22] p. 15 or Bourbaki [7].

THEOREM (2.3): *Let $\{X_i, f_{ij}\}$ be a projective limit system, where the X_i are coset varieties over $\mathfrak{K}$ equipped with the $\mathscr{W}$-topology, and the f_{ij} are affine maps. Suppose the X_i are non-empty. Then conclusions* (*a*), (*b*), (*c*) *and* (*d*) *of lemma* 2.1 *hold.*

The topology induced on proj lim X_i will also be called the $\mathscr{W}$-topology.

In applications in this paper the X_i will be either cosets xH of an algebraic group G acting on a vector space over $\mathfrak{K}$, where H is the stabilizer of a subspace U of V; or else homogeneous spaces G/H where H is such a stabilizer. For instance, the set of Cartan subalgebras of a finite-dimensional Lie algebra over $\mathfrak{K}$ is in natural bijective correspondence with G/H where G is the automorphism group of the Lie algebra and H is the stabilizer of a Cartan subalgebra; this because the Cartan subalgebras are G-conjugate (Jacobson [17] p. 273).

We are uncertain to what extent the hypotheses of theorem 2.3 may be weakened. Some restriction on the field or the varieties is required, as the following example (Deligne [11]) makes clear. Let $\mathfrak{k} = \mathbb{Q}(T_i)_{i \in I}$ where the T_i are an arbitrary set of indeterminates, and consider the affine cubic $v \subseteq \mathfrak{k}^2$ defined by

$$y^2 = x^3 + 1.$$

The $\mathfrak{k}$-points of V have rational coordinates, so are countable. We may omit them one by one to give a projective limit system with empty limit. By choosing I large enough we can make the cardinality of $\mathfrak{k}$ arbitrarily large. The best general results on projective limits of algebraic varieties are in Grothendieck [14], but they involve passage to an algebraically closed extension field.

For arbitrary fields the above methods will prove the (doubtless well known) result that a projective limit of finite-dimensional vector spaces, with affine *linear* maps, is non-empty. In place of the $\mathscr{W}$-topology take one with subbase the affine linear subspaces.

3. Groups of automorphisms

Let L be a Lie algebra, not necessarily of finite dimension, over $\mathfrak{k}$. If $x \in L$ is a *nil* element, that is, given any finite-dimensional subspace V of L there exists an integer n such that $Vx^{*n} = 0$, then we may define the exponential

$$\exp(x^*) = 1 + x^* + \frac{1}{2!}x^{*2} + \cdots$$

and this is an automorphism of L (Hartley [15] p. 262, Jacobson [17] p. 9). If X is a subset of a group, define $\langle X \rangle$ to be the subgroup generated by X. We define several groups of automorphisms of L:

$$\begin{aligned}
\mathscr{A}(L) &= \{\text{all automorphisms of } L\} \\
\mathscr{I}(L) &= \{\alpha \in \mathscr{A}(L) : I^\alpha = I \text{ for all } I \lhd L\} \\
\mathscr{N}(L) &= \langle \exp(x^*) : x \text{ nil} \rangle \\
\mathscr{R}(L) &= \langle \exp(x^*) : x \in \rho(L) \rangle.
\end{aligned}$$

Clearly $\mathscr{R}(L) \subseteq \mathscr{N}(L) \subseteq \mathscr{I}(L) \subseteq \mathscr{A}(L)$. The group $\mathscr{R}(L)$ was used in [26] to prove a limited conjugacy theorem for Levi subalgebras.

More important for us is a further group introduced by Winter [40] p. 93 (see also Humphreys [16] p. 82). Suppose that L is finite-dimensional over $\mathfrak{K}$. For $x \in L$, $\lambda \in \mathfrak{K}$ define

$$L_\lambda(x^*) = \{y \in L : y(x^* - \lambda)^n = 0 \text{ for some } n > 0\}.$$

An element $x \in L$ is *strongly ad-nilpotent* if there exists $y \in L$ such that y^* has a non-zero eigenvalue λ with $x \in L_\lambda(y^*)$. This forces x^* to be nilpotent (Humphreys [16] p. 82) and so we may define

$$\mathscr{E}(L) = \langle \exp(x^*) : x \in L, x \text{ strongly ad-nilpotent} \rangle.$$

The group $\mathscr{E}(L)$ possesses several useful properties which the other groups listed do not.

(a) *Extension*: If $H \leqq L$ and x is strongly ad-nilpotent in H, then x is strongly ad-nilpotent in L. The map

$$\exp(x_H^*) \mapsto \exp(x_L^*)$$

extends in an obvious way to a map $\mathscr{E}(H) \to \mathscr{E}(L)$.

(b) *Lifting*: Every epimorphism $L \to L'$ induces an epimorphism $\mathscr{E}(L) \to \mathscr{E}(L')$.

For proofs see Humphreys [16] p. 82 or Winter [40] p. 93.

By virtue of these properties the elements of $\mathscr{E}(L)$ are good substitutes for inner automorphisms of groups. Finally, for technical reasons we must ensure that we are working with algebraic groups. Now if L has finite dimension then $\mathscr{A}(L)$ is affine algebraic (Chevalley [10] p. 143). If G is an algebraic group acting morphically on a variety V and if W is a closed subvariety of V then $N_G(W)$ is a closed subgroup of G, by Borel [6] p. 97. Hence

$$\mathscr{I}(L) = \bigcap_{I \lhd L} N_{\mathscr{A}(L)}(I)$$

is closed, thus algebraic. Now $\mathscr{E}(L)$ is generated by 1-parameter subgroups

$$\{\exp(\lambda x^*) : \lambda \in \mathfrak{K}\}$$

each of which is a connected algebraic group. Finitely many of these therefore generate a connected algebraic group. But $\mathscr{E}(L)$ is connected, and satisfies the ascending chain condition for closed connected subgroups (Borel [6] p. 5), so $\mathscr{E}(L)$ is generated by finitely many 1-parameter subgroups, so is algebraic. Similarly $\mathscr{N}(L)$ and $\mathscr{R}(L)$ are algebraic groups.

(I am indebted to David Winter for this argument).

4. Levi subalgebras

A *Levi subalgebra* of a locally finite Lie algebra L is a semisimple subalgebra complementing the radical $\sigma(L)$. From [27] it follows that every semisimple ideally finite Lie algebra is a direct sum of finite-dimensional simple algebras, and Levi subalgebras of ideally finite algebras exist and are precisely the maximal semisimple subalgebras. In this section we shall use the results of Section 2 to prove a conjugacy theorem for Levi subalgebras. The considerations surrounding $\bar{\mathscr{E}}$ are not needed in this case (except for a sharper result which we give later) so the method is particularly transparent.

Let L be an ideally finite Lie algebra over $\mathfrak{K}$, and let $\{F_i\}_{i\in I}$ be the set of all finite-dimensional ideals of L. We can partially order I by letting $i \leqq j$ if $F_i \leqq F_j$ $(i, j \in I)$, and clearly this makes I a directed set. For each $i \in I$ we define

$$\mathscr{I}_i = \mathscr{I}(F_i)$$

so that $\mathscr{I}_i$ is an algebraic group acting linearly on F_i and fixing setwise all smaller F_j. Now for $j \leqq i$ the restriction map defines a function

$$f_{ij} : \mathscr{I}_i \to \mathscr{I}_j$$

and one verifies that

$$\{\mathscr{I}_i, f_{ij}\}$$

is a projective limit system. We define

$$\hat{\mathscr{I}}(L) = \text{proj lim } \mathscr{I}_i.$$

This is a group, with a natural action by automorphisms of L. For if $\alpha = (\alpha_i) \in \hat{\mathscr{I}}(L)$ we may define, for $x \in F_j$,

$$x^\alpha = x^{\alpha_j}.$$

This is well defined by the projective property, and has inverse (α_i^{-1}); it is an automorphism sonce any pair of elements of L lie in some F_i. We might say that $\hat{\mathscr{I}}(L)$ is a *proalgebraic group* (although Serre [22] and others use this term in a more restricted sense), that is, a projective limit of algebraic groups. It has a natural compact topology, namely the $\mathscr{W}$-topology of § 2. We identify $\hat{\mathscr{I}}(L)$ with its image as a subgroup of $\mathscr{A}(L)$, and it then contains $\mathscr{R}(L)$.

THEOREM (4.1): *Let L be ideally finite over $\mathfrak{K}$. Then all Levi subalgebras of L are $\hat{\mathscr{I}}(L)$-conjugate.*

PROOF: Let $\{F_i\}_{i\in I}$ be the set of all finite-dimensional ideals of L, and order I as above. First we show that for any Levi subalgebra Λ and any finite-dimensional ideal F of L, the intersection $F \cap \Lambda$ is a Levi subalgebra of F. For let M be any Levi subalgebra of F. Then M is finite-dimensional, so by [26] lemma 5.5 p. 93 there exists $\alpha \in \mathscr{R}(L)$ such that $M^\alpha \leqq \Lambda$. But $F^\alpha = F$ so that $M^\alpha \leqq \Lambda \cap F$. Now M is maximal semisimple in F, and $\Lambda \cap F \lhd \Lambda$ so is semisimple ([26] lemma 4.8 p. 90); hence $\Lambda \cap F = M^\alpha$ and so is a Levi subalgebra of F.

Now let Λ_1 and Λ_2 be Levi subalgebras of L. By the above, $\Lambda_{1i} = \Lambda_1 \cap F_i$ and $\Lambda_{2i} = \Lambda_2 \cap F_i$ are Levi subalgebras of F_i for all $i \in I$. Let

$$\mathscr{B}_i = \{\alpha \in \mathscr{I}(F_i) : \Lambda_{1i}^\alpha = \Lambda_{2i}\}.$$

The classical conjugacy theorem for Levi subalgebras (Jacobson [17] p. 92) shows that $\mathscr{B}_i \neq \emptyset$ for each $i \in I$. Further, if we choose $\alpha \in \mathscr{B}_i$ then it is clear that $\mathscr{B}_i = \alpha N_i$, where $N_i = N_{\mathscr{I}(F_i)}(\Lambda_{2i})$ is a closed subgroup of $\mathscr{I}(F_i)$ by Borel [6] p. 97. Hence each $\mathscr{B}_i$ is a coset variety. If $j \leqq i$ then $F_j \leqq F_i$, so $\Lambda_{1j} \leqq \Lambda_{1i}$ and $\Lambda_{2j} \leqq \Lambda_{2i}$. Hence, with f_{ij} as above, we have $f_{ij}(\mathscr{B}_i) \subseteq \mathscr{B}_j$. Clearly f_{ij} is affine, so by theorem 2.3

$$\mathscr{B} = \text{proj lim } \mathscr{B}_i \neq \emptyset.$$

Now if $\alpha \in \mathscr{B} \subseteq \hat{\mathscr{I}}(L)$ it follows that $\Lambda_{1i}^\alpha \leqq \Lambda_{2i}$ for all $i \in I$, so that $\Lambda_1^\alpha \leqq \Lambda_2$. By maximality we have $\Lambda_1^\alpha = \Lambda_2$, and the theorem is proved.

This result has a corollary for the class $\acute{\mathrm{N}}\mathfrak{F}$ of [26]. Following Amayo and Stewart [2] we define, for any Lie algebra L, the *$\mathfrak{F}$-radical* $\rho_{\mathfrak{F}}(L)$ to be the sum of the finite-dimensional ideals of L. This is always a characteristic ideal of L ([2] corollary 4.2), even for fields of non-zero characteristic. We also define the *locally nilpotent residual* $\lambda_{\mathrm{L}\mathfrak{N}}(L)$ to be the intersection of the ideals I of L for which L/I is locally nilpotent. It is easy to see (by considering finite-dimensional subalgebras) that in a locally finite Lie algebra L we have $L/\lambda_{\mathrm{L}\mathfrak{N}}(L)$ locally nilpotent. Hence $\lambda_{\mathrm{L}\mathfrak{N}}(L)$ is the unique smallest ideal with locally nilpotent factor.

Suppose now that $L \in \acute{\mathrm{N}}\mathfrak{F}$ over $\mathfrak{k}$, and let $\mathfrak{F}^*(L)$ be the set of ascendant finite-dimensional subalgebras, as in [26]. By Simonjan [23] each

$$F^\omega = \bigcap_{n=1}^{\infty} F^n$$

is an ideal of L. Let

$$L^* = \sum \{F^\omega : F \in \mathfrak{F}^*(L)\}.$$

Then it is shown in [26] theorem 6.5 p. 97 that L/L^* is locally nilpotent. Clearly $L^* \leqq \rho_{\mathfrak{F}}(L)$, so we have

$$\lambda_{\mathrm{L}\mathfrak{N}}(L) \leqq L^* \leqq \rho_{\mathfrak{F}}(L).$$

We now have:

Corollary (4.2): *Let L be an $\acute{\mathrm{N}}\mathfrak{F}$-algebra over $\mathfrak{K}$. If Λ_1 and Λ_2 are Levi subalgebras of L then they lie inside $\rho_{\mathfrak{F}}(L)$, and are conjugate by an element of $\hat{\mathscr{I}}(\rho_{\mathfrak{F}}(L))$.*

Proof: If Λ is a Levi subalgebra of L then Λ is a direct sum of finite-dimensional simple algebras Λ_k, for which $\Lambda_k^2 = \Lambda_k$. Hence $\Lambda \leqq L^* \leqq \rho_{\mathfrak{F}}(L)$. But the latter is ideally finite, and Λ is a Levi subalgebra. The result follows.

To obtain a conjugacy theorem for Levi subalgebras of $\acute{\mathrm{N}}\mathfrak{F}$-algebras using this result (if indeed such a theorem is true) it is necessary to extend automorphisms from $\rho_{\mathfrak{F}}(L)$ to L. We are unable to do this in general, even using $\bar{\mathscr{E}}(L)$, but under more restrictive hypotheses it is possible.

Theorem (4.3): *Let L be an $\acute{\mathrm{N}}\mathfrak{F}$-algebra over $\mathfrak{K}$ having a Levi subalgebra Λ_0 such that $[\Lambda_0, \rho(L), \sigma(L)] = 0$. Then all Levi subalgebras of L are $\mathscr{A}$ (L)-conjugate.*

Before giving the proof, we note:

Corollary (4.4): *If L is an $\acute{\mathrm{N}}\mathfrak{F}$-algebra over $\mathfrak{K}$, and if $\sigma(L)$ is abelian, then all Levi subalgebras of L are $\mathscr{A}(L)$-conjugate.*

Proof: $[\Lambda_0, \rho(L), \sigma(L)] \leqq \sigma(L)^2 = 0$.

In fact, corollary 4.4 is true over $\mathfrak{k}$ rather than $\mathfrak{K}$, as is proved without using projective limits in Amayo and Stewart [1] theorem 13.5.10 p. 272. However theorem 4.3 is more general. For the proof of this theorem we need a lemma from [1] (lemma 13.5.9 p. 272): to state this we need a definition. Let L be an $\acute{\mathrm{N}}\mathfrak{F}$-algebra over $\mathfrak{k}$. A subalgebra T of L is *tame* if T is semisimple and $\sigma(L)+T \lhd L$, or equivalently T is a direct summand of a Levi subalgebra of L.

LEMMA (4.5): *Let L be an $\acute{\mathrm{N}}\mathfrak{F}$-algebra over $\mathfrak{k}$, T a tame subalgebra of L, and Λ a Levi subalgebra of L. Suppose that α and β are two automorphisms belonging to $\mathscr{R}(L)$, such that T^{α} and T^{β} are subalgebras of Λ. Then $\alpha|_T = \beta|_T$.*

PROOF OF THEOREM: 4.3: Let $\{F_j\}_{j\in J}$ be the set of ascendant finite-dimensional subalgebras of L, and let Λ_1 be any Levi subalgebra of L. Define

$$\Lambda_{0j} = \Lambda_0 \cap F_j, \qquad \Lambda_{1j} = \Lambda_1 \cap F_j \qquad (j \in J)$$

which are Levi subalgebras of F_j^{ω}, and hence of F_j. Let $Z_j = \zeta_1(\sigma(F_j))$. We claim that

$$Z_j+\Lambda_{0j} = Z_j+\Lambda_{1j} \tag{*}$$

for all $j \in J$. Certainly there exists $\alpha \in \mathscr{R}(F_j^{\omega})$ such that $\Lambda_{0j}^{\alpha} = \Lambda_{1j}$, and hence

$$(Z_j+\Lambda_{0j})^{\alpha} = Z_j^{\alpha}+\Lambda_{0j}^{\alpha} = Z_j+\Lambda_{1j}.$$

Further, $Z_j+\Lambda_{0j}$ is idealized by F_j^{ω}. For if we let $R_j = \sigma(F_j^{\omega})$ we have $F_j^{\omega} = R_j+\Lambda_{0j}$. Further, R_j is nilpotent (Jacobson [17] p. 91) so that $R_j \leqq \rho(L)$ (see [26] corollary 3.15 p. 87). Hence

$$\begin{aligned}[][Z_j+\Lambda_{0j}, F_j^{\omega}] &= [Z_j+\Lambda_{0j}, R_j+\Lambda_{0j}] \\ &\leqq Z_j+\Lambda_{0j}+[\Lambda_{0j}, R_j].\end{aligned}$$

But

$$[[\Lambda_{0j}, R_j], \sigma(F_j)] \leqq [\Lambda_0, \rho(L), \sigma(L)] = 0$$

by hypothesis, and $[\Lambda_{0j}, R_j] \leqq \sigma(F_j)$, so we have

$$[\Lambda_{0j}, R_j] \leqq \zeta_1(\sigma(F_j)) = Z_j$$

and therefore

$$[Z_j+\Lambda_{0j}, F_j^{\omega}] \leqq Z_j+\Lambda_{0j}$$

as claimed. But then $\mathscr{R}(F_j^\omega)$ fixes setwise $Z_j+\Lambda_{0j}$, and so

$$Z_j+\Lambda_{1j} = (Z_j+\Lambda_{0j})^\alpha = Z_j+\Lambda_{0j}$$

and (*) is proved.

It follows that there exists $\beta\in\mathscr{R}(Z_j)$ such that

$$\Lambda_{0j}^\beta = \Lambda_{1j}. \tag{**}$$

For each $j\in J$ we let $\mathscr{B}_j$ be the set of automorphisms β of F_j such that

(i) $(\Lambda_0 \cap F_j)^\beta = \Lambda_1 \cap F_j$,

(ii) If $F_i \leqq F_j$ for $i\in I$ then $F_i^\beta = F_i$.

We show that $\mathscr{B}_i \neq \emptyset$. Since $\Lambda_{0j} \leqq F_j^\omega$ it follows that $\Lambda_0 \cap F_j = \Lambda_{0j}$, and similarly $\Lambda_1 \cap F_j = \Lambda_{1j}$. By (**) there exists $\beta\in\mathscr{R}(Z_j) \subseteq \mathscr{R}(F_j)$ such that (i) holds, and β is the identity on $\sigma(F_j)$. We prove that β satisfies (ii) as well. We have $F_i = \sigma(F_i)+\Lambda_{0i}$, and there exists $\gamma\in\mathscr{R}(F_i)$ such that $\Lambda_{0i}^\gamma = \Lambda_{1i} \leqq \Lambda_1$. But $\Lambda_{0i}^\beta \leqq \Lambda_{0j}^\beta = \Lambda_{1j} \leqq \Lambda_1$; further Λ_{0i} is tame. Since $\mathscr{R}(F_i) \subseteq \mathscr{R}(F_j)$ we may apply lemma 4.5 to conclude that

$$\Lambda_{0i}^\beta = \Lambda_{0i}^\gamma \leqq \mathrm{F}_i.$$

Therefore

$$\begin{aligned} F_i^\beta &= \sigma(F_i)^\beta + \Lambda_{0i}^\beta \\ &\leqq \sigma(F_i)+F_i \\ &= F_i \end{aligned}$$

since $\sigma(F_i) \leqq \sigma(F_j)$, on which β is the identity. Hence (ii) holds.

The set of automorphisms β of F_j satisfying (ii) is an algebraic group, and hence $\mathscr{B}_j$ is a coset variety. We partially order J by setting $i \leqq j$ if $F_i \leqq F_j$ $(i, j\in J)$ which makes J a directed set by virtue of Hartley [15] theorem 6 p. 259. If $f_{ij}:\mathscr{B}_i \to \mathscr{B}_j$ is restriction $(j \leqq i)$ then we obtain a projective limit system $\{\mathscr{B}_i, \mathrm{f}_{ij}\}$ of non-empty coset varieties. Exactly as in theorem 4.1 we see that if $\alpha \in\mathscr{B} = \text{proj lim } \mathscr{B}_i$ then $\Lambda_0^\alpha = \Lambda_1$. Thus any Levi subalgebra is conjugate to Λ_0, and it follows that all Levi subalgebras are conjugate.

Unlike the finite-dimensional case, we do not in general have conjugacy of Levi subalgebras under $\mathscr{R}(L)$, or even $\mathscr{N}(L)$. This is easy to see. For let K be any finite-dimensional Lie algebra over $\mathfrak{K}$ having distinct Levi subalgebras Λ, Λ'. Let M be an infinite index set, let K_μ be an isomorphic copy of K for each $\mu\in M$, and let Λ_μ and Λ'_μ be the images of Λ and Λ'

under such an isomorphism. Then the direct sum

$$L = \bigoplus_{\mu \in M} K_\mu$$

has Levi subalgebras

$$\Lambda_1 = \bigoplus_{\mu \in M} \Lambda_\mu, \qquad \Lambda_2 = \bigoplus_{\mu \in M} \Lambda'_\mu.$$

Obviously L is ideally finite. But any element of $\mathscr{R}(L)$ or $\mathscr{N}(L)$ is the identity on all but a finite number of the K_μ, hence Λ_1 and Λ_2 cannot be conjugate under $\mathscr{R}(L)$ or $\mathscr{N}(L)$.

5. Borel subalgebras

By a *Borel subalgebra* of a Lie algebra we shall mean a maximal locally soluble subalgebra. By Zorn's lemma, Borel subalgebras always exist, and every locally soluble subalgebra is contained in a Borel subalgebra. We can easily characterize the Borel subalgebras of $\acute{\mathrm{N}}\mathfrak{F}$-algebras:

PROPOSITION (5.1): *Let L be an $\acute{\mathrm{N}}\mathfrak{F}$-algebra over $\mathfrak{k}$, and let $L/\sigma(L)$ be canonically decomposed as a direct sum $\bigoplus_{i \in I} S_i$ of simple finite-dimensional ideals. Then the Borel subalgebras of L are precisely the complete inverse images in L of algebras $\bigoplus_{i \in I} B_i$, where for each $i \in I$, B_i is a Borel subalgebra of S_i.*

PROOF: It is clear that subalgebras of this type are Borel. If B is a Borel subalgebra of L then $\sigma(L)+B$ is locally soluble, hence $\sigma(L) \leqq B$. Now $B/\sigma(L)$ is a Borel subalgebra of $L/\sigma(L)$. By considering projections onto the S_i it follows that $B/\sigma(L)$ is contained in some $\bigoplus_{i \in I} B_i$, and hence by maximality is of the desired form.

COROLLARY (5.2): *Let L be an $\acute{\mathrm{N}}\mathfrak{F}$-algebra over $\mathfrak{k}$, Λ any Levi subalgebra of L, with canonical decomposition $\Lambda = \bigoplus_{i \in I} S_i$. Then as the B_i range over all Borel subalgebras of S_i, the algebras $\sigma(L) + \bigoplus_{i \in I} B_i$ are precisely the Borel subalgebras of L.*

Examples similar to that in §4 show that Borel subalgebras of ideally finite algebras need not be $\mathscr{N}(L)$-conjugate. However, in the algebraically closed case, they will be $\hat{\mathscr{I}}(L)$-conjugate. To prove this we need:

LEMMA (5.3): *Let L be ideally finite over $\mathfrak{k}$, having a Borel subalgebra B;*

and let F be a finite-dimensional ideal of L. Then $B \cap F$ is a Borel subalgebra of F.

PROOF: As in 5.2 we have $B = \sigma(L) + \bigoplus_{i \in I} B_i$ relative to a Levi subalgebra Λ of L. Now $F = \sigma(F) + (\Lambda \cap F)$ by the usual argument. Since $\Lambda \cap F \lhd F$ we have $\Lambda \cap F = \bigoplus_{i \in J} S_i$ for a finite subset $J \subseteq I$, where $\Lambda = \bigoplus_{i \in I} S_i$ is the canonical decomposition. But now

$$B \cap F = \sigma(F) + \bigoplus_{i \in J} S_i$$

which is a Borel subalgebra of F.

THEOREM (5.4): *Let L be ideally finite over $\mathfrak{K}$. Then all Borel subalgebras of L are $\hat{\mathscr{I}}(L)$-conjugate.*

PROOF: The argument is exactly that used for Levi subalgebras in theorem 4.1, using lemma 5.3 and the conjugacy of Borel subalgebras in the finite-dimensional situation (Borel [5], Humphreys [16] p. 84).

We also have a homomorphism-invariance theorem for Borel subalgebras:

LEMMA (5.5): *Let L be ideally finite over $\mathfrak{k}$, with a Borel subalgebra B, and let $I \lhd L$. Then $B + I/I$ is a Borel subalgebra of L/I.*

PROOF: The radical $\sigma(I) = \sigma(L) \cap I$ ([26] p. 87 lemma 3.18) which is an ideal of L. Since $\sigma(I) \leqq B$ we may pass to a quotient, and hence assume $\sigma(I) = 0$. Then I is a semisimple ideal of L and it follows from [26] §5 that I is a direct summand of L, complemented by $\sigma(L) + \Lambda$ where Λ is semisimple. Then $\Lambda + I$ is a Levi subalgebra of L, and using the characterization of Borel subalgebras given in proposition 5.1 we obtain the lemma.

Note that from theorem 4.1 it is easy to show that if Λ is a Levi subalgebra of an ideally finite algebra over $\mathfrak{K}$ and if $I \lhd L$, then $\Lambda \cap I$ is a Levi subalgebra of I. Hence in lemma 5.3 we may drop the requirement that F be of finite dimension, provided we replace $\mathfrak{k}$ by $\mathfrak{K}$.

6. Further properties of automorphisms

In this section we develop methods of constructing automorphisms of ideally finite algebras over $\mathfrak{K}$, needed to prove existence and conjugacy of Cartan subalgebras in §7. Recall that a Lie algebra L is *residually finite* if it has a collection of ideals I_λ such that $\bigcap_\lambda I_\lambda = 0$ and L/I_λ is finite-dimensional for all λ. It is trivial to see that every ideally finite Lie algebra is residually finite modulo its centre (lemma 7.2). In this section we first deal with the residually finite case, and then find a method for lifting the resulting automorphisms from a quotient algebra to the whole algebra.

From now on let L be ideally finite over $\mathfrak{K}$. If $I \lhd L$ and L/I has finite dimension we shall say that I is a *cofinite* ideal of L. A *finite residual system* in L is a set of ideals K_j $(j \in J)$ of L, such that

(i) $\bigcap_{j\in J} K_j = 0$,

(ii) If $i, j \in J$ then there exists $k \in J$ such that $K_i \cap K_j \geqq K_k$,

(iii) L/K_j is finite-dimensional for all $j \in J$.

Clearly L has such a system if and only if L is residually finite, and then thes set of *all* cofinite ideals is a finite residual system. There may sometimes be advantages in using a smaller system, however.

We partially order J by letting $i \leqq j$ if $K_i \geqq K_j$, so that (ii) implies that J is directed. Whenever $i \leqq j$ there is a natural homomorphism

$$\pi_{ji} : L/K_j \to L/K_i.$$

For each $j \in J$ we let $\mathscr{B}_j$ be the set of automorphisms of L/K_j leaving invariant all K_i/K_j for $i \leqq j$. Clearly $\mathscr{B}_j$ is an algebraic group. Further, if $i \leqq j$ then π_{ji} induces a morphism

$$p_{ji} : \mathscr{B}_j \to \mathscr{B}_i$$

and it is clear that

$$\{\mathscr{B}_j, p_{ij}\}$$

is a projective limit system. We define

$$\mathscr{B}(L) = \text{proj lim } \mathscr{B}_j.$$

Further, we let $\mathscr{C}_j$ be the set of $\beta \in \mathscr{B}_j$ such that β leaves invariant I/K_j for every $I \lhd L$ with $I \geqq K_j$. This is an algebraic subgroup of $\mathscr{B}_j$, and $\{\mathscr{C}_j, p_{ji}|\mathscr{C}_j\}$ is also a projective limit system. We set

$$\mathscr{C}(L) = \text{proj lim } \mathscr{C}_j \subseteq \mathscr{B}(L).$$

Because L is ideally finite it transpires that elements of $\mathscr{C}(L)$ act naturally as automorphisms of L, and in fact there is a natural injection $\mathscr{C}(L) \to \hat{\mathscr{I}}(L)$. The reason is:

LEMMA (6.1): *Let L be an ideally finite algebra over $\mathfrak{K}$ with a finite residual system $\{K_j\}_{j \in J}$. If F is a finite-dimensional subspace of L then $K_k \cap F = 0$ for some $k \in J$.*

PROOF: Let $0 \neq x \in F$. Then $x \notin K_j$ for some $j \in J$, and so

$$\dim K_j \cap F < \dim F.$$

Inductively there exists $i \in J$ such that $K_i \cap (K_j \cap F) = 0$. If we take $k \in J$ such that $K_k \leqq K_i \cap K_j$ then $K_k \cap F = 0$.

Now if $\gamma \in \mathscr{C}(L)$ then $\gamma = (\gamma_i)_{i \in J}$ where each γ_i is an automorphism of L/K_i fixing all ideals of L/K_i; and if $i \leqq j$ then

$$p_{ji}(\gamma_j) = \gamma_i. \tag{*}$$

Let $x_1, \cdots, x_n \in L$. We can find a finite-dimensional ideal X of L containing $x_1, \cdots, x_n$; and there exists $k \in J$ such that $K_k \cap X = 0$. Hence there is a natural injection

$$e : X \to L/K_k$$

and $e(X)$ is an ideal of L/K_k. Hence γ_k induces an automorphism of $e(X)$, and we can use e to pull this back to give an automorphism of X. Abusing notation, this defines $x_1^\gamma, \cdots, x_n^\gamma$. The action of γ is well defined by (*), and it is easy to check that it yields an automorphism of L. Since γ fixes every finite-dimensional ideal of L, it follows without difficulty that $\gamma \in \hat{\mathscr{I}}(L)$.

To cope with lifting problems we need a little more. Let us call an automorphism α of L *locally inner* if, given any finite set of elements $x_1, \cdots, x_n \in L$, we can find a finite-dimensional subalgebra X of L containing $x_1, \cdots, x_n$, and an automorphism $\beta \in \bar{\mathscr{E}}(X)$, such that $x_i^\alpha = x_i^\beta$ for $i = 1, 2, \cdots, n$. Using the extension property of $\bar{\mathscr{E}}$ we may even assume X to be an ideal of L. We let

$$\mathscr{D}_j = \bar{\mathscr{E}}(L/K_j) \subseteq \mathscr{C}_j.$$

The lifting property of $\bar{\mathscr{E}}$ (specifically, the surjectivity of the induced map) shows that $\{\mathscr{D}_j, p_{ji}|_{\mathscr{D}_j}\}$ is a projective limit system, and we let

$$\mathscr{D}(L) = \operatorname{proj} \lim \mathscr{D}_j \subseteq \mathscr{C}(L).$$

It is now easy to check that under the injection $\mathscr{C}(L) \to \hat{\mathscr{I}}(L)$ the elements of $\mathscr{D}(L)$ act as locally inner automorphisms of L. We define

$$\mathscr{L}(L)$$

to be the group of all locally inner automorphisms of L. So we may assume that $\mathscr{D}(L) \subseteq \mathscr{L}(L)$.

Similar results for groups may be found in Tomkinson [36] pp. 684–685, but with more complicated proofs. The method above could be used to simplify them.

We can now give a useful description of locally inner automorphisms in terms of projective limits, as follows. For each $i \in I$ and $j \geqq i$ the group $\bar{\mathscr{E}}(F_j)$ induces automorphisms on F_i. Let $\mathscr{G}_{ji}$ be the resulting subgroup of $\mathscr{A}(F_i)$, and let the restriction map be denoted by

$$r_{ji} : \bar{\mathscr{E}}(F_j) \to \mathscr{G}_{ji}.$$

By the extension property, $\bar{\mathscr{E}}(F_i) \subseteq \mathscr{G}_{ji}$ for all $j \geqq i$. We define

$$\mathscr{G}_i = \bigcup_{j \geqq i} \mathscr{G}_{ji}.$$

It is easy to see from the extension property that the set of all $\mathscr{G}_{ji}$, for $j \geqq i$, is directed by inclusion; and so $\mathscr{G}_i$ is a subgroup of $\mathscr{A}(F_i)$ fixing setwise all smaller F_k. Now $\mathscr{E}(F_j)$ is generated by 1-parameter subgroups

$$\{\exp(\lambda x^*) : \lambda \in \mathfrak{K}\}$$

where x is strongly ad-nilpotent. Hence $\mathscr{E}(F_j)$ is connected (in the Zariski topology. It follows (Borel [6] p. 88) that the $\mathscr{G}_{ji}$ are closed connected subgroups of $\mathscr{A}(F_i)$. Now an algebraic group is connected if and only if it is irreducible as an algebraic variety (Borel [6] p. 87), and then the finiteness of combinatorial dimension (Borel [6] p. 5) of $\mathscr{A}(L)$ implies that

$$\mathscr{G}_i = \mathscr{G}_{i_0 i}$$

for some $i_0 \geqq i$. Hence in fact $\mathscr{G}_i$ is a connected algebraic group. If f_{ji} is the map induced by restriction $F_j \to F_i$ then we clearly have

$$f_{ji}(\mathscr{G}_j) = \mathscr{G}_i$$

for all $j \geqq i$. This yields a projective limit system $\{\mathscr{G}_i, f_{ji}\}$ and evidently

$$\mathscr{L}(L) = \text{proj lim } \mathscr{G}_i.$$

We can use this description to prove lifting and extension properties for locally inner automorphisms (cf. Stonehewer [34]).

THEOREM (6.2): *Let L be ideally finite over $\mathfrak{K}$, and $H \leqq L$.*
(i) *Every element $\sigma' \in \mathscr{L}(H)$ extends to an element $\sigma \in \mathscr{L}(L)$.*
(ii) *Every epimorphism $L \to L'$ induces an epimorphism $\mathscr{L}(L) \to \mathscr{L}(L')$.*

PROOF: Let $\{F_i\}_{i \in I}$ be the set of all finite-dimensional ideals of L, with I ordered as usual.

(i) For each $i \in I$ define

$$\mathscr{B}_i = \{\alpha \in \mathscr{G}_i : \alpha|_{H \cap F_i} = \sigma'|_{H \cap F_i}\}$$

noting that $H \cap F_i$ is σ'-invariant. Obviously $\mathscr{B}_i$ is a coset variety. We prove $\mathscr{B}_i \neq \emptyset$. If we choose $x_1, \cdots, x_n$ spanning $H \cap F_i$ then we can find $j \in I$ such that $x_1, \cdots, x_n \in H \cap F_j$, and there exists $\tau \in \bar{\mathscr{E}}(H \cap F_j)$ such that $x_1^{\sigma'} = x_1^{\tau}, \cdots, x_n^{\sigma'} = x_n^{\tau}$. If we extend τ to τ' on F_j and put $\alpha = \tau'|F_i$ *then* $\alpha \in \mathscr{B}_i$. Letting f_{ji} denote restriction, we obtain a projective limit system $\{\mathscr{B}_i, f_{ji}\}$. If we pick

$$\sigma \in \text{proj lim } \mathscr{B}_i$$

then $\sigma|H = \sigma'$ and $\sigma \in \mathscr{L}(L)$.

(ii) The proof is similar: we let $\mathscr{C}_i$ be the set of all $\alpha \in \mathscr{G}_i$ such that α induces the same automorphism as a given σ' on L', and choose $\sigma \in \text{proj lim } \mathscr{C}_i$. Surjectivity of the induced map is obvious.

If instead of $\bar{\mathscr{E}}$ we use $\mathscr{N}$ it is still possible to prove (ii) on the assumption that the kernel of the epimorphism $L \to L'$ is contained in the centre (even the hypercentre) of L.

7. Cartan subalgebras

Many equivalent, but superficially different, properties may be used to define a Cartan subalgebra of a finite-dimensional Lie algebra. When generalized to infinite dimensions these properties may cease to be equivalent. The definition of a Cartan subalgebra best adapted to the present circumstances seems to be that suggested by formation theory. Thus, let L be ideally finite. We say that a subalgebra C of L is a *Cartan subalgebra* of L if

(i) C is locally nilpotent,

(ii) If $C \leqq H \leqq L$, $K \lhd H$, and H/K is locally nilpotent, then $K+C=H$.

In formation-theoretic terms, C is a *locally nilpotent projector* of L. That this property, in finite dimensions, is equivalent to the usual definition of a Cartan subalgebra (nilpotent and self-idealizing) is proved in Barnes and Gastineau-Hills [3] p. 343 example 1.3. (We shall show in [29] that for ideally finite algebras over $\mathfrak{K}$ the Cartan subalgebras are precisely the locally nilpotent self-idealizing subalgebras, but this is less convenient as a definition.) Immediate consequences of the definition are:

LEMMA (7.1): *Let L be an ideally finite Lie algebra over $\mathfrak{K}$, with $X \lhd L$, and let C be a Cartan subalgebra of L. Then*

(i) *C is a maximal locally nilpotent subalgebra of L.*
(ii) *$C+X/X$ is a Cartan subalgebra of L/X.*
(iii) *If $C \leqq H \leqq L$ then C is a Cartan subalgebra of H.*
(iv) *If C'/X is a Cartan subalgebra of L/X and C'' is a Cartan subalgebra of C', then C'' is a Cartan subalgebra of L.*
(v) *C contains the centre of L.*
(vi) *If C' contains the centre of L and if $C'/\zeta_1(L)$ is a Cartan subalgebra of $L/\zeta_1(L)$, then C' is a Cartan subalgebra of L.*
(vii) *If $C \leqq H \leqq L$ then H is self-idealizing in L.*

PROOF: (i), (ii), (iii) follow as in Barnes and Gastineau-Hills [3] lemmas 1.4 and 1.5, p. 344. For (iv) argue as in Gardiner, Hartley, and Tomkinson [12] lemma 5.3 p. 201. Now (v) follows from (i) and (vi) from (iv) noting that if a Lie algebra is locally nilpotent modulo its centre then it is locally nilpotent. Finally (vii) follows as in Barnes and Gastineau-Hills [3] lemma 1.7 p. 344.

Part (vi) of this lemma allows us to work modulo the centre of L. The next lemma opens the way to the methods of § 6.

LEMMA (7.2): *If L is ideally finite then $L/\zeta_1(L)$ is residually finite.*

PROOF: If F is a finite-dimensional ideal of L then $L/C_L(F)$ is finite-dimensional ([24] p. 302). The intersection of the $C_L(F)$ over all such F is precisely the centre of L.

We use this result to construct Cartan subalgebras 'from the top down', rather than working 'from the bottom up' as for Levi and Borel subalgebras. This is because Cartan subalgebras do not behave well on intersecting with ideals, but are suitably behaved under quotients.

Following the argument of Tomkinson [36] we obtain the following situation.

THEOREM (7.3): *Let L be a residually finite ideally finite algebra over* $\mathfrak{K}$, *and let* $\{K_j\}_{j\in J}$ *be the set of cofinite ideals of* L. *Then there exist subalgebras* C_j *of* L, *for each* $j \in J$, *such that*

(i) $K_j \leqq C_j$,

(ii) *If* $K_i \geqq K_j$ *then* $C_i \geqq C_j$,

(iii) C_j/K_j *is a Cartan subalgebra of* L/K_j.

PROOF: Let $\mathscr{C}_j$ be the set of Cartan subalgebras C_j/K_j of L/K_j. The natural homomorphisms induce maps

$$p_{ij} : \mathscr{C}_i \to \mathscr{C}_j$$

whenever $j \leqq i$ (that is, $K_j \geqq K_i$), because of homomorphism-invariance in finite dimensions (cf. lemma 7.1(ii)). We shall give the $\mathscr{C}_j$ the structure of coset varieties, in such a way that the p_{ij} become affine. Now suppose C is a Cartan subalgebra of a finite-dimensional Lie algebra F over $\mathfrak{K}$. Since $\mathscr{A}(F)$ acts transitively on the set of Cartan subalgebras of F, this set is in bijective correspondence with the points of the homogeneous space $\mathscr{A}(F)/N_{\mathscr{A}(F)}(C)$. This carries the $\mathscr{W}$-topology: if we can show that the topology induced on $\mathscr{C}_j$ is independent of the choice of C it will follow (choosing C's related by a homomorphism) that the p_{ij} are affine. If we choose a different Cartan subalgebra it will be of the form C^α where $\alpha \in \mathscr{A}(F)$. Now $N_{\mathscr{A}(F)}(C^\alpha) = (N_{\mathscr{A}(F)}(C))^{\alpha^{-1}}$. Since conjugation gives an affine automorphism of $\mathscr{A}(F)$, which is both continuous and closed, the $\mathscr{W}$-topology is unchanged on $\mathscr{C}_j$. Therefore $\{\mathscr{C}_j, p_{ij}\}$ is a projective limit system of coset varieties and affine maps, and we can find

$$(C_j) \in \text{proj lim } \mathscr{C}_j.$$

Then the C_j satisfy (i), (ii), and (iii).

We now show (using an argument of the referee) that under these conditions $\bigcap_{j\in J} C_j$ is a Cartan subalgebra of L.

THEOREM (7.4): *With the hypotheses of theorem* 7.3, *and with subalgebras* C_j *satisfying* (i), (ii), (iii) *of that theorem, then* $\bigcap C_j$ *is a Cartan subalgebra of* L.

PROOF: We use some finite-dimensional terminology, to be found in

Jacobson [17]. If A is any finite-dimensional Lie algebra over K and B is a Cartan subalgebra of A, then $B = A_0(x)$ for some $x \in L$, where $A_0(x)$ denotes the null-component. (Take x to be a regular element.) Further if $I \lhd A$ then $B+I/I = (A/I)_0(x+I)$. In fact if $D \lhd A$, $y \in A$, and ϕ is any homomorphism of A, then $D_0(y)\phi = (D\phi)_0(y\phi)$.

Let F be any finite-dimensional ideal of L. Choose $i \in J$ such that $F \cap K_i = 0$ and define $C_F = C_i \cap F$. Let x_i be any element of L such that $C_i/K_i = (L/K_i)_0(x_i + K_i)$. Then for large enough n,

$$(C_i \cap F)x_i^{*n} \subseteq K_i \cap F = 0,$$

and so $C_i \cap F \leqq F_0(x_i)$. But

$$F_0(x_i) + K_i/K_i \leqq (L/K_i)_0(x_i + K_i) = C_i/K_i,$$

so $F_0(x_i) \leqq C_i \cap F$, and we get

$$C_F = C_i \cap F = F_0(x_i),$$

this being therefore independent of the choice of x_i. Let $j \geqq i$ and let $x_j + K_j$ be an element of L/K_j such that

$$(L/K_j)_0(x_j + K_j) = C_j/K_j.$$

Applying the natural homomorphism $L/K_j \to L/K_i$, we get that

$$C_i/K_i = (L/K_i)_0(x_j + K_i),$$

and

$$C_i \cap F = F_0(x_j) = C_j \cap F.$$

Thus C_F is independent of the choice of i. It follows that if $F_1 \leqq F_2$ then $C_{F_1} \leqq C_{F_2}$. Hence $C = \bigcup_F C_F$ is a locally nilpotent subalgebra of L.

Let F be any finite-dimensional ideal of L such that $F + K_i = L$, choose j such that $K_j \cap F = 0$, and let $x_j + K_j$ be an element of L/K_j with null-component C_j/K_j. Then $x_j + K_i$ belongs to L/K_i and C_i/K_i is its null-component. The natural map $F \to L/K_i$ is an epimorphism mapping $F_0(x_j)$ onto C_i/K_i. In other words,

$$C_i = C_F + K_i = C + K_i$$

for all $i \in J$.

Suppose now that $C \leqq H \rhd K$, and H/K is locally nilpotent. Let F be a finite-dimensional ideal of L, choose $i \in J$ so that $F \cap K_i = 0$. Since $C+K_i = C_i$, there exists $x_i \in C$ such that x_i+K_i belongs to L/K_i and C_i/K_i is its null-component. Then x_i^* induces a nil endomorphism on H/K and so induces a nilpotent endomorphism on $H \cap F/K \cap F$. Thus $H \cap F = (K \cap F)+(H \cap F)_0(x_i) \leqq (K \cap F)+C_F$, and obviously equality holds. Since L is the union of all such F, it follows that $H = K+C$. Therefore C is a Cartan subalgebra. Clearly $C = \bigcap C_i$.

Corollary (7.5): *Let L be ideally finite over $\mathfrak{K}$. Then L possesses at least one Cartan subalgebra.*

Proof: If $Z = \zeta_1(L)$ then L/Z is residually finite. By theorems 7.3 and 7.4, L/Z has a Cartan subalgebra C/Z. By lemma 7.1(vi) C is a Cartan subalgebra of L.

The next corollary will be useful in future work:

Corollary (7.6): *Let L be ideally finite over $\mathfrak{K}$, and let C be a subalgebra of L such that $C+K_j/K_j$ is a Cartan subalgebra of L/K_j for every cofinite ideal K_j of L. Then $C+\zeta_1(L)$ is a Cartan subalgebra of L.*

Proof: This is an immediate consequence of theorem 7.4 and lemma 7.1(vi).

Next we turn to conjugacy.

Theorem (7.7): *Let L be ideally finite over $\mathfrak{K}$. Then any two Cartan subalgebras of L are $\mathscr{L}(L)$-conjugate.*

Proof: Both Cartan subalgebras contain $\zeta_1(L)$. Since elements of $\mathscr{L}(L/\zeta_1(L))$ lift to $\mathscr{L}(L)$ by theorem 6.2(ii) we may work modulo $\zeta_1(L)$ and hence assume L is residually finite. The methods of § 6 now apply, as follows. Let C_1 and C_2 be the two Cartan subalgebras. For each cofinite ideal K_j of L it follows that C_1+K_j/K_j and C_2+K_j/K_j are Cartan subalgebras of L/K_j. Define

$$\mathscr{B}_j = \{\alpha \in \bar{\mathscr{E}}(L/K_j) : (C_1+K_j/K_j)^\alpha = C_2+K_j/K_j\}.$$

This is a non-empty coset variety, non-emptiness being 16.4 of Humphreys

[16] p. 84. Therefore we can find a locally inner automorphism

$$\beta \in \text{proj lim } \mathscr{B}_j.$$

Then

$$(C_1 + K_j)^\beta = C_2 + K_j \qquad (j \in J)$$

and so

$$(\bigcap_j C_1 + K_j)^\beta = \bigcap_j (C_1 + K_j)^\beta = \bigcap_j C_2 + K_j.$$

The left-hand side contains C_1. Considering its image modulo each K_j it is locally nilpotent; maximality (lemma 7.1(i) implies it equals C_1. Similarly the right-hand side equals C_2. This proves the theorem.

We intend to give an alternative approach to the existence of Cartan subalgebras in [29], and to develop their properties in more detail in [30].

8. Borel-Cartan pairs

A *Borel-Cartan* pair of L is a pair (B, C) where B is a Borel subalgebra of L and C is a Cartan subalgebra of B.

LEMMA (8.1): *If L is ideally finite over $\mathfrak{K}$ and (B, C) is a Borel-Cartan pair, then C is a Cartan subalgebra of L.*

PROOF: It is easy to give a proof based on corollary 7.6. For applications in [29] we wish to avoid using theorem 7.4, to yield an alternative proof. Therefore we proceed as follows.

Consider first the case where L is semisimple. Then $L = \bigoplus_{i \in I} S_i$ is a direct sum of finite-dimensional simple ideals, and $B = \bigoplus_{i \in I} B_i$ where B_i is a Borel subalgebra of S_i. The projection C_i of C on B_i is a Cartan subalgebra of B_i. Since C is maximal locally nilpotent, $C = \bigoplus_{i \in I} C_i$. The lemma is true in finite dimensions, so each C_i is a Cartan subalgebra of S_i. Suppose $C \leqq H \rhd K$ where H/K is locally nilpotent. For each finite-dimensional ideal

$$S = S_{i_1} \oplus \cdots \oplus S_{i_t}$$

of L, we know that $C \cap S$ is a Cartan subalgebra of S, so that $H \cap S =$

$(K \cap S)+(C \cap S)$. Therefore $H = K+C$ and C is a Cartan subalgebra.

In the general case, let $R = \sigma(L)$. Then $B \geqq R$ and B/R is a Borel subalgebra of L/R. Now $C+R/R$ is a Cartan subalgebra of L/R by lemma 7.1(ii). Since C is a Cartan subalgebra of $C+R \leqq B$ by lemma 7.1(iii) it follows by lemma 7.1(iv) that C is a Cartan subalgebra of L.

The argument of theorem 7.7, applied to Borel subalgebras, yields a strengthening of theorem 5.4:

THEOREM (8.2): *If L is ideally finite over $\mathfrak{K}$ then any two Borel subalgebras of L are $\mathscr{L}(L)$-conjugate.*

We may even combine theorems 7.7 and 8.2:

THEOREM (8.3): *If L is ideally finite over $\mathfrak{K}$, then any two Borel-Cartan pairs of L are $\mathscr{L}(L)$-conjugate.*

PROOF: Let (B, C) and (B', C') be Borel-Cartan pairs of L. By theorem 8.2 there exists $\alpha \in \mathscr{L}(L)$ such that $B^{\alpha} = B'$. But now C^{α} and C' are Cartan subalgebras of B'. By theorem 7.7 there exists $\beta \in \mathscr{L}(B')$ such that $C^{\alpha\beta} = C'$. By theorem 6.2 we can extend β to $\beta' \in \mathscr{L}(L)$, and since $B'^{\beta} = B'$ it follows that $B^{\alpha\beta'} = B'$, $C^{\alpha\beta'} = C'$.

This result generalizes Winter [40] p. 99.

Theorem 4.1 can also be improved in a manner similar to that whereby theorem 5.4 can be improved to theorem 8.2. The important case is in finite dimensions:

LEMMA (8.4): *Let L be a finite-dimensional Lie algebra over $\mathfrak{K}$. Then all Levi subalgebras of L are $\mathscr{E}(L)$-conjugate.*

PROOF: By induction on the dimension, using the lifting and extension properties of $\mathscr{E}(L)$, we may assume that $L = A + \Lambda$ (split extension) where A is an abelian ideal, Λ is semisimple, and A is irreducible as Λ-module. The theorem of Mal'cev and Harish-Chandra (Jacobson [17] p. 92) asserts that all Levi subalgebras of L are conjugate under the group generated by all $\exp(a^*)$ for $a \in A$.

Let B be the additive subgroup of A generated by all strongly ad-nilpotent elements of L lying inside A. Then B is a vector subspace invariant under all automorphisms of L. It follows that B is an ideal of L (either by using corollary 3.2 of Towers [37] p. 443, proved also by Tuck [38], or by looking at the ad-nilpotent elements of Λ which remain ad-nilpotent in their action on A by Humphreys [16] p. 30, so can be exponentiated). Irreducibility implies $B = 0$ or $B = A$. If $B = 0$ then every ad-semisimple element of A centralizes A, so A centralizes A (being generated by such elements), and now A is the unique Levi subalgebra and conjugacy is trivial. Otherwise $B = A$, so every $a \in A$ can be written

$$a = a_1 + \cdots + a_n$$

for strongly ad-nilpotent $a_1, \cdots, a_n$. Then

$$\exp(a^*) = 1 + a^* = \prod_{i=1}^{n} (1 + a_i^*) = \prod_{i=1}^{n} \exp(a_i^*)$$

belongs to $\mathscr{E}(L)$, and the theorem of Mal'cev and Harish-Chandra implies $\mathscr{E}(L)$-conjugacy.

The above methods of using projective limits easily yield:

THEOREM (8.5): *Let L be ideally finite over $\mathfrak{K}$. Then any two Levi subalgebras of L are $\mathscr{L}(L)$-conjugate.*

An analogue of this result holds for associative algebras. It is hoped to publish details as [31].

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Mathematics Institute
University of Warwick
Coventry
England