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REAL-VALUED FUNCTIONS ON CERTAIN SEMI-METRIC SPACES

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In [1], H. Blumberg showed that if f is a real-valued function on Euclidean n-space E_n, then E_n contains a dense subspace Y (depending on f) such that f restricted to Y is continuous. In this paper it is shown that if f is a real-valued function on a regular semi-metrizable Baire space X, then X has a dense subspace Y such that f restricted to Y is continuous. Other questions and extensions of Blumberg's theorem are in [2], [6] and [7].

In proving the indicated result, the concepts of First Category sets and Second Category sets are crucial. The following theorem (found in [5], page 82) is implicitely used: If $\{X_\alpha\}$ is a family of sets open relative to the union $S = \bigcup X_\alpha$ and if each X_α is of the First Category, then S is also of the First Category.

All undefined terms and notations are as in [4].

1. Preliminaries

In the following definitions let f be a real-valued function on a topological space X and let $x \in X$.

DEFINITION (1.1): The function f is said to approach x First Categorically (written $f1 \to x$) if there is an $\varepsilon > 0$ and a neighborhood $N(x, \varepsilon)$ of x such that $M(x, \varepsilon) = \{z \in N(x, \varepsilon) : |f(z) - f(x)| < \varepsilon\}$ is a First Category set in X.

DEFINITION (1.2): The function f is said to approach x Second Categorically (written $f2 \to x$) if given $\varepsilon > 0$ then there exists a neighborhood $N(x, \varepsilon)$ of x such that $M(x, \varepsilon) = \{z \,\varepsilon\, N(x, \varepsilon) : |f(z) - f(x)| < \varepsilon\}$ is a Second Category set in X. The function f is said to approach x Second Categorically via R (written $f2 \to x$ via R) if given $\varepsilon > 0$, there is a neighborhood $N(x, \varepsilon)$ such that $M(x, \varepsilon) \cap R$ is a Second Category set in X.

DEFINITION (1.3): An open set U is a partial neighborhood of a point x if either x is in U or x is a limit point of U.

It follows from Definition 1.2 that $f2 \to x$ if given $\varepsilon > 0$ there is a partial U of x such that for any open subset V of U $\{z \varepsilon V : |f(z) - f(x)| < \varepsilon\}$ is a Second Category subset of U.

DEFINITION (1.4): A function f is said to approach x densely (written $f \to x$ densely) if given $\varepsilon > 0$ there is a neighborhood $N(x, \varepsilon)$ of x such that $M(x, \varepsilon) = \{z \in N(x, \varepsilon) : |f(z) - f(x)| < \varepsilon\}$ is dense in $N(x, \varepsilon)$. If x is a limit point of R, then f is said to approach x densely via R (written $f \to x$ densely via R) if $M(x, \varepsilon) \cap R$ is dense in $N(x, \varepsilon) \cap R$.

The following is a useful characterization of Definition 1.4.

THEOREM (1.5): *Let f be a real-valued function on a topological space X. If $x \in X$, then $f \to x$ densely if and only if for each partial neighborhood U of x, $f(x)$ is a limit point of $f(U)$.*

PROOF: Suppose $f \to x$ densely and U is any partial neighborhood of x. Let $\varepsilon > 0$ be given, then x has a neighborhood $N(x, \varepsilon)$ such that $M(x, \varepsilon)$ is dense in $M(x, \varepsilon)$. Thus there exists $z \in M(x, \varepsilon) \cap U$ such that

$$|f(z) - f(x)| < \varepsilon.$$

Hence $f(x)$ is a limit point of $f(U)$.

To show the converse, suppose f does not approach x densely. Then there is an $\varepsilon > 0$ such that for each neighborhood N of x, the set $\{z \in N : |f(x) - f(z)| < \varepsilon\}$ is not dense in N. Thus, there is a non-empty open set U_N contained in N such that for all $y \varepsilon U_N$, $|f(y) - f(x)| \geqq \varepsilon$. Then $U = \bigcup \{U_N : N$ a neighborhood of $x\}$ is a partial neighborhood of x such that $f(x)$ is not a limit point of $f(U)$.

Let Z^+ denote the set of natural numbers.

THEOREM (1.6): *Let f be a real-valued function on a topological space X. Then $F_1 = \{x \in X : f1 \to x\}$ and $F_2 = \{x \in X : f$ does not densely approach $x\}$ are sets of the First Category in X.*

PROOF: If $x \in F_1$, then there is an $\varepsilon(x) > 0$ and a neighborhood $N(x, \varepsilon(x))$ of x such that $M(x, \varepsilon(x))$ is a First Category set. There is no generality lost if it is assumed that $\varepsilon(x) = 1/m(x)$ for some $m(x) \in Z^+$. For each $k \in Z^+$ let $C(k) = \{x \in F_1 : m(x) = k\}$ and let $D(k) = \{d(k, i) : i \in Z^+\}$ be a countable dense subset of $f(C(k))$. Let $D = \bigcup\{D(k) : k \in Z^+\}$. If $d(m, i) \in D$ let

$$R(m, i) = \{x \in C(m) : d(m, i) \leqq f(x) < d(m, i) + 1/m\}$$

and if $x \in R(m, i)$, let

$$RM(x, i) = \{z \in M(x, 1/m) : d(m, i) \leqq f(z) < d(m, i) + 1/m\}.$$

Similarly define

$$L(m, i) = \{x \in C(m) : d(m, i) - 1/m < f(x) \leqq d(m, i)\}$$

and if $x \in L(m, i)$, let

$$LM(x, i) = \{z \in M(x, 1/m) : d(m, i) - 1/m < f(z) \leqq d(m, i)\}.$$

If x and y are in $R(m, i)$, then

$$RM(x, i) \cap N(y, 1/m) \subseteq RM(y, i).$$

For if $z \in RM(x, i) \cap N(y, 1/m)$, then $|f(z) - f(y)| < 1/m$ and

$$d(m, i) \leqq f(z) < d(m, i) + 1/m$$

and hence, $z \in RM(y, i)$. Thus

$$T(m, i) = \bigcup\{RM(x, i) : x \in R(m, i)\}$$

and $S(m, i) = \bigcup\{LM(x, i) : x \in L(m, i)\}$ are First Category sets. Since

$$F_1 \subset [\bigcup\{T(m, i) : m, i \in Z^+\}] \cup [\bigcup\{S(m, i) : m, i \in Z^+\}],$$

it follows that F_1 is a First Category set. The theorem mentioned in the introduction was used in the proof of this theorem.

If $x \in F_2$, then there exists $\varepsilon(x) = \varepsilon > 0$ such that for each neighborhood $N(x, \varepsilon)$ of x, $M(x, \varepsilon)$ is not dense in $N(x, \varepsilon)$. Let $\{r_1, r_2, \cdots\}$ be the set of rational numbers and if $r_i < r_j$, let

$$F(i, j) = \{x \in F_2 : f(x) - \varepsilon(x) < r_i < f(x) < r_j < f(x) + \varepsilon(x)\}.$$

It follows that $F(i, j)$ is nowhere dense for suppose 0 is an open set such that $F(i, j)^- \supset 0$. If p and q are in $F(i, j) \cap 0$, then $|f(p) - f(q)| < \varepsilon(p)$.

Thus $\{q \in 0 : |f(p) - f(q)| < \varepsilon(p)\}$ is dense in 0. This contradiction shows that $F(i, j)$ is nowhere dense and it follows that

$$F_2 = \bigcup \{F(i, j) : i, j \in Z^+, r_i < r_j\}$$

is a First Category set.

2. Semi-metrizable Baire spaces

In the following let all spaces be T_1 spaces.

DEFINITION (2.1): A topological space is a Baire Space if the countable intersection of open dense sets is a dense set.

THEOREM (2.2): *If X is a Baire space and f is a real-valued function on X, then there is a dense set D (depending on f) such that if $x \in D$, then $f \to x$ densely via D.*

PROOF: Let $F_1 = \{x \in X : f1 \to x\}$. By Theorem 1.6, F_1 is a First Category set. Let $R_1 = X - F_1$. If $f2 \to x$, then $f2 \to x$ via R_1. Let $F_2 = \{x \in R_1 : f$ does not approach x densely via $R_1\}$. Again by Theorem 1.6, F_2 is a First Category set. Thus $D = X - (F_1 \cup F_2)$ is a residual set and, since X is a Baire space, D is dense in X. If $f2 \to x$ via R_1, then $f2 \to x$ via D and if $x \in D$ then $f \to x$ densely via R_1. Let $x \in D$. If $\varepsilon > 0$ is given and $U \cap D$ is any partial neighborhood of x in D (U is a partial neighborhood of x in X), then, since $f \to x$ densely via R_1 there exists $q \in U \cap R_1$ such that $|f(q) - f(x)| < \varepsilon/2$. Since U is a neighborhood of q, $\{z \in U : |f(z) - f(q)| < \varepsilon/2\} \cap D$ is a nonempty Second Category set. Let y be any one of its elements. Then

$$|f(y) - f(x)| \leqq |f(y) - f(q)| + |f(q) - f(x)| < \varepsilon/2 + \varepsilon/2 < \varepsilon.$$

Thus $f(x)$ is a limit point of $f(U \cap D)$ and, by Theorem 1.5, $f \to x$ densely via D.

DEFINITION (2.3): A topological space is a semi-metric space if there is a function d with domain $X \times X$ and range a subset of the non-negative real numbers such that

(i) $d(x, y) = d(y, x) \geqq 0$,

(ii) $d(x, y) = 0$ if and only if $x = y$, and
(iii) x is a limit point of a set M if and only if

$$\inf\{d(x, y) : y \in M\} = d(x, M) = 0 \quad \text{(See [3])}.$$

In [3], by letting $g(n, x) = \operatorname{int}\{y \in X : d(x, y) < 1/n\}$, R. W. Heath has shown the following equivalent condition for a space to be a semi-metric space.

THEOREM (2.4): *Let X be a regular space and $G = \{g(n, x) : n \in Z^+, x \in X\}$ a collection of open subset of X. If G satisfies*

(i) *for each $x \in X$, $\{g(m, x) : m \in Z^+\}$ is a non-increasing local base at x, and*
(ii) *if $y \in X$ and, for each $n \in Z^+$, $y \in g(n, x_n)$, then the point sequence $x_1, x_2, \cdots$ converges to y.*

Then X is a semi-metric space.

This theorem is a useful tool in the following theorem.

THEOREM (2.5): *If f is a real valued function on a regular semi-metrizable Baire space X, then there is a dense subset Y of X such that f restricted to Y is continuous.*

PROOF: Since X is semi-metrizable there exists a collection

$$G = \{g(m, x) : m \in Z^+, x \in X\}$$

of open subsets of X satisfying parts (i) and (ii) of Theorem 2.5. Let D be a dense set in X such that if $x \in D$, then $f \to x$ densely via D. The existence of D is guaranteed by Theorem 2.2. Construct a discrete subset $B(1) = \{x(1, \alpha) : \alpha \in A(1)\}$ of X and a pairwise disjoint subcollection $G(1) = \{g(n(1, \alpha), x(1, \alpha)) : \alpha \in A(1)\}$ of G such that

(i) $(\bigcup\{g \in G(1)\})^- = X$, and
(ii) for each $\alpha \in A(1)$, $g(n(1, \alpha), x(1, \alpha))$ contains a dense subset $h(1, \alpha) \subseteq D$ such that if $z \in h(1, \alpha)$, *then* $|f(z) - f(x(1, \alpha))| < 1$.

To obtain $B(1)$ and $G(1)$ let η be a well ordering of D and let $\varepsilon = 1$ be given. Let $x(1, 1)$ be the first element of η and let $n(1, 1)$ be the first element of Z^+ such that

$$h(1, 1) = \{z \in g(n(1, 1), x(1, 1)) \cap D : |f(z) - f(x(1, 1))| < 1\}$$

is dense in $g(n(1, 1), x(1, 1))$. Suppose that $x(1, \beta)$ has been chosen for each

$\beta < \delta$ such that

$$g(n(1, \beta), x(1, \beta)) \cap g(n(1, \alpha), x(1, \alpha)) = \emptyset$$

if $\alpha < \delta$, $\beta < \delta$ and $\alpha \neq \beta$. Let $x(1, \delta)$ be the first element of η such that $x(1, \delta) \notin (\bigcup \{g(n(1, \beta), x(1, \beta)) : \beta < \delta\})^-$. Let $n(1, \delta)$ be the first element of Z^+ such that

$$g(n(1, \delta), x(1, \delta)) \cap (\bigcup \{g(n(1, \beta), x(1, \beta) : \beta < \delta\})^- = \emptyset$$

and $h(1, \delta) = \{z \in g(n(1, \delta), x(1, \delta)) \cap D : |f(z) - f(x(1, \delta))| < 1\}$ is dense in $g(n(1, \delta), x(1, \delta))$.

Let $A(1)$ be the set of all α which have been chosen in the process described above. Let $B(1) = \{x(1, \alpha) : \alpha \in A(1)\}$ and let

$$G(1) = \{g(n(1, \alpha), x(1, \alpha) : \alpha \in A(1)\}.$$

Let $H(1) = \bigcup \{h(1, \alpha) : \alpha \in A(1)\}$. It follows that if $x \in H(1)$, then $f \to x$ densely via $H(1)$. For if $x \in H(1)$, then there exists $\alpha \in A(1)$ such that $x \in h(1, \alpha)$. Thus $|f(x) - f(x(1, \alpha))| = 1 - \delta$ for some $\delta > 0$. But if $x \in H(1)$, then $x \in D$. Thus given $\delta > 0$, there is a neighborhood $N(x, \delta)$ of x such that

$$\{z \in N(x, \delta) \cap D : |f(z) - f(x)| < \delta\}$$

is dense in $n(x, \delta)$. If

$$z \in N(x, \delta) \cap D \cap g(n(1, \alpha), x(1, \alpha)),$$

then

$$|f(z) - f(x(1, \alpha))| \leqq |f(z) - f(x)| + |f(x) - f(x(1, \alpha))| < \delta + 1 - \delta = 1.$$

Thus $z \in h(1, \alpha) \subseteq H(1)$.

Suppose $B(1), \cdots, B(k), G(1), \cdots G(k), H(1), \cdots, H(k)$ have been chosen such that for $1 \leqq i \leqq k$

(i) $B(1) \subseteq \cdots \subseteq B(k)$,

(ii) if $g \in G(i)$, then g is a member of the local base for some element of $B(i)$,

(iii) $(\bigcup \{g \in G(i)\})^- = X$,

(iv) if $g \in G(i+1)$, then there is a $g' \in G(i)$ such that $g' \supseteq g^-$,

(v) the elements of $G(i)$ are pairwise disjoint,

(vi) $D \supseteq H(1) \supseteq \cdots \supseteq H(k)$,
(vii) $H(i) = \bigcup \{h(i, \alpha) : \alpha \in A(i)\}$ where $h(i, \alpha) \subseteq H(i-1)$ and $h(i, \alpha)$ is a dense subset of $g(n(i, \alpha), x(i, \alpha))$ such that if $z \in h(i, \alpha)$, then $|f(z) - f(x(i, \alpha))| < 1/i$ and
(viii) if $x \in H(i)$, then $f \to x$ densely via $H(i)$.

To obtain $B(k+1)$, $G(k+1)$, and $H(k+1)$, let $g(n(k, \alpha), x(k, \alpha)) \in G(k)$. Let $x(k, \alpha) \in B(k+1)$ and let $n(k+1, \alpha)$ be the first element of Z^+ such that

$$g(n(k, \alpha), x(k, \alpha)) \supset (g(n(k+1, \alpha), x(k, \alpha)))^-$$

and

$$\{z \in g(n(k+1, \alpha), x(k, \alpha)) \cap H(k) : |f(z) - f(x(k, \alpha))| < 1/k+1\}$$

is dense in $g(n(k+1, \alpha), x(k, \alpha))$. Select from

$$U = g(n(k, \alpha), x(k, \alpha)) - [g(n(k+1, \alpha), x(k, \alpha))]^-$$

a discrete subset $B(k+1, \alpha)' = \{x(k+1, \beta) : \beta \in A(k+1, \alpha)\}$ and select from G a pairwise disjoint collection

$$G(k+1, \alpha)' = \{g(n(k+1, \beta), x(k+1, \beta)) : \beta \in A(k+1, \alpha)\}$$

such that
(i) if $g \in G(k+1, \alpha)'$, then $g \subset U$,
(ii) $(\bigcup \{g \in G(k+1, \alpha)'\})^- = U^-$, and
(iii) for each $\beta \in A(k+1, \alpha)$, $g(n(k+1, \beta), x(k+1, \beta))$ contains a dense subset $h(k+1, \beta) \subset H(k)$ such that if $z \in h(k+1, \beta)$, then

$$|f(z) - f(x(k+1, \beta))| < 1/k+1,$$

and
(iv) $B(k+1, \alpha)' \subset H(k)$.

Let $B(k+1, \alpha) = B(k+1, \alpha)' \cup \{x(k, \alpha)\}$ and let

$$G(k+1, \alpha) = G(k+1, \alpha)' \cup \{g(n(k+1, \alpha), x(k, \alpha))\}.$$

Then let

$$B(k+1) = \bigcup \{B(k+1, \alpha) : \alpha \in A(k)\},$$

$$G(k+1) = \bigcup \{G(k+1, \alpha) : \alpha \in A(k)\},$$

$$H(k+1, \alpha) = \bigcup \{h(k+1, \beta) : \beta \in A(k+1, \alpha)\},$$

and

$$H(k+1) = \bigcup \{H(k+1, \alpha) : \alpha \in A(k)\}.$$

It clearly follows that the induction hypothesis is satisfied.

Let $Y = \bigcup \{B(n) : n \in Z^+\}$ and for each $n \in Z^+$, let $K(n) = \bigcup \{g \in G(n)\}$. It follows that $K = \bigcap \{K(n) : n \in Z^+\}$ is dense since each $K(n)$ is an open dense subset of X. Notice that Y is a dense subset of K for if $z \in K$, then, for each $i \in Z^+$, there is an $x(i, \alpha_i)$ such that $z \in g(n(i, \alpha_i), x(i, \alpha_i))$ and, sine X is a semi-metric space, the point sequence $x(1, \alpha_1), x(2, \alpha_2), \cdots$ converges to z. Thus Y is dense in X.

Let $x \in Y$ and let $\varepsilon > 0$ be given. Since $x \in Y$, there exists $i \in Z^+$ such that $x \in B(j)$ for each $j \geqq i$ and there exists $k \in Z^+$ such that $1/k < \epsilon$. Let $m = \max \{i, k\}$. Since $x \in B(m)$, $g(n, x) \in G(m)$ for some $n \in Z^+$ and if $z \in g(n, x) \cap Y$, then $|f(z) - f(x)| < 1/m < \varepsilon$. Thus f restricted to Y is a continuous function.

DEFINITION (2.6): A semi-metric space X is said to be weakly complete provided there is a distance function d such that the topology of X is invariant with respect to d and if $\{M_i : i \in Z^+\}$ is a monotonic decreasing sequence of non-empty closed sets such that, for each $n \in Z^+$, there exists a $1/n$-neighborhood of a point $P_n \in M_n$ which contains M_n, then $\bigcap \{M_n : n \in Z^+\}$ is non-void.

Standard arguments show that a regular weakly complete semi-metric space is a Baire space. Thus the following is established.

COROLLARY (2.7): *If f is a real-valued function in a regular, weakly complete semi-metric space X, then X has a dense subset Y such that f restricted to Y is continuous.*

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