COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 29, nº 2 (1974), p. 179-190
<http://www.numdam.org/item?id=CM_1974__29_2_179_0>

COMPOSITIO MATHEMATICA, Vol. 29, Fasc. 2, 1974, pag. 179–190
Noordhoff International Publishing
Printed in the Netherlands

A UNIFORMLY CONVEX BANACH SPACE WHICH CONTAINS NO l_p

T. Figiel and W. B. Johnson [1]

Abstract

There is a uniformly convex Banach space with unconditional basis which contains no subspace isomorphic to any l_p ($1 < p < \infty$). The space may be chosen either to have a symmetric basis, or so that it contains no subsymmetric basic sequence.

It is proved that a super-reflexive space with local unconditional structure can be equivalently normed so that its modulus of convexity is of power type.

1. Introduction

Recently Tsirelson [11] constructed an example of a reflexive Banach space with unconditional basis which contains no super-reflexive subspace and no subsymmetric basic sequence. This was the first example of a Banach space which contains no subspace isomorphic to any l_p ($1 \leqq p < \infty$) or c_0. In section 2 we construct the space, hereafter called T, conjugate to that of Tsirelson's and give another proof that T and T^* have the aforementioned properties. We present this alternate approach to Tsirelson's space for three reasons: (1) There is a simple analytical description of the norm of T which allows a more elementary argument than that used by Tsirelson and gives an analytical rather than geometrical description of the example. (2) The 'forcing method' employed by Tsirelson to construct his example is more clearly evident in the analytic construction of the conjugate space. (3) In section 4 we apply a convexity procedure to the norm of T to construct a uniformly convex space. The proof that this space contains no subsymmetric basic sequence makes use of some inequalities involving the norm on T.

At the end of section 2, we use a variation of the construction used in [1] to show that there is a reflexive space with symmetric basis which contains no super-reflexive subspace (and hence no isomorphic copy

[1] The second named author was supported in part by NSF GP-33578.

of any l_p). This symmetrization procedure is modified in section 4 to yield a uniformly convex space with symmetric basis which contains no l_p.

In section 3 we show that if Y has an unconditionally monotone basis which satisfies a lower l_q estimate ($1 < q < \infty$) and there is a $p > 1$ so that the norm on Y is p-convex (cf. Section 3 for definitions) then Y is uniformly convex and in fact the modulus of convexity for Y dominates $K_p \varepsilon^{2+q}$ for some positive constant K_p. This result and a renorming lemma yield the last result mentioned in the abstract.

Our notation is standard for Banach space theory. We recall only that a basis (e_n) is unconditionally monotone provided

$$\|\sum \alpha_n e_n\| \leqq \|\sum \beta_n e_n\| \quad \text{whenever} \quad |\alpha_n| \leqq |\beta_n|.$$

A basis (e_n) is: *subsymmetric* if it is equivalent to each of its subsequences; *symmetric* if it is equivalent to each of its permutations. $d(X, Y)$ denotes the Banach-Mazur distance coefficient $\inf\{\|L\| \cdot \|L^{-1}\|\}$, the inf over all invertible linear operators from X onto Y. For $1 \leqq p \leqq \infty$, X is said to contain l_p^n's uniformly for large n provided there exists a sequence (X_n) of linear manifolds in X with $\sup d(X_n, l_p^n) < \infty$. If also there are projections P_n from X onto X_n with $\sup \|P_n\| < \infty$, then X is said to contain uniformly complemented l_p^n's for large n.

Finally, 'subspace' means 'infinite dimensional closed linear subspace.'

We would like to thank Professor W. J. Davis for several useful discussions on the symmetrization procedure used in sections 2 and 4, and Professor H. P. Rosenthal for drawing our attention to Proposition 3.4 in [3].

2. The conjugate to Tsirelson's example

We work with unconditionally monotone norms on X, the space of scalar sequences which have only finitely many non-zero terms. Given $x \in X$ and E a set of integers, Ex is the sequence which agrees with x at coordinates in E and is zero in other coordinates. Thus $\|Ex\| \leqq \|x\|$ for any unconditionally monotone norm $\|\cdot\|$ on X.

If E, F are finite sets of integers, we write $E < F$ if $\max E < \min F$. Given finite sets $E_1, E_2, \cdots, E_k$ of integers, we say $(E_j)_{j=1}^k$ is *admissible* provided the E_j's are increasing (in the sense that $E_i < E_{i+1}$), and $\{k\} < E_1$.

We define a sequence of norms on X as follows:

$$\|x\|_0 = \|x\|_{c_0}$$

$$\|x\|_{n+1} = \max\left[\|x\|_n, \tfrac{1}{2}\max\left\{\sum_{i=1}^{k} \|E_i x\|_n : (E_i)_{i=1}^k \text{ is admissible}\right\}\right].$$

$\|\cdot\|_n$ is an increasing sequence of monotonely unconditional norms on X, so $\|x\| = \lim \|x\|_n$ is a monotonely unconditional norm on X. It can be shown that the completion T of $(X, \|\cdot\|)$ is the conjugate to the space constructed by Tsirelson. It is clear that $\|\cdot\|$ has the following property (in fact, $\|\cdot\|$ is the unique norm on X with this property):

$$(*) \qquad \|x\| = \max \left(\|x\|_{c_0}, \tfrac{1}{2} \max \left\{ \sum_{i=1}^{k} \|E_i x\| : (E_i)_{i=1}^{k} \text{ is admissible}\right\}\right).$$

Suppose that $(x_i)_{i=1}^{k} \subseteq X$ with $\operatorname{supp} x_i$ increasing, $\|x_i\| = 1$, and $\operatorname{supp} x_1 \subseteq [k+1, \infty)$. It follows from the definition of $\|\cdot\|$ that, for any scalars

$$(a_i)_{i=1}^{k}, \; \Big\| \sum_{i=1}^{k} a_i x_i \Big\| \geqq \tfrac{1}{2} \sum_{i=1}^{k} |a_i|.$$

Thus, $(x_i)_{i=1}^{k}$ is 2-equivalent to the unit vector basis of l_1^k. From this it follows that any normalized block basic sequence of the unit vector basis of $(X, \|\cdot\|)$ has, for each k, subsequences of length k which are 2-equivalent to the unit vector basis of l_1^k. Thus, no subspace of the completion T of $(X, \|\cdot\|)$ is super-reflexive, and T does not contain an isomorphic copy of $l_p \, (1 < p < \infty)$ or c_0. We have also that any subsymmetric basic sequence in T must be equivalent to the unit vector basis of l_1. Since T has an unconditional basis, we can show that T is reflexive and contains no subsymmetric basic sequence by proving that l_1 is not isomorphic to a subspace of T.

To this end we prove the following

LEMMA 2.1: *For every $\alpha > 1$ there is a $\beta < 2$ (in fact $\beta \leqq \frac{1}{2}(3+\alpha^{-1})$) so that if $x_0, x_1, \cdots, x_m$ are in X with*

$$\operatorname{supp} x_0 \subseteq [1, k] < \operatorname{supp} x_1 < \operatorname{supp} x_2 < \cdots$$

and $m \geqq \alpha k$, then

$$\Big\|x_0 + m^{-1} \sum_{i=1}^{m} x_i\Big\| \leqq \beta \max_{0 \leqq i \leqq m} \|x_i\|.$$

Once Lemma 2.1 has been proved, we complete the proof that T does not contain any subspace isomorphic to l_1 as follows: By a result of James' [7], if T contains an isomorphic copy of l_1, then there is a sequence (z_n) of unit vectors in T which is $\frac{9}{8}$-equivalent to the unit vector basis of l_1. By a standard gliding hump argument, we can assume that (z_n) is a block basic sequence of the unit vectors in X. If $\operatorname{supp} z_1 \subseteq [1, \cdots, k]$, we have from Lemma 2.1 that

$$\|z_1+(2k)^{-1}\sum_{i=2}^{2k+1} z_i\| \leqq \tfrac{7}{4}$$

and thus (z_n) cannot be $\frac{9}{8}$-equivalent to the unit vector basis of l_1.

We turn to the proof of Lemma 2.1. In view of (*) it is enough to show that, whenever $(E_j)_{j=1}^n$ is admissible,

$$\sum_{j=1}^{n} \|E_j(x_0+m^{-1}\sum_{i=1}^{m} x_i)\| \leqq 2\beta \max_{0\leqq i\leqq m} \|x_i\|. \tag{1}$$

(1) is clear with any $\beta \geqq 1$ when $n \geqq k$, because then $E_j x_0 = 0$ for each $1 \leqq j \leqq n$ by admissibility of (E_j). So assume that $n < k$ and let

$$A = \{i : \|E_j x_i\| \neq 0 \text{ for at least 2 values of } j\}$$

$$B = \{i : \|E_j x_i\| \neq 0 \text{ for at most 1 value of } j\}.$$

Of course, the cardinality of A is at most $n-1$. Using this, the triangle inequality, and (*), we have that

$$\begin{aligned}\sum_{j=1}^{n} \|E_j(x_0+m^{-1}\sum_{i=1}^{m} x_i)\| &\leqq \sum_{j=1}^{n}\|E_j x_0\| + m^{-1}(\sum_{i\in A}\sum_{j=1}^{n}\|E_j x_i\| + \sum_{i\in B}\sum_{j=1}^{n}\|E_j x_i\|)\\ &\leqq 2\|x_0\| + m^{-1}(2\sum_{i\in A}\|x_i\| + \sum_{i\in B}\|x_i\|)\\ &\leqq 2\|x_0\| + m^{-1}[2(n-1)+m-n+1]\max_{1\leqq i\leqq m}\|x_i\|\\ &\leqq 2\|x_0\| + m^{-1}(m-1+k)\max_{1\leqq i\leqq m}\|x_i\|\\ &\leqq (3+\alpha^{-1})\max_{0\leqq i\leqq m}\|x_i\| (\text{since } m^{-1}k \leqq \alpha).\end{aligned}$$

This completes the proof.

REMARK 2.1: A simple duality argument shows that every infinite dimensional subspace of T^* contains l_∞^n uniformly for large n, hence T^* also contains no subsymmetric basic sequence.

The rest of this section is devoted to the construction of a reflexive space Y with symmetric basis which contains no l_p. We use the factorization technique of [1] and follow the notation used there.

Let W be the unit ball of l_1 considered as a subset of c_0. For $n = 1, 2, \cdots$, let $U_n = 2^n W + 2^{-n}\,\mathrm{Ball}_{c_0}$ and let $\|\cdot\|_n$ be the gauge of U_n. Set

$$Y = \{y \in c_0 : \|(\|y\|_n)_{n=1}^{\infty}\| < \infty\},$$

where $\|\cdot\|$ is the norm in the space T. The remark after Lemma 1 in [1] yields that Y is reflexive and the unit vectors form an unconditional basis for Y. Now if π is a permutation of the integers and $x = (x(i))$ is in W (respectively, Ball_{c_0}), then $(x(\pi(i)))$ is also in W (respectively, Ball_{c_0}). From this it follows easily that the unit vectors form a symmetric basis for Y.

Let

$$Z = (\textstyle\sum (c_0, \|\cdot\|_n))_T = \{(x_n) : x_n \in c_0 \quad \text{and} \quad (\|x_n\|_n)_{n=1}^{\infty} \in T\}.$$

Note that the map $Y \to Z$ defined by $y \to (y, y, y, \cdots)$ is an isometry. So in order to show that Y contains no isomorph of any l_p it is sufficient to prove that each reflexive subspace of Z contains a subspace which embeds into T. Since c_0 has no reflexive subspaces, this is an immediate consequence of the following known lemma:

LEMMA 2.2: *Let (X_n) be Banach spaces, let E be a space with normalized unconditional basis, and suppose V is a subspace of $(\sum X_n)_E$. Assume that for each n, $P_{n|V}$ is strictly singular, where $P_m : (\sum X_n)_E \to X_m$ is the natural projection. Then V has a subspace which embeds into E.*

PROOF: The hypothesis yields that the natural projections $Q_m : (\sum X_n)_E \to X_1 + \cdots + X_m$ are strictly singular, hence for each m and $\varepsilon > 0$ there is $v_m \in V$ with $\|v_m\| = 1$ and $\|Q_m v_m\| < \varepsilon$. Thus by standard gliding hump and stability arguments, there is an increasing sequence (n_k) of positive integers and vectors $x_n \in X_n$ so that

$$z_k = \sum_{i=n_k}^{n_{k+1}-1} x_i$$

are unit vectors and $\overline{\mathrm{span}}\,(z_k)$ is isomorphic to a subspace of V. But let $y_k = y_k(i) \in E$ be defined by

$$y_k(i) = \begin{cases} \|x_i\|, & \text{if } n_k \leqq i < n_{k+1} \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that (z_k) is equivalent to (y_k). This completes the proof.

3. Renorming of super-reflexive spaces with unconditional bases

We digress from our study of Tsirelson-type examples and study general spaces with unconditional bases. First we need some notation. Throughout this section $(E, \|\cdot\|)$ will be a space with monotonely unconditional basis (e_n, e_n^*). The norm on E is *p-convex* (respectively, *p-concave*) provided that $\|\sum (|x_i|^p + |y_i|^p)^{1/p} e_i\|^p \leqq \|\sum x_i e_i\|^p + \|\sum y_i e_i\|^p$ (respectively, $\|\sum (|x_i|^p + |y_i|^p)^{1/p} e_i\|^p \geqq \|\sum x_i e_i\|^p + \|\sum y_i e_i\|^p$) for all scalars

(x_i) and (y_i). $\|\cdot\|$ satisfies an *upper l_p estimate* (respectively, *lower l_p estimate*) provided $\|x+y\|^p \leqq \|x\|^p + \|y\|^p$ (respectively, $\|x+y\|^p \geqq \|x\|^p + \|y\|^p$) whenever x and y are disjointly supported relative to (e_i). It is clear that if $\|\cdot\|$ is p-convex (respectively, p-concave), then it satisfies an upper (respectively, lower) l_p estimate.

The renorming lemma we prove is that if E is super-reflexive then E admits an equivalent unconditionally monotone norm which is p-convex and q-concave for some $1 < p, q < \infty$. This result (which is a very easy consequence of an important result of Rosenthal/Maurey [10], [9] and an observation of Dubinsky, Pełczyński, and Rosenthal [3]) justifies the main result of this section:

THEOREM 3.1: *Assume that the norm on E is p-convex and satisfies a lower l_q estimate ($1 < p \leqq q < \infty$). Then E is uniformly convex and the modulus of convexity for E satisfies $\delta_E(\varepsilon) \geqq K\varepsilon^{2+q}$ for some positive constant $K = K(p)$.*

In order to prove the renorming lemma, we need

LEMMA 3.1: *Suppose $(E, \|\cdot\|)$ does not contain l^n_∞ uniformly for large n. Then there is $q < \infty$ so that E admits an equivalent unconditionally monotone q-concave norm.*

PROOF: By Maurey's extension [9] of Rosenthal's theorem [10], there is $q < \infty$ so that every operator from c_0 into E is q-absolutely summing. The proof of Proposition 3.4 of [3] then yields that there is a constant K so that

$$(3.1) \qquad \Big(\sum_{i=1}^{m} \|x_i\|^q\Big)^{1/q} \leqq K \Big\| \sum_{j=1}^{\infty} \Big(\sum_{i=1}^{m} |e_j^*(x_i)|^q\Big)^{1/q} e_j \Big\|$$

for every sequence

$$(x_i)_{i=1}^m \text{in } E.$$

Define $|\cdot|$ on E by

$$|x| = \sup \Big\{\Big(\sum_{i=1}^{m} \|x_i\|^q\Big)^{1/q} : |e_j^*(x)|^q = \sum_{i=1}^{m} |e_j^*(x_i)|^q \text{ for each } j\Big\}.$$

It is clear that $|\cdot|$ is unconditionally monotone (since $\|\cdot\|$ is), symmetric, positive homogeneous, and, by (3.1), is equivalent to $\|\cdot\|$. The q-concavity of $|\cdot|$ is also obvious. There are various ways of seeing that $|\cdot|$ satisfies the triangle inequality. For example, let $\mathscr{A}$ be the set of all

$$a = (a_{ij})_{i=1}^{m}{}_{j=1}^{\infty}$$

for which $a_{ij} \geqq 0$ and $\sum_{i=1}^{m} a_{ij}^q = 1$ for each $j = 1, 2, \cdots$, and let

$$|x|_a = (\sum_{i=1}^{m} \| \sum_{j=1}^{\infty} a_{ij} e_j^*(x) e_j \|^q)^{1/q}.$$

Clearly $|x|_a$ is a semi-norm and $|x| = \sup \{|x|_a : a \in \mathscr{A}\}$. This completes the proof.

By duality (cf. the proof of Proposition 3.4 in [3]) we have

REMARK 3.1: Assume that E does not contain uniformly complemented l_1^n's for large n. Then by Lemma 3.1 there is an equivalent unconditionally monotone norm on E which is p-convex for some $p > 1$.

REMARK 3.2: *Suppose that E does not contain l_∞^n uniformly for large n and E does not contain uniformly complemented l_1^n's for large n. Then there is an equivalent unconditionally monotone norm on E which is p-convex and q-concave for some $1 < p \leqq q < \infty$.*

PROOF: By Remark 3.1 we may assume that $\|\cdot\|$ is p-convex. Let $q \geqq p$ so that (3.1) from Lemma 3.1 is satisfied for some constant K and renorm E as in Lemma 3.1. We need only verify that $|\cdot|$ is p-convex. Suppose x, y are in E and

$$a = (a_{ij})_{i=1}^{m}{}_{j=1}^{\infty}$$

is in $\mathscr{A}$ ($\mathscr{A}$ as in the proof of Lemma 3.1). We have

$$[\sum_{i=1}^{m} \| \sum_{j=1}^{\infty} a_{ij}(|e_j^*(x)|^p + |e_j^*(y)|^p)^{1/p} e_j \|^q]^{p/q}$$

$$\leqq [\sum_{i=1}^{m} (\| \sum_{j=1}^{\infty} a_{ij} e_j^*(x) e_j \|^p + \| \sum_{j=1}^{\infty} a_{ij} e_j^*(y) e_j \|^p)^{q/p}]^{p/q}$$

(by p-convexity of $\|\cdot\|$)

$$\leqq [\sum_{i=1}^{m} \| \sum_{j=1}^{\infty} a_{ij} e_j^*(x) e_j \|^q]^{p/q} + [\sum_{i=1}^{m} \| \sum_{j=1}^{\infty} a_{ij} e_j^*(y) e_j \|^q]^{p/q} \text{ (since } p < q).$$

This gives that $|\sum (|e_j^*(x)|^p + |e_j^*(y)|^p)^{1/p} e_j|_a^p \leqq |x|_a^p + |y|_a^p$ for each a in $\mathscr{A}$, and hence $|\cdot|$ is p-convex. This completes the proof.

In preparation for the proof of Theorem 3.1, we make a simple observation: given $p > 1$ there is a constant $M = M(p)$ so that for all reals a and b and $\mathcal{N} > 0$,

$$(3.2)\quad |a|^p + |b|^p = 2, \qquad |a+b| > 2(1-\mathcal{N}) \quad \text{imply} \quad |a-b| \leqq M\mathcal{N}^{\frac{1}{2}}.$$

Certainly it is enough to verify (3.2) for small $\mathcal{N}$ and for such $\mathcal{N}$ the hypothesis of (3.2) imply that $ab > 0$, so we may assume $a > 1 > b > 0$. Write $a = (1+\beta)^{1/p}$, $b = (1-\beta)^{1/p}$ for suitable $\beta \in (0, 1)$ and observe that there are positive constants c_1 and c_2 so that $(1+\beta)^{1/p} - (1-\beta)^{1/p} \leqq c_1\beta$ and $2 - [(1+\beta)^{1/p} + (1-\beta)^{1/p}] \geqq c_2\beta^2$.

Proof of Theorem 3.1: Let x, y be in E, $\|x\| \leqq 1$, $\|y\| \leqq 1$, and $\|x+y\| \geqq 2(1-\delta)$. Let

$$u = \sum_{j=1}^{\infty} \tfrac{1}{2}|e_j^*(x+y)|e_j, \qquad v = \sum_{j=1}^{\infty} [\tfrac{1}{2}(|e_j^*(x)|^p + |e_j^*(y)|^p)]^{1/p} e_j.$$

Clearly $0 \leqq u \leqq v$ and $1-\delta \leqq \|u\|$, while $\|v\| \leqq 1$ by p-convexity of the norm. Fix $\mathcal{N}$, $\delta < \mathcal{N} < 1$ and let $S = \{j : e_j^*(u) > (1-\mathcal{N})e_j^*(v)\}$. A simple calculation (cf. the argument for Lemma III.7 of [8]) yields that $\|Sv\| \geqq 1 - \delta\mathcal{N}^{-1}$, so that by the lower l_q estimate

$$1 \geqq \|v\|^q \geqq \|(\sim S)v\|^q + (1-\delta\mathcal{N}^{-1})^q.$$

Thus

$$\begin{aligned}\|(\sim S)(x-y)\| &\leqq 2\|(\sim S)v\| \\ &\leqq 2[1-(1-\delta\mathcal{N}^{-1})]^q \leqq 2(q\delta\mathcal{N}^{-1})^{1/q} \leqq 4(\delta\mathcal{N}^{-1})^{1/q}\end{aligned}$$

To estimate $\|S(x-y)\|$ we use (3.2) to get that for j in S,

$$|e_j^*(x) - e_j^*(y)| < M\mathcal{N}^{\frac{1}{2}}|e_j^*(v)|,$$

so that

$$\|S(x-y)\| \leqq M\mathcal{N}^{\frac{1}{2}}\|Sv\| \leqq M\mathcal{N}^{\frac{1}{2}}.$$

Thus

$$\|x-y\| \leqq \|S(x-y)\| + \|(\sim S)(x-y)\| \leqq M\mathcal{N}^{\frac{1}{2}} + 4(\delta\mathcal{N}^{-1})^{1/q}.$$

Setting $\mathcal{N} = \delta^{2/(q+2)}$ we get the desired estimate. This completes the proof.

We now indicate how the results of this section can be extended to spaces with local unconditional structure (l.u.st.). Recall that a Banach space Y has l.u.st. if $Y = \cup E_\alpha$ where the E_α's are finite dimensional, directed by inclusion, and E_α has a basis $(e_i^\alpha)_{i=1}^{n_\alpha}$ so that $\sup_\alpha\{$unconditional constant of $(e_i^\alpha)\} < \infty$. Assume that $(Y, \|\cdot\|)$ has l.u.st. given by (E_α) and Y is super-reflexive; in particular, there is a k so that for any α, $d(F, l_1^k) \geqq 2$ if F is any supspace of E_α or of E_α^*. By the Rosenthal-Maurey theorem, there are $p < \infty$ and a constant K (p and K depend on k but not on α) so that $\pi_p(T) \leqq K\|T\|$ for any operator T from c_0 into any E_α or E_α^*. ($\pi_p(T)$ is the p-absolutely summing norm of T.) Thus by the proof of Theorem 3.1, there are $1 < q < \infty$, positive constants c_1 and c_2, and norms $|\ |_\alpha$ on E_α for each α so that $\|y\| \leqq |y|_\alpha \leqq c_1\|y\|$ $(y \in E_\alpha)$, and $\delta_\alpha(\varepsilon) \geqq c_2\varepsilon^q$, $\delta_\alpha^*(\varepsilon) \geqq c_2\varepsilon^q$, where δ_α (respectively, δ_α^*) is the modulus of convexity of $(E_\alpha, |\ |_\alpha)$ (respectively, $(E_\alpha^*, |\ |_\alpha)$). By passing to a subnet of $(|\cdot|_\alpha)$ we may assume that $|y| = \lim_\alpha |y|_\alpha$ exists for each $y \in Y$. Clearly $|\cdot|$ is equivalent to the original norm on Y and the modulus of convexity of $(Y, |\cdot|)$ and $(Y^*, |\cdot|)$ both dominate $c_2\varepsilon^q$. We have thus proved:

THEOREM 3.3: *If Y is super-reflexive and has l.u.st. then there is an equivalent norm $|\cdot|$ on Y so that the moduli of convexity of $(Y, |\cdot|)$ and $(Y^*, |\cdot|)$ both dominate $c\varepsilon^p$ for some $c > 0$ and $p < \infty$.*

REMARK 3.3: If Y has l.u.st., Y does not contain l_∞^n for large n, and Y does not contain uniformly complemented l_1^n's for large n, then Y is super-reflexive by the results of [8]. We do not see how to derive this from the techniques of this section. However, if Y has an unconditional basis, this result follows from the results of this section. Also, if Y has l.u.st. and does not contain l_1^n uniformly for large n, then the proof of Theorem 3.3 shows that Y is super-reflexive.

REMARK 3.4: Enflo [4] proved that every super-reflexive space Y admits an equivalent uniformly convex norm. Theorem 3.3 shows that if Y has l.u.st., the equivalent norm can be chosen so that its modulus of convexity dominates a power function. We do not know whether this is true if Y does not have l.u.st.

4. Uniformly convex examples of Tsirelson type

Let $\|\cdot\|$ be the Tsirelson norm on X constructed in Section 2. We recall that $\|\cdot\|$ satisfies

$$(*)\qquad \|x\| = \max\left[\|x\|_{c_0}, \sup\left\{\tfrac{1}{2}\sum_{i=1}^{n}\|E_i x\| : (E_i)_{i=1}^n \text{ is admissible}\right\}\right].$$

Let (e_i) denote the unit vectors in X. Fix $1 < p$, $r < \infty$ and define, for $x = (x_i)$ in X,

$$\|x\|_1 = \|\sum |e_i^*(x)|^p e_i\|^{1/p}.$$

Let $\|\cdot\|_2$ be the dual norm to $\|\cdot\|_1$ (considered still as a norm on X)

and let $|||x||| = \|\sum |e_i^*(x)|^r e_i\|_2^{1/r}$ for x in X. Observe that $\| \ \|_1$ is p-convex, hence $\| \ \|_2$ is q-concave for

$$\frac{1}{p} + \frac{1}{q} = 1,$$

(cf., e.g. the proof of Proposition 3.4 of [3]). A trivial computation yields that $|||\cdot|||$ is rq concave and, of course, $|||\cdot|||$ is r-convex. Thus by Theorem 3.1, the completion Z of $(X, |||\cdot|||)$ is uniformly convex.

In order to see that Z contains no copy of any l_p, we trace what happens to (*) and the inequality of Lemma 1 in this convexification of $\|\cdot\|$.

Suppose $(E_i)_{i=1}^n$ is admissible. Then for every x in X we have from (*) and the definition of $\|\cdot\|_1$ that

$$\Big(\sum_{i=1}^n \|E_i x\|_1^p\Big)^{1/p} \leqq 2^{1/p}\|x\|_1$$

Hence by duality we have for every x in X

$$\|x\|_2 \leqq 2^{1/p}\Big(\sum_{i=1}^n \|E_i x\|_2^q\Big)^{1/q}$$

whence for every x in X

$$|||x||| \leqq 2^{1/pr}\Big(\sum_{i=1}^n |||E_i x|||^{rq}\Big)^{1/rq}. \tag{4.1}$$

From (4.1) and the rq concavity of $|||\cdot|||$ we have that any subsymmetric basic sequence in Z is equivalent to the unit vector basis of l_{rq}.

To prove that l_{rq} does not embed into A, we need to analyze the inequality in Lemma 2.1 in the context of $|||\cdot|||$. Assuming d, k, m, and β are as in Lemma 2.1, we have for $(x_i)_{i=0}^m$ as in Lemma 2.1 that

$$\|x_0 + m^{-1/p}\sum_{i=1}^m x_i\|_1 \leqq \beta^{1/p} \max_{0 \leqq i \leqq m} \|x_i\|_1.$$

A simple duality argument then yields for $(x_i)_{i=0}^m$ as in Lemma 2.1 that

$$\|x_0\|_2 + m^{-1/p}\sum_{i=1}^m \|x_i\|_2 \leqq \beta^{1/p}\|\sum_{i=0}^m x_i\|_2.$$

Interpreting this in terms of $|||\cdot|||$ we have for $(x_i)_{i=0}^m$ as in Lemma 2.1 that

$$|||x_0|||^r + m^{-1/p}\sum_{i=1}^m |||x_i|||^r \leqq \beta^{1/p}|||\sum_{i=0}^m x_i|||^r. \tag{4.2}$$

From an argument of James' [7] (cf. Lemma 2.2 of [5] for an explicit statement and proof) we have that if Z contains a subspace isomorphic to l_{rq}, then for every $\varepsilon > 0$ there is a sequence (z_n) of vectors in X with with $|||z_n||| = 1$ and

$$(4.3) \qquad (1+\varepsilon)(\sum |\alpha_n|^{rq}) > |||\sum \alpha_n z_n|||^{rq}.$$

Clearly (z_n) my be taken with $\operatorname{supp} z_n < \operatorname{supp} z_{n+1}$. Choose k with $\operatorname{supp} z_0 \subset [1, k]$, set $\alpha = 2$, choose $m \geqq \alpha k$, and let $\beta = \frac{1}{2}(3+2^{-1})$ as in Lemma 2.1. Set $x_0 = z_0$, $x_i = m^{-1/rq} z_i$. Substitution into (4.2) and (4.3) leads to a contradiction when $(1+\varepsilon)^{p-1} < 2\beta^{-1}$.

Our next goal is the construction of a uniformly convex space with symmetric basis which contains no l_p. To accomplish this we modify the the construction at the end of Section 2.

Fix p, q, r in $(1, \infty)$ with $p < q$ and let $X_n = (l_q, \|\cdot\|_n)$ where

$$\|x\|_n = \inf\{(\|y\|_{l_p}^r + \|z\|_{l_q}^r)^{1/r} : 2^n y + 2^{-n} z = x\}.$$

We will prove that if E has a normalized unconditionally monotone basis, then the diagonal subspace $V = \{(x_n) \in (\sum X_n)_E : x_n = x_1 \text{ for all } n\}$ satisfies the hypothesis of Lemma 2. Letting E be the space Z constructed above, we will thus have that V contains no copy of any l_p. The argument at the end of Section 2 shows that V has a symmetric basis. Also, V is uniformly convex. To see this, observe that the unit ball of X_n is the image of the unit ball of $F = (l_p \oplus l_q)_r$ under the linear operator $Q_n : F \to l_q$ defined by $Q_n(y, z) = 2^n y + 2^{-n} z$, hence the modulus of convexity of X_n (and incidentally also of X_n^*) is not worse than that of F. Thus by [2] or [6], $(\sum X_n)_Z$ is uniformly convex. (Alternatively, one can check directly that Theorem 3.1 can be applied to $(\sum X_n)_{Z^*}$.) Let j be the projection of V onto the first coordinate; $jv = x$ for $v = (x, x, x, \cdots)$ in V. Since all the norms $\| \ \|_n$ are equivalent to $\| \ \|_{l_q}$, it is sufficient to show that j is strictly singular when considered as an operator into l_q. Suppose to the contrary that there is an infinite dimensional subspace U of V and a constant c so that $c\|jv\|_{l_q} \geqq \|v\|_V$ for all v in V. It is convenient for us to identify the element $v = (x, x, \cdots)$ of V with $jv = x$ in l_q and write $|||x|||$ for $\|v\|_V$ and $\|x\|$ for $\|x\|_{l_q}$. We thus have for $x \in jV$ that $d\|x\| \leqq |||x|||$, where $d = \|j\|^{-1}$, and for $x \in jU$ that $|||x||| \leqq c\|x\|$.

Now fix positive integers n and N. A standard gliding hump argument yields a sequence (x_i) in jU and mutually disjoint finite sets (E_i) of integers so that

$$|||x_i||| = 2c \qquad |||x_i - E_i x_i||| \leqq (2N)^{-1} d.$$

We have

$$2cd^{-1} \geqq \|x_i\| \geqq \|E_i x_i\| \geqq \|x_i\| - \|x_i - E_i x_i\| \geqq 2 - d^{-1}|||x_i - E_i x_i||| \geqq 1.$$

Consider a decomposition

$$\sum_{i=1}^{2N} E_i x_i = 2^n w + 2^{-n} z \qquad (w \in l_p, z \in l_q).$$

Since $E_i x_i = 2^n E_i w + 2^{-n} E_i z$, we have $1 \leqq \|E_i x_i\| \leqq 2^n \|E_i w\| + 2^{-n} \|E_i z\|$, and thus one of the events $\|E_i w\| \geqq 2^{-n-1}$ or $\|E_i z\| \geqq 2^{n-1}$ must occur at least N times, so by disjointness of the E_i's we have that

$$(\|w\|_{l_p}^r + \|z\|_{l_q}^r)^{1/r} \geqq \min(N^{1/p} 2^{-n-1}, N^{1/q} 2^{n-1}).$$

From the definition of $\|\cdot\|_n$ and the fact that $p < q$ it thus follows that if N is large relative to n, then

$$\|\sum_{i=1}^{2N} E_i x_i\|_n \geqq 2^{n-1} N^{1/q} \tag{4.3}$$

On the other hand, for all N

$$\begin{aligned}\|\sum_{i=1}^{2N} E_i x_i\|_n &\leqq |||\sum_{i=1}^{2N} E_i x_i||| \leqq |||\sum_{i=1}^{2N} x_i||| + \sum_{i=1}^{2N} |||x_i - E_i x_i||| \\ &\leqq c\|\sum_{i=1}^{2N} x_i\| + d \leqq c\|\sum_{i=1}^{2N} E_i x_i\| + c\sum_{i=1}^{2N} \|x_i - E_i x_i\| + d \\ &\leqq c(2N)^{1/q}(2cd^{-1}) + c + d.\end{aligned}$$

Of course, this estimate on $\|\sum_{i=1}^{2N} E_i x_i\|_n$ is incompatible with (4.3) if n is large enough relative to c and d^{-1}. This completes the proof that j is strictly singular.

REFERENCES

[1] W. J. Davis, T. Figiel, W. B. Johnson, and A. Pelczynski: Factoring weakly compact operators. *J. Functional Anal. 17* (1974).

[2] M. M. Day: Some more uniformly convex spaces. *Bull. Amer. Math. Soc. 47* (1941) 504–507.

[3] E. Dubinsky, A. Pelczynski, and H. P. Rosenthal: On Banach spaces X for which $\pi_2(\mathcal{L}_\infty, X) = B(\mathcal{L}_\infty, X)$. *Studia Math. 44* (1972) 617–648.

[4] P. Enflo: Banach spaces which can be given an equivalent uniformly convex norm. *Israel J. Math. 13* (1972) 281–288.

[5] P. Enflo and H. P. Rosenthal: Some results concerning $L^p(\mu)$ spaces. *J. Functional Anal. 14* (1973) 325–348.

[6] T. Figiel: An example of an infinite dimensional Banach space non-isomorphic to its Cartesian square. *Studia Math. 42* (1972) 295–306.

[7] R. C. James: Uniformly non-square Banach spaces. *Ann. of Math. 80* (1964) 542–550.

[8] W. B. Johnson: On finite dimensional subspaces of Banach spaces with local unconditional structure. *Studia Math. 51* (1974).

[9] B. Maurey: Théorémes de factorisation pour les opérateurs linéaires á valeurs dans les espaces L^p. *Société Mathématique de France* (1974).

[10] H. P. Rosenthal: On subspaces of L^p. *Ann. of Math. 97* (1973) 344–373.

[11] B. S. Tsirelson: Not every Banach space contains l_p or c_0.

(Oblatum 6–V–1974)

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