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SEMI-CHARACTERISTICS AND FREE GROUP ACTIONS

R. E. Stong

1. Introduction

Recently, Ronnie Lee [5] has introduced a semi-characteristic homomorphism

$$\chi_{\frac{1}{2}} : \mathfrak{N}_{2n+1}(G) \to \tilde{R}_{GL,\mathrm{ev}}(G)$$

from the unoriented bordism group of free G actions, G a finite group, into a Grothendieck group of representations of G over a finite field K of characteristic 2. One of the questions he raises is to compute this invariant in terms of Stiefel-Whitney numbers, and that question will be answered here.

Perhaps more interesting is the fact that $\chi_{\frac{1}{2}}$ can be computed quite simply. Specifically, there is a class $i_*(K) \in \tilde{R}_{GL,\mathrm{ev}}(G)$ obtained by extension from the Sylow 2 subgroup of G, so that for any free G action (M, ϕ),

$$\chi_{\frac{1}{2}}(M; K) = s\chi(M) \cdot i_*(K)$$

where $s\chi(M)$ is the Kervaire semi-characteristic [4]

$$s\chi(M) = \sum_{i=0}^{n} (-1)^i \dim H^i(M; Z_2)$$

in Z_2, $\dim M = 2n+1$. Except when G has odd order, so that $i_*(K) = 0$, Lee's invariant then reduces to the usual semicharacteristic.

A direct proof that $s\chi(M)$ is a cobordism invariant of (M, ϕ), for G of even order, will be given. This involves showing that for a free involution $T : M^{2n+1} \to M^{2n+1} s\chi(M)$ is just the Euler characteristic of the submanifold $N^{2n} \subset M^{2n+1}/T$ which defines the double cover of M/T by M.

An analogous result holds for arbitrary sphere bundles, and this will be used to show that for even dimensional manifolds with involution which is free on the boundary,

$$s\chi(\partial V) = \chi(F) + \chi(F \cap F)$$

where T is an involution on V with F the fixed set of T, and $F \cap F$ the self intersection of F in V.

As a corollary, one obtains a more geometric proof of a result of Conner and Floyd [2]: If $T: M^{2n} \to M^{2n}$ is an involution on a manifold of odd Euler characteristic, then some component of the fixed set has dimension at least n.

Finally, the semicharacteristics for oriented manifolds introduced by Lee will be examined. Unfortunately, the algebraic problems are much harder, and the results are far from complete. For groups with abelian Sylow 2 subgroup, the invariants always vanish (Proposition 5.4) for $4k+3$ dimensional manifolds. For abelian groups and manifolds of dimension $4k+1$, the invariants are determined in Propositions 5.5 and 5.6.

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2. Lee's invariant

In order to understand Lee's invariant, one needs primarily to define the Grothendieck group $\tilde{R}_{GL,\mathrm{ev}}(G)$. Let K be a finite field of characteristic 2, and G a finite group. $R_K(G)$ denotes the Grothendieck group of finite dimensional G representations over K.

If V is a G-representation over K, a G quadratic form (V, ϕ) is a symmetric bilinear pairing $\phi: V \times V \to K$ such that

$$\phi(gx, gy) = \phi(x, y).$$

The form is *even* if for all $t \in G$, $t \neq e$ and t of order 2,

$$\phi(x, tx) = 0$$

for every $x \in V$. The form is non-singular if the homomorphism $ad\ \phi: V \to V^*$ given by $(ad\ \phi)(x)(y) = \phi(x, y)$ is an isomorphism.

$R_{GL,\mathrm{ev}}(G)$ is the quotient group of $R_K(G)$ obtained by dividing out the subgroup generated by the classes of those V which admit a non-singular even quadratic form.

If $H \subset G$, one has a transfer homomorphism

$$i^*: R_{GL,\mathrm{ev}}(G) \to R_{GL,\mathrm{ev}}(H)$$

obtained by considering a G representation as an H-representation, and an extension homomorphism

$$i_*: R_{GL,\mathrm{ev}}(H) \to R_{GL,\mathrm{ev}}(G)$$

obtained by sending W to $KG \otimes_{KH} W$.

Then $\tilde{R}_{GL,\mathrm{ev}}(G)$ is defined to be the cokernel of

$$i_* : R_{GL,\mathrm{ev}}(\{e\}) \to R_{GL,\mathrm{ev}}(G).$$

Thus $\tilde{R}_{GL,\mathrm{ev}}(G)$ is obtained from $R_K(G)$ by dividing out the subgroup generated by the non-singular even forms and the free KG modules.

The homomorphism

$$\chi_{\frac{1}{2}} : \mathfrak{N}_{2n+1}(G) \to \tilde{R}_{GL,\mathrm{ev}}(G)$$

assigns to (M^{2n+1}, ϕ) the class $\sum_{i=0}^{n}(-1)^i[H^i(M;K)]$, where G acts on $H^i(M;K)$ via ϕ.

Now for $H \subset G$, i^* and i_* induce homomorphisms

$$i^* : \tilde{R}_{GL,\mathrm{ev}}(G) \to \tilde{R}_{GL,\mathrm{ev}}(H)$$

and

$$i_* : \tilde{R}_{GL,\mathrm{ev}}(H) \to \tilde{R}_{GL,\mathrm{ev}}(G).$$

Letting

$$i^* : \mathfrak{N}_*(G) \to \mathfrak{N}_*(H)$$

by sending (M, ϕ) to $(M, \phi/H \times M)$ and

$$i_* : \mathfrak{N}_*(H) \to \mathfrak{N}_*(G)$$

by sending (N, ψ) to the class of $G \times N/(gh^{-1}, hx) \sim (g, x)$ with action $g'(g, x) = (g'g, x)$, one has a commutative diagram (Lemma 4.10 of [5])

$$\begin{array}{ccccc}
\mathfrak{N}_{2n+1}(H) & \xrightarrow{i_*} & \mathfrak{N}_{2n+1}(G) & \xrightarrow{i^*} & \mathfrak{N}_{2n+1}(H) \\
\downarrow{\scriptstyle \chi_{\frac{1}{2}}} & & \downarrow{\scriptstyle \chi_{\frac{1}{2}}} & & \downarrow{\scriptstyle \chi_{\frac{1}{2}}} \\
\tilde{R}_{GL,\mathrm{ev}}(H) & \xrightarrow{i_*} & \tilde{R}_{GL,\mathrm{ev}}(G) & \xrightarrow{i^*} & \tilde{R}_{GL,\mathrm{ev}}(H).
\end{array}$$

The other fact needed here is that if $S \subset G$ is the Sylow 2-subgroup of G, then the composite

$$i_* \circ i^* : \mathfrak{N}_*(G) \to \mathfrak{N}_*(S) \to \mathfrak{N}_*(G)$$

is the identity. (Note: This is Lemma 4.11 (3) of [5]; beware that parts (1) and (2) of the Lemma do not hold for arbitrary G). To see this one notes that if $f : M \to BG$ represents $\alpha \in \mathfrak{N}_*(G)$ then $i_* \circ i^*(\alpha)$ is represented by $f \circ \pi : \tilde{M} \to BG$ where $\tilde{M}$ is the bundle induced by

$$\begin{array}{ccc}
\tilde{M} & \xrightarrow{\tilde{f}} & BS \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\
M & \xrightarrow{f} & BG
\end{array}$$

Then for $x \in H^*(BG; Z_2)$,

$$\begin{aligned}\langle w_\omega(\tilde{M})(f \circ \pi)^*(x), [\tilde{M}]\rangle &= \langle \pi^*(w_\omega(M)f^*(x)), [\tilde{M}]\rangle \\ &= [G:S]\langle w_\omega(M)f^*(x), [M]\rangle\end{aligned}$$

and $[G:S]$ = index of S in $G = 1 \pmod 2$.).

LEMMA 2.1: *If S is a 2 group, then $\tilde{R}_{GL,\mathrm{ev}}(S)$ is isomorphic to Z_2 if $S \neq \{e\}$ and is the zero group if $S = \{e\}$.*

PROOF: If $S = \{e\}$, $i_* : R_{GL,\mathrm{ev}}(\{e\}) \to R_{GL,\mathrm{ev}}(S)$ is the identity, so the cokernel, $\tilde{R}_{GL,\mathrm{ev}}(S)$, is the zero group.

Thus suppose $S \neq \{e\}$. If V is any representation space for S, S acts on the underlying set of V which has an even number of elements, and each orbit has 2^j elements for some j. Since S fixes $\{0\}$, S must also fix a nonzero vector x. Thus V contains a trivial representation, Kx. Then $[V] = [K]+[V/Kx]$, and inductively $R_K(S) \cong Z$ assigning to V its dimension over K.

On $K \oplus K$ with trivial S action one has the hyperbolic form $\phi((x_1, x_2), (y_1, y_2)) = x_1 y_2 + x_2 y_1$, which is even. On the other hand, $KS \oplus_K W$ has dimension divisible by 2^S = order of S, and any even form has even dimension, so $\tilde{R}_{GL,\mathrm{ev}}(S) \cong Z_2$.

To see that any even form has even dimension, it suffices to restrict (V, ϕ) to some subgroup of order 2 in S. If t is the element of order 2, the form $\psi : V \times V \to K$ defined by $\psi(x, y) = \phi(x, ty) = \phi(tx, y)$ is then non-singular and $\psi(x, x) = 0$. One may then choose a symplectic base for (V, ψ). $*$

PROPOSITION 2.2: *The homomorphism*

$$\chi_{\frac{1}{2}} : \mathfrak{N}_{2n+1}(G) \to \tilde{R}_{GL,\mathrm{ev}}(G)$$

sends (M, ϕ) to $s\chi(M) \cdot i_*(K)$ where

$$s\chi(M) = \sum_{i=0}^{n} (-1)^i \dim H^i(M; Z_2)$$

and $i_*(K)$ is the class obtained by applying

$$i_* : \tilde{R}_{GL,\mathrm{ev}}(S) \to \tilde{R}_{GL,\mathrm{ev}}(G),$$

S the Sylow 2-subgroup of G to the 1-dimensional trivial S representation.

PROOF: This is essentially the proof given in Theorem 4.13 of [5]. First, $H^i(M; K) \cong H^i(M; Z_2) \otimes_{Z_2} K$, so

$$\begin{aligned}
\chi_{\frac{1}{2}}(M; K) &= \chi_{\frac{1}{2}}(i_* i^* M; K) \\
&= i_* \chi_{\frac{1}{2}}(i^* M; K) \\
&= i_*\Big(\sum_0^n (-1)^i [H^i(M; K)]\Big) \\
&= i_*\Big(\sum_0^n (-1)^i \dim_K H^i(M; K) \cdot [K]\Big) \\
&= i_*(s\chi(M) \cdot [K]) \\
&= s\chi(M) \cdot i_*([K]). \qquad *
\end{aligned}$$

Note: If G has odd order, $S = \{e\}$, and $i_*(K) = 0$. If G has even order, $i^* i_*(K)$ is represented by $KG \otimes_{KS} K$ which has dimension $[G : S] =$ odd. Thus $i^* i_*(K) \neq 0$ and so $i_*(K) \neq 0$. Thus, the Kervaire semi-characteristic is an invariant of free G bordism, if G has even order. It is definitely not an invariant when G has odd order.

It should be remarked that Lee's invariant is stronger than just the Kervaire semi-characteristic. His arguments make heavy use of the fact that $i_*(K)$ is not in general the class of the trivial G representation. The formula $\chi_{\frac{1}{2}}(M, \phi) = s\chi \cdot (M) i_*(K)$ contains more geometric information that the value of the semicharacteristic alone.

3. Kervaire's semicharacteristic

The basic result needed to analyze the Kervaire semicharacteristic will be:

PROPOSITION 3.1: *Let M be a closed manifold of dimension $2n + r$ and ξ an r-plane bundle over M. Then the Kervaire semicharacteristic of the sphere bundle of ξ, $s\chi(S(\xi))$, is the sum of the Euler characteristics of M and N, where $N \subset M$ is the submanifold dual to ξ; i.e. $s\chi(S(\xi)) = \chi(M) + \chi(N)$.*

PROOF: The Gysin sequence of the bundle ξ gives an exact sequence

$$\begin{aligned}
0 &\leftarrow A \leftarrow H^{n+r-1}(S(\xi)) \leftarrow H^{n+r-1}(M) \leftarrow H^{n-1}(M) \leftarrow H^{n+r-2}(S(\xi)) \leftarrow \\
&\cdots \leftarrow H^r(S(\xi)) \leftarrow H^r(M) \leftarrow H^\circ(M) \leftarrow H^{r-1}(S(\xi)) \leftarrow H^{r-1}(M) \leftarrow 0 \leftarrow \\
&\leftarrow H^{r-2}(S(\xi)) \leftarrow H^{r-2}(M) \leftarrow \cdots \leftarrow 0 \leftarrow H^\circ(S(\xi)) \leftarrow H^\circ(M) \leftarrow 0.
\end{aligned}$$

where

$$A = \ker \{\cup\, w_r(\xi) : H^n(M) \to H^{n+r}(M)\}.$$

The usual rule for Euler characteristics in an exact sequence gives

$$\begin{aligned} s\chi(S(\xi)) &= \sum_{0}^{n+r-1} (-1)^i \dim H^i(S(\xi)) \\ &= \sum_{0}^{n+r-1} (-1)^i \dim H^i(M) + (-1)^{n+r-1} \dim A \\ &\qquad\qquad + (-1)^{r-1} \sum_{0}^{n-1} (-1)^i \dim H^i(M) \\ &= \chi(M) - \dim H^n(M) + \dim A \pmod 2 \\ &= \chi(M) + \dim im\{\cup w_r(\xi) : H^n(M) \to H^{n+r}(M)\} \end{aligned}$$

Now consider the symmetric quadratic form

$$\phi : H^n(M) \times H^n(M) \to Z_2$$

defined by $\phi(x, y) = \langle w_r(\xi) \cup x \cup y, [M]\rangle = \langle f^*(x) \cup f^*(y), [N]\rangle$. where $f : N \to M$ is the inclusion. Clearly, the rank of ϕ is equal to the dimension of the image of $\{\cup w_r(\xi) : H^n(M) \to H^{n+r}(M)\}$. On the other hand, there exist classes $v \in H^n(M)$ so that $\phi(x, x) = \phi(x, v)$ for all $x \in H^n(M)$, and for any such v, rank $(\phi) = \phi(v, v)$ in Z_2. Now the Stiefel-Whitney class of N is given by $f^*(w(M)/w(\xi))$, and so there is a class $v' \in H^n(M)$ with $f^*(v') = v_n(N)$ being the n-th Wu class of N. Thus, for any $x \in H^n(M)$,

$$\begin{aligned} \phi(x, x) &= \langle f^*(x) \cup f^*(x), [N]\rangle = \langle v_n(N) \cup f^*(x), [N]\rangle \\ &= \langle f^*(x) \cup f^*(v'), [N]\rangle = \phi(x, v') \end{aligned}$$

and

$$\begin{aligned} \text{rank } (\phi) &= \langle f^*(v') \cup f^*(v'), [N]\rangle = \langle v_n(N) \cup v_n(N), [N]\rangle \\ &= \langle w_{2n}(N), [N]\rangle = \chi(N). \end{aligned}$$

Hence, $s\chi(S(\xi)) = \chi(M) + \chi(N)$. *

Note: One would like to prove this using only the cohomology structure, but it seems to depend heavily on the fact that the Wu class $v_n(N)$ belongs to the image of f^*.

COROLLARY 3.2: *If M^{2n+1} is a closed manifold and $T : M \to M$ is a free involution, then $s\chi(M) = \chi(N)$ where $N^{2n} \subset M^{2n+1}/T$ is the submanifold which defines the double cover of M/T by M.*
(*See* [1], *Prop* (3.4), *and* [3], *Cor.* 2.7).

PROOF: $M = S(\lambda)$ where $\lambda \to M/T$ is the line bundle associated to the double cover of M/T by M, and N is the submanifold dual to λ. Since M/T has odd dimension, $\chi(M/T) = 0$. *

COROLLARY 3.3: *If G is a finite group of even order, then assigning to (M^{2n+1}, ϕ) the semi-characteristic $s\chi(M)$ defines a homomorphism*

$$s\chi : \mathfrak{N}_{2n+1}(G) \to Z_2.$$

PROOF: Letting $Z_2 \subset G$ be any subgroup of order 2, $s\chi$ is given by the composite of

$$i^* : \mathfrak{N}_{2n+1}(G) \to \mathfrak{N}_{2n+1}(Z_2)$$

and the Smith homomorphism ([1] §26)

$$\Delta : \mathfrak{N}_{2n+1}(Z_2) \to \mathfrak{N}_{2n}(Z_2)$$

and the usual isomorphism

$$\mathfrak{N}_{2n}(Z_2) \cong \mathfrak{N}_{2n}(BZ_2)$$

and the augmentation

$$\varepsilon : \mathfrak{N}_{2n}(BZ_2) \to \mathfrak{N}_{2n}$$

and the Euler characteristic

$$\chi : \mathfrak{N}_{2n} \to Z_2. \qquad *$$

One may now write down a characteristic number description of the semi-characteristic, as was asked for by Lee. Being given (M^{2n+1}, ϕ), let $h : M/G \to BG$ classify the principal bundle $M \to M/G$. Let $Z_2 \subset G$ be any subgroup of order 2, $c \in H^1(BZ_2, Z_2)$ the nonzero class, and $i_* : H^*(BZ_2, Z_2) \to H^*(BG; Z_2)$ the extension homomorphism. Then

$$s\chi(M) = \left\langle \sum_{j=0}^{2n+1} w_{2n+1-j}(M/G) h^* i_*(c^j); [M/G] \right\rangle$$

i.e. $s\chi$ is associated with the characteristic class

$$\sum_{j=0}^{2n+1} w_{2n+1-j} i_*(c^j).$$

To see this, one notes that the diagram

$$\begin{tikzcd} M/Z_2 \arrow[r, "\tau \times \tilde{h}"] \arrow[d, "\pi'"] & BO \times BZ_2 \arrow[d, "1 \times \pi"] \\ M/G \arrow[r, "\tau \times h"] & BO \times BG \end{tikzcd}$$

commutes. Thus

$$\langle \sum_{0}^{2n+1} w_{2n+1-j}(M/G)h^*i_*(c^j); [M/G]\rangle$$

$$= \langle \sum_{0}^{2n+1} w_{2n+1-j} \otimes i_*(c^j), (\tau \times h)_*([M/G])\rangle$$

$$= \langle (1 \times \pi)_*(\sum_{0}^{2n+1} w_{2n+1-j} \otimes c^j), (\tau \times h)_*([M/G])\rangle$$

$$= \langle \sum_{0}^{2n+1} w_{2n+1-j} \otimes c^j, (\tau \times \tilde{h})_*([M/Z_2])\rangle$$

$$= \langle \sum_{0}^{2n+1} w_{2n+1-j}(M/Z_2)\tilde{h}^*(c^j), [M/Z_2]\rangle$$

where

$$(1 \times \pi)_* : H^*(BO \times BZ_2; Z_2) \to H^*(BO \times BG; Z_2)$$

is the cohomology 'transfer' of a finite cover. Now

$$\langle w_{2n+1}(M/Z_2), [M/Z_2]\rangle = \chi(M/Z_2),$$

and

$$\langle \sum_{1}^{2n+1} w_{2n+1-j}(M/Z_2)\tilde{h}^*(c^j), [M/Z_2]\rangle$$

$$= \langle h^*(c) \cdot \sum_{1}^{2n+1} w_{2n+1-j}(M/Z_2)h^*(c^{j-1}), [M/Z_2]\rangle$$

$$= \langle f^*(\sum_{1}^{2n+1} w_{2n+1-j}(M/Z_2)h^*(c^{j-1})), [N]\rangle$$

$$= \langle w_{2n}(N), [N]\rangle$$

$$= \chi(N).$$

Since $\chi(M/Z_2)+\chi(N) = s\chi(M)$, the result follows.

The characteristic number formulation seems to depend heavily on the choice of the subgroup Z_2; in fact it does not.

LEMMA 3.4: *If M^{2n+1} admits a free action of $Z_2 \times Z_2$, then $s\chi(M) = 0$.*

PROOF: Take T_1, T_2 as generators of $Z_2 \times Z_2$. Then $s\chi(M) = \chi(N_1)$ where $N_1 \subset M/T_1$ is dual to the double cover. However in $M/Z_2 \times Z_2$, one may take N_2 dual to the double cover by M/T_2 and if

$$\pi : M/T_1 \to M/Z_2 \times Z_2,$$

$\pi^{-1}(N_2)$ may be taken to be N_1; thus N_1 may be taken to have a free involution induced by T_2, so N_1 bounds and $\chi(N_1) = 0$. $*$

Thus if the semi-characteristic is non-trivial on free G bordism, then G can contain no subgroup $Z_2 \times Z_2$, in particular, the Sylow 2 subgroup S of G can contain no such subgroup. Thus, every abelian subgroup of S is cyclic which implies that S is either cyclic or generalized quaternion. If S is cyclic or generalized quaternion, it contains a unique element of order 2, and since any two Sylow 2 subgroups are conjugate, any two elements of order 2 in G are conjugate.

Restated, either the semi-characteristic is trivial for G or up to conjugacy, there is a unique element of order 2.

If G contains a subgroup $Z_2 \times Z_2$, and H is a subgroup of order 2 lying in the Sylow subgroup S, then S contains a central subgroup K of order 2. If $H = K$, and L is any other subgroup of order 2 in S, $H \times L \subset S$, while if $H \neq K$, $H \times K \subset S$. Thus H lies in a subgroup isomorphic to $Z_2 \times Z_2$. Now $i^* : H^*(B(Z_2 \times Z_2); Z_2) \to H^*(BZ_2, Z_2)$ is epic so i_* is zero ($i_* i^* = 0$), but $i_* : H^*(BZ_2, Z_2) \to H^*(BG; Z_2)$ factors through $B(Z_2 \times Z_2)$, hence is zero.

If G contains no subgroup $Z_2 \times Z_2$, then the classes $i_*(c^j)$ and $i_*(\bar{c}^j)$ for two different subgroups Z_2 differ by the action of an inner automorphism on G, but inner automorphisms are trivial on cohomology, so $i_*(c^j) = i_*(\bar{c}^j)$.

4. Self-intersections

The cobordism invariance of the semi-characteristic for free involutions on odd dimensional manifolds gives rise to a cobordism invariant of even dimensional manifolds with involution which is free on the boundary. Denoting this cobordism group by $\mathfrak{N}_*^{Z_2}(\text{Free } \partial)$, the composite

$$\mathfrak{N}_{2n}^{Z_2} (\text{Free } \partial) \xrightarrow{\partial} \mathfrak{N}_{2n-1}(Z_2) \xrightarrow{s\chi} Z_2$$

is the homomorphism of interest.

The cobordism group $\mathfrak{N}_{2n}^{Z_2} (\text{Free } \partial)$ has been analyzed thoroughly by Conner and Floyd [2] (28.1). It may be identified via the fixed point homomorphism with $\oplus_{j=0}^{2n} \mathfrak{N}_{2n-j}(BO_j)$, by assigning to (V^{2n}, T) the cobordism classes $F^{2n-j} \xrightarrow{\nu} BO_j$ of the maps classifying the normal bundle to the codimension j part of the fixed set of T.

From Corollary 3.3, $s\chi(\partial V)$ is given as the sum of the semi-characteristics of the sphere bundles of the normal bundles of the F^{2n-j}, and by Proposition 3.1, these semi-characteristics are the sum of the Euler characteristics of F^{2n-j} and the submanifold dual to ν. The submanifold dual to ν may also be described as the self-intersection of F^{2n-j} in the disc of ν.

Being given (V^{2n}, T) with fixed set F, one may consider the self-intersection $F \cap F$ of F in V, i.e. the submanifold of F obtained by deforming F to be transverse regular to itself within V, and taking the intersection. The cobordism class of $F \cap F$ is a cobordism invariant of (V, T). (To see this, make the fixed set of a cobordism from (V, T) to (V', T') transverse to itself). In fact, the self-intersection of F^{2n-j} with itself is the submanifold dual to v. Thus one has:

Proposition 4.1: *If (V^{2n}, T) is a manifold with involution which is free on ∂V, then*

$$s\chi(\partial V) = \chi(F) + \chi(F \cap F),$$

where F is the fixed set of T and $F \cap F$ is the self-intersection of F in V.

In particular, if V is closed, $s\chi(\partial V) = 0$, and $\chi(F) \equiv \chi(F \cap F) \bmod 2$. Combining this with $\chi(V) \equiv \chi(F) \pmod 2$, one has $\chi(F \cap F) \equiv \chi(V)$. (See Conner and Floyd [2] (27.2), or note that if T is simplicial on V, the simplices of V consist of pairs σ, $T\sigma \neq \sigma$ and simplices of F). Thus one has:

Proposition 4.2: ([2], (27.4)]. *If $T : M^{2n} \to M^{2n}$ is an involution on a closed manifold of odd Euler characteristic, then some component of the fixed set of T has dimension at least n.*

Proof: If the fixed set has dimension less than n, then the normal bundle of the fixed component F^i has dimension greater than i, so has a section. Thus, $F \cap F$ can be taken empty, and $\chi(F \cap F) = 0$. Then $\chi(M) \equiv \chi(F \cap F)$ and M has even Euler characteristic. *

5. Lee's oriented invariants

Lee also introduced semicharacteristic invariants

$$\chi_{\frac{1}{2}} : \Omega_{2n+1}(G, \omega) \to \tilde{R}_{GL,Sp}(G, \omega) \qquad n \text{ even}$$

and

$$\chi_{\frac{1}{2}} : \Omega_{2n+1}(G, \omega) \to \tilde{R}_{GL,0}(G, \omega) \qquad n \text{ odd}$$

for free G actions on oriented manifolds, using cohomology with K coefficients, where K is a field of characteristic not 2. He characterizes these invariants as 'remarkably useless' and yet they are far from trivial.

Being given a finite group G and homomorphism $\omega : G \to Z_2 = \{+1, -1\}$, $\Omega_*(G, \omega)$ denotes the cobordism group of free G actions on oriented manifolds for which each $g \in G$ preserves or reverses orientation as $\omega(g)$ is respectively $+1$ or -1. When ω is trivial, this is the usual oriented G bordism group $\Omega_*(BG)$; when ω is non-trivial, the kernel of ω is a normal subgroup $H \subset G$ of index 2 giving a double cover $BH \xrightarrow{\pi} BG$, and the group $\Omega_*(G, \omega)$ is the oriented bordism group $\tilde{\Omega}_{*+1}(M_\pi, BH)$ where M_π is the mapping cone of π. (*Note*: given $V \xrightarrow{f} M_\pi$, $\partial V \xrightarrow{f} BH$, f may be made transverse to BG giving an unoriented manifold N with principal G bundle P so that P/H is the orientation cover of N; thus $[V, f]$ gives the action of G on P).

One has a restriction homomorphism $i^* : \Omega_*(G, \omega) \to \Omega_*(S, \omega/S)$ for a subgroup $S \subset G$ by restricting the action to S, and an extension homomorphism $i_* : \Omega_*(S, \omega/S) \to \Omega_*(G, \omega)$ assigning to (M, S) the action on $G \times M/(g, m) \sim (gs^{-1}, sm)$ given by $g'(g, m) = (g'g, m)$, where G is oriented by ω so that $g \in G$ is a positively oriented point if $\omega(g) = +1$, and is negatively oriented if $\omega(g) = -1$. (*Note*: The S action $s_*(g, m) = (gs^{-1}, sm)$ is then orientation preserving making $G \times M/\sim$ oriented).

PROPOSITION 5.1: *The semicharacteristic*

$$\chi_{\frac{1}{2}} : \Omega_*(G, \omega) \to \tilde{R}_{GL,x}(G, \omega)$$

depends only on the Sylow 2-subgroup of G; *specifically*

$$\chi_{\frac{1}{2}}(M; K) = i_* \chi_{\frac{1}{2}}(i^*M; K)$$

where i_*, i^* *are extension and restriction from a Sylow 2-subgroup* S *of* G.

PROOF: One has a commutative diagram

$$\begin{array}{ccccc}
\Omega_*(S, \omega/S) & \xrightarrow{i_*} & \Omega_*(G, \omega) & \xrightarrow{i^*} & \Omega_*(S, \omega/S) \\
\downarrow{\scriptstyle \chi_{\frac{1}{2}}} & & \downarrow{\scriptstyle \chi_{\frac{1}{2}}} & & \downarrow{\scriptstyle \chi_{\frac{1}{2}}} \\
\tilde{R}_{GL,x}(S, \omega/S) & \xrightarrow{i_*} & \tilde{R}_{GL,x}(G, \omega) & \xrightarrow{i^*} & \tilde{R}_{GL,x}(S, \omega/S)
\end{array}$$

and so one wants $M \equiv i_* i^* M$ mod kernel $\{\chi_{\frac{1}{2}}(\ ; K)\}$. Now Lee notes that $\chi_{\frac{1}{2}}$ has image in the subgroup of $\tilde{R}_{GL,x}(G, \omega)$ consisting of elements of order 2, so kernel $\{\chi_{\frac{1}{2}}(\ ; K)\} \supset 2\Omega_*(G, \omega)$.

One now has a commutative diagram

$$\begin{array}{ccccc}
\Omega_*(S, \omega/S) & \xrightarrow{i_*} & \Omega_*(G, \omega) & \xrightarrow{i^*} & \Omega_*(S, \omega/S) \\
{\scriptstyle 2}\downarrow & & {\scriptstyle 2}\downarrow & & {\scriptstyle 2}\downarrow \\
\Omega_*(S, \omega/S) & \xrightarrow{i_*} & \Omega_*(G, \omega) & \xrightarrow{i^*} & \Omega_*(S, \omega/S) \\
{\scriptstyle \rho}\downarrow & & {\scriptstyle \rho}\downarrow & & {\scriptstyle \rho}\downarrow \\
\mathfrak{N}_*(S) & \xrightarrow{i_*} & \mathfrak{N}_*(G) & \xrightarrow{i^*} & \mathfrak{N}_*(S)
\end{array}$$

where ρ is reduction, and the columns are exact (when ω is trivial, this is the exact Rohlin sequence ([2] (16.2)) $\Omega_*(BG) \xrightarrow{2} \Omega_*(BG) \xrightarrow{f} \mathfrak{N}_*(BG)$, while if ω is non-trivial, it is the Rohlin sequence for (M_π, BH) combined with the Thom isomorphism $\tilde{\mathfrak{N}}_{*+1}(M_\pi, BH) \cong \mathfrak{N}_*(BG)$).

Since $i_* i^* = 1$ on $\mathfrak{N}_*(G)$, $i_* i^* = 1 \bmod 2\Omega_*(G, \omega)$ on $\Omega_*(G, \omega)$. *

Note: There are no non-trivial semicharacteristic invariants for a group of odd order, for $\tilde{R}_{GL,x}([1], \omega/1)$ is the zero group.

The major advantage of this result is that one need only consider ordinary representations; i.e. representations of a 2-group on a field of characteristic different from 2, and may largely ignore the odd part of G which might have led to modular representations.

PROPOSITION 5.2: *If G is a finite group with non-trivial cyclic Sylow 2-subgroup S, and $1 : G \to Z_2$ is the trivial homomorphism, then*

$$\chi_{\frac{1}{2}} : \Omega_{2n+1}(G, 1) \to \tilde{R}_{GL,0}(G, 1) \qquad n \text{ odd}$$

is the zero homomorphism, and

$$\chi_{\frac{1}{2}} : \Omega_{2n+1}(G, 1) \to \tilde{R}_{GL,Sp}(G, 1) \qquad n \text{ even}$$

is given by

$$\chi_{\frac{1}{2}}(M; K) = s\chi(M) \cdot i_*(K)$$

where i_* is the extension from S.

Note: It will be shown that $i_*(K) \neq 0$.

PROOF: The proof will be somewhat involved, needing first the case $G = Z_2$.

Let K be a field of characteristic not equal to 2. The irreducible K representations of Z_2 are K_+, K_-, the one dimensional representations with $tx = x$ and $tx = -x$ respectively, where t is the non-trivial element of Z_2 and $x \in K$. $R_K(Z_2)$ is then isomorphic to $Z \oplus Z$, where the isomorphism assigns the dimensions of image $(\frac{1}{2}(1+t))$ and image $(\frac{1}{2}(1-t))$; i.e. the number of copies of K_+ and K_-.

Each of K_+ and K_- has the nonsingular symmetric form $\phi : K \times K \to K$ given by $\phi(x, y) = xy$, and so $R_{GL,0}(Z_2, 1) = 0$.

A skew form which is nonsingular on V makes im $(\frac{1}{2}(1+t))$ and im $(\frac{1}{2}(1-t))$ orthogonal and induces nonsingular skew forms on each, so each is even dimensional, with $2K_+$ and $2K_-$ having the hyperbolic forms. Thus $R_{GL,Sp}(Z_2, 1) \cong Z_2 \oplus Z_2$. Extending K from the trivial group to Z_2 gives $K_+ \oplus K_-$, so $\tilde{R}_{GL,Sp}(Z_2, 1) \cong Z_2$ and the isomorphism sends V to dim $V \cdot [K]$, where $K = K_+$ is the trivial representation.

Thus for $G = Z_2$, $\chi_{\frac{1}{2}}$ is zero on $\Omega_{2n+1}(Z_2, 1)$ if n is odd, and on $\Omega_{2n+1}(Z_2, 1)$, with n even,

$$\chi_{\frac{1}{2}}(M; K) = \sum_0^n (-1)^i [H^i(M; K)]$$

$$= \{\sum_0^i (-1)^i \dim_K H^i(M; K)\} \cdot [K]$$

By the work of Lusztig, Milnor, and Peterson [6] an oriented manifold of dimension $4r+1$ which bounds as an unoriented manifold has the property that its semicharacteristic is independent of the field with which it is computed. Thus, the equation becomes $\chi_{\frac{1}{2}}(M; K) = s\chi(M) \cdot [K]$.

Now let $G = Z_{2^s}$, $s \geq 1$. Let γ denote the standard complex line bundle over $CP(\infty) = BS^1$. Then the sphere bundle of $\gamma^{2^s} = \gamma \otimes_C \cdots \otimes_C \gamma$ (2^s times) may be identified with BZ_{2^s} and the cofibration

$$S(\gamma^{2^s}) \to D(\gamma^{2^s}) \to T(\gamma^{2^s})$$

gives an exact sequence

$$\begin{tikzcd} \Omega_*(S(\gamma^{2^s})) \arrow[r] & \Omega_*(D(\gamma^{2^s})) \arrow[r] & \tilde{\Omega}_*(T(\gamma^{2^s})) \arrow[ll, bend left] \end{tikzcd}$$

Projection is a homotopy equivalence, and identifies $\Omega_*(D(\gamma^{2^s}))$ with $\Omega_*(CP(\gamma))$, while the Thom isomorphism identifies $\tilde{\Omega}_*(T(\gamma^{2^s}))$ with $\Omega_{*-2}(CP(\infty))$. Thus, one has an exact sequence

$$\begin{tikzcd} \Omega_*(BZ_{2^s}) \arrow[r, "\pi_*"] & \Omega_*(BS^1) \arrow[r, "\alpha"] & \Omega_*(BS^1) \arrow[ll, bend left, "\beta"] \end{tikzcd}$$

Now $\Omega_*(BZ_{2^s}) \cong \Omega_* \oplus \tilde{\Omega}_*(BZ_{2^s})$, where the Ω_* summand is obtained from the inclusion of a point and $\tilde{\Omega}_*(BZ_{2^s})$ consists of 2-torsion. The Ω_* summand maps isomorphically to the similar Ω_* summand of $\Omega_*(BS^1)$.

In the special case $s = 1$, $\pi_* : \Omega_*(BZ_2) \to \Omega_*(BS^1)$ maps onto the torsion subgroup (Note: The torsion in $\Omega_*(BS^1)$ maps monomorphically into unoriented bordism of BS^1, but $\pi^* : H^*(BS^1; Z_2) \to H^*(BZ_2; Z_2)$ is monic, so π_* is epic in unoriented bordism, and α is zero. Thus if x is a torsion class $\rho\alpha x = \alpha\rho x = 0$, but αx is torsion so $\rho\alpha x = 0$ implies $\alpha x = 0$). One then has, for any s,

$$\Omega_*(BZ_2) \xrightarrow{\pi'_*} \Omega_*(BZ_{2^s}) \xrightarrow{\pi_*} \Omega_*(BS^1)$$

and the image of π_* is contained in the image of $\pi_* \circ \pi'_*$. Thus

$$\beta + \pi'_* : \Omega_*(BS^1) \oplus \Omega_*(BZ_2) \to \Omega_*(BZ_{2^s})$$

is epic; i.e. every free Z_{2^s} action is bordant to a sum of restrictions of free S^1 actions and extensions of free Z_2 actions.

Note: For further discussion of the cofibration, one may see [7]. The fact that $\beta + \pi'_*$ is epic was worked out in a joint discussion with Russell J. Rowlett, for a theorem on which he was working.

Now consider an element in $\Omega_{2n+1}(Z_{2^s}, 1)$ with n odd, and write it as $(M, \phi)+(N, \psi)$ where (M, ϕ) is the restriction of an S^1 action, and (N, ψ) is the extension of a Z_2 action (N', ψ'). Then $\chi_{\frac{1}{2}}(N; K) = i_* \chi_{\frac{1}{2}}(N', K)$, but $\chi_{\frac{1}{2}}(N', K) = 0$. Also $\chi_{\frac{1}{2}}(M, K) = \{\sum_0^n (-1)^i \dim H^i(M, K)\}$. $[K]$ for Z_{2^s} acts trivially on $H^*(M; K)$, being the restriction of an S^1 action. Since the trivial representation admits the nonsingular symmetric form $\phi : K \times K \to K : (x, y) \to xy$, $[K] = 0$. Thus

$$\chi_{\frac{1}{2}} : \Omega_{2n+1}(Z_{2^s}, 1) \to \tilde{R}_{GL,0}(Z_{2^s}, 1)$$

is the zero homomorphism, (n odd).

Letting n be even, an element in $\Omega_{2n+1}(Z_{2^s}, 1)$, $s > 1$, may be written as $(M, \phi)+(N, \psi)$ as above. Then

$$\chi_{\frac{1}{2}}(N, K) = i_* \chi_{\frac{1}{2}}(N', K) = i_*(s\chi(N') \cdot [K]) = s\chi(N') i_*[K].$$

In particular, if N' is the sphere S^{2n+1} with antipodal action,

$$i_*[K] = \chi_{\frac{1}{2}}(i_*(S^{2n+1}); K) = \chi_{\frac{1}{2}}(i_* i^*(S^{2n+1}, \theta); K)$$

where θ is the standard free Z_{2^s} action, but $i_* i^*$ is trivial on unoriented bordism, so $i_* i^*(S^{2n+1}, \theta)$ is divisible by 2. Thus $i_*[K] = 0$ and $\chi_{\frac{1}{2}}(N, K) = 0$. Note that $s\chi(N) = 2^{s-1} s\chi(N') = 0$. Since Z_{2^s} acts trivially on $H^*(M; K)$, one has $\chi_{\frac{1}{2}}(M; K) = s\chi(M) \cdot [K]$, and combining

$$\chi_{\frac{1}{2}}(M \cup N; K) = s\chi(M \cup N) \cdot [K].$$

Thus the proposition is true for $G = Z_{2^s}$, and applying Proposition 5.1 gives the result for all G with cyclic Sylow 2-subgroup.

To see that $i_*[K] \neq 0$, consider the restriction to $Z_2 \subset G$. $KG \otimes_{KS} K$ has dimension $[G:S] =$ odd over K, so restricts to the nonzero class in $\tilde{R}_{GL,Sp}(Z_2, 1)$. *

Now turning to homomorphisms $\omega : G \to Z_2$ which are non-trivial, one has

PROPOSITION 5.3: *If $\omega : G \to Z_2$ is non-trivial, then the composite*

$$\Omega_{2n+1}(G, \omega) \xrightarrow{\rho} \mathfrak{N}_{2n+1}(G) \xrightarrow{\chi_{\frac{1}{2}}} \tilde{R}_{GL,\mathrm{ev}}(G)$$

is the zero homomorphism.

PROOF: $\chi_{\frac{1}{2}}(\rho M; K) = s\chi(M) i_*[K]$, and so one wants $s\chi(M) = 0$. Since

ω is non-trivial, there is an x with $\omega(x) = -1$, and $\omega(x^{2j+1}) = -1$ so by taking a suitable odd power of x, one may find x with $\omega(x) = -1$ and $x^{2^s} = 1$; i.e. it is sufficient to consider G cyclic of order 2^s.

If $s = 1$, $M \xrightarrow{\pi} M/Z_2$ is the orientation cover, and

$$\begin{aligned} s\chi(M) &= \langle w_{2n}c + w_{2n-1}c^2 + \cdots + c^{2n+1}, [M/Z_2]\rangle \\ &= \langle cv'v', \quad [M/Z_2]\rangle = \langle w_1 v'v', [M/Z_2]\rangle \\ &= \langle S_q{}^1((v')^2), [M/Z_2]\rangle = 0, \end{aligned}$$

or alternately, the submanifold $N \subset M/Z_2$ dual to w_1 is a torsion element of Ω_*, but $\chi(N) = \text{Index}(N) \pmod 2$ and the index vanishes on torsion classes.

If $s > 1$, one has a diagram

$$\begin{tikzcd} M \arrow[d] & \\ M/Z_2 \arrow[r] \arrow[d] & BZ_2 \arrow[d] \\ M/Z_{2^{s-1}} \arrow[r] \arrow[d] & BZ_{2^{s-1}} \arrow[d] \\ M' = M/Z_{2^s} \arrow[r] & BZ_{2^s} \arrow[d] \\ & BS^1 = CP(\infty) \end{tikzcd}$$

and

$$\begin{aligned} s\chi(M) &= \langle w_{2n}c + w_{2n-1}c^2 + \cdots + c^{2n+1}, [M/Z_2]\rangle \\ &= \langle w_{2n}i_*(c) + w_{2n-1}i_*(c^2) + \cdots + i_*(c^{2n+1}), [M']\rangle. \end{aligned}$$

Now $H^*(BZ_{2^s}; Z_2)$ is generated by a 1-dimensional class d and a 2-dimensional class α (a Bockstein of d) with $d^2 = 0$. The class α comes from $CP(\infty)$ and restricts to c^2 in BZ_2. One then has $i_*(c^{2j}) = 0$ and $i_*(c^{2j+1}) = d\alpha^j$. The condition that ω is non-trivial is that $M/Z_{2^{s-1}}$ is the orientation cover of M', so d restricts to w_1. Thus

$$s\chi(M) = \langle w_{2n}w_1 + w_{2n-2}w_1\alpha + \cdots + w_1\alpha^n, [M']\rangle.$$

Letting $N \subset M'$ be the codimension 2 submanifold dual to the complex line bundle coming from $CP(\infty)$,

$$w(N) = w(M)/1+\alpha$$

so

$$w_1(N) = w_1,\ w_{2n-2}(N) = w_{2n-2} + w_{2n-4}\alpha + \cdots + \alpha^{n-1}$$

and

$$s\chi(M) = \langle w_{2n}w_1, [M']\rangle + \langle w_{2n-2}w_1, [N]\rangle.$$

For a manifold V of dimension $2j+1$, $w_{2j} = v_j^2$ so

$$\langle w_{2j}w_1, [V]\rangle = \langle w_1 v_j^2, [V]\rangle = \langle S_q{}^1(v_j^2), [V]\rangle = 0,$$

and so $s\chi(M) = 0$. $*$

Now consider an *abelian* group G with $\omega : G \to Z_2$ a homomorphism, and let K be a field having characteristic zero or relatively prime to the order of G.

If V is an irreducible K representation of G, then V is a module over the commutative ring KG and has the property that if $x \neq 0$ is an element of V, then $(KG)x = V$. For any nonzero element x in V, $Ix = \{\lambda \in KG | \lambda x = 0\}$ is a (two sided) ideal in KG, and KG/Ix is a field (Note: If $\mu \notin Ix$, $\mu x \neq 0$ and $(KG)\mu x = V$ so there is a $\lambda \in KG$ with $\lambda\mu x = x$). Further, Ix is independent of x. One may then identify V with a finite extension $\tilde{K} = KG/I$ of the field K.

Letting $1 \in \tilde{K}$ be the multiplicative unit, let $H \subset G$ be the isotropy group $\{g \in G/g1 = 1\}$, so that the orbit $G \cdot 1$ is identifiable with G/H and consists of $[G:H] = [G/H:1]$ elements of $\tilde{K}$. If $g \cdot 1 = \lambda_g \in \tilde{K}$, action by g on V is given by multiplication by $\lambda_g \in \tilde{K}$. In particular, if e is the exponent of G/H, i.e. $z^e = 1$ for all $z \in G/H$, then $G \cdot 1$ consists of e-th roots of unity in $\tilde{K}$, but there are at most e e-th roots of unity. Thus the exponent and order of G/H are the same, and G/H is cyclic.

Then $\tilde{K}$ is a splitting field for $x^e - 1$ over K, i.e. $x^e - 1$ factors as $\Pi(x-\rho)$ where $\rho \in G \cdot 1$ and $\tilde{K}$ is generated over K by G and hence by the elements in $G \cdot 1$. Further, the polynomial $x^e - 1$ is separable over K for the roots $\rho \in G \cdot 1$ are distinct. Thus $\tilde{K}$ is a finite dimensional Galois extension of K and hence is a separable extension. In particular, $\tilde{K}$ has a non-singular symmetric bilinear form given by $\phi(x, y) = \mathrm{trace}_{\tilde{K}/K}(xy)$, the trace of the K-linear map given by multiplication by xy.

Now define an automorphism $\sigma : KG \to KG$ by

$$\sigma(\textstyle\sum \alpha_g g) = \sum \omega(g)\alpha_g g^{-1}$$

(an anti-automorphism if G is nonabelian), so that the KG module structure on the ω-dual of V is given by $(\lambda f)(x) = f(\sigma(\lambda)x)$ for $f \in \mathrm{Hom}\,(V, K)$.

CLAIM: If V is isomorphic to its ω-dual V^*, then $\sigma(I) = I$, where $I = \{\lambda \in KG | \lambda x = 0 \forall x \in V\}$. To see this, let $\psi : V \to V^*$ be an isomorphism of KG modules. Then for $v, v' \in V$, $\lambda \in KG$,

$$\psi(\lambda v)(v') = \{\lambda\psi(v)\}(v') = \psi(v)(\sigma(\lambda)v')$$

so if $\lambda \in I$, $\psi(v)(\sigma(\lambda)v') = 0$ for all v and so $\sigma(\lambda)v' = 0$ and $\sigma(\lambda) \in I$, while if $\sigma(\lambda) \in I$, $\psi(\lambda v)(v') = 0$ for all v' and so $\psi(\lambda v) = 0$ or $\lambda v = 0$ and so $\lambda \in I$.

Thus, if $V \cong V^*$, σ induces an automorphism $\sigma \cdot \tilde{K} \to \tilde{K}$.

CLAIM: The form $\theta(x, y) = \text{trace}_{\tilde{K}/K}(x \cdot \sigma(y))$ on $\tilde{K}$ is a symmetric non-singular ω-form on $\tilde{K}$. To see this,

$$\begin{aligned}\theta(y, x) = \text{trace}_{\tilde{K}/K}(y \cdot \sigma(x)) &= \text{trace}_{\tilde{K}/K}(\sigma(x \cdot \sigma(y)) \\ &= \text{trace}_{\tilde{K}/K}(x\sigma(y)) = \theta(x, y)\end{aligned}$$

and

$$\begin{aligned}\theta(gx, gy) = \text{trace}_{\tilde{K}/K}(gx\sigma(y)\omega(g)g^{-1}) &= \omega(g)\,\text{trace}_{\tilde{K}/K}(x\sigma(y)) \\ &= \omega(g)\theta(x, y)\end{aligned}$$

while $\{x | \theta(x, y) = 0 \text{ for all } y\}$ is a G invariant subspace of V and is proper since $\text{trace}_{\tilde{K}/K}(xy)$ is nonsingular, so is the zero subspace.

From this one has:

PROPOSITION 5.4: *If the Sylow 2 subgroup of G is abelian, then*

$$\chi_{\frac{1}{2}} : \Omega_{2n+1}(G, \omega) \to \tilde{R}_{GL,0}(G, \omega)$$

is the zero homomorphism.

PROOF: It suffices to verify this on the Sylow 2 subgroup, S. Then $R_K(S)$ is the free abelian group with base the irreducible representations, which one may list as $\{[V] | V \cong V^*\} = T_0$ and $\{[V] | V \not\cong V^*\} = T_1$. Divide T_1 into two disjoint classes T_+ and T_- so that if $[V] \in T_+$ then $[V^*] \in T_-$. By the above discussion, $[V] = 0$ in $R_{GL,0}(S, \omega|S)$ if $[V] \in T_0$, and thus $R_{GL,0}(S, \omega/S)$ is the free abelian group with base the classes $[V]$ with $[V] \in T_+$ (and $[V^*] = -[V]$). Since $(KG)^* = KG$, KG is zero in $R_{GL,0}(S, \omega|S)$, and so $\tilde{R}_{GL,0}(S, \omega|S) = R_{GL,0}(S, \omega|S)$ is torsion free. Since $\chi_{\frac{1}{2}}(\Omega_{2n+1}(S, \omega/S))$ consists of 2 torsion, it is the zero group. *

Note: To see that $(KG)^* = KG$, one need only consider the form $\theta(\sum \alpha_g g, \sum \beta_g g) = \sum \omega(g)\alpha_g \beta_{g^{-1}}$, which is an orthogonal form.

Now returning to an irreducible representation V of G with $V \cong V^*$, suppose there is an element $\zeta \in \tilde{K}$ with $\sigma(\zeta) = -\zeta$. Then

$$\tau(x, y) = \text{trace}_{\tilde{K}/K}(\zeta x \sigma(y))$$

is a nonsingular skew ω-form on V. To see this,

$$\tau(y, x) = \operatorname{trace}_{\tilde{K}/K}(\zeta y\sigma(x)) = \operatorname{trace}_{\tilde{K}/K}(\sigma(\zeta y\sigma(x))) = \operatorname{trace}_{\tilde{K}/K}(\sigma(\zeta)x\sigma(y)) = -\operatorname{trace}_{\tilde{K}/K}(\zeta x\sigma(y)) = -\tau(x, y)$$

and

$$\tau(gx, gy) = \operatorname{trace}_{\tilde{K}/K}(\zeta gx\sigma(y)\omega(g)g^{-1}) = \omega(g)\tau(x, y),$$

while $\{x|\tau(x, y) = 0\forall y\}$ is a proper G invariant subspace of V and so is zero.

Now $\sigma : \tilde{K} \to \tilde{K}$ is an involution, so decomposes $\tilde{K}$ into ± 1 eigenspaces. Thus if $\sigma(\zeta) = -\zeta$ has no solution, then $\sigma(\lambda) = \lambda$ for all λ. Applying this to $g \in G$, $gx = \omega(g)g^{-1}x$ for all $x \in V$ or $g^2x = \omega(g)x$, i.e. g^2 acts on V as multiplication by ω(g).

There are now several cases to consider.

First, suppose $\omega : G \to Z_2 = \{1, -1\}$ is the trivial homomorphism. Then supposing $V \cong V^*$ and that there is no element $\zeta \in \tilde{K}$ with $\sigma(\zeta) = -\zeta$, g^2 acts trivially on V for all G. Thus $H = \{g|g1 = 1\}$ is a subgroup of index 2 in G or G itself and there is a homomorphism $\phi : G \to Z_2$ with kernel H so that the representation V is the representation K_ϕ of G on K given by $gx = \phi(g)\cdot x$.

In order to analyze $\tilde{R}_{GL,Sp}(G, 1)$, divide the irreducible K representations into four classes, T_+ and T_- consisting of two disjoint collections of V with $V \ncong V^*$, so that if $V \in T_+$, $V^* \in T_-$, T_0 the collection of those $V \cong V^*$ for which there is a $\zeta \in \tilde{K}$ with $\sigma(\zeta) = -\zeta$, and Φ, the collection of K_ϕ with $\phi \in \operatorname{Hom}(G; Z_2)$. Then $R_K(G, 1)$ is free abelian with base $[V]$, with V in $\Phi \cup T_0 \cup T_+ \cup T_-$. Any representation W with a symplectic form decomposes into sums of irreducible summands corresponding to the different irreducibles and must pair nV against nV^*, V being irreducible. In particular, if $V \in T_+$, the number of copies of V and V^* in W is the same, and of course $V \oplus V^*$ has a hyperbolic form, and the number of copies of K_ϕ in W is even, for a nonsingular skew form on a K vector space must have even rank, while $K_\phi \oplus K_\phi$ has a hyperbolic form. Thus $R_{GL,Sp}(G, 1)$ is the direct sum of a free abelian group on $[V]$, $V \in T_+$ (with $[V^*] = -[V]$) and a Z_2 vector space with base the $[V]$, $V \in \Phi$.

Now turning to KG, $(KG)^* \cong KG$ so the number of occurrences of V and V^* in KG is the same. Further, K_ϕ is one-dimensional so absolutely irreducible and hence occurs exactly once in KG. Thus

$$[KG] = \sum [K_\phi] \in R_{GL,Sp}(G, 1)$$

and $\tilde{R}_{GL,Sp}(G, 1)$ is the direct sum of a free abelian group on the classes $[V]$ for $V \in T_+$ and a Z_2 vector space on the classes $[K_\phi]$ for $\Phi \in \operatorname{Hom}(G; Z_2)$ a *nontrivial* homomorphism. The class of $[K_1] = [K]$, the trivial representation is $\sum_{\phi \neq 1}[K_\phi]$.

Being given a manifold M^{2n+1} with free G action, the coefficient of $[K_\phi] \in \tilde{R}_{GL,Sp}(G, 1)$ is the sum of the dimensions of the subspaces of the $H^i(M, K)$ on which G acts trivially (the number of copies of K_1) and as multiplication via ϕ (the number of copies of K_ϕ), which is the dimension of the subspace on which the kernel of ϕ acts trivially. However, the projection $\pi : M \to M/\ker \phi$ onto the orbit space of the action of the kernel of ϕ induces an isomorphism of $H^i(M/\ker \phi; K)$ onto the elements of $H^i(M; K)$ invariant under $\ker \phi$. Thus one has:

Proposition 5.5: *If G is abelian and K is a field of characteristic zero or prime to the order of G, then the 2-torsion subgroup of $\tilde{R}_{GL,Sp}(G, 1)$ is a Z_2 vector space with a base $\{[K_\phi]\}$ where ϕ is a nontrivial homomorphism of G to Z_2. The homomorphism*

$$\chi_{\frac{1}{2}} : \Omega_{2n+1}(G, 1) \to \tilde{R}_{GL,Sp}(G, 1)$$

sends the class of M^{2n+1} into

$$\sum_{\phi} s\chi(M/\ker \phi) \cdot [K_\phi].$$

Notes:

(1) This applies via 5.1 to any G with abelian Sylow 2 subgroup. However, the $s\chi(M/\ker \phi)$ may satisfy dependence relations for the action of the normalizer of S may carry ϕ into some other homomorphism. When G is abelian, $i_*[K_{\phi/S}] = [K_\phi]$, and the result looks nicer.

(2) This shows that Lee's impressions were incorrect; one can obtain nontrivial invariants from these semicharacteristics. Taking G to be $Z_2 \times Z_2$, the unoriented invariants were trivial, but these are not. In particular, if M is a manifold with involution t and $\tilde{M}$ is its extension to $Z_2 \times Z_2$, then $s\chi(\tilde{M}/\ker \phi) = s\chi(M)$ if $\phi(t) \neq 1$, while

$$s\chi(\tilde{M}/\ker \phi) = s\chi(2(M/Z_2)) = 0$$

if $\phi(t) = 1$.

(3) This result should be compared with 5.2 for $G = Z_{2^s}$, for the two results give $s\chi(M) \cdot [K]$ and $s\chi(M/Z_{2^{s-1}})[K_\phi]$ where

$$\phi : Z_{2^s} \to Z_{2^s}/Z_{2^{s-1}} \cong Z_2$$

is the unique non-trivial homomorphism. Since $[K] = [K_\phi]$, this simply asserts equality of the semicharacteristics. One may obtain this equality using either approach.

From a cobordism point of view M may be written as a sum of terms $N^{2j} \times (S^{2k+1}, \theta)$ with N oriented and $2j+2k = 2n$, n odd and $\tilde{M}$ where $\tilde{M}$ is an extension from $Z_{2^{s-1}}$ (in fact from Z_2). Now the semicharacteristic of $\tilde{M}$ is trivial, and $\tilde{M}/Z_{2^{s-1}}$ is two copies of the same manifold so has trivial semicharacteristic. Now $s\chi(N \times S^{2k+1}) = \chi(N) \cdot s\chi(S^{2k+1})$ vanishes if j is odd (for an oriented manifold has $\chi(N) \equiv \text{Index}\,(N)$ which vanishes if j is odd) and similarly $s\chi(N \times (S^{2k+1}/Z_{2^{s-1}}))$ vanishes. Thus it suffices to show $s\chi(S^{2k+1}/Z_{2^{s-1}}) = 1$ if k is odd, but this is trivial.

One may also give a purely representation theoretic proof of the result, computing $s\chi(M)$ and $s\chi(M/Z_{2^{s-1}})$ over any field K of characteristic not 2. From Lee's result ([5], Lemma 2.4), $\chi_{\frac{1}{2}}(M; K) \cong \chi_{\frac{1}{2}}(M, K)^*$ in $\tilde{R}_K(Z_2)$ and $(KZ_{2^s})^* = KZ_{2^s}$, so writing $\chi_{\frac{1}{2}}(M; K)$ in $R_K(Z_{2^s})$ as

$$nK_1 + mK_\phi + p_v V + \sum (q_{v'} V' + r_{v'} V'^*)$$

with $V \in T_0$, $V' \in T_+$, $q_{v'} = r_{v'} \bmod 2$, giving $s\chi(M) = n + m + \sum p_v \dim V$. On the other hand $s\chi(M/Z_{2^{s-1}}) = n + m$ and so it suffices to show that $\dim V$ is even for all $V \in T_0$; i.e. that every self dual irreducible representation of Z_{2^s} other than K and K_ϕ is even dimensional. (*Note*: If $s = 1$, K and K_ϕ are the only irreducibles, so there is nothing to prove. Thus one may suppose $s > 1$.)

First, if $x^{2^{s-1}} = -1$ is solvable in K, then every irreducible representation has the form K_β and is given by K with the generator of Z_{2^s} acting as multiplication by β where $\beta^{2^s} = 1$. Since $(K_\beta)^* = K_{\beta^{-1}}$, K_β is self dual only if $\beta = \beta^{-1}$ or $\beta^2 = 1$. Thus only K_1 and K_ϕ are self dual.

Thus, one may suppose $x^{2^{r-1}} = -1$ is solvable in K but $x^{2^r} = -1$ is not, where $1 \leqq r < s$. The irreducible representations of K are then of the form K_β, $\beta^{2^r} = 1$, or have a base $x, tx, t^2x, \cdots, t^{2^p - 1}x$ with $t^{2^p}x = \theta x$ where $\theta^{2^{r-1}} = -1$, $\theta \in K$, and $p + r \leqq s$, $p \geqq 1$. The dual of the latter may be similarly described but corresponds to θ^{-1}, so is self dual only if $\theta = \theta^{-1}$ or $\theta^2 = 1$ and $r = 1$. Similarly, $(K_\beta)^* = K_{\beta^{-1}}$ and K_β is self dual only if $\beta^2 = 1$. Thus $r = 1$ or the only self duals are K_1 and K_ϕ.

Assuming $r = 1$, the irreducibles are K_1, K_ϕ or of the form with a base $x, tx, \cdots, t^{2^p - 1}x$ with $t^{2^p}x = -x$ and with $1 \leqq p < s$. In this case, all are self dual, but only K_1 and K_ϕ have odd dimension. $*$

The referee observes that $S\chi$ is invariant under field extension, and by [6], is independent of the characteristic for manifolds of dimension $4k+1$. Thus, one may compute over the reals. Considering the representation of Z_{2^s} on $H^i(M; R)$ and splitting into irreducible representations, $H^i(M/Z_{2^{s-1}}; R)$ is clearly isomorphic to the sum of the representation spaces where the generator acts as multiplication by ± 1. The remaining components are all two dimensional.

Now returning to the general situation, consider the case with $\omega : G \to Z_2$ nontrivial, with $V \cong V^*$ and $\tilde{K}$ containing no element ζ with $\sigma(\zeta) = -\zeta$, so that $g^2x = \omega(g)x$ for all g in G. In particular, $g^4x = x$ and for some g, $g^2x = -x$. Letting $H = \{g | g1 = 1\}$, it follows that G/H

is cyclic of order 4, and that V is given by a representation of $G/H = Z_4$ for which the subgroup Z_2 acts as multiplication by -1.

The first obvious case is when there is no homomorphism $\theta : G \to Z_4$ for which $\theta(g^2) = \omega(g) \in Z_2$. Noting that the epimorphism $\pi : Z_4 \to Z_2$ is given by $\pi(x) = x^2$ (considering $Z_2 \subset Z_4$ as the squares), this is the case in which $\omega : G \to Z_2$ cannot be written in the form $\pi \circ \phi$ with $\phi : G \to Z_4$. Then every self dual representation is symplectic and letting the set of irreducible representations of G be decomposed into T_0,T_+ and T_-, $\tilde{R}_{GL,Sp}(G, \omega)$ is free abelian on the classes $[V]$ with V in T_+, and so $\chi_{\frac{1}{2}}$ is zero.

If there is an element $t \in G$ of order 2 with $\omega(t) \neq 1$, there can be no homomorphism $\phi : G \to Z_4$ with $\pi \circ \phi = \omega$. The converse is also true; if there is no element $t \in G$ of order 2 with $\omega(t) \neq 1$, then there is a homomorphism $\phi : G \to Z_4$ with $\pi \circ \phi = \omega$. (To see this, write

$$G = Z_{2^{s_1}} \oplus \cdots \oplus Z_{2^{s_n}} \oplus Z_{r_1} \oplus \cdots \oplus Z_{r_j}$$

where r_i are odd. If t_i generates the summand $Z_{2^{s_i}}$, there is a t_i of minimal order for which $\omega(t_i) \neq 1$. If $\omega(t_j) \neq 1$ for some other t_j, t_j may be replaced by $t_j t_i$ giving a new generator for a summand on which ω is trivial. After iterating, ω factors through projection on the t_i summand.)

Suppose there is a homomorphism $\phi : G \to Z_4$ with $\pi \circ \phi = \omega$. The irreducible representations of Z_4 may be described as follows:

Case I: If the equation $x^2 = -1$ is solvable in K then every irreducible representation of Z_4 is of the form K_β with the generator of Z_4 acting on K as multiplication by β, where $\beta^4 = 1$. Those β with $\beta^2 = -1$ give representations with Z_2 acting as -1. K_β is its own π-dual. Choosing one specific $\beta \in K$ with $\beta^2 = -1$ as generator of Z_4, the nonsymplectic self dual irreducible representations of G are then in one-to-one correspondence with $\{\phi : G \to Z_4 | \pi \circ \phi = \omega\} = \Phi$ with G acting on K by $gx = \phi(g) \cdot x$. This will be denoted $K\langle\phi\rangle$. Now $R_K(G)$ is free abelian with a base given by the $K\langle\phi\rangle$, $\phi \in \Phi$, those $V \cong V^*$ not in Φ, called T_0, and T_+, T_- which decompose those $V \ncong V^*$. $R_{GL,Sp}(G, \omega)$ is the direct sum of the free abelian group on T_+ and the Z_2 vector space on Φ (a skew form on W makes W self dual so V and V^* occur with the same multiplicity: if $nK\langle\phi\rangle$ occurs in W $nK\langle\phi\rangle$ has a skew form so n is even). Each $K\langle\phi\rangle$ occurs once in KG, since $K\langle\phi\rangle$ is absolutely irreducible, and so $[KG] = \sum [K\langle\phi\rangle]$.

Note: Writing Z_4 additively, ϕ and θ taking G into Z_4 with $\pi \circ \phi = \pi \circ \theta = \omega$ differ by a homomorphism of G into Z_2 i.e. $\theta = \phi + \lambda$. Thus fixing one $\phi_0 : G \to Z_4$, $\phi \to \phi - \phi_0$ defines a one-to-one correspondence between Φ and Hom $(G; Z_2)$. Thus $\tilde{R}_{GL,Sp}(G, \omega)$ is the direct sum of the free abelian group on T_+ and the Z_2 vector space with base the $K\langle\phi_0+\lambda\rangle$ where $\lambda \in \operatorname{Hom}(G; Z_2)$ is nontrivial, and $[K\langle\phi_0\rangle] = \sum_\lambda [K\langle\phi_0+\lambda\rangle]$. Notice that $\phi_0+\lambda+\omega$ is the negative of $\phi_0+\lambda$.

Being given a manifold M^{2n+1}, n even, with a free G action and $\phi: G \to Z_4$ with $\pi \circ \phi = \omega$, $H^*(M/\ker \phi; K)$ may be identified with the elements of $H^*(M; K)$ invariant under $\ker \phi$, i.e. with the summands $K_1, K_\omega, K\langle\phi\rangle$ and $K\langle\phi+\omega\rangle$, while $H^*(M/\ker \omega; K)$ is identifiable with the summands K_1 and K_ω. Thus letting $n\langle\phi\rangle$ be the number of summands of $K\langle\phi\rangle$ in

$$\sum_0^n (-1)^i H^i(M; K), \quad n\langle\phi\rangle + n\langle\phi+\omega\rangle = s\chi(M/\ker \phi) - s\chi(M/\ker \omega).$$

Now $M/\ker \phi$ and $M/\ker \omega$ admit free orientation reversing Z_4 and Z_2 actions, so by 5.3 $n\langle\phi\rangle \equiv n\langle\phi+\omega\rangle$ in Z_2. Letting ϕ_0 be fixed as above, the coefficient of $[K\langle\phi_0+\omega\rangle]$ in $\chi_{\frac{1}{2}}(M; K)$ is $n\langle\phi_0\rangle + n\langle\phi_0+\omega\rangle = 0$, while for $\lambda \neq 1, \omega$, the coefficients of $[K\langle\phi_0+\lambda\rangle]$ and $[K\langle\phi_0+\lambda+\omega\rangle]$ are equal and are given by

$$\begin{aligned}\tfrac{1}{2}\{ &\sum_0^n (-1)^i \dim H^i(M/\ker \phi_0; K) - \sum_0^n (-1)^i \dim H^i(M/\ker \omega; K) \\ &+ \sum_0^n (-1)^i \dim H^i(M/\ker(\phi_0+\lambda); K) - \sum_0^n (-1)^i \dim (M/\ker \omega; K)\}.\end{aligned}$$

Letting

$$s\chi_K(M) = \sum_0^n (-1)^i \dim H^i(M; K)$$

in Z, this gives

$$\begin{aligned}\chi_{\frac{1}{2}}(M; K) \sum \tfrac{1}{2}(s\chi_K(M/\ker \phi_0) + s\chi_K(M/\ker(\phi_0+\lambda)) \\ \times \{[K\langle\phi_0+\lambda\rangle] + [K\langle\phi_0+\lambda+\omega\rangle]\}\end{aligned}$$

where the sum is over representatives λ for the pairs λ, $\lambda+\omega$, where $\lambda \neq 1, \omega$.

Case II: If the equation $x^2 = -1$ is not solvable in K, then every irreducible representation of Z_4 is one of the forms K_1, K_{-1} or V where V is the 2 dimensional K representation given by $t(x, y) = (-y, x)$ (Note: If $c(x, y) = (x, -y)$, $tc = -ct$, so this is equivalent to the representation with the generator of Z_4 acting as $-t$). Thus, for each pair of homomorphisms ϕ and $\phi+\omega$ sending G to Z_4 and lifting ω there is an irreducible 2 dimensional representation, $V\langle\phi, \phi+\omega\rangle$. Decomposing the non-self duals into T_+ and T_- and letting $\Phi = \{\phi: G \to Z_4 | \pi \circ \phi = \omega\}$,

$R_{GL,Sp}(G, \omega)$ is the direct sum of the free abelian group on T_+ and a Z_2 vector space with base the $V\langle\phi, \phi+\omega\rangle$ for the pairs $\{\phi, \phi+\omega\}$ of elements of Φ. (*Note*: If $nV\langle\phi, \phi+\omega\rangle$ admits a symplectic form, then extending K to a splitting field K' for x^2+1, $nK'_\phi+nK'_{\phi+\omega}$ has a symplectic form, so n is even.) Now KG has each $V\langle\phi, \phi+\omega\rangle$ appearing exactly once (extending to K', K'_ϕ and $K'_{\phi+\omega}$ appear exactly once in $K'G$) so $\tilde{R}_{GL,Sp}(G, \omega)$ is the direct sum of a free abelian group on T_+ and a Z_2 vector space with base the $[V\langle\phi_0+\lambda, \phi_0+\lambda+\omega\rangle]$. $\lambda \neq 1, \omega$, and with

$$[V\langle\phi_0, \phi_0+\omega\rangle] = \sum_\lambda [V\langle\phi_0+\lambda, \phi_0+\lambda+\omega\rangle].$$

Since the number of copies of $V\langle\phi, \phi+\omega\rangle$ in $\sum(-1)^i H^i(M; K)$ is $\frac{1}{2}(s\chi_K(M/\ker\phi) - s\chi_K(M/\ker\omega))$, one has

$$\chi_{\frac{1}{2}}(M, K) = \sum \{\tfrac{1}{2}(s\chi_K(M/\ker\phi_0) + s\chi_K(M)/\ker(\phi_0+\lambda)))\} \\ [V\langle\phi_0+\lambda, \phi_0+\lambda+\omega\rangle].$$

This completes the list of cases, with a full understanding of each of the $\tilde{R}_{GL,Sp}(G, \omega)$, but with several cases. One may obtain a clean result:

PROPOSITION 5.6: *If G is abelian and K is a field of characteristic zero or prime to the order of G and $\omega : G \to Z_2$ is a nontrivial homomorphism then $\chi_{\frac{1}{2}}(M^{2n+1}, K) \in \tilde{R}_{GL,Sp}(G, \omega)$ is determined by the numbers*

$$\tfrac{1}{2}\{s\chi_K(M/\ker\phi) + s\chi_K(M/\ker\phi')\} \in Z_2$$

where

$$s\chi_K(M^{2n+1}) = \sum_0^n (-1)^i \dim H^i(M; K) \in Z$$

and where $\phi, \phi' : G \to Z_4$ are liftings of ω.

COROLLARY 5.7: *If the Sylow 2 subgroup of G is either $Z_2 \times \cdots \times Z_2$ or cyclic, and if $\omega : G \to Z_2$ is nontrivial, then*

$$\chi_{\frac{1}{2}} : \Omega_{2n+1}(G, \omega) \to \tilde{R}_{GL,Sp}(G, \omega)$$

is zero.

Notes:

(1) $\chi_{\frac{1}{2}}$ can be nontrivial. Let $G = Z_4 \times Z_2$ generated by t, s with $t^4 = s^2 = 1$, $ts = st$. Let $\omega(t) = -1$, $\omega(s) = 1$. If M_0^{2n+1} is a manifold with free involution s', consider $Z_4 \times M_0$ with $t(x, y) = (tx, y)$ and $s(x, y) = (x, s'y)$ and the obvious ω orientation; i.e. the extension from Z_2 to G of M_0. There are two classes of liftings of ω, ϕ_0 with kernel

$\{s\}$ and ϕ_1 with kernel $\{st^2\}$. One has $M/\ker\phi_0 \cong Z_4 \times (M_0/Z_2)$ and $M/\ker\phi_1 \cong 2$ copies of M, so $\frac{1}{2}\{s\chi_K(M/\ker\phi_0) + s\chi_K(M/\ker\phi_1)\}$ is $2s\chi(M_0/Z_2) + s\chi(M_0) \equiv s\chi(M_0)$.

(2) It would be nice to know if the expression

$$\tfrac{1}{2}\{s\chi_K(M/\ker\phi) + s\chi_K(M/\ker\phi')\}$$

is independent of K. This is in fact true. First consider $\omega: G \to Z_2$ and two liftings $\phi, \phi': G \to Z_4$. Let $H = \ker\phi \cap \ker\phi'$, and then G/H acts on M/H and is a free action of $Z_4 \times Z_2$ of the sort in Note 1 above. Thus one need only check this on $Z_4 \times Z_2$ actions.

First, one needs to compute $\Omega_*(Z_4 \times Z_2, \omega)$. If $\rho: BZ_2 \to BZ_4$, $\Omega_*(Z_4 \times Z_2, \omega) \cong \Omega_{*+1}(D(\rho) \times BZ_2, S(\rho) \times BZ_2)$ where D, S denote disc and sphere of the line bundle of ρ. The homomorphism given by inclusion of $(D(\rho) \times pt, S(\rho) \times pt)$ may be identified with the extension from $\Omega_*(Z_4, \pi)$, and the complementary summand is identifiable with

$$\begin{aligned} \tilde{\Omega}_{*+1}(M(\rho) \wedge BZ_2) &= \lim \pi_{*+r+1}(M(\rho) \wedge BZ_2 \wedge MSO(r)) \\ &= \lim \pi_{*-r+1}(M(\rho) \wedge MO(r+1)) \\ &= \tilde{\mathfrak{N}}_*(M(\rho)) \\ &\cong \mathfrak{N}_{*-1}(BZ_4) \end{aligned}$$

where the homomorphism to $\tilde{\mathfrak{N}}_*(M(\rho))$ is obtained by dualizing the line bundle given by the map into BZ_2 and the last is the Thom isomorphism.

Now $\mathfrak{N}_*(BZ_4)$ is generated as $\mathfrak{N}_*$ module by the spheres (S^{2n+1}, i) and by the extensions from Z_2 of (S^{2n}, a) which will be denoted $2S^{2n}$, $t(x, 0) = (x, 1)$, $t(x, 1) = (-x, 0)$ giving the action. Now let M be a closed manifold, not necessarily orientable and consider $S(\det\tau \oplus 1) \times S^{2n+1}$ or $S(\det\tau \oplus 1) \times 2S^{2n}$, where $\det\tau$ is the determinant of the tangent bundle of M. Let s act as the antipodal map in the fibers of $S(\det\tau \oplus 1)$ and let t act diagonally, by multiplication by -1 in the fibers of $\det\tau$, 1 in those of the trivial bundle and with the given action on S^{2n+1} or $2S^{2n}$. The double cover of the action of $Z_2 = \{s\}$ has base $RP(\det\tau \oplus 1) \times X$ and dualizing this line bundle gives $RP(\det\tau) \times X$; i.e. $M \times X$ and in $\mathfrak{N}_{*-1}(BZ_4)$ this gives the class $M \times (S^{2n+1}, i)$ or $M \times (2S^{2n}, t)$. Thus these classes in $\Omega_*(Z_4 \times Z_2, \omega)$ are generators modulo extensions from (Z_4, π).

For $S(\det\tau \oplus 1) \times S^{2n+1} = N$, the cohomology of $N/\ker\phi_0$ and $N/\ker\phi_1$ are identifiable with the elements in $H^*(N; K)$ invariant under s and st^2, but t^2 is trivial on cohomology, so these quotients have the same K cohomology. Thus

$$\tfrac{1}{2}\{s\chi_K(N/\ker\phi_0) + s\chi_K(N/\ker\phi_1)\} = s\chi_K(N/\ker\phi_0)$$

which is even; i.e. $\chi_{\frac{1}{2}}(N, K)$ is zero.

For $S(\det \tau \oplus 1) \times 2S^{2n} = N$, s and st^2 act preserving the components of N. Thus $N/\ker \phi_0$ consists of 2 copies of $RP(\det \tau \oplus 1) \times S^{2n}$ and $N/\ker \phi_1$ consists of 2 copies of $S((\det \tau \oplus 1) \otimes \gamma)$ over $M \times RP(2n)$, where γ is the nontrivial line bundle over $RP(2n)$. Thus

$$\tfrac{1}{2}\{s\chi_K(N/\ker \phi_0) + s\chi_K(N/\ker \phi_1)\}$$

is

$$s\chi_K(RP(\det \tau \oplus 1) \times S^{2n}) + s\chi_K(S((\det \tau \oplus 1) \otimes \gamma).$$

These bound $RP(\det \tau \oplus 1) \times D^{2n+1}$ and $D((\det \tau \oplus 1) \otimes \gamma)$ unorientedly and so the semicharacteristics are independent of K.

For an extension, let M_0 have a free Z_4 action and let $M = M_0 \times Z_2$ with $t(x, y) = (tx, y)$, $s(x, y) = (x, -y)$ which gives the extension. Then $M/\ker \phi_0$ and $M/\ker \phi_1$ may each be identified with M_0 for s and st^2 interchange components. Thus

$$\tfrac{1}{2}\{s\chi_K(M/\ker \phi_0) + s\chi_K(M/\ker \phi_1)\} = s\chi_K(M_0)$$

which is even since M_0 has an orientation reversing Z_4 action.

Since the invariants $\frac{1}{2}\{s\chi_K(M/\ker \phi_0) + s\chi_K(M/\ker \phi_1)\}$ are cobordism invariants and agree on a base of $\Omega_*(Z_4 \times Z_2, \omega)$ they agree. Thus the value is independent of K.

Beware: The independence of K assumed throughout that the characteristic of K is *not* 2. The expression

$$\tfrac{1}{2}\{s\chi_{Z_2}(M/\ker \phi_0) + s\chi_{Z_2}(M/\ker \phi_1)\}$$

is not a cobordism invariant, as one may verify by considering $S(\det \tau \oplus 1) \times S^1 = M$ for the bundle over $S^6 \times S^7 \times RP(2)$; the invariant is 1, but the manifold bounds – bounding $S(\det \tau \oplus 1) \times S^1$ for the bundle over $D^7 \times S^7 \times RP(2)$.

To compute the invariant, $M/\{s\} = RP(\det \tau \oplus 1) \times S^1$ has mod 2 cohomology a free module over that of $S^6 \times S^7 \times RP(2) \times S^1$ on a 1-dimensional class. Thus, $\dim H^i(M/\{s\}; Z_2)$ is given by 1, 3, 4, 3, 1, 0, 1, 4, 7 in dimensions 0 through 8 and $s\chi_{Z_2}(M/\{s\}) = 4$. For $M/\{st^2\}$, one has $S^6 \times S^7 \times S((\det \tau \oplus 1) \oplus \gamma)$ where the sphere bundle is over $RP(2) \times RP(1)$. In the spectral sequence for the sphere bundle the fiber class transgresses to $\alpha \cdot \sigma$ (the product of the generators, so $\dim H^i(S((\det \tau \oplus 1) \otimes \gamma); Z_2)$ is 1, 2, 2, 2, 1 in dimensions 0 through 4, and $\dim H^i(M/\{st^2\}; Z_2)$ is 1, 2, 2, 2, 1, 0, 1, 3, 4 so $s\chi_{Z_2}(M/\{st^2\}) = 2$.

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