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A LIMIT LAW FOR RANDOM WALK IN A RANDOM ENVIRONMENT

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Introduction

M. Kozlov [6] and F. Solomon [10] considered the following model: Let $\{\alpha_i\}_{-\infty<i<\infty}$ be a doubly infinite sequence of independent identically distributed random variables with values in $[0, 1]$ and let

$$\mathscr{A} = \sigma\{\alpha_i : -\infty < i < +\infty\},$$

the σ-field generated by the α_i. $\{X_t\}_{t\geqq 0}$ is a sequence of integer valued random variables with

$$\begin{aligned}(1.1)\quad X_0 = 0,\ P\{X_{t+1} &= X_t+1|\mathscr{A}, X_0,\cdots, X_t\} = \alpha_i \quad \text{on} \quad \{X_t = i\},\\ P\{X_{t+1} &= X_t-1|\mathscr{A}, X_0,\cdots, X_t\} = \beta_i \equiv 1-\alpha_i\\ &\qquad\qquad\qquad\qquad \text{on} \quad \{X_t = i\}.\end{aligned}$$

Here $\{\alpha_i\}$ represents the "random environment". Once this is chosen it remains fixed for all time, and the process $\{X_t\}$ is a random walk which can move only one step to the right or left at a time. The probability of the X process moving to the right depends on its last position and on the environment. Alternatively, for fixed $\{\alpha_i\}$ one can describe $\{X_t\}_{t\geqq 0}$ as the sequence of states of a birth and death process with birth, respectively death parameters α_i and $\beta_i = i-\alpha_i$. Note that $\{X_t\}$ is not Markovian when $\{\alpha_i\}$ is not fixed. As a matter of fact, one finds out more and more about the environment by taking more and more observations of X_t. A closely related model had been introduced on physical grounds by Chernov [2] and Temkin [11].

In the above named papers the remarkable phenomenon was discovered that one may have $X_t \to \infty$ w.p.l. but $(1/t)X_t \to 0$ w.p.l. as well. I.e., it is possible that X_t grows indefinitely, but slower than linearly.

In the above model this occurs when

$$E \log \frac{\beta_0}{\alpha_0} < 0, \quad \text{but} \quad E \frac{\beta_0}{\alpha_0} \geqq 1 \tag{1.2}$$

(see [10]). It was conjectured by Kolmogorov and by the third author that in these cases $n^{-1/\kappa} T_n$ would have a stable limit distribution, where

$$T_n = \min \{t : X_t = n\} = \text{first hitting time of } \{n\},$$

and κ is the unique positive number for which

$$E \left(\frac{\beta_0}{\alpha_0} \right)^{\kappa} = 1.$$

This limit result would be equivalent to saying that $t^{-\kappa} X_t$ has a certain limit distribution which is closely related to that of $n^{-1/\kappa} T_n$ (see the theorem below). The purpose of this paper is to prove the above conjecture under the hypothesis that $\log(\beta_0/\alpha_0)$ has a non-arithmetic distribution. Unfortunately [6] and [10] considered special examples in which $\log(\beta_0/\alpha_0)$ does have an arithmetic distribution so that they were unable to prove the conjecture. Our precise result follows. Note that it also gives the limit distribution for T_n and X_t even when $E(\beta_0/\alpha_0) < 1$, i.e., when $\kappa > 1$, in which case $n^{-1} T_n$ and $t^{-1} X_t$ converge with probability one to a positive limit (see [10]).

THEOREM: *Let $\{\alpha_i\}_{-\infty < i < \infty}$ be independent identically distributed such that*

$$-\infty \leqq E \log \frac{\beta_0}{\alpha_0} < 0 \qquad (\beta_0 = 1 - \alpha_0), \tag{1.3}$$

(1.4) *there exists a* $0 < \kappa < \infty$ *for which*

$$E \left(\frac{\beta_0}{\alpha_0} \right)^{\kappa} = 1 \quad \text{and} \quad E \left(\frac{\beta_0}{\alpha_0} \right)^{\kappa} \log^+ \frac{\beta_0}{\alpha_0} < \infty,$$

(1.5) *the distribution of* $\log \beta_0/\alpha_0$ *(excluding the possible atom at* $-\infty$*) is non-arithmetic.*

Then, the following limit laws hold for T_n *and* X_t *with* $0 < A_\kappa, B_i < \infty$ *suitable constants and* $L_\kappa(\cdot)$ *a stable law of index* $\kappa (L_\kappa$ *is concentrated on* $[0, \infty)$ *if* $\kappa < 1$ *and has mean zero if* $\kappa > 1)$:

(i) *If* $\kappa < 1$,

$$\lim_{n\to\infty} P\{n^{-1/\kappa} T_n \leqq x\} = L_\kappa(x)$$

and

$$\lim_{t\to\infty} P\{t^{-\kappa} X_t \leqq x\} = 1 - L_\kappa(x^{-1/\kappa}),$$

(ii) *If* $\kappa = 1$, then for suitable $D(n) \sim \log n$ and $\delta(t) \sim (A_1 \log t)^{-1} t$

$$\lim_{n\to\infty} P\{n^{-1}(T_n - A_1 n D\,(n\mu^{-1})) \leqq x\} = L_1(x)$$

and

$$\lim_{t\to\infty} P\{t^{-1}(\log t)^2 (X_t - \delta(t)) \leqq x\} = 1 - L_1(-A_1^2 x),$$

(iii) *If* $1 < \kappa < 2$

$$\lim_{n\to\infty} P\{n^{-1/\kappa}(T_n - A_\kappa n) \leqq x\} = L_\kappa(x)$$

and

$$\lim_{t\to\infty} P\left\{t^{-1/\kappa}\left(X_t - \frac{t}{A_\kappa}\right) \leqq x\right\} = 1 - L_\kappa(-x A_\kappa^{1+1/\kappa}),$$

(iv) *If* $\kappa = 2$

$$\lim_{n\to\infty} P\{B_1^{-1}(n \log n)^{-\frac{1}{2}}(T_n - A_2 n) \leqq x\} = \Phi(x) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-s^2/2} ds$$

and

$$\lim_{t\to\infty} P\left\{A_2^{\frac{3}{2}} B_1^{-1}(t \log t)^{-\frac{1}{2}}\left(X_t - \frac{t}{A_2}\right) \leqq x\right\} = \Phi(x),$$

(v) *If* $\kappa > 2$

$$\lim_{n\to\infty} P\{B_2^{-1} n^{-\frac{1}{2}}(T_n - B_3 n) \leqq x\} = \Phi(x)$$

and

$$\lim_{n\to\infty} P\left\{B_3^{\frac{3}{2}} B_2^{-1} n^{-\frac{1}{2}} \left(X_t - \frac{t}{B_3}\right) \leqq x\right\} = \emptyset(x).$$

REMARK 1: The theorem remains valid if everywhere T_n is replaced by

$T_n^* = \# \{t : X_t \leqq n\} =$ total time spent by the X process in $(-\infty, n]$.

REMARK 2: As the comments below and the proof in sect. 2 show, the limit theorems for T_n are equivalent to limit theorems for certain branching processes in random environment with immigration.

REMARK 3: (1.3) allows $P\{\beta_0 = 0\} = P\{\alpha_0 = 1\} > 0$, i.e. the distribution of $\log \beta_0/\alpha_0$ may have an atom at $-\infty$. It cannot have an atom at $+\infty$ by (1.4). The precise meaning of (1.5) is that the group generated by

$$(-\infty, +\infty) \cap \operatorname{supp}\left(\log \frac{\beta_0}{\alpha_0}\right)$$

is dense in $(-\infty, +\infty)$.

We give an indication of the proof, whose details will be carried out in the next section. Introduce,

$$U_i^n = \# \text{ of steps by } \{X_t\} \text{ from } i \text{ to } i-1 \text{ during } [0, T_n) = \# \{t < T_n : X_t = i, X_{t+1} = i-1\}.$$

Clearly

$$n = X_{T_n} - X_0 = \# \text{ of steps to the right during } [0, T_n) - \# \text{ of steps to the left during } [0, T_n),$$

so that

$$\begin{aligned} T_n &= \# \text{ of steps during } [0, T_n) = \# \text{ of steps to the right during } [0, T_n) + \# \text{ of steps to the left during } [0, T_n) \\ &= n + 2\{\# \text{ of steps to the left during } [0, T_n)\} \\ &= n + 2\sum_i U_i^n. \end{aligned}$$

By definition of U_i^n and T_n, $U_i^n = 0$ for $i > n$ and

$$\sum_{i \leqq 0} U_i^n \leqq \text{total time spent by } X_t \text{ in } [-\infty, 0] < \infty$$

w.p.l. since $X_t \to \infty$ w.p.l. under (1.3) (see [10]). This implies

$$\lim_{n \to \infty} n^{-1/\kappa}\{\sum_{i \leqq 0} U_i^n + \sum_{i > n} U_i^n\} = 0 \text{ w.p.l.}$$

and it suffices to show that

$$\sum_{i=1}^{n} U_i^n \tag{1.6}$$

converges to L_κ in distribution after suitable normalization. Now fix $\{\alpha_i\}$ for the moment, so that conditionally on this $\{\alpha_i\} X_t$ is a Markov chain. Observe that a step from j to $j-1$ has to occur either between T_j and the first step from j to $j+1$ or between two successive steps from j to $j+1$. When $X_{t_0} = j$ for some t_0, then the conditional probability, given $\{\alpha_i\}$ and $X_0, \cdots X_{t_0}$, of going k times from j to $j-1$ before the next move from j to $j+1$ is $\alpha_j \beta_j^k$. From this one can see that the conditional distribution of U_j^n, given $\mathscr{A}$ and U_{j+1}^n, $U_{j+2}^n, \cdots U_{n-1}^n$ is precisely the distribution of the sum of $1 + U_{j+1}^n$ independent random variables $V_1, V_2, \cdots$, each with the geometric distribution

$$P\{V_l = k\} = \alpha_j \beta_j^k. \tag{1.7}$$

In other words, for fixed $\{\alpha_i\}$ and n the sequence $U_n^n = 0$, $U_{n-1}^n, \cdots, U_1^n$ has the distribution of the first n generations of an inhomogeneous branching process with one immigrant in each generation and with offspring distribution (1.7) for all particles in the $(n-j-1)^{\text{th}}$ generation (including the immigrant entering at time $n-j-1$.) (see [1], Ch. 6.7 or [9], Ch. 7). When $\{\alpha_i\}$ is random as well then $U_n^n = 0$, $U_{n-1}^n, \cdots, U_1^n$ from n generations of a branching process in random environment with one immigrant each unit of time (see [1], Ch. 6.5). Since $\alpha_{n-1}, \cdots, \alpha_1$ have the same joint distribution as $\alpha_0, \cdots, \alpha_{n-2}$ it follows that (1.6) has the same distribution as

$$\sum_{t=0}^{n-1} Z_t, \tag{1.8}$$

where $Z_0 = 0$, $Z_1, Z_2, \cdots$ forms a branching process in random environment with one immigrant each unit of time and offspring distribution (1.7) for each particle (including the immigrant) present at time j. The environmental variables α_i are independent identically distributed. The equivalence of the distributions of (1.6) and (1.8) was already discussed in [6] and may be well known in the theory of birth and death processes. As in [6] we introduce the stopping times

$$\nu_0 = 0, \qquad \nu_{k+1} = \min\{t > \nu_k : Z_t = 0\}.$$

The ν_k are the successive times at which no offspring from previous generations is left so that the Z_t process starts afresh at those times with one new immigrant. In particular the random variables $((\nu_{k+1}-\nu_k), W_k)$, where

$$W_k = \sum_{\nu_k \leqq t < \nu_{k+1}} Z_t, \qquad k = 0, 1, 2, \cdots,$$

are independent and identically distributed (when the α_i are also random). As we shall see $\mu \equiv E(\nu_{k+1}-\nu_k) < \infty$ so that

$$\sum_{t=0}^{n-1} Z_t \approx \sum_{0 \leqq k \leqq \mu^{-1}n} W_k.$$

If we can show that W_0 is in the domain of attraction of a stable law of index κ, then the theorem will follow from these observations by standard arguments. W_0 is the total number of particles which were born before time ν_1. Its randomness is due to randomness in the environment plus additional fluctuations in the number of progeny of each particle, once the environment is fixed. It will turn out that the latter fluctuations mainly have influence in the beginning and we will be able to approximate W_k by random variables of the form $\gamma_k(R_k+1)$ where all the $\gamma_0, \gamma_1, \gamma_2, \cdots, R_0, R_1, \cdots$ are independent, all the γ_i have the same distribution and all the R_i have the distribution of

$$\eta_0 = \sum_{t=1}^{\infty} \prod_{i=0}^{t-1} \left(\frac{\beta_i}{\alpha_i}\right).$$

Note that $\eta_0 =$ expected number of total progeny of the immigrant at time 0, given α_i, $i \geqq 0$. It was shown in [5] that

$$P\{\eta_0 > x\} \sim Kx^{-\kappa}, \qquad x \to \infty,$$

for some $0 < K < \infty$ so that all the R_i are in the domain of attraction of a stable law of index κ. Once this point has been reached, the remainder of the proof is straight sailing.

2. Proof of theorem

Throughout this section all hypotheses of our Theorem will be in force and we use the following notation: $Z_0 = 0$, $Z_1, Z_2, \cdots$ is a branching process in random environment with one immigrant entering each generation. When $Z_0, \cdots, Z_t$, $\alpha_0, \cdots, \alpha_t$ are given, Z_{t+1} is the sum of $Z_t + 1$ independent identically distributed random variables which take the value k with probability $\alpha_t \beta_t^k$ $(k = 0, 1, 2, \cdots)$.

(2.1) $Z_{s,t}$ = number of progeny alive at time t of the immigrant who entered at time s, $s < t$.

Several times we use the representation

$$Z_t = \sum_{s=0}^{t-1} Z_{s,t}, \tag{2.2}$$

which is obvious from (2.1). The total number of progeny of the immigrant at time s is denoted by

$$Y_s = \sum_{t=s+1}^{\infty} Z_{s,t}. \tag{2.3}$$

$$\nu = \min\{t > 0 : Z_t = 0\}, \tag{2.4}$$

$$W = \sum_{t=0}^{\nu-1} Z_t, \tag{2.5}$$

$$m_i = \frac{\beta_i}{\alpha_i}, \tag{2.6}$$

$$\eta_s = E\{Y_s | \mathscr{A}\} = \sum_{t=s+1}^{\infty} \prod_{i=s}^{t-1} m_i. \tag{2.7}$$

Finally we introduce the stopping time

$$\sigma = \sigma(A) = \min\{t : Z_t > A\}. \tag{2.8}$$

The principal tool in the proof is

LEMMA 1: *For some constant $K > 0$*

$$P\{\eta_0 \geqq x\} \sim Kx^{-\kappa}, \qquad x \to \infty. \tag{2.9}$$

(2.9) is just a special case (with $Q_i \equiv 1$ and $M_i \geqq 0$) of Theorem 5 in [5] (Note that in the one-dimensional case this theorem is fairly simple and one only needs Section 3 of [5].)

LEMMA 2:

$$P\{\nu > t\} \leqq K_1 e^{-K_2 t}, \qquad t \geqq 0, \tag{2.10}$$

for suitable K_1, $K_2 > 0$, and $E\nu < \infty$.

PROOF: Even though a simpler proof can be given in the special case under consideration (where the α_i are independent, identically distributed) we prefer to give a proof which appears adaptable to more general situations (e.g. where the α_i form a finite Markov chain; the models in [2] and [11] can be formulated in this way.) We divide the proof into separate steps. Only the case $E \log m_0 > -\infty$ is treated in detail[1].

Step 1. Let

$$a = E \log m_i \text{ (note } a < 0, \text{ by (1.3))},$$

$$S_n = \sum_{i=0}^{n-1} \left\{\log m_i - \frac{a}{2}\right\} \qquad (S_0 = 0),$$

$$N_0 = 0, \qquad N_{k+1} = \inf\{n > N_k : S_n \leqq S_{N_k}\}.$$

S_n is a random walk whose increments have expectation $\frac{1}{2}a < 0$ (by (1.3)) and N_k is the sequence of its successive downward ladder indices. We have for any $\theta \geqq 0$

$$P\{N_1 > n\} \leqq P\{S_n > 0\} \leqq Ee^{\theta S_n} = \{e^{-\frac{1}{2}\theta a} Em_0^\theta\}^n.$$

Since

$$\lim_{\theta \downarrow 0} (Em_0^\theta)^{1/\theta} = e^{E(\log m_0)} = e^a$$

we may assume that $\theta > 0$ is fixed such that

[1] If $P\{m_0 = 0\}$ 0 then (2.10) is immediate from $\nu = \min\{i : m_{i-1} = 0\}$. If $P\{m_0 = 0\} = 0$, but still $E \log m_0 = -\infty$, then one can take any negative number for a in the proof below.

$$Em_0^\theta \leqq (e^{\frac{3}{4}a})^\theta < 1,$$

and hence

$$P\{N_1 > n\} \leqq e^{(\frac{1}{4}\theta a)n}.$$

In particular

$$b = EN_1 < \infty$$

and Ee^{tN_1} exists for $|t| < |\frac{1}{4}\theta\alpha|$. By Bernstein's inequality (see [12], problems 10.12–10.14) this implies the existence of $K_3, K_4 > 0$ such that

(2.11) $$P\{N_k \geqq 2kb\} \leqq K_3 e^{-K_4 k}, \qquad k \geqq 0.$$

In view of (2.11) it suffices to prove

(2.12) $$P\{v > N_k|\mathscr{A}\} \leqq K_5 e^{-K_6 k}, \qquad k \geqq 0,$$

for some constants $K_5, K_6 > 0$ (which do not depend on $\alpha_0, \alpha_1, \cdots$).

Step 2. Next we prove two estimates. Let

$$c = \frac{e^{\frac{1}{2}a}}{1-e^{\frac{1}{2}a}},$$

$$d = \tfrac{1}{4}e^{-2c}.$$

Then for all $k \geqq 1, j \geqq 0$

(2.13) $$P\{v \leqq N_k|\mathscr{A}\} \geqq P\{Z_{N_k} = 0|\mathscr{A}\} \geqq d > 0,$$

(2.14) $$P\{\sum_{s \leqq N_j} Z_{s, N_{j+k}} > 0|\mathscr{A}\} \leqq (1-e^{\frac{1}{2}a})^{-1}e^{\frac{1}{2}ka}.$$

The first inequality in (2.13) is obvious. Also, by standard branching process formulae

(2.15) $$E\{Z_{s,t}|\mathscr{A}\} = \prod_{i=s}^{t-1} m_i = \exp\sum_{i=s}^{t-1} \log m_i$$

and for $s \leqq N_k - 1$

$$\sum_{i=s}^{N_k-1} \log m_i = \sum_{i=s}^{N_k-1} \{\log m_i - \tfrac{1}{2}a\} + (N_k - s)\tfrac{1}{2}a \leqq (N_k - s)\tfrac{1}{2}a \tag{2.16}$$

because N_k is a downward ladder index of S_n. It follows that

$$E\{Z_{N_k}|\mathscr{A}\} = \sum_{s=0}^{N_k-1} E\{Z_{s,N_k}|\mathscr{A}\} \leqq \frac{e^{\frac{1}{2}a}}{1-e^{\frac{1}{2}a}} = c,$$

and since

$$E\{Z_{N_k}|\mathscr{A}\} \geqq E\{Z_{N_k-1}|\mathscr{A}\}m_{N_k-1}$$

we have

$$E\{Z_{N_k-1}|\mathscr{A}\} \leqq c(m_{N_k-1})^{-1}.$$

Also

$$P\{Z_{N_k} = 0|\mathscr{A}, Z_0, \cdots, Z_{N_k-1}\} = (\alpha_{N_k-1})^{Z_{N_k-1}+1},$$

and a fortiori

$$P\{Z_{N_k} = 0|\mathscr{A}\} \geqq P\{Z_{N_k-1} \leqq L|\mathscr{A}\}\alpha_{N_k-1}^{L+1}. \tag{2.17}$$

We take

$$L = 2c(m_{N_k-1})^{-1} \geqq 2E\{Z_{N_k-1}|\mathscr{A}\}$$

so that the first factor in the right hand side of (2.17) is at least $\frac{1}{2}$. Also, by definition of N_k we must have $S_{N_k} \leqq S_{N_k-1}$ and hence

$$\log \beta_{N_k-1}(1-\beta_{N_k-1})^{-1} = \log m_{N_k-1} \leqq \tfrac{1}{2}a < 0$$

so that

$$\alpha_{N_k-1} \geqq \tfrac{1}{2}, \qquad \beta_{N_k-1} \leqq \tfrac{1}{2} \text{ and } \alpha_{N_k-1} = 1-\beta_{N_k-1} \geqq e^{-2\beta_{N_k-1}}.$$

Finally, therefore, the right hand side of (2.17) is at least

$$\tfrac{1}{4}\exp\{-2\beta_{N_k-1}\cdot 2c(m_{N_k-1})^{-1}\} \geqq \tfrac{1}{4}e^{-2c},$$

which proves (2.13). (2.14) is much easier. In fact, by (2.15)–(2.16) the left hand side of (2.14) equals

$$
\begin{aligned}
P\{\sum_{s \leqq N_j} Z_{s, N_j+k} \geqq 1 | \mathscr{A}\} &\leqq E\{\sum_{s \leqq N_j} Z_{s, N_j+k} | \mathscr{A}\} \\
&= E\{\sum_{s \leqq N_j} \exp \sum_{i=s}^{N_j+k-1} \log m_i | \mathscr{A}\} \\
&\leqq E\{\sum_{s \leqq N_j} \exp (N_{j+k} - s)\tfrac{1}{2}a | \mathscr{A}\} \leqq \sum_{l=k}^{\infty} e^{\frac{1}{2}la} = \frac{e^{\frac{1}{2}ka}}{1-e^{\frac{1}{2}a}}.
\end{aligned}
$$

Step 3. We turn to (2.12). Fix k, and the sequence $\alpha_0, \alpha_1, \cdots$. All the probabilities in this step will be conditioned on this sequence being fixed and for brevity we denote the corresponding conditional probabilities by $\bar{P}$. Take $k_0 = k$ and if $k_0, \cdots, k_i$ have been found and $k_i > 0$, take

$$
k_{i+1} = \max \{l < k_i : \sum_{N_l}^{N_{k_i}-1} Z_{s, N_{k_i}} > 0, \text{ but } \sum_{N_{l+1}}^{N_{k_i}-1} Z_{s, N_{k_i}} = 0\}
$$

on the set $\{Z_{N_{k_i}} > 0\}$, and

$$
k_{i+1} = k_i - 1 \quad \text{on} \quad \{Z_{N_{k_i}} = 0\}.
$$

If $k_i = 0$ take $k_{i+1} = 0$ as well. The occurrence of $k_{i+1} = l$ depends only on the $Z_{s,t}$ with $N_l \leqq s < t \leqq N_k$. Since $Z_{s_1, t_1}, s_1 < t_1 \leqq N_{k_i}$ is independent of all $Z_{s_2, t_2}, N_{k_i} \leqq s_2 < t_2$ we see from (2.14)

$$
\begin{aligned}
\bar{P}\{k_i - k_{i+1} > r | k_0, \cdots, k_i\} &\leqq \bar{P}\{\sum_{s < N_{k_i - r}} Z_{s, N_{k_i}} > 0\} \\
&\leqq (1-e^{\frac{1}{2}a})^{-1} e^{\frac{1}{2}ra} \quad \text{on} \quad \{k_i > 0\}.
\end{aligned}
$$

This estimate remains trivially valid on $\{k_i = 0\}$ as well. Because $a < 0$ this implies the existence of a $K_7 < \infty$ and $\lambda_0 > 0$ such that

$$
\bar{E}\{e^{\lambda(k_i - k_{i+1})} | k_0, \cdots, k_i\} \leqq e^{K_7 \lambda}, \qquad 0 \leqq \lambda \leqq \lambda_0.
$$

Consequently, for $l_0 = [(1/2K_7)k]$

$$
\begin{aligned}
\bar{P}\{k_i = 0 \text{ for some } i \leqq l_0\} &\leqq \bar{P}\{k_0 - k_{l_0} \geqq k\} \\
&\leqq e^{-\lambda_0 k} \bar{E} e^{\lambda_0 (k_0 - k_{l_0})} \leqq e^{-\lambda_0 k + K_7 \lambda_0 l_0} \leqq e^{-(\frac{1}{2}\lambda_0)k}.
\end{aligned}
$$

Again using the conditional independence of Z_{s_1, t_1} and Z_{s_2, t_2} for given

$k_0, k_1, \cdots, k_i, s_1 < t_1 \leqq k_i \leqq s_2 < t_2$ we conclude from (2.13) that $\bar{P}\{Z_{N_{k_i}} = 0 | k_0, \cdots, k_i, Z_{s,t} \text{ for } N_{k_i} \leqq s < t \leqq k\} \geqq d$. Thus

$$\bar{P}\{k_l > 0 \text{ but } Z_{N_{k_i}} > 0 \text{ for } i = 0, \cdots, l\}$$

$$= \sum_{r=1}^{k-1} \bar{P}\{Z_{N_k} > 0, k_1 = r\} P\{k_l > 0 \text{ but } Z_{N_{k_i}} > 0 \text{ for } i = 1, \cdots, l | Z_{N_k} > 0, k_1 = r\}$$

$$\leqq (1-d) \max_{1 \leqq r < k} \bar{P}\{k_l > 0 \text{ but } Z_{N_{k_i}} > 0 \text{ for } i = 1, \cdots, l | Z_{N_k} > 0, k_1 = r\}$$

$$= (1-d) \max_{1 \leqq r < k} \bar{P}\{k_l > 0 \text{ but } Z_{N_{k_i}} > 0 \text{ for } i = 1, \cdots, l | \sum_{s=N_r}^{N_k - 1} Z_{s, N_k} > 0 \text{ but } \sum_{s=N_{r+1}}^{N_k - 1} Z_{s, N_k} = 0\} \leqq \cdots \leqq (1-d)^{l+1}.$$

Finally $v \leqq N_k$ on the set where $Z_{N_{k_i}} = 0$ for some $k_i > 0$, so that

$$\bar{P}\{v > N_k\} \leqq \bar{P}\{k_i = 0 \text{ for some } i \leqq l_0\}$$
$$+ \bar{P}\{k_0 > k_1 > \cdots > k_{l_0} > 0 \text{ but } Z_{N_{k_i}} > 0 \text{ for } i = 0, \cdots, l_0\}$$
$$\leqq e^{-(\frac{1}{2}\lambda_0)k} + (1-d)^{l_0} \leqq K_5 e^{-K_6 k}$$

for some $K_5, K_6 > 0$ which do not depend on the specific sequence $\{\alpha_i\}$. This proves (2.12) and hence the lemma. ∎

As an immediate corollary of Lemma 2 we have for all $\varepsilon > 0$, $A > 0$

$$\text{(2.18)} \quad P\{W \geqq \varepsilon x, \sigma(A) \geqq v\} \leqq P\{Av \geqq \varepsilon x\} = o(x^{-\kappa}), \qquad x \to \infty,$$

because

$$W = \sum_{t=0}^{v-1} Z_t \leqq Av \quad \text{on} \quad v \leqq \sigma(A).$$

For the same reason

$$\text{(2.19)} \qquad P\{\sum_{t=0}^{\sigma-1} Z_t \geqq \varepsilon x, \sigma(A) < v\} \leqq P\{v \geqq \varepsilon A^{-1} x\} = o(x^{-\kappa}).$$

LEMMA 3: *If $\kappa \leqq 2$, then there exists for all $\varepsilon > 0$ an $A_0 = A_0(\varepsilon) < \infty$ such that*

$$P\{\sum_{\sigma \leqq s < v} Y_s \geqq \varepsilon x\} \leqq \varepsilon x^{-\kappa} \qquad \text{for} \quad A \geqq A_0.$$

PROOF: Taking into account that $\sum s^{-2} = \pi^2/6 \leqq 2$ we have

$$(2.20) \quad P\{\sum_{\sigma \leqq s < v} Y_s \geqq \varepsilon x\} = P\{\sum_{s=1}^{\infty} I[\sigma \leqq s < v] Y_s \geqq 6\pi^{-2}\varepsilon x \sum_{s=1}^{\infty} s^{-2}\}$$
$$\leqq \sum_{s=1}^{\infty} P\{I[\sigma \leqq s < v] Y_s \geqq \tfrac{1}{2}\varepsilon x s^{-2}\}.$$

But Y_s depends only on $\alpha_s, \alpha_{s+1}, \cdots$ and the numbers of offspring of particles in generations $s, s+1, \cdots$, whereas the event $\{\sigma \leqq s < v\}$ is defined in terms of $Z_0, \cdots, Z_s$. Thus Y_s and $I[\sigma \leqq s < v]$ are independent and Y_s has the same distribution as Y_0, so that the last sum in (2.20) equals

$$\sum_{s=1}^{\infty} P\{\sigma \leqq s < v\} P\{Y_0 \geqq \tfrac{1}{2}\varepsilon x s^{-2}\}.$$

Thus, if we can prove that

$$(2.21) \qquad P\{Y_0 \geqq x\} \leqq K_8 x^{-\kappa}$$

for some $K_8 < \infty$, then it follows that (2.20) is at most

$$x^{-\kappa} 2^{\kappa} \varepsilon^{-\kappa} K_8 \sum_{s=1}^{\infty} s^{2\kappa} P\{\sigma \leqq s < v\}$$
$$\leqq x^{-\kappa} 2^{\kappa} \varepsilon^{-\kappa} K_8 E\{v^{2\kappa+1}; \sigma < v\} \leqq \varepsilon x^{-\kappa}$$

for $A \geqq A_0(\varepsilon)$ (because $Ev^{2\kappa+1} < \infty$ and $\sigma(A) \uparrow \infty$ in probability as $A \to \infty$). Now observe that $\eta_t = m_t(1+\eta_{t+1})$ and consequently (with $Z_{0,0} = 0$)

$$(2.22) \qquad Y_0 = \sum_{t=1}^{\infty} Z_{0,t} = \sum_{t=1}^{\infty} Z_{0,t}(1+\eta_t - m_t(1+\eta_{t+1}))$$
$$= \sum_{t=1}^{\infty} (Z_{0,t} - Z_{0,t-1} m_{t-1})(1+\eta_t);$$

the manipulations with these sums are justified because the sums only run till v which is finite w.p.l. Again using the independence of $(1+\eta_t)$

and $(m_{t-1}, Z_{0,t-1}, Z_{0,t})$ we have as above

$$P\{Y_0 \geqq x\} \leqq \sum_{t=1}^{\infty} P\{|Z_{0,t}-Z_{0,t-1}m_{t-1}|(1+\eta_t) \geqq \tfrac{1}{2}t^{-2}x\}$$

$$\leqq \sum_{t=1}^{\infty} \int P\{|Z_{0,t}-Z_{0,t-1}m_{t-1}| \in \mathrm{d}s\} P\{1+\eta_0 \geqq (2st^2)^{-1}x\}.$$

By (2.9) there exists a $K_9 < \infty$ for which

$$P\{1+\eta_0 \geqq (2st^2)^{-1}x\} \leqq K_9(2st^2)^{\kappa}x^{-\kappa},$$

and there results

$$P\{Y_0 \geqq x\} \leqq x^{-\kappa}2^{\kappa}K_9 \sum_{t=1}^{\infty} t^{2\kappa}E|Z_{0,t}-Z_{0,t-1}m_{t-1}|^{\kappa} \tag{2.23}$$

$$\leqq x^{-\kappa}2^{\kappa}K_9 \sum_{t=1}^{\infty} t^{2\kappa}E\{E\{|Z_{0,t}-Z_{0,t-1}m_{t-1}|^2|\mathscr{A}\}^{\kappa/2}\}$$

(Jensen's inequality; recall $\kappa \leqq 2$ in this lemma). We complete the proof of (2.21) and the lemma by proving the convergence of the last series in (2.23). For this purpose we observe that for $t \geqq 2$, when $Z_{0,t-1}$ and α_i, $i \geqq 0$, are given, then $Z_{0,t}$ can be written as

$$Z_{0,t} = \sum_{j=1}^{Z_{0,t-1}} V_j,$$

where V_j represents the number of children of the j^{th} particle among the $Z_{0,t-1}$ descendants at time $(t-1)$ of the immigrant at time zero. The V_j are conditionally independent and for each j

$$P\{V_j = k|\mathscr{A}, Z_{0,t-1}\} = \alpha_{t-1}\beta_{t-1}^{k}, \qquad k \geqq 0.$$

Thus

$$E\{|Z_{0,t}-Z_{0,t-1}m_{t-1}|^2|Z_{0,t-1}, \mathscr{A}\}$$
$$= Z_{0,t-1}\sigma^2(V_1|\mathscr{A}) = Z_{0,t-1}(m_{t-1}+m_{t-1}^2)$$

and

$$
\begin{aligned}
(2.24)\quad E\{(E\{|Z_{0,t}-Z_{0,t-1}m_{t-1}|^2|\mathscr{A}\})^{\kappa/2}\} \\
&\leqq E\{(E\{Z_{0,t-1}|\mathscr{A}\})^{\kappa/2}(m_{t-1}^{\kappa/2}+m_{t-1}^{\kappa})\} \\
&= E\prod_{i=0}^{t-2} m_i^{\kappa/2}(m_{t-1}^{\kappa/2}+m_{t-1}^{\kappa}) \leqq 2(Em_0^{\kappa/2})^{t-2}.
\end{aligned}
$$

Moreover

$$
Em_0^{\kappa/2} < 1 \tag{2.25}
$$

because Em_0^x is a convex function of x which equals one at $x = 0$ and $x = \kappa$. For $t = 1$ we obtain

$$
EZ_{0,1}^{\kappa} \leqq E\{E\{Z_{0,1}^2|\mathscr{A}\}^{\kappa/2}\} \leqq E(m_0+2m_0^2)^{\kappa/2} \leqq 3.
$$

The convergence of the series in (2.23) is now evident. ■

Next we introduce

$S_{\sigma,t}$ = number of progeny alive at time t of the Z_σ particles present at σ

provided $\sigma < t$. We take $S_{\sigma,\sigma} = Z_\sigma$ and

$$
S_\sigma = \sum_{t=\sigma}^{\infty} S_{\sigma,t} = Z_\sigma + \text{total progeny of the } Z_\sigma \text{ particles present at } \sigma.
$$

The interpretation of W as the number of particles born before v immediately shows that on $\{\sigma < v\}$

$$
W = \sum_{s=0}^{\sigma-1} Z_s + S_\sigma + \sum_{\sigma \leqq s < v} Y_s.
$$

(2.18), (2.19) and Lemma 3 and the fact that $W \geqq S_\sigma$ therefore allow us to write for sufficiently large A and x

$$
\begin{aligned}
(2.26)\quad P\{\sigma < v, S_\sigma \geqq x\} \leqq P\{W \geqq x\} \\
&\leqq P\{\sigma < v, S_\sigma \geqq x(1-2\varepsilon)\} + 3\varepsilon x^{-\kappa}.
\end{aligned}
$$

We shall now compare S_σ to $Z_\sigma(1+\eta_\sigma)$. We can expect these to be not very different because $Z_\sigma \geqq A$ is large (for A large) and $S_\sigma - Z_\sigma$ counts the progeny of this large number of independent particles, and

$$
E\{S_\sigma|\sigma < v, Z_\sigma, \mathscr{A}\} = Z_\sigma(1+\eta_\sigma).
$$

LEMMA 4: *If $\kappa \leqq 2$, then for fixed A*

$$E\{Z_\sigma^\kappa; \sigma < v\} < \infty.^1 \tag{2.27}$$

If $\kappa > 2$, then

$$EW^2 < \infty. \tag{2.28}$$

PROOF: We have on $\{\sigma < v\}$

$$Z_\sigma = (Z_{\sigma-1}+1)\frac{Z_\sigma}{Z_{\sigma-1}+1} \leqq (A+1)\frac{Z_\sigma}{Z_{\sigma-1}+1} \leqq (A+1)\sum_{1 \leqq t \leqq v} \frac{Z_t}{Z_{t-1}+1}.$$

Therefore, if $\kappa \geqq 1$

$$(E\{Z_\sigma^\kappa; \sigma < v\})^{1/\kappa} \leqq (A+1)\left(E\left\{\sum_{t \geqq 1} \frac{Z_t}{Z_{t-1}+1} I[t \leqq v]\right\}^\kappa\right)^{1/\kappa} \tag{2.29}$$

$$\leqq \text{(Minkowski's inequality)}(A+1)\sum_{t \geqq 1}\left(E\left\{\left(\frac{Z_t}{Z_{t-1}+1}\right)^\kappa I[t \leqq v]\right\}\right)^{1/\kappa}.$$

As in the argument leading to (2.24) we can write Z_t as

$$\sum_{j=1}^{Z_{t-1}+1} V_j,$$

where V_j represents the number of children of the j^{th} particle of the $(t-1)^{\text{st}}$ generation if $j \leqq Z_{t-1}$, and of the immigrant at time $t-1$ if $j = Z_{t-1}+1$. Again the conditional probability of $\{V_j = k\}$ given $Z_0, \cdots, Z_{t-1}$ and $\alpha_i, i \geqq 0$, is $\alpha_{t-1}\beta_{t-1}^k$ and the V_j are conditionally independent. By Jensen's and Minkowski's inequality we get for $1 \leqq \kappa \leqq 2$, as in (2.24),

$$(E\{Z_t^\kappa | \mathscr{A}, Z_0, \cdots, Z_{t-1}\})^{1/\kappa} \leqq \sum_{j=1}^{Z_{t-1}+1} (E\{V_j^2 | \mathscr{A}, Z_0, \cdots, Z_{t-1}\})^{\frac{1}{2}}$$
$$= (Z_{t-1}+1)(m_{t-1}+2m_{t-1}^2)^{\frac{1}{2}}.$$

[1] When the proof is completed we shall see that (2.27) actually has a finite limit as $A \to \infty$ (see Lemma 6, especially (2.35) and (2.36).

It follows that

$$E\left\{\left(\frac{Z_t}{Z_{t-1}+1}\right)^{\kappa} I[t \leqq v]\right\}$$

$$= E\left\{E\left\{\left(\frac{Z_t}{Z_{t-1}+1}\right)^{\kappa} | Z_0, \cdots, Z_{t-1}\mathscr{A}\right\}; t \leqq v\right\}$$

$$\leqq E\{(m_{t-1}+2m_{t-1}^2)^{\kappa/2}; t \leqq v\}$$

$$\leqq 2^{\kappa}(E\{m_{t-1}^{\kappa/2}; t \leqq v\}+E\{m_{t-1}^{\kappa}; t \leqq v\})$$

$$= 2^{\kappa}(Em_0^{\kappa/2}+Em_0^{\kappa})P\{v > t-1\} \leqq 2^{\kappa+1}P\{v > t-1\}$$

(for the one but last step, note that m_{t-1} is independent of $I[t \leqq v] = I[t-1 < v]$, and for the last step use (2.25)). (2.27) now follows from (2.29) and Lemma 2 if $1 \leqq \kappa \leqq 2$. For $\kappa < 1$ we use the inequality

$$(\textstyle\sum a_i)^{\kappa_1} \leqq \sum a_i^{\kappa_1}, \tag{2.30}$$

valid for $a_i \geqq 0$ and $0 \leqq \kappa_1 \leqq 1$. Now

$$E\{Z_\sigma^{\kappa}; \sigma < v\} \leqq (A+1)^{\kappa} \sum_{t \geqq 1} E\left\{\left(\frac{Z_t}{Z_{t-1}+1}\right)^{\kappa} I[t \leqq v]\right\}$$

$$\leqq (A+1)^{\kappa} \sum_{t \geqq 1} E\{I[t \leqq v](Z_{t-1}+1)^{-\kappa}(E\{Z_t|\mathscr{A}, Z_0, \cdots, Z_{t-1}\})^{\kappa}\}$$

$$\text{(Jensen's inequality)} \leqq (A+1)^{\kappa} \sum_{t \geqq 1} E\{I[t \leqq v]m_{t-1}^{\kappa}\}$$

$$= (A+1)^{\kappa} \sum_{t \geqq 1} P\{v \geqq t\} < \infty.$$

To estimate EW^2 for $\kappa > 2$ we write

$$W = \text{total number of particles born up till time } v$$

$$= \sum_{0 \leqq s < v} Y_s = \sum_{s=0}^{\infty} Y_s I[s < v]$$

Thus, by Minkowski's inequality and the independence of Y_s and

$I[s < v] = I[s-1 \leqq v]$

$$(EW^2)^{\frac{1}{2}} \leqq \sum_{s=0}^{\infty} (E\{Y_s I[s < v]\}^2)^{\frac{1}{2}}$$
$$= \sum_{s=0}^{\infty} (EY_0^2)^{\frac{1}{2}} (P\{v > s\})^{\frac{1}{2}}.$$

By Lemma 2

$$\sum_{s=0}^{\infty} (P\{v > s\})^{\frac{1}{2}} < \infty$$

so that we merely have to prove $EY_0^2 < \infty$ when $\kappa > 2$. But $Y_0 = \sum Z_{0,t}$, so that

$$(2.31) \qquad (EY_0^2)^{\frac{1}{2}} = (E\{\sum_{t=1}^{\infty} Z_{0,t}\}^2)^{\frac{1}{2}}$$
$$\leqq \sum_{t=1}^{\infty} (EZ_{0,t}^2)^{\frac{1}{2}} = \sum_{t=1}^{\infty} (E\{E\{Z_{0,t}^2 | \mathscr{A}\}\})^{\frac{1}{2}}$$

However, for fixed $\alpha_0, \alpha_1, \cdots Z_{0,t}$ is just an inhomogeneous branching process with a geometric offspring distribution with mean m_i for the particles in the i^{th} generation. The second moment for such a process can be computed by standard methods for branching processes (see [1], Ch. 1.2 or [9], Ch. 1.6).

$$(2.32)\ E\{Z_{0,t}^2 | \mathscr{A}\} = (E\{Z_{0,t} | \mathscr{A}\})^2 + \sigma^2\{Z_{0,t} | \mathscr{A}\}$$
$$= (\prod_{i=0}^{t-1} m_i)^2 + \sum_{k=0}^{t-1} \prod_{i=0}^{k-1} m_i (m_k^2 + m_k) \prod_{j=k+1}^{t-1} m_i^2$$
$$\leqq 3 \sum_{k=0}^{t} \prod_{i=0}^{k-1} m_i \prod_{j=k}^{t-1} m_j^2 \text{ (empty products equal one).}$$

Hence

$$(2.33) \qquad (EY_0^2)^{\frac{1}{2}} \leqq 3^{\frac{1}{2}} \sum_{t=1}^{\infty} (E \sum_{k=0}^{t} \prod_{i=0}^{k-1} m_i \prod_{j=k}^{t-1} m_j^2)^{\frac{1}{2}}$$
$$= 3^{\frac{1}{2}} \sum_{t=1}^{\infty} (\sum_{k=0}^{t} (Em_0)^k (Em_0^2)^{t-k})^{\frac{1}{2}}$$
$$\leqq 3^{\frac{1}{2}} \sum_{t=1}^{\infty} (t+1)^{\frac{1}{2}} \{\max (Em_0, Em_0^2)\}^{t/2}.$$

As in the proof of Lemma 3 we conclude from the convexity of $x \to Em_0^x$ that

$$\max(Em_0, Em_0^2) < 1$$

so that indeed $EY_0^2 < \infty$ and $EW^2 < \infty$. ■

LEMMA 5: *If $\kappa \leqq 2$ then there exists for all $\varepsilon > 0$ on $A_1 = A_1(\varepsilon)$ such that*

$$P\{|\sum_{t=\sigma}^{\infty}(S_{\sigma,t}-Z_\sigma\prod_{i=\sigma}^{t-1}m_i)| \geqq \varepsilon x, \sigma < v\} \leqq \varepsilon x^{-\kappa}E\{Z_\sigma^\kappa; \sigma < v\}$$

for $A \geqq A_1$.

PROOF: This is quite analogous to the proof of Lemma 3. We have

$$S_{\sigma,t}-Z_\sigma\prod_{i=\sigma}^{t-1}m_i = \sum_{\sigma+1\leqq l\leqq t}(S_{\sigma,l}\prod_{i=l}^{t-1}m_i - S_{\sigma,l-1}\prod_{i=l-1}^{t-1}m_i),$$

and therefore

$$\sum_{t=\sigma}^{\infty}(S_{\sigma,l}-Z_\sigma\prod_{i=\sigma}^{t-1}m_i) = \sum_{l=\sigma+1}^{\infty}\sum_{t=l}^{\infty}(S_{\sigma,l}\prod_{i=l}^{t-1}m_i - S_{\sigma,l-1}\prod_{i=l-1}^{t-1}m_i)$$

$$= \sum_{l=\sigma+1}^{\infty}(S_{\sigma,l}-S_{\sigma,l-1}m_{l-1})\sum_{t=l}^{\infty}\prod_{i=l}^{t-1}m_i = \sum_{l=\sigma+1}^{\infty}(S_{\sigma,l}-S_{\sigma,l-1}m_{l-1})(1+\eta_l).$$

Just as in (2.20)–(2.23) we have on the set $\{\sigma < v\}$

$$P\{|\sum_{t=\sigma}^{\infty}(S_{\sigma,t}-Z_\sigma\prod_{i=\sigma}^{t-1}m_i)| \geqq \varepsilon x, |\sigma, Z_0, \cdots, Z_\sigma, \mathscr{A}\}$$

$$\leqq \sum_{l=\sigma+1}^{\infty}P\{|S_{\sigma,l}-S_{\sigma,l-1}m_{l-1}|(1+\eta_l) \geqq \tfrac{1}{2}\varepsilon x(l-\sigma)^{-2}|\sigma, Z_0, \cdots, Z_\sigma\mathscr{A}\}$$

$$\leqq \sum_{l=\sigma+1}^{\infty}\int P\{|S_{\sigma,l}-S_{\sigma,l-1}m_{l-1}| \in \mathrm{d}s|\sigma, Z_0, \cdots, Z_\sigma, \mathscr{A}\} \cdot K_9 2^\kappa(l-\sigma)^{2\kappa}\varepsilon^{-\kappa}x^{-\kappa}s^\kappa$$

$$\leqq K_9\left(\frac{2}{\varepsilon}\right)^\kappa x^{-\kappa}\sum_{l=\sigma+1}^{\infty}(l-\sigma)^{2\kappa}(E\{|S_{\sigma,l}-S_{\sigma,l-1}m_{l-1}|^2|\sigma, Z_0, \cdots, Z_\sigma, \mathscr{A}\})^{\kappa/2}.$$

Also, analogously to (2.24),

$$E\{|S_{\sigma,l}-S_{\sigma,l-1}m_{l-1}|^2|\sigma, Z_0,\cdots, Z_\sigma, S_{\sigma,l-1}, \mathscr{A}\} = S_{\sigma,l-1}(m_{l-1}+m_{l-1}^2)$$

and

$$\begin{aligned}
&(E\{|S_{\sigma,l}-S_{\sigma,l-1}m_{l-1}|^2|\sigma, Z_0,\cdots, Z_\sigma, \mathscr{A}\})^{\kappa/2} \\
&\qquad = (E\{S_{\sigma,l-1}|\sigma, Z_0,\cdots, Z_\sigma, \mathscr{A}\})^{\kappa/2}(m_{l-1}+m_{l-1}^2)^{\kappa/2} \\
&\qquad \leqq (Z_\sigma \prod_{i=\sigma}^{l-2} m_i)^{\kappa/2}(m_{l-1}^{\kappa/2}+m_{l-1}^{\kappa}).
\end{aligned}$$

Finally, for a suitable $K_{10} < \infty$

$$\begin{aligned}
&P\{|\sum_{t=\sigma}^{\infty} S_{\sigma,t}-Z_\sigma \prod_{i=\sigma}^{t-1} m_i| \geqq \varepsilon x, \sigma < v\} \\
&\qquad \leqq K_9\left(\frac{2}{\varepsilon}\right)^{\kappa} x^{-\kappa}E\{Z_\sigma^{\kappa/2}\sum_{l=\sigma+1}^{\infty}(l-\sigma)^{2\kappa}\prod_{i=\sigma}^{l-2} m_i^{\kappa/2}(m_{l-1}^{\kappa/2}+m_{l-1}^{\kappa}); \sigma < v\} \\
&\qquad \leqq 2K_9\left(\frac{2}{\varepsilon}\right)^{\kappa} x^{-\kappa}E\{Z_\sigma^{\kappa/2}\sum_{l=\sigma+1}^{\infty}(l-\sigma)^{2\kappa}(Em_0^{\kappa/2})^{l-\sigma-1}; \sigma < v\} \\
&\qquad \leqq K_{10}(\varepsilon x)^{-\kappa}E\{Z_\sigma^{\kappa/2}; \sigma < v\}(\text{recall } Em_0^{\kappa/2} < 1) \\
&\qquad \leqq K_{10}(\varepsilon x)^{-\kappa}A^{-\kappa/2}E\{Z_\sigma^{\kappa}; \sigma < v\} \leqq \varepsilon x^{-\kappa}E\{Z_\sigma^{\kappa}; \sigma < v\}
\end{aligned}$$

for $A \geqq A_1(\varepsilon)$. (In one but last inequality we used the fact that $Z_\sigma \geqq A$.) ■

LEMMA 6: *If $\kappa \leqq 2$, then there exists a $0 < K_{11} < \infty$ such that*

$$\lim_{x\to\infty} x^\kappa P\{W \geqq x\} = K_{11}.$$

PROOF: This is merely a combination of (2.26) and Lemma 5. Since

$$W \geqq S_\sigma = \sum_{t=\sigma}^{\infty} S_{\sigma,t}$$

(2.26) and Lemma 5 give for $A \geqq A_2(\varepsilon)$

$$(2.34)\quad P\{\sigma < \nu, Z_\sigma \sum_{t=\sigma}^{\infty} \prod_{i=\sigma}^{t-1} m_i \geqq (1+\varepsilon)x\} - \varepsilon x^{-\kappa} E\{Z_\sigma^\kappa; \sigma < \nu\}$$

$$\leqq P\{W \geqq x\}$$

$$\leqq P\{\sigma < \nu, Z_\sigma \sum_{t=\sigma}^{\infty} \prod_{i=\sigma}^{t-1} m_i \geqq (1-3\varepsilon)x\} + \varepsilon x^{-\kappa}(3 + E\{Z_\sigma^\kappa; \sigma < \nu\}).$$

Since

$$\sum_{t=\sigma}^{\infty} \prod_{i=\sigma}^{t-1} m_i = 1 + \eta_\sigma$$

(2.34) can be rewritten as

$$P\{\sigma < \nu, Z_\sigma(1+\eta_\sigma) \geqq (1+\varepsilon)x\} - \varepsilon x^{-\kappa} E\{Z_\sigma^\kappa; \sigma < \nu\} \leqq P\{W \geqq x\}$$
$$\leqq P\{\sigma < \nu, Z_\sigma(1+\eta_\sigma) \geqq (1-3\varepsilon)x\} + \varepsilon x^{-\kappa}(3 + E\{Z_\sigma^\kappa; \sigma < \nu\}).$$

Consequently, it suffices to show that for each fixed A

$$(2.35)\quad 0 < \lim_{x\to\infty} x^\kappa P\{\sigma = \sigma(A) < \nu, Z_\sigma(1+\eta_\sigma) \geqq x\}$$
$$= KE\{Z_\sigma^\kappa; \sigma < \nu\} < \infty.$$

However, (2.35) is immediate from (2.9), because the conditional distribution of η_σ, given $\sigma < \nu$ and Z_σ is again the unconditional distribution of η_0. Thus, by virtue of (2.9) and (2.27)

$$(2.36)\quad \lim_{x\to\infty} x^\kappa P\{\sigma < \nu, Z_\sigma(1+\eta_\sigma) \geqq x\}$$

$$= \lim_{x\to\infty} x^\kappa \int_A^\infty P\{\sigma < \nu, Z_\sigma \in \mathrm{d}s\} P\left\{1+\eta_0 \geqq \frac{x}{s}\right\}$$

$$= K \int_A^\infty P\{\sigma < \nu, Z_\sigma \in \mathrm{d}s\} s^\kappa = KE\{Z_\sigma^\kappa; \sigma < \nu\} < \infty.$$

Also

$$E\{Z_\sigma^\kappa; \sigma < \nu\} \geqq A^\kappa P\{Z_1 > A\} \geqq A^\kappa E\beta_0^{A+1} > 0. \qquad \blacksquare$$

From here on the *proof of the theorem* is standard. We already showed in the introduction that the limit distribution of T_n is the same as that of $n + 2\sum_{t=0}^{n-1} Z_t$, provided the latter exists. But if we define, as in the introduction, $\nu_0 = 0 < \nu_1 < \nu_2, \cdots$ as the successive times at which

$Z_t = 0$, and put

$$W_k = \sum_{\nu_k \leqq t < \nu_{k+1}} Z_t,$$

then the pairs $\{(\nu_{k+1}-\nu_k), W_k\}_{k\geqq 0}$ are independent, all with distribution of the (ν, W) of (2.4) and (2.5), because (ν, W) coincides with $(\nu_1-\nu_0, W_0)$. The limit distribution for $\sum_{t=1}^{n} Z_t$ is therefore obtainable as the limit distribution of the sum of a random number of W_k. Many theorems of this nature are known (see for instance [7], [8] and [13]) and we therefore only indicate how to handle case (ii) of our theorem, when $\kappa = 1$. By Lemma 6 and Theorem 7.35.2 in [4] or Theorem 17.5.3 in [3b] there exists a constant $0 < C_1 < \infty$ and a stable law L of index 1 such that

$$P\{n^{-1}(\sum_{k=0}^{n-1} W_k - C_1 nD(n)) \leqq x\} \to L(x), \tag{2.37}$$

where $D(n) = K_{11}^{-1}\int_0^n x\,\mathrm{d}P\{W \leqq x\} \sim \log n$.

Let

$$\rho(n) = \max\{i : \nu_i < n\}.$$

Since, by Lemma 2

$$\mu \equiv E(\nu_{i+1}-\nu_i) = E\nu < \infty, \sigma^2(\nu_{i+1}-\nu_i) = \sigma^2(\nu) < \infty,$$

we have from renewal theory (see [3a], Ch. 13.6 and [3b], p. 372)

$$\lim_{n\to\infty} P\{|\rho(n)-n\mu^{-1}| \leqq C\sqrt{n}\} \geqq 1-\varepsilon$$

as soon as $C = C(\varepsilon)$ is sufficiently large. Since the Z_t and W_k are non-negative

$$\sum_{k=0}^{\rho(n)-1} W_k \leqq \sum_{t=0}^{n-1} Z_t \leqq \sum_{k=0}^{\rho(n)} W_k,$$

and for sufficiently large n

$$P\left\{n^{-1}\left(n+2\sum_{t=0}^{n-1} Z_t - 2C_1\mu^{-1}nD\left(\frac{n}{\mu}\right)\right) \leqq x\right\} \leqq P\{\rho(n) < n\mu^{-1}-C\sqrt{n}\}$$

$$+P\left\{n^{-1}\left(n+2\sum_{k<n\mu^{-1}-C\sqrt{n}} W_k - 2C_1\mu^{-1}nD\left(\frac{n}{\mu}\right)\right) \leqq x\right\}$$

$$\leqq \varepsilon + P\{n^{-1}\{\sum_{k<n\mu^{-1}-C\sqrt{n}} W_k - C_1(n\mu^{-1}-C\sqrt{n})\, D(n\mu^{-1}-C\sqrt{n})\}$$

$$\leqq \tfrac{1}{2}(x-1)+o(1)\}.$$

This holds for any $\varepsilon > 0$ and therefore, by (2.37)

$$\limsup P\{n^{-1}(T_n - 2C_1\mu^{-1}nD(n\mu^{-1}) \leqq x\}$$
$$= \limsup P\{n^{-1}(n+2\sum_{t=0}^{n-1} Z_t - 2C_1\mu^{-1}nD(n\mu^{-1}) \leqq x\} \leqq L(\tfrac{1}{2}\mu(x-1)).$$

In the same way one proves that $L(\frac{1}{2}\mu(x-1))$ is a lower bound for $\liminf P\{n^{-1}(T_n - 2C_1\mu^{-1}nD(n\mu^{-1}) \leqq x\}$ so that the limit theorem for T_n in case (ii) follows, with $A_1 = 2C_1\mu^{-1}$ and $L_1(x) = L(\frac{1}{2}\mu(x-1))$. To obtain from this the limit theorem for X_t we observe that for any positive integers t, γ, Γ

$$(2.38) \qquad \{T_\gamma \geqq t\} \subset \{X_t \leqq \gamma\}$$
$$\subset \{T_{\gamma+\Gamma} \geqq t\} \cup \{\inf_{s \geqq T_{\gamma+\Gamma}} X_s - (\gamma+\Gamma) \leqq -\Gamma\}$$

Now

$$\inf_{s \geqq T_{\gamma+\Gamma}} X_s - (\gamma+\Gamma) \quad \text{and} \quad \inf_{s \geqq 0} X_s$$

have the same distribution, because

$$X_{T_{\gamma+\Gamma}} = \gamma + \Gamma$$

and, even though $T_{\gamma+\Gamma}$ is a stopping time there is no information on the sample path obtainable from the fact that X reached $\gamma+\Gamma$ at some time; indeed $X_s \to \infty$ w.p.l. under (1.3) (see [10]). Thus, the probability of the last event in (2.38) can be made small uniformly in t and γ by fixing Γ large. In particular, we take $\delta = \delta(t)$ such that

$$(2.39) \qquad A_1\delta(t) \cdot D(\mu^{-1}\delta(t)) = t + o(1)$$

and

$$(2.40) \qquad \gamma(t) = \delta(t) + t(\log t)^{-2}x.$$

Then it can be shown from (2.39) and the definition of $D(\cdot)$ that

$$\gamma(t) \sim \delta(t) \sim (A_1 \log t)^{-1} t$$

and

$$(2.41) \qquad \gamma(t)^{-1}\{t - A_1\gamma(t)D(\mu^{-1}\gamma(t))\} \to -A_1^2 x.$$

(2.38) and (2.41) now imply

$$\lim_{t \to \infty} P\{X_t \leqq \gamma(t)\} = \lim_{t \to \infty} P\{T_{\gamma(t)} \geqq t\} = \lim_{t \to \infty} P\{T_{\gamma(t)+r} \geqq t\} =$$

$$= \lim_{t - \infty} P\{\gamma^{-1}(t)(T_{\gamma(t)} - A_1 \gamma(t) D(\mu^{-1}\gamma(t)) \geqq -A_1^2 x\},$$

so that case (ii) is proved completely. Case (i)–(iv) are handled in the same way (compare also [13] sect. 5 and [8], sect. 5 and 6.) Case (v) follows very quickly from the central limit theorem applied to the random variables

$$W_k - C_2(v_{k+1} - v_k), \qquad k = 0, 1, \cdots$$

where C_2 is chosen such that

$$0 = EW_k - C_2 \mu = E\{W_k - C_2(v_{k+1} - v_k)\}.$$

(See [7], sect. 7 and [8], Cor. 5.2.) Note that these random variables have finite variance by Lemmas 2 and 4, when $\kappa > 2$.

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