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THE HAUPTVERMUTUNG FOR C^∞ HOMEOMORPHISMS II A PROOF VALID FOR OPEN 4-MANIFOLDS

M. G. Scharlemann and L. C. Siebenmann [1]

Introduction

It has often been observed that every twisted sphere $M^m = B^m_+ \cup_f B^m_-$ of Milnor is C^∞ homeomorphic to the standard sphere S^m, although in general it is not diffeomorphic to S^m. Recall that a twisted sphere is put together from copies of the standard hemispheres $B^m_\pm$ *of* S^m by reidentifying boundaries ∂B^m_+ to ∂B^m_- under a diffeomorphism *f*. One obtains a homeomorphism $h: M^m \to S^m$ by setting $h|B^m_- =$ identity and $h|B^m_+ = \{\text{cone on } f: S^{m-1} \to S^{m-1}\}$, the latter regarded as a self-homeomorphism of $B^m_+ = \text{cone}(S^{m-1})$. This is C^∞ and non singular, except at the origin in B^m_+ (= cone vertex). Composing *h* with a suitable C^∞ homeomorphism λ whose derivatives vanish at the origin of B^m_+ yields a C^∞ homeomorphism $h: M^m \to S^m$ (Appendix A).

Since the twisted spheres represent the classical obstructions to smoothing a PL homeomorphism to a diffeomorphism, it is not surprising to find (§4 of preprint)[2] that if *M* is any PL manifold and σ, σ' are two compatible smoothness structures on it, then one can obtain a C^∞ smooth homeomorphism $h: M_\sigma \to M_{\sigma'}$. It would be reasonable to guess that the same is true for arbitrary smoothings σ, σ' of *M*. However, we prove the following.

HAUPTVERMUTUNG FOR C^∞ HOMEOMORPHISMS: *Let $f: M' \to M$ be a C^∞ homeomorphism of connected metrizable smooth manifolds without boundary. If M and M' are of dimension* 4 *suppose they are non-compact. Let M and M' be given Whitehead compatible*[3] PL *structures* [Mu_2].

[1] Supported in part by NSF Grant GP-34006.

[2] We there used classical smoothing theory and a TOP/C^∞ handle lemma for index $\geqq 6$. Surely a more direct proof exists!?

[3] A PL manifold structure Σ on *M* is (C^∞) *Whitehead compatible* with the smooth (C^∞) structure of *M* if for some PL triangulation of M_Σ as a simplicial complex, the inclusion of each closed simplex is smooth and nonsingular as a map to *M*.

Then there exists a topological isotopy of f to a PL *homeomorphism.*

Our purpose here is to give a handle by handle proof of this result which uses no obstruction theory and which does succeed with the specified four-manifolds.

Note that the singularities of the differential Df may form a nasty closed set in M' of dimension as high as $m-1$. The one pleasant property which for us distinguishes f from a mere homeomorphism is the fact that the critical values are meager by the Sard-Brown Theorem [11], both for f and for the *composition* of f with any smooth map $M \to X$. In fact our result follows with astonishing ease from this fact.

In dimension $\leqq 6$ the PL homeomorphism asserted by the C^∞ Hauptvermutung is equivalent to diffeomorphism since there is no obstruction to smoothing a PL homeomorphism [12] [8].

Ordinary homeomorphism in dimension $\leqq 6$ does not imply diffeomorphism. Thus the following example may clarify the meaning of our theorem. The second author shows in [17, § 2] how to construct a homeomorphism

$$h\colon T(\beta) \to T^6$$

of a smooth [1] manifold $T(\beta)$ that is known not to be diffeomorphic to T^6. By construction h is a diffeomorphism [2] over the complement of a standard subtorus $T^3 \subset T^6$, and also over T^3 itself. The C^∞ Hauptvermutung shows that there is no way of making h smooth – say by squeezing towards the singularity set T^3 as one does for twisted spheres. The homeomorphisms that disprove the Hauptvermutung are thus measurably more complex than those known previously.

The C^∞ Hauptvermutung lends credence to the following seemingly difficult conjecture due to Kirby and Scharlemann [5]. Consider the least pseudo-group MCCG_n of homeomorphisms on R^n which contains all C^∞ homeomorphisms of open subsets of R^n.

CONJECTURE: *The isomorphism classification of* MCCG_n *manifolds coincides naturally with the isomorphism classification of* PL *n-manifolds without boundary.*

It can be shown that every PL homeomorphism of open subsets of R^n is in MCCG_n, see [5], [ ][3]. Thus MCCG can be regarded as an enlargement of PL to contain DIFF, an enlargement which might eventually be useful in dynamics, group action theory, smoothing theory, etc. – espe-

[1] In [17, § 2] one can replace PL everywhere by DIFF with no essential change in proofs.

[2] In [17, § 2] one should choose the DIFF pseudo-isotopy $H\colon (I; 0, 1) \times B^2 \times T^n$ to be used to build h constant near 0 and 1 so as to prevent unwanted kinks in h.

[3] Mistrust this assertion, as no proof has been written down. (Oct. 1974).

cially at points where mere homeomorphism seems too coarse a notion.

The organization of this article is as follows:

Section 1. A C^∞/DIFF handle lemma for index $\leqq 3$ in any dimension;
Section 2. A weak C^∞/DIFF handle lemma for index 4 in dimension 4;
Section 3. Proof of an elaborated C^∞ Hauptvermutung;
Appendix A. C^∞-smoothing an isolated singularity;
Appendix B. Potential counterexamples in dimension 4.

1. A C^∞/DIFF handle lemma for index $\leqq 3$

The proof of the C^∞ Hauptvermutung will be based on two handle-smoothing lemmas 1.1 and 2.1 below.

DATA: Let B^k be the unit ball in R^k and let $f: M \to B^k \times R^n$ be a C^∞ homeomorphism which is nonsingular near the boundary.

DEFINITION: *A* C^∞ isotopy f_t, $0 \leqq t \leqq 1$, of f will be called *allowable* if it fixes all points outside some compactum in $(\operatorname{int} B^k) \times R^n$ – i.e. it has compact support in $(\operatorname{int} B^k) \times R^n$.

1.1. C^∞/DIFF HANDLE LEMMA (index $\leqq 3$): *For* $f: M \to B^k \times R^n$ *as above and* $k = 0, 1, 2, 3$, *there is an allowable isotopy of* f *to a* C^∞ *homeomorphism* f_1 *which is non-singular near* $f_1^{-1}(B^k \times 0)$.

Recall that for index $k = 3$, the C^0 version of this lemma is false, a key failure of the C^0 Hauptvermutung [6].

PROOF of 1.1: Our first step is to allowably isotop f so that $0 \in R^n$ is a regular value of the projection $p_2 f: M \to R^n$. Choose a regular value y_0 in R^n with $|y_0| < \frac{1}{2}$. Let ψ_t, $0 \leqq t \leqq 1$, be a diffeotopy (non-singular C^∞ isotopy) of $\mathrm{id}|R^n$ with support in $\mathring{B}^n$ carrying y_0 to 0. Let $\gamma: B^k \to [0, 1]$ be a C^∞ map such that $\gamma = 0$ near ∂B^k and f is nonsingular over $\{\gamma^{-1}[0, 1)\} \times B^n$. Now

$$\Psi_t: B^k \times R^n \to B^k \times R^n$$

defined by $\Psi_t(x, y) = (x, \psi_{t\gamma(y)}(y))$ for $0 \leqq t \leqq 1$ gives an allowable isotopy $f_t = \Psi_t f$ as desired. See figure 1a, which illustrates this manoeuvre for $k = n = 1$.

Revert to f as notation for $f_1 = \Psi_1 f$.

As a second step we will allowably isotop f by a squeeze so that the structure imposed by f on $B^k \times R^n$ is a product along R^n near $B^k \times 0$. Choose a small closed ε-ball B_ε about 0 in R^n such that $p_2 f$ is nonsingular over B_ε, hence a (trivial) smooth bundle projection over B_ε. Choose a trivialization φ of this bundle in a commutative diagram

$$\begin{array}{ccccc} N\times B_\varepsilon & \xrightarrow[\cong]{\varphi} & f^{-1}(B^k\times B_\varepsilon) & \xrightarrow{f} & B^k\times B_\varepsilon \\ & {\scriptstyle p_2}\searrow & \downarrow & \swarrow{\scriptstyle p_2} & \\ & & B_\varepsilon & & \end{array}$$

With the help of a collar of ∂N we can arrange that on a neighborhood of $\partial N\times B_\varepsilon$, Φ coinsides with f^{-1}. See figure 1b, which illustrates the behavior of $f\varphi(x\times B_\varepsilon)$ for 5 values of x in N.

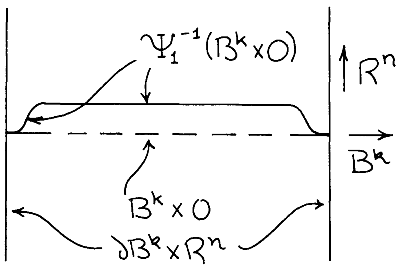

Figure 1a.

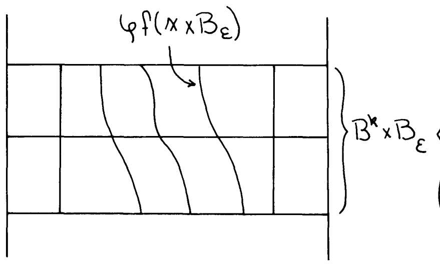

Figure 1b.

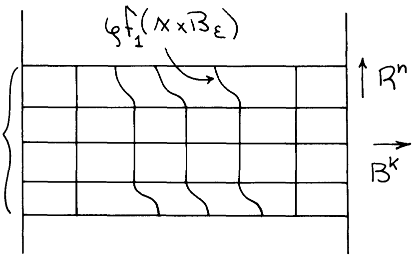

Figure 1c.

Let $\Lambda\colon[0,\infty)\to[0,\infty)$ be a smooth map such that $\Lambda([0,\varepsilon/2])=0$ while $\Lambda\colon(\varepsilon/2,\infty)\to(0,\infty)$ is a diffeomorphism equal to the identity on $[\varepsilon,\infty)$; then define a C^∞ homotopy $\lambda_t\colon R^n\to R^n$, $0\leqq t\leqq 1$, by

$$\lambda_t(y) = (1-t)y+t\,\frac{\Lambda(|y|)}{|y|}\,y$$

where $\Lambda(|y|)/|y|$ is understood to be zero for $y=0$. Define an allowable isotopy (see figure 1c)

$$f_t\colon M\to B^k\times R^n$$

to be fixed outside $f^{-1}(B^k\times B_\varepsilon)$ and to send $\varphi(x,y)\in\varphi(N\times B_\varepsilon)=f^{-1}(B^k\times B_\varepsilon)$ to $(p_1 f(x,\lambda_t(y)),y)\in B^k\times R^n$. It is not difficult to see that this completes the second step. Again revert to f as notation for f_1.

The handle lemma is now clearly reduced to the handle problem posed by $f^{-1}(B^k\times 0)\to B^k\times 0$. Thus it remains only to prove

1.2. Lemma: *If $f: M \to B^k$, $k = 0, 1, 2, 3$, is a C^∞ homeomorphism which is nonsingular near ∂M, then f is C^∞ isotopic rel ∂M* [1] *to a diffeomorphism.*

Proof of Lemma 1.2: By relative uniqueness of smooth structures in dimension $\leqq 3$, [10] [12] [8] there is a diffeomorphism $\alpha: B^k \to M$ which is inverse to f near the boundary. Then $f' = f\alpha: B^k \to B^k$ extends by the identity map to a C^∞-homeomorphism $S^k \to S^k$ where we identify B^k to B^k_+ in S^k. This map in turn extends to a C^∞-homeomorphism $B^{k+1} \to B^{k+1}$ by the smoothing lemma of Appendix A.

We now have a C^∞-homeomorphism $B^{k+1} \to B^{k+1}$ which is the identity near $B^k_- \subset \partial B^{k+1}$ and $f\alpha$ on $B^k_+ \subset \partial B^{k+1}$. Let $\theta: B^{k+1} \to B^k \times I$ be a homeomorphism which sends B^k_+ onto $B^k \times \{0\}$ and is a diffeomorphism except where corners are added in B^k_-. Then $\theta F \theta^{-1}: B^k \times I \to B^k \times I$ is the identity near $B^k \times \{0\} \cup \partial B^k \times I$ and hence a C^∞-homeomorphism everywhere. Now $\theta F \theta^{-1}(\alpha^{-1} \times \mathrm{id}_I)$ is the required C^∞-isotopy from f to a diffeomorphism.

This completes the proof of Lemma 1.2 and with it the proof of the C^∞/DIFF handle lemma for index $\leqq 3$.

Assertion: *In the above proofs the use of relative uniqueness theorems for smooth structures in dimension $\leqq 3$ can be replaced by the smooth Alexander-Schoenflies theorems in dimension $\leqq 3$* (the latter are easily proved, c.f. Cerf [1, Appendix]).

Proof of Assertion: First note that these Schoenflies theorems suffice to prove Lemma 1.2 in case M is known to embed smoothly and nonsingularly in R^k.

Next suppose the assertion established for index $< k$. (It is trivial for index 0.) Then deal with index k by establishing Lemma 1.2 for index k using the smooth Schoenflies theorem in dimension k, as follows. Smoothly triangulate B^k so finely that

(*) *For each k-simplex σ of B^k, $f^{-1}(\sigma)$ lies in a co-ordinate chart of M.*

The index $< k$ case suffices to get a C^∞ isotopy of f rel ∂M to an f_1 that is nonsingular over the $(k-1)$-skeleton and still satisfies (*). Then the smooth Schoenflies theorem suffices, by our first remark, to establish Lemma 1.2 for index k.

2. A weak C^∞/DIFF handle lemma for index 4

The C^∞/DIFF handle problem for index 4 and dimension 4 admits a

[1] i.e. isotopic fixing a neighborhood of ∂M.

weak solution based on the weak Schoenflies theorem for dimension 4 (given by Rourke and Sanderson [14, 3.38]) [1]:

THEOREM: *Let* $S \subset R^4 - 0$ *be a smoothly embedded* 3-*sphere, and let* T *be the closure of the bounded component of* $R^4 - S$. *Then* $T - 0$ *is diffeomorphic to* $B^4 - 0$.

DEFINITION: We call a homotopy h_t, $0 \leqq t \leqq 1$, *almost compact* if, for each $\tau < 1$, the homotopy h_t, $0 \leqq t \leqq \tau$, has compact support.

2.1. PROPOSITION: *Suppose* M^4 *is a smooth submanifold of* R^4, *and* $f: M \to B^4$ *is a* C^∞ *homeomorphism which is a diffeomorphism over a neighborhood of the boundary* ∂B^4. *Then there is an isotopy rel boundary* $f_t: M \to B^4$, $0 \leqq t \leqq 1$, *such that*:

(i) $f_o = f$ *and* f_1 *is a diffeomorphism over* $B^k - \{p\}$ *for some point* $p \in \operatorname{int} B^4$.

(ii) f_t *restricts to a* C^∞ *almost compact isotopy* $M - f^{-1}\{p\} \to B^k - \{p\}$.

(iii) f_t *is fixed over some smooth path from* p *to* ∂B^4.

PROOF OF 2.1: Without loss of generality we may assume there is a radius of B^4 over which f is nonsingular. In this case we will make $p = \{0\} \in B^4$ and cause the path mentioned in (iii) to be this radius. By the weak Schoenflies theorem, we can find a homeomorphism $\alpha: B^4 \to M$ such that $f\alpha: B^4 \to B^4$ restricts to a diffeomorphism $(B^4 - 0) \to (B^4 - 0)$ and is the identity near ∂B^4. We can alter α rel boundary by a diffeotopy of $(B^4 - 0) \xrightarrow{\alpha} M^4 - f^{-1}\{0\}$, so that $f\alpha$ is also the identity on the chosen radius. This requires just a proper version, applied to $\alpha|$(open radius), of Whitney's (ambient) isotopy theorem cf. [2].

Identifying $B^4 - \{0\}$ naturally to $\partial B^4 \times R_+ = \partial B^4 \times [0, \infty)$ we are only required to find, for a certain $\{q\} \in \partial B^4$, an almost compact C^∞-isotopy f_t', $0 \leqq t \leqq 1$, fixing $\{q\} \times R_+$ and a neighborhood of $\partial B^4 \times \{0\}$, from $f' = f \circ \alpha: \partial B^4 \times R_+ \to \partial B^4 \times R_+$ to a diffeomorphism. Once this is accomplished the required isotopy f_t of f will be $f_t(f^{-1}(0)) = 0$ and $f_t(x) = f_t' \circ \alpha^{-1}(x)$ for $x \in M - f^{-1}(0)$.

Let $\mu_t: [0, \infty) \to [0, \infty)$ be an almost compact smooth (into) isotopy from the identity to a diffeomorphism $\mu_1: [0, \infty) \to [0, \varepsilon)$. (Only μ_1 is not onto.) Let $\varepsilon > 0$ be so small that f' is a diffeomorphism on $S^3 \times [0, \varepsilon)$. Define $f_t': \partial B^4 \times [0, \infty) \to \partial B^4 \times [0, \infty)$ to be

$$\{(\mathrm{id}|\partial B^4) \times \mu_t\} \circ f' \circ \{(\mathrm{id}|\partial B^4) \times \mu_t^{-1}\}.$$

[1] It is a down to earth version of Mazur's proof of the topological Schoenflies theorem [9].

It clearly has the right properties and completes the proof of Proposition 2.1.

3. Proof of an elaborated C^∞ Hauptvermutung

3.1. THEOREM: (C^∞ Hauptvermutung). *Consider a C^∞ homeomorphism $f: M' \to M$ of smooth m-dimensional manifolds equipped with Whitehead triangulations. Suppose f is also a* PL *equivalence over a neighborhood of some closed subset C of M.*

In case $\dim M = 4$ *or* $\dim \partial M = 4$ *we make some provisos. If* $\dim M = 4$ *we suppose that each component of the complement of C in M has noncompact closure in M. In case* $\dim \partial M = 4$ *we suppose that each component of $\partial M - C$ has noncompact closure in ∂M.*

(I) *Then, for $m \leqq 4$, there exists a C^∞ isotopy rel C from f to a diffeomorphism.*

(II) *For $m = 5$ or 6, there exists a topological isotopy rel C from f to a diffeomorphism.*

(III) *For all m, there exists a topological isotopy rel C from f to a* PL *homeomorphism.*

The salient advance beyond [15] is clearly the case of open 4-manifolds in (III). Note that (II) is implied by (III) and classical smoothing theory (but we naturally get to (II) first).

REMARK 1: If f is a C^∞ homeomorphism which is a PL equivalence near C, then f will be non-singular near C. Indeed f PL implies that for each (closed) principal simplex σ of a suitable subdivision of M', f maps σ linearly into a principal simplex of M, hence C^∞ non-singularly with rank m into M as a C^∞ manifold. Thus, in the above theorem, f is actually nonsingular near C.

REMARK 2: The provisos concerning dimension 4 can be eliminated if and only if the smooth 4-dimensional Schoenflies conjecture is true. (See Appendix B and Lemma 1.2.)

REMARK 3: It is easy to believe that in (II) the isotopy can be C^∞.

REMARK 4: The isotopies produced by 3.1 can be made as small as we please for the strong (majorant) topology – except possibly where dimension 4 manifolds or boundaries intervene. This is accomplished merely by using sufficiently *fine* Whitehead C^1 triangulations in the proofs to follow.

3.2. *Proof of* 3.1 *Part I: Manifolds of dimension* $\leqq 4$.

This is by far the most delicate part.

Exploit smooth collars of $\partial M'$ and ∂M corresponding under f near C to C^∞ isotope f rel C by a classical squeezing argument (cf. proof of 1.1) so that f becomes a product near the boundary along the collaring interval factor. This property is to be preserved carefully through all changes of f.

Select a smooth Whitehead triangulation of M so fine that f is nonsingular over a subcomplex containing C, and the preimage of each 4-simplex lies in a co-ordinate chart. With no loss of generality we suppose now that C is a subcomplex.

Apply the C^∞/DIFF handle lemma 1.1, around the smooth open k-simplices $\mathring{\sigma} \cong R^k$ of M in order of increasing dimension for $k = 0, 1, 2, 3$, to make f nonsingular over a neighborhood of the 3-skeleton of M. When $\mathring{\sigma}$ lies in ∂M the handle lemma gives a C^∞ isotopy of $f|: \partial M' \to \partial M$ which we must damp out along the collaring interval factor to get a C^∞ isotopy of f. The proof is now complete for $m \leqq 3$.

Suppose now that $m = 4$. It is easy to choose the handles so near to the open simplices that for each 4-simplex σ, the preimage of σ remains in its co-ordinate chart throughout the isotopy constructed thus far.

Using the index 4 weak C^∞/DIFF handle lemma 2.1, we could give an *isotopy* of f over smooth 4-handles in the open 4-simplices to obtain a homeomorphism which is a diffeomorphism on the complement of center points of these 4-handles. There is a well-known *trick* that then provides a diffeomorphism homotopic to f when M is open. But, to ensure the C^∞ isotopy asserted by 3.1 we must now take some care and execute the isotopy and the trick *simultaneously*.

After making f nonsingular over a neighborhood of the 3-skeleton, we have a C^∞ homeomorphism $f: M' \to M$ which is nonsingular except well within the interior of the preimage of a smooth 4-handle B_i inside each 4-simplex $\mathring{\sigma}_i$. We extend the smooth arcs given by the weak C^∞/DIFF handle lemma 2.1 obtaining, for each 4-handle B_i, a point p_i in $\operatorname{int} B_i$ and a smooth arc α_i from p_i to ∞ in the complement of C. Here we use the curious proviso that these components are unbounded in M. We can arrange that $\alpha_i \cap \partial M = \phi$, that $\alpha_i \cap \alpha_j = \phi = \alpha_i \cap B_j$ for $i \neq j$ and that the union of the α_i is a properly embedded smooth submanifold of M.

The weak index 4 handle lemma provides an isotopy $f_t: M' \to M$ such that

(a) $f_0 = f$ and f_1 is a diffeomorphism over $M - \bigcup_i \{p_i\}$

(b) $f_t(M' - \bigcup_i f^{-1}\{p_i\})$ is an almost compact C^∞ isotopy in $M - \bigcup_i \{p_i\}$.

(c) f_t is constant over each smooth arc α_i.

Extend the smooth arcs α_i and $f_1^{-1}\alpha_i = f^{-1}\alpha_i$ slightly to smooth arcs $\beta_i: R_+ \to M$ and $\beta_i': R_+ \to M'$ parametrized so that $\beta_i(1) = p_i$.

Choose disjoint closed tubular neighborhoods $\bar{\beta}_i$: $R_+ \times B^3 \to M$ and $\bar{\beta}_i': R_+ \times B^3 \to M'$ of β_i and β_i' such that their sum over i is a properly embedded submanifold of M and M' respectively.

Define an isotopy $g_t: M \to M$, $0 \leqq t \leqq 1$, by

(i) $g_t(x) = x$ if $t = 0$ or x is outside the normal tubes $\operatorname{Im}(\bar{\beta}_i)$.

(ii) For x in $\operatorname{Im}(\bar{\beta}_i)$, say $x = \bar{\beta}_i(u, v)$,

$$g_t(x) = \bar{\beta}_i(\bar{\mu}_{t(1-|v|)}(u), v).$$

where $\bar{\mu}_t: R_+ \to R_+$ is an almost compact smooth nonsingular (into) isotopy with $\bar{\mu}_1(R_+) = [0, 1)$, adjusted to be constant near $t = 0$ and $t = 1$. This is an almost compact smooth into isotopy of $\mathrm{id}|M$ with

$$g_1 M = M - \bigcup_i \alpha_i = f_1(M' - \bigcup_i \alpha'_i).$$

Define $g_t': M' \to M'$ similarly.

Consider the composed isotopy $f_t^* = g_t^{-1} \circ f_t \circ g_t': M' \to M$, $0 \leqq t \leqq 1$. Since $f_1 g_1' M' = g_1 M$ and f_t is a C^∞ isotopy for $t < 1$ while f_1 is a diffeomorphism over $g_1 M$, this f_t^* is a C^∞ isotopy. It runs from f to a diffeomorphism and finally establishes Part I.

3.3. *Proof of* 3.1, *Part II*: 5- *and* 6-*manifolds*

As in the proof of Part I we can find an isotopy of f rel C to make f a diffeomorphism over a neighborhood of the 3-skeleton of M.

If $\dim \partial M = 4$, we can even use Part I to make f a diffeomorphism over a neighborhood of ∂M.

As in part I, f can be, near the boundary, always a product along the interval factor of collarings of the boundaries.

Applying a TOP/DIFF handle lemma to handles of index 4, 5, and 6 with cores in the open simplices of M of increasing dimension 4, 5, and 6 we can now topologically isotop f rel C and rel the 3-skeleton to a diffeomorphism. More precisely the TOP/DIFF version of the TOP/PL handle straightening theorem of [6] is to be used. No immersion theory is required; the associated torus problem – presented by an exotic structure

$$(B^k \times T^n)_\Sigma, \qquad k+n = m, \qquad k = 4, 5, 6,$$

standard near the boundary – may be solved by simply connected surgery. To do this, first use the Product Structure Theorem [7, § 5] to reduce to the two cases (i) $k = k+n \geqq 5$; (ii) $k = 4$, $n = 1$. Then for $k = k+n \geqq 5$ we solve by the smooth Poincaré Theorem [3]. The remaining case $k = 4$,

$n = 1$ is reduced by [18, § 5] to a surgery problem rel boundary with target $B^4 \times [-1, 1]$ – which is just the smooth Poincaré Theorem for dimension 5 [3]. Compare [16] [4].

3.4. *Proof* 3.1, *Part III, the* C^∞ *Hauptvermutung*

Following the proof for part II, we isotop f rel C to make f a diffeomorphism over a neighborhood of $C \cup M^{(6)}$. As f is already PL over a neighborhood of C the (relative) Whitehead triangulation uniqueness theorem [13] provides an isotopy of f rel C making f PL over a neighborhood of $C \cup M^{(6)}$.

Now we can further isotop f rel $C \cup M^{(6)}$ to a PL homeomorphism using the TOP/PL handle straightening lemma of [6] for handle index values $\geqq 6$. We note no sophisticated techniques are required here; for example the Product Structure Theorem of [7] (based on handlebody theory) reduces the straightening lemma of [6] for index $k \geqq 6$ to the PL Poincaré theorem for a disc of dimension k.

Appendix A. C^∞-smoothing an isolated singularity

The proof of the following proposition was given to us by C. T. C. Wall, when we had proved just a special case sufficient for the C^∞ Hauptvermutung.

Proposition A.1: *Let $f: R^r \to R^s$ be a continuous map that is C^∞ on $R^r - 0$. There exists a C^∞ homeomorphism $\mu: [0, \infty) \to [0, \infty)$ (depending on f) such that the map $h: R^r \to R^s$, $h(x) = \mu(\|x\|^2) f(x)$ is a C^∞ mapping.*

Proof of A.1: Write

$$N_{n,r}(f) = \sup \left\{ \left\| \frac{\partial^I f}{\partial x^I} \right\| : \frac{1}{n+1} \leqq \|x\|^2 \leqq \frac{1}{n-1}, |I| = r \right\}.$$

Choose a decreasing sequence c_n with $c_n N_{n,r}(f) \to 0$ as $n \to \infty$ for each r (easily done by diagonal process). If μ is C^∞-homeomorphism of $[0, \infty)$, nonsingular on $(0, \infty)$ and flat at 0, with $\mu^{(s)}(y)/c_n y \to 0$ as $y \to 0$ for all s (where n depends on y by $1/(n+1) \leqq y \leqq 1/(n-1)$ then $g(x) = \mu(\|x\|^2)$ is C^∞, and as

$$D^I(fg) = \Sigma(D^J f D^K g: J + K = I\} \text{ by Leibnitz' theorem,}$$

$D^J f$ is estimated by an $N_{n,r}(f)$ and $D^K g$ by $c_n \|x\|^2$, we have $D^I(fg) \to 0$ as $\|x\| \to 0$. Thus by induction if we define $h(x) = f(x)g(x)$ $(x \neq 0)$ $h(0) = 0$, h is flat at 0 as required.

We construct $\mu(y) = \int_0^y \mu'$ defining first μ' so that, for small y, $\mu'(y) = \sum_2^\infty 2^{-n} c_{n+1} B\{(n^2-1)y-n\}$, where $B(x) > 0$ for $\|x\| < 1$ and $= 0$ otherwise. At most 2 terms in the summation can be nonzero, and since each $B^{(s)}(x)$ is bounded, the desired estimates follow easily.

Appendix B. Potential Counterexamples in Dimension 4

It is clear that a positive solution to the smooth (or PL) Schoenflies conjecture in dimension 4 would eliminate the conditions concerning dimension 4 in the C^∞ Hauptvermutung 3.1. Conversely we show now that a counterexample to this conjecture would give a counterexample to the C^∞ Hauptvermutung for compact (even closed) 4-manifolds.

PROPOSITION: *Suppose S is a smoothly embedded 3-sphere in $R^4 - 0$, and T is the closure in R^4 of the bounded component of $R^4 - S$. Then there exist C^∞ homeomorphisms $B^4 \to T$ and $T \to B^4$ each with one singular point, at* 0.

DISCUSSION: The 4-dimensional smooth Schoenflies conjecture asserts that every such T is in fact diffeomorphic to B^4, (equivalently PL isomorphic to B^4, cf [12]). So it is immediate that a counterexample T to this conjecture would yield a counterexample $T \to B^4_-$ to the C^∞ Hauptvermutung. By capping off with 4-discs it also yields a counterexample $M = T \cup B^4_+ \overset{C^\infty}{\to} S^4$ for closed 4-manifolds. In each case there is just one singularity.

PROOF OF PROPOSITION: Mazur's Schoenflies argument (as reworked in [14, 3.38] yields a diffeomorphism $f\colon (R^4 - 0) \to (R^4 - 0)$ with $f(B^4 - 0) = T - 0$. Then application of Lemma A.1 to f and f^{-1} respectively yields the asserted homeomorphisms.

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