COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 29, nº 3 (1974), p. 213-222
<http://www.numdam.org/item?id=CM_1974__29_3_213_0>

COMPOSITIO MATHEMATICA, Vol. 29, Fasc. 3, 1974, pag. 213–222
Noordhoff International Publishing
Printed in the Netherlands

REPRESENTABLE EPIMORPHISMS OF MONOIDS[1]

Matthew Gould

Introduction

Given a universal algebra $\mathfrak{A}$, its endomorphism monoid $E(\mathfrak{A})$ induces a monoid $E^*(\mathfrak{A})$ of mappings of the subalgebra lattice $S(\mathfrak{A})$ into itself. Specifically, for $\alpha \in E(\mathfrak{A})$ define $\alpha^* : S(\mathfrak{A}) \to S(\mathfrak{A})$ by setting $X\alpha^* = \{x\alpha | x \in X\}$ for all $X \in S(\mathfrak{A})$. (Note that α^* is determined by its action on the singleton-generated subalgebras.) The monoid of *closure endomorphisms* (cf. [1], [3], [4]) is then defined as $E^*(\mathfrak{A}) = \{\alpha^* | \alpha \in E(\mathfrak{A})\}$. Clearly $E^*(\mathfrak{A})$ is an epimorphic image of $E(\mathfrak{A})$ under the map $\alpha \mapsto \alpha^*$; this epimorphism shall be denoted $\varepsilon(\mathfrak{A})$, or simply ε when there is no risk of confusion.

Given an epimorphism of monoids, $\Psi : M \to M\Psi$, let us say that Ψ is *representable* if there exist an algebra $\mathfrak{A}$ and isomorphisms $\sigma : M \to E(\mathfrak{A})$ and $\tau : M\Psi \to E^*(\mathfrak{A})$ such that $\Psi\tau = \sigma\varepsilon$. We shall characterize all representable epimorphisms defined on a given monoid M. The representability of an epimorphism Ψ will be shown to depend on the location of its kernel in the congruence lattice of M. More specifically, we shall define on M a congruence ρ having the property that the representable epimorphisms on M are precisely those Ψ for which $\ker \Psi \subseteq \rho$.

Notation and terminology of universal algebra used here will generally conform to Grätzer [5]. Exceptions will be the terms epimorphism and monomorphism for onto and one-to-one homomorphism respectively, and the notation $\ker \Psi$ for the congruence $\{\langle m, n\rangle \in M^2 | m\Psi = n\Psi\}$ where Ψ is an epimorphism on M. Also, we adopt the convention that for any set A, any map $\sigma : M \to A^A$, and any element $m \in M$, the symbol σ_m will be used (in place of the customary $m\sigma$) to denote the image of m under σ. Additional terminology relating to monoids (semigroups with identity) will conform to Clifford and Preston [2].

[1] This work was partially supported by a grant from the Research Council of Vanderbilt University. Subsequent revision was in part supported by a grant from the National Research Council of Canada.

1. A preliminary characterization

Before defining the relation ρ we note that representability of an epimorphism is more easily expressed in terms of its kernel.

LEMMA 1.1: *An epimorphism Ψ on a monoid M is representable if and only if there exist an algebra $\mathfrak{A}$ and an isomorphism $\sigma : M \to E(\mathfrak{A})$ such that ker Ψ = ker $\sigma\varepsilon$.*

PROOF: The reverse implication being obvious, let $\mathfrak{A}$ and σ be given as indicated, and define $\tau : M\Psi \to E^*(\mathfrak{A})$ by setting $(m\Psi)\tau = \sigma_m^*$ for all $m \in M$. The inclusion ker $\Psi \subseteq$ ker $\sigma\varepsilon$ guarantees that τ is well-defined, while the reverse inclusion insures that τ is one-to-one. It is readily verified that τ is an epimorphism, and clearly $\Psi\tau = \sigma\varepsilon$, whence Ψ is representable.

2. Statement of main result and proof of necessity

Define on the monoid M a binary relation ρ^0 to consist of those pairs $\langle m, n\rangle \in M^2$ having the property that

$$\{\langle u, v\rangle \in M^2 | mu = mv\} = \{\langle u, v\rangle \in M^2 | nu = nv\}.$$

It is easily seen that ρ^0 is a right-congruence but generally not a congruence. Thus we define ρ as the largest congruence contained in ρ^0. The existence of ρ follows from the fact that the congruences on M constitute a complete sublattice of the lattice of all equivalence relations on M. However, it is easily checked that ρ can be given explicitly as the set of those pairs $\langle m, n\rangle \in M^2$ such that $\langle tm, tn\rangle \in \rho^0$ for all $t \in M$.

Since the kernel of an epimorphism is a congruence, it is immediate from the definition of ρ that the statements (ii) and (iii) of the following theorem are equivalent. The equivalence of (i) and (ii) constitutes the main result of this paper.

THEOREM 2.1: *For an epimorphism Ψ defined on a monoid M the following assertions are equivalent.*

(i) *Ψ is representable.*
(ii) *ker $\Psi \subseteq \rho$.*
(iii) *ker $\Psi \subseteq \rho^0$.*

PROOF that (i) implies (iii): Given an algebra $\mathfrak{A} = \langle A; F\rangle$ and an isomorphism $\sigma : M \to E(\mathfrak{A})$ with ker Ψ = ker $\sigma\varepsilon$, let $\langle m, n\rangle \in$ ker Ψ. Then $\sigma_m^* = \sigma_n^*$, and so for each $x \in A$ we have $[x]\sigma_m^* = [x]\sigma_n^*$ (where $[x]$ denotes the subalgebra generated by $\{x\}$), whence there is a unary polynomial p_x such that $x\sigma_n = p_x(x\sigma_m)$.

Now let $\langle u, v\rangle \in M^2$ such that $mu = mv$. Then $x\sigma_m\sigma_u = x\sigma_m\sigma_v$ for all $x \in A$ and an application of the polynomial p_x yields $x\sigma_n\sigma_u = x\sigma_n\sigma_v$. Since this holds for all x, we have $\sigma_n\sigma_u = \sigma_n\sigma_v$ whereupon $nu = nv$. Likewise $nu = nv$ implies $mu = mv$, so $\langle m, n\rangle \in \rho^0$ and (iii) is established.

The next three sections will be devoted to proving that (ii) implies (i).

3. Preliminaries to proof of sufficiency

We shall need to know that when only finitely generated algebras are considered an apparently weaker form of the condition in Lemma 1.1 is still equivalent to representability. The desired result is Lemma 3.2 below. It is proved with the aid of a lemma whose proof is not given here because it is virtually contained in the proof of the Lemma of [3].

LEMMA 3.1: *An epimorphism Ψ on a monoid M is representable by means of a finitely generated algebra if and only if there exist a finitely generated algebra $\mathfrak{A}$ and a monomorphism $\sigma : M \to E(\mathfrak{A})$ such that $\ker \Psi = \ker \sigma\varepsilon$.*

LEMMA 3.2: *An epimorphism Ψ on a monoid M is representable by means of a finitely generated algebra if and only if there exist a finitely generated algebra $\mathfrak{A}$ and a monomorphism $\sigma : M \to E(\mathfrak{A})$ such that $\ker \Psi \subseteq \ker \sigma\varepsilon$.*

PROOF: The reverse implication being obvious, let $\mathfrak{A} = \langle A; F\rangle$ and σ be given as indicated. Since the nullary operations of an algebra can always be replaced by constant unary operations without altering the endomorphisms or the non-void subalgebras, we may assume that all operations of $\mathfrak{A}$ have positive rank.

Let 0 be any object not an element of $M\Psi$ and set $R = M\Psi \cup \{0\}$. View R as a monoid containing $M\Psi$ as a submonoid, and such that the binary operation in $M\Psi$ has been extended to all of R in the obvious way: $r0 = 0r = 0$ for all $r \in R$.

Set $B = A \times R$ and observe the following notational conventions.

(1) A pair in B will be denoted x_r instead of $\langle x, r\rangle$.

(2) For all $C \subseteq A$ the notation C_r will be used in place of $C \times \{r\}$.

(3) For $m \in M$ the element $m\Psi$ will be denoted m'.

(4) For $x \in A$ the symbol $[x]$ will denote the subalgebra of $\mathfrak{A}$ generated by $\{x\}$, while for $x_r \in B$ the subalgebra generated by $\{x_r\}$ in the algebra $\mathfrak{B}$ (defined below) will be denoted $\|x_r\|$.

(5) $\varepsilon(\mathfrak{A})$ will be denoted ε, while $\varepsilon(\mathfrak{B})$ will be denoted ε_1.

We construct on B the algebra $\mathfrak{B}$ by defining operations as follows.

For each $f \in F$, define on B a k-ary operation $\bar{f}$, where k is the rank of f, by setting

$$\bar{f}(x^1_{r_1}, \cdots, x^k_{r_k}) = f(x^1, \cdots, x^k)_{r_1} \qquad \text{for all } x^1, \cdots, x^k \in A$$

and $r_1, \cdots r_k \in R$.

Also, define for each $t \in R$ a binary operation g_t by:

$$g_t(x_r, y_s) = \begin{cases} x_r & \text{if } s \in M\Psi \\ x_{tr} & \text{if } s = 0, \end{cases}$$

for all $x, y \in A$ and $r, s \in R$. Set $\mathfrak{B} = \langle B; \{\bar{f} | f \in F\} \cup \{g_t | t \in R\}\rangle$.

It is routine to verify that $\|x_r\| = [x]_r$ for all $x_r \in B$.

Now define a map $\beta : M \to B^B$ by stipulating that $x_r \beta_m = (x\sigma_m)_{rm'}$ for all $m \in M$ and $x_r \in B$. It is straightforward to verify that β is a monomorphism of M into $E(\mathfrak{B})$.

To apply Lemma 3.1 it remains to show that ker $\beta\varepsilon_1$ = ker Ψ and $\mathfrak{B}$ is finitely generated. We use the fact, noted earlier, that closure endomorphisms are determined by their action on singleton-generated subalgebras.

Now, $\langle m, n\rangle \in \ker \beta\varepsilon_1$ means $\|x_r\|\beta^*_m = \|x_r\|\beta^*_n$ for all $x_r \in B$: equivalently, $([x]\sigma^*_m)_{rm'} = ([x]\sigma^*_n)_{rn'}$ for all $x \in A$ and $r \in R$, which holds if and only if $[x]\sigma^*_m = [x]\sigma^*_n$ for all $x \in A$ and $rm' = rn'$ for all $r \in R$. But the last statement says precisely that $\sigma^*_m = \sigma^*_n$ and $m' = n'$, i.e.,

$$\langle m, n\rangle \in \ker \sigma\varepsilon \cap \ker \Psi = \ker \Psi.$$

Thus ker $\beta\varepsilon_1$ = ker Ψ.

To see that $\mathfrak{B}$ is finitely generated, let W be a finite generating set for $\mathfrak{A}$ and set $\bar{W} = W \times \{e', 0\}$ where e is the identity element of M. Clearly $\mathfrak{B}$ is generated by the set $W \times R$, which in turn can be generated from $\bar{W}$ because $w_t = g_t(w_{e'}, w_0)$ for all $w_t \in W \times R$. Thus $\bar{W}$ is a finite generating set for $\mathfrak{B}$, and the lemma is proved by appeal to the preceding lemma.

4. Intermediate construction for proof of sufficiency

Let M be a fixed monoid and let ρ be the congruence defined in Section 3; i.e., ρ is the set of all pairs $\langle m, n\rangle \in M^2$ such that

$$\{\langle t, u, v\rangle \in M^3 | tmu = tmv\} = \{\langle t, u, v\rangle \in M^3 | tnu = tnv\}.$$

We define a *ρ-system* to be a pair $\langle \mathfrak{A}, \sigma\rangle$, where $\mathfrak{A}$ is a finitely generated multi-unary partial algebra, σ is a monomorphism of M into $E(\mathfrak{A})$, and: (*) For all $x \in A$, $\langle m, n\rangle \in \rho$, $\langle u, v\rangle \in M^2$, and $p \in P_1(\mathfrak{A})$, $p(x\sigma_{mu}) = p(x\sigma_{mv})$ implies $p(x\sigma_{nu}) = p(x\sigma_{nv})$, where $P_1(\mathfrak{A})$ denotes the set of all unary polynomials of $\mathfrak{A}$, and the implication is to be taken in the weakest possible sense: whenever all four terms are defined and the first two are equal, then the second two are equal.

We first show that there exists a ρ-system and then that any ρ-system $\langle \mathfrak{A}_0, \sigma^0 \rangle$ can be built up into a pair $\langle \mathfrak{A}, \sigma \rangle$ where $\mathfrak{A}$ is an algebra and σ is a monomorphism of M into $E(\mathfrak{A})$ such that $\rho \subseteq \ker \sigma\varepsilon$. By Lemma 3.2 this will complete the proof of Theorem 2.1.

The process by which $\langle \mathfrak{A}, \sigma \rangle$ is constructed from $\langle \mathfrak{A}_0, \sigma^0 \rangle$ involves a modification of the technique, discussed in Chapters 2 and 4 of [5], of freely extending a partial algebra to an algebra. Roughly speaking, we form the union $\mathfrak{A} = \langle A; F \rangle$ of an infinite sequence of successive partial-algebra extensions constructed in such a way that each extension remedies the defects of the former while introducing new defects that must be remedied by the next extension.

To ensure that $\rho \subseteq \ker \sigma\varepsilon$ we will need a kind of 'transitivity' for the unary polynomials of $\mathfrak{A}$; i.e., we will need to have enough unary polynomials so that whenever $\langle m, n \rangle \in \rho$ and $x \in A$, there is a unary polynomial taking $x\sigma_m$ to $x\sigma_n$. The condition (*) enables us to define the needed polynomials as partial operations at each step, and these partial operations become fully defined in the union. Thus it is necessary for each successive partial-algebra extension to perform four tasks: (1) to extend the domain of each previously defined partial operation to include all of the previous set; (2) to extend to the new elements introduced in (1) those endomorphisms that correspond to members of M under the given monomorphism; (3) to introduce new partial operations so that the 'transitivity' is maintained with regard to the extended endomorphisms; and (4) to preserve (*) so that the process may be repeated.

LEMMA 4.1: *There exists a ρ-system.*

PROOF: For each $t \in M$ define a unary operation f_t on M by: $f_t(x) = tx$ for all $x \in M$. Set $F = \{f_t | t \in M\}$ and $\mathfrak{A} = \langle M, F \rangle$. (It will later be of incidental interest to note that F includes the identity function on M.)

Define $\sigma : M \to M^M$ by stipulating that $x\sigma_m = xm$ for all $m, x \in M$. It is easy to verify (and also quite well known: see, e.g., Theorem 1.12.3 of [5]) that σ maps M into $E(\mathfrak{A})$ and is a monomorphism (in fact, an isomorphism).

Clearly $\mathfrak{A}$ is generated by the identity element of M, so to show that $\langle \mathfrak{A}, \sigma \rangle$ is a ρ-system it remains only to verify (*). Obviously $P_1(\mathfrak{A}) = F$, so take $f_t \in F$, $\langle m, n \rangle \in \rho$, $\langle u, v \rangle \in M^2$, and $x \in M$. Then $f_t(x\sigma_{mu}) = f_t(x\sigma_{mv})$ implies $txmu = txmv$, which by the nature of ρ implies $txnu = txnv$, whence $f_t(x\sigma_{nu}) = f_t(x\sigma_{nv})$ and (*) has been verified.

In the following lemma the domain of a partial operation or polynomial f will be denoted $D(f)$, and 1_X will denote the identity function on a set X.

Lemma 4.2: *If* $\langle\mathfrak{A}, \sigma\rangle$ *is a* ρ*-system then so is* $\langle\bar{\mathfrak{A}}, \bar{\sigma}\rangle$, *where* $\bar{\mathfrak{A}}$ *and* $\bar{\sigma}$ *are defined as follows.*

For each $f \in F$ *and each* $a \in A \backslash D(f)$, *choose a symbol* $s(f, a)$ *in such a way that* $s(f, a) = s(g, b)$ *implies* $\langle f, a\rangle = \langle g, b\rangle$, *and the set* S *of these symbols is disjoint from* A.

Set $\bar{A} = A \cup S$ *and define partial unary operations on* $\bar{A}$ *as follows.*

For $f \in F$ *define* $\bar{f}$ *with* $D(\bar{f}) = A$, *by:* $\bar{f}|_{D(f)} = f$ *and* $\bar{f}(a) = s(f, a)$ *for* $a \in A \backslash D(f)$.

For all $a \in A$ *and* $\langle m, n\rangle \in \rho$ *define* $T[a, m, n]$ *so that* $D(T[a, m, n]) = \{a\sigma_{mu} | u \in M\}$ *and* $T[a, m, n](a\sigma_{mu}) = a\sigma_{nu}$.

Set $\bar{F} = \{\bar{f} | f \in F\} \cup \{T[a, m, n] | a \in A, \langle m, n\rangle \in \rho\}$ *and* $\bar{\mathfrak{A}} = \langle\bar{A}; \bar{F}\rangle$.

Define $\bar{\sigma} : M \to \bar{A}^{\bar{A}}$ *by stipulating that for every* $m \in M$, $\bar{\sigma}_m|_A = \sigma_m$ *and* $s(f, a)\bar{\sigma}_m = \bar{f}(a\sigma_m)$ *for all* $s(f, a) \in S$.

Proof: The operations $T[a, m, n]$ are well-defined because $\langle\mathfrak{A}, \sigma\rangle$ satisfies (*). Also, it is easy to check that $\bar{\sigma}$ is a monomorphism of M into $E(\bar{\mathfrak{A}})$. In five steps we verify (*) for $\langle\bar{\mathfrak{A}}, \bar{\sigma}\rangle$. First we fix $\langle m, n\rangle \in \rho$, $\langle u, v\rangle \in M^2$, and $x \in \bar{A}$.

Step 1. If $g \in P_1(\mathfrak{A})$ then $g(x\bar{\sigma}_{mu}) = g(x\bar{\sigma}_{mv})$ implies $g(x\bar{\sigma}_{nu}) = g(x\bar{\sigma}_{nv})$ provided all four terms are defined.

If $x \in A$ there is nothing to prove, so suppose $x = s(f, a) \in S$. Then $\bar{f}(a\sigma_{mu}) = x\bar{\sigma}_{mu} \in D(g) \subseteq A$, whence $a\sigma_{mu} \in D(f)$ and likewise for $a\sigma_{nu}$.

Thus $g(x\bar{\sigma}_{mu}) = g(x\bar{\sigma}_{mv})$ implies $gf(a\sigma_{mu}) = gf(a\sigma_{mv})$ and the result follows because (*) holds in $\langle\mathfrak{A}, \sigma\rangle$.

Step 2. (*) holds for $1_{\bar{A}}$.

Again, there is nothing to prove if $x \in A$, so suppose $x = s(f, a) \in S$ and $x\bar{\sigma}_{mu} = x\bar{\sigma}_{mv}$, i.e., $\bar{f}(a\sigma_{mu}) = \bar{f}(a\sigma_{mv})$. Then either $a\sigma_{mu} \in D(f)$ and $f(a\sigma_{mu}) = f(a\sigma_{mv})$, or else $a\sigma_{mu} \in A \backslash D(f)$ and $a\sigma_{mu} = a\sigma_{mv}$. In either case the result follows because (*) holds in $\langle\mathfrak{A}, \sigma\rangle$.

Step 3. (*) holds for every $p \in P_1(\bar{\mathfrak{A}})$ having the form $p = \bar{f}_1\bar{f}_2 \cdots \bar{f}_k$.

First note that if $k > 1$, then for any $y \in D(p)$ we must have

$$\bar{f}_k(y) \in D(\bar{f}_{k-1}) = A,$$

and therefore $y \in D(f_k)$, i.e., $\bar{f}_k(y) = f_k(y)$. Continuing in this way we see that $p = \bar{f}_1 f_2 \cdots f_k$. Set $g = f_2 \cdots f_k$ if $k > 1$, and $g = 1_A$ if $k = 1$. Thus $p = \bar{f}_1 g$ and $g \in P_1(\mathfrak{A})$.

If $\bar{f}_1 g(x\bar{\sigma}_{mu}) = \bar{f}_1 g(x\bar{\sigma}_{mv})$, then either $g(x\bar{\sigma}_{mu}) \in D(f_1)$ and $f_1 g(x\bar{\sigma}_{mu}) = f_1 g(x\bar{\sigma}_{mv})$, or else $g(x\bar{\sigma}_{mu}) \in A \backslash D(f)$ and $g(x\bar{\sigma}_{mu}) = g(x\bar{\sigma}_{mv})$. In either case the result follows from Step 1.

Step. 4. Let $g \in P_1(\bar{\mathfrak{A}})$ and suppose (*) holds for g. Let $a \in A$ and $x_1, x_2 \in \bar{A}$ and $\langle r, s\rangle \in \rho$. Set $T = T[a, r, s]$. Then (whenever all four terms are defined) $gT(x_1) = gT(x_2)$ if and only if $g(x_1) = g(x_2)$.

Assuming $x_1, x_2 \in D(T)$, choose $c, d \in M$ such that $x_1 = a\sigma_{rc}$ and $x_2 = a\sigma_{rd}$. Then $gT(x_1) = gT(a\sigma_{rc}) = g(a\sigma_{sc})$ and similarly $gT(x_2) = g(a\sigma_{sd})$. Therefore, since (*) holds in $\langle \mathfrak{A}, \sigma \rangle$, we have $gT(x_1) = gT(x_2)$ if and only if $g(a\sigma_{rc}) = g(a\sigma_{rd})$, i.e., $g(x_1) = g(x_2)$.

Step 5. Let $p \in P_1(\bar{\mathfrak{A}})$ and suppose p is neither $1_{\bar{A}}$ nor of the form considered in Step 3. Then (*) holds for p.

To see this, note that p can be written as $p = p_0 T_0 p_1 T_1 \cdots p_{k-1} T_{k-1} p_k$ where each p_i is either $1_{\bar{A}}$ or of the form considered in Step 3, and each T_i is of the form $T[a, s, t]$. Set $p' = p_0 p_1 \cdots p_k$. Since (*) holds for p' by Steps 2 and 3, it suffices to show for all $y_1, y_2 \in \bar{A}$ that $p(y_1) = p(y_2)$ if and only if $p'(y_1) = p'(y_2)$. However, this follows from Step 4 by a straightforward induction on k.

5. Proof of sufficiency concluded

We now complete the proof of Theorem 2.1, that is, we show that $\ker \Psi \subseteq \rho$ implies Ψ is representable. By Lemma 3.2 it suffices to construct a finitely generated algebra $\mathfrak{A}$ and a monomorphism $\sigma : M \to E(\mathfrak{A})$ such that $\rho \subseteq \ker \sigma\varepsilon$. Actually this inclusion will be equality, since $\sigma\varepsilon$ is obviously representable and we have proved that (i) implies (ii) in Theorem 2.1. Moreover, the σ we construct will even be an isomorphism.

Let $\langle \mathfrak{A}_0, \sigma^0 \rangle$ be a ρ-system, as given in Lemma 4.1, and set $\mathfrak{A}_0 = \langle A_0; F_0 \rangle$. Using Lemma 4.2 we inductively define for all $k < \omega$ partial algebras $\mathfrak{A}_k = \langle A_k; F_k \rangle$ by setting $A_k = \overline{A_{k-1}}$ and $F_k = \overline{F_{k-1}}$ for $0 < k < \omega$. Set $A = \bigcup(A_k | k < \omega)$ and define operations on A as follows.

For each $f_0 \in F_0$ define $f_k = \overline{f_{k-1}}$ for all $0 < k < \omega$, and set $f = \bigcup(f_k | k < \omega)$.

For each $x \in A$ choose the smallest i for which $x \in A_i$, and for all $\langle m, n \rangle \in \rho$ let $R_0[x, m, n]$ denote the operation $T[x, m, n] \in F_{i+1}$. For $0 < k < \omega$ define $R_k[x, m, n] = \overline{R_{k-1}[x, m, n]}$, and set $R[x, m, n] = \bigcup(R_k[x, m, n] | k < \omega)$.

Set $F = \{f | f_0 \in F_0\} \cup \{R[x, m, n] | x \in A, \langle m, n \rangle \in \rho\}$ and $\mathfrak{A} = \langle A; F \rangle$.

Finally, we define $\sigma : M \to A^A$ as follows. For $0 < k < \omega$ set $\sigma^k = \overline{\sigma^{k-1}}$, and for each $m \in M$ define $\sigma_m = \bigcup(\sigma^k_m | k < \omega)$.

Clearly $\mathfrak{A}$ is an algebra (i.e., the operations in F are defined on all of A) and is generated by any generating set for $\mathfrak{A}_0$. Moreover, it is straightforward to show that σ is a monomorphism of M into $E(\mathfrak{A})$.

It remains to show that $\rho \subseteq \ker \sigma\varepsilon$, i.e., that $\langle m, n \rangle \in \rho$ implies $X\sigma^*_m = X\sigma^*_n$ for all $X \in S(\mathfrak{A})$. Since it suffices to consider only singleton-generated subalgebras, we show that $\langle m, n \rangle \in \rho$ implies $[x\sigma_m] = [x\sigma_n]$ for all $x \in A$. By symmetry it is enough to show that $x\sigma_n \in [x\sigma_m]$. But this is clear, since $x\sigma_n = R[x, m, n](x\sigma_m)$.

Although the proof of Theorem 2.1 is now complete, it is of incidental interest to observe that the monomorphism σ is even an isomorphism. To see this, recall from the construction of $\langle \mathfrak{A}_0, \sigma^0 \rangle$ in Lemma 4.1 that σ^0 is an isomorphism and that 1_{A_0} is an operation in F_0. Setting $f_0 = 1_{A_0}$, note that $A_0 = \{x \in A | f(x) = x\}$. It follows that every endomorphism α of $\mathfrak{A}$ takes A_0 into itself, whereupon $\alpha|_{A_0} \in E(\mathfrak{A}_0)$, whence $\alpha|_{A_0} = \sigma^0_m$ for some $m \in M$. Since A_0 generates $\mathfrak{A}$, the fact that α and σ_m agree on A_0 implies $\alpha = \sigma_m$. Hence the monomorphism σ maps M onto $E(\mathfrak{A})$.

6. Corrolaries and questions

Since the representability of an epimorphism depends only on its kernel, let us define a *representor* of monoid M to be a congruence θ that is the kernel of a representable epimorphism on M. Equivalently, a congruence θ is a representor of M if and only if the natural epimorphism $M \to M/\theta$ is representable, i.e., there exist an algebra $\mathfrak{A}$ and an isomorphism $\sigma : M \to E(\mathfrak{A})$ such that $\theta = \ker \sigma\varepsilon = \{\langle m, n \rangle \in M^2 | \sigma^*_m = \sigma^*_n\}$. In view of the homomorphism theorem for monoids, our main result states that *the representors of M are precisely the congruences of M contained in* ρ. It is immediate that the equality relation on M is a representor (i.e., every isomorphism defined on M is representable), and this fact generalizes a result of [4].

It is natural to classify representors in terms of properties of the factor monoids. Given a class K of monoids, call a representor θ of M a K-representor if $M/\theta \in K$. In the following corollaries and questions only group and semilattice representors will be considered.

The first corollary notes that the algebras we have constructed are finitely generated.

Corollary 6.1. *Every representable epimorphism is representable by means of a finitely generated algebra.*

The following corollary generalizes a result of [3].

Corollary 6.2: *Given a monoid M, the following are equivalent.*

(1) *M is left cancellative.*

(2) *The universal congruence $M \times M$ is a representor.*

(3) *Every congruence on M is a representor.*

(4) *There exists a group-representor of M.*

Proof: (1) implies (2) because $\rho = M \times M$ if M is left cancellative. From Theorem 2.1 it is immediate that (2) implies (3), and obviously (3) implies (4). To see that (4) implies (1), consider an algebra $\mathfrak{A}$ for which $E^*(\mathfrak{A})$ is a group. Then $\alpha \in E(\mathfrak{A})$ implies α^* is invertible, whence α^* maps

$S(\mathfrak{A})$ onto $S(\mathfrak{A})$. Thus there is some $X \in S(\mathfrak{A})$ with $A = X\alpha^*$. But then $A = X\alpha^* \subseteq A\alpha^* \subseteq A$ implies $A = A\alpha$, i.e., α is onto. Hence α is a left cancellative transformation of A, and so $E(\mathfrak{A})$ is a left cancellative monoid.

COROLLARY 6.3: *If the monoid M is a semilattice, then the only representor of M is the equality relation.*

PROOF: Suppose $\langle m, n\rangle \in \rho$ and let e denote the identity element of M. Then $eme = emm$ implies $ene = enm$, so $n = nm$. By symmetry $m = mn$. By commutativity $n = m$, whence ρ is the equality relation and the corollary is an immediate consequence of the theorem.

COROLLARY 6.4: *Let M be a periodic, commutative monoid that has a semilattice-representor. Then ρ is the unique semilattice-represenor of M.*

PROOF: Let θ be a semilattice-representor of M; then $\theta \subseteq \rho$ by the theorem. Thus M/ρ is a homomorphic image of M/θ, and so M/ρ is a semilattice.

To show that $\theta \supseteq \rho$ we show that $\rho = \eta$, where η is the smallest congruence on M whose factor monoid is a semilattice. Clearly $\eta \subseteq \rho$. To see that $\eta \supseteq \rho$ we use the following explicit description of η for commutative M (see [6] or [2]): $\eta = \{\langle \mathrm{m}, \mathrm{n}\rangle \in \mathrm{M}^2 |$ There exist positive integers i and j such that m divides n^i and n divides m^j.$\}$.

Now, let $\langle m, n\rangle \in \rho$. Since M is periodic, m has finite order, and so m^k is idempotent for some $k > 0$. Because ρ is a congruence, $\langle m^k, n^k\rangle \in \rho$ and therefore $em^km^k = em^ke$ implies $en^km^k = en^ke$ (where e is the identity), whence m divides n^k. By symmetry n divides a positive power of m, so $\langle m, n\rangle \in \eta$ and the corollary is proved.

QUESTION 6.5: If M is a finite monoid, is every representable epimorphism on M representable by means of a finite algebra?

QUESTION 6.6: Given a monoid M, set $M_1 = M/\rho$ and let ρ_1 denote the relation ρ as defined on M_1. Is it always true that ρ_1 is the equality relation on M_1? If not, then for which M is it true?

QUESTION 6.7: Is representability transitive? That is, if $\Psi : M \to N$ and $\phi : N \to P$ are representable epimorphisms, is $\Psi\phi$ representable?

Note that an affirmative answer to 6.7 implies an affirmative answer to 6.6. To see this, take $N = M_1$ and $P = M_1/\rho_1$, and let Ψ and ϕ be the respective natural epimorphisms. Then representability of $\Psi\phi$ implies $\ker \Psi\phi \subseteq \rho$, and it follows that ρ_1 is equality.

In a letter to the author dated June 1, 1974 Professor B. M. Schein states that 6.6 'is known to have an affirmative answer for inverse semigroups.'

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(Oblatum 13–III–1973)

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